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Hermite-Hadamard type inequalities for multiplicatively p-convex functions

Serap Özcan[1*]

*Correspondence:
serapozcan87@gmail.com;
serapozcan@klu.edu.tr
[1]Department of Mathematics, Faculty of Sciences and Arts, Kırklareli University, Kırklareli, Turkey

Abstract

In this paper, we introduced the concept of multiplicatively p-convex functions and established Hermite-Hadamard type integral inequalities in the setting of multiplicative calculus for this newly created class of functions. We also gave integral inequalities of Hermite-Hadamard type for product and quotient of multiplicatively p-convex functions. Furthermore, we obtained novel multiplicative integral-based inequalities for the product and quotient of convex and multiplicatively p-convex functions. Additionally, we derived certain upper limits for this new class of functions. The findings we proved are generalizations of the results in the literature. The results obtained in this study may inspire further research in various scientific areas.

Mathematics Subject Classification: 26D15; 26A51

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1 Introduction and preliminaries

The classical or usual convexity is defined as follows:

The function $\varphi : [\theta_1, \theta_2] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex in the classical sense if

$$\varphi\big(t\varkappa + (1-t)y\big) \le t\varphi(\varkappa) + (1-t)\varphi(y)$$

for all $\varkappa, y \in [\theta_1, \theta_2]$ and $t \in [0,1]$. The function φ is said to be concave if $-\varphi$ is convex.

Many studies have indicated that several inequalities can be derived from convex functions. One of the most renowned inequalities that pertain to the integral mean of a convex function is the Hermite-Hadamard inequality. This double inequality is stated as follows (see, [1–3]).

Theorem 1.1 *Let $\varphi : I = [\theta_1, \theta_2] \subset \mathbb{R} \to \mathbb{R}$ be an integrable convex function. Then*

$$\varphi\left(\frac{\theta_1 + \theta_2}{2}\right) \le \frac{1}{\theta_2 - \theta_1}\int_{\theta_1}^{\theta_2} \varphi(\varkappa)\, d\varkappa \le \frac{\varphi(\theta_1) + \varphi(\theta_2)}{2}.$$

Both inequalities hold in the reversed direction if φ is concave.

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The Hermite-Hadamard inequality can be seen as a refinement of the concept of convexity. Numerous researchers have been extensively investigated this concept since it was independently discovered by Hermite in 1883 and Hadamard in 1896. In particular, over the past two decades, many researchers have devoted their efforts to finding new bounds for both the left- and right-hand sides of the Hermite-Hadamard inequality. Several studies have proposed novel approaches to enhance, improve, and extend this inequality.

2 Preliminaries

Now we present basic definitions and results, where I and $\Im$ are intervals.

In [4], Iscan defined the concept of harmonically convexity and proved the Hermite–Hadamard inequality for harmonically convex functions as following:

Definition 2.1 Let $I \subseteq \mathbb{R} \setminus \{0\}$ be an interval. A function $\varphi : I \to \mathbb{R}$ is said to be harmonically convex function if

$$\varphi\left(\frac{\varkappa y}{t\varkappa + (1-t)y}\right) \le (1-t)\varphi(\varkappa) + t\varphi(y)$$

for all $\varkappa, y \in I$, $t \in [0,1]$.

Theorem 2.1 *Let $\varphi : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a harmonically convex function and $\theta_1, \theta_2 \in I$ with $\theta_1 < \theta_2$. If $\varphi \in L[\theta_1, \theta_2]$, then following inequalities hold*:

$$\varphi\left(\frac{2\theta_1\theta_2}{\theta_1+\theta_2}\right) \le \frac{\theta_1\theta_2}{\theta_2-\theta_1}\int_{\theta_1}^{\theta_2}\frac{\varphi(\varkappa)}{\varkappa^2}\,d\varkappa \le \frac{\varphi(\theta_1)+\varphi(\theta_2)}{2}. \tag{1}$$

Some recent results for Hermite–Hadamard inequalities for harmonically type convex function can be seen in [5–8].

Definition 2.2 ([9]) Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R} \setminus \{0\}$. A function $\varphi : I \to \mathbb{R}$ is said to be a p-convex function if

$$\varphi\left(\left[t\varkappa^p + (1-t)y^p\right]^{1/p}\right) \le t\varphi(\varkappa) + (1-t)\varphi(y)$$

for all $\varkappa, y \in I$ and $t \in [0,1]$.

According to Definition (2.2), it is obvious that for $p = 1$ and $p = -1$, p-convexity reduces to classical convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.

In Theorem 5 of [10], if we take $I \subset (0,\infty)$, $p \in \mathbb{R} \setminus \{0\}$ and $h(t) = t$, then we have the following theorem:

Theorem 2.2 *Let $\varphi : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R} \setminus \{0\}$ and $\theta_1, \theta_2 \in I$ with $\theta_1 < \theta_2$. If $\varphi \in L[\theta_1, \theta_2]$, then*

$$\varphi\left(\left[\frac{\theta_1^p+\theta_2^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)d\varkappa \le \varphi\left(\frac{\theta_1+\theta_2}{2}\right).$$

Definition 2.3 ([3]) A function $\varphi : \Im \to (0,\infty)$ is said to be log or multiplicatively convex, if

$$\varphi\big(t\varkappa + (1-t)y\big) \le \big[\varphi(\varkappa)\big]^t\big[\varphi(y)\big]^{(1-t)}$$

for all $\varkappa, y \in \Im$ and $t \in [0,1]$.

2.1 Multiplicative calculus

Recall that the multiplicative integral, which is a type of integral that involves the product of terms raised to certain powers, is represented by $\int_{\theta_1}^{\theta_2}(\varphi(\varkappa))^{d\varkappa}$, while the ordinary integral, which is a type of integral that involves the sum of terms, is typically denoted by $\int_{\theta_1}^{\theta_2}(\varphi(\varkappa))\,d\varkappa$. Using distinct symbols helps to differentiate between these two types of integrals.

There is the following relation between the Riemann integral and the multiplicative integral [11].

Proposition 2.1 *If φ is Riemann integrable on $[\theta_1,\theta_2]$, then φ is multiplicative integrable on $[\theta_1,\theta_2]$, and*

$$\int_{\theta_1}^{\theta_2}\big(\varphi(\varkappa)\big)^{d\varkappa} = e^{\int_{\theta_1}^{\theta_2}\ln(\varphi(\varkappa))\,d\varkappa}.$$

In [11], Bashirov et al. show that multiplicative integral has the following results and notations:

Proposition 2.2 *If φ is positive and Riemann integrable on $[\theta_1,\theta_2]$, then φ is multiplicative integrable on $[\theta_1,\theta_2]$ and*

1 $\int_{\theta_1}^{\theta_2}((\varphi(\varkappa))^p)^{d\varkappa} = \int_{\theta_1}^{\theta_2}((\varphi(\varkappa))^{d\varkappa})^p$,
2 $\int_{\theta_1}^{\theta_2}(\varphi(\varkappa)\psi(\varkappa))^{d\varkappa} = \int_{\theta_1}^{\theta_2}(\varphi(\varkappa))^{d\varkappa}.\int_{\theta_1}^{\theta_2}(\psi(\varkappa))^{d\varkappa}$,
3 $\int_{\theta_1}^{\theta_2}(\frac{\varphi(\varkappa)}{\psi(\varkappa)})^{d\varkappa} = \frac{\int_{\theta_1}^{\theta_2}(\varphi(\varkappa))^{d\varkappa}}{\int_{\theta_1}^{\theta_2}(\psi(\varkappa))^{d\varkappa}}$,
4 $\int_{\theta_1}^{\theta_2}(\varphi(\varkappa))^{d\varkappa} = \int_{\theta_1}^{\mu}(\varphi(\varkappa))^{d\varkappa}.\int_{\mu}^{\theta_2}(\varphi(\varkappa))^{d\varkappa}, \theta_1 \le \mu \le \theta_2$.
5 $\int_{\theta_1}^{\theta_1}(\varphi(\varkappa))^{d\varkappa} = 1$ and $\int_{\theta_1}^{\theta_2}(\varphi(\varkappa))^{d\varkappa} = (\int_{\theta_2}^{\theta_1}(\varphi(\varkappa))^{d\varkappa})^{-1}$.

In [12], Ali et al. established the Hermite–Hadamard inequality for multiplicatively convex functions in the setting of multiplicative calculus as follows:

Theorem 2.3 *Let φ be a positive and multiplicatively convex function on $[\theta_1,\theta_2]$. Then*

$$\varphi\left(\frac{\theta_1+\theta_2}{2}\right) \le \left(\int_{\theta_1}^{\theta_2}\big(\varphi(\varkappa)\big)^{d\varkappa}\right)^{\frac{1}{\theta_2-\theta_1}} \le G\big(\varphi(\theta_1),\varphi(\theta_2)\big),$$

where $G(\cdot,\cdot)$ is the geometric mean.

One of the first studies of multiplicative calculus was made by [13] in the 1970s. Since then, a number of interesting results has been obtained due to its many applications in various fields. For example, in [14], Bashirov and Riza introduced complex multiplicative calculus. In [15] and [16], some properties of stochastic multiplicative calculus have been studied. For some applications and other aspects of this discipline, (see [17–22]) and the references cited therein.

3 Main results

In this section, we give a new definition, called multiplicatively p-convex function, and obtain some Hermite-Hadamard type integral inequalities in the setting of multiplicative calculus for multiplicatively p-convex and convex functions.

Definition 3.1 A nonnegative function $\varphi : I \to \mathbb{R}$ is said to be multiplicatively p-convex if

$$\varphi\big(\big[t\varkappa^p + (1-t)y^p\big]^{1/p}\big) \le \big[\varphi(\varkappa)\big]^t\big[\varphi(y)\big]^{(1-t)}$$

holds for all $\varkappa, y \in I$, $t \in [0,1]$ and $p \in \mathbb{R} \setminus \{0\}$.

Theorem 3.1 *Let φ be a multiplicatively p-convex function on* $[\theta_1,\theta_2]$. *Then*

$$\varphi\left(\left[\frac{\theta_1^p+\theta_2^p}{2}\right]^{\frac{1}{p}}\right) \le \left(\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \le G\big(\varphi(\theta_1),\varphi(\theta_2)\big).$$

Proof Note that

$$\begin{aligned}
&\ln\varphi\left(\left[\frac{\theta_1^p+\theta_2^p}{2}\right]^{\frac{1}{p}}\right)\\
&\quad= \ln\varphi\left(\left[\frac{t\theta_1^p+(1-t)\theta_2^p+(1-t)\theta_1^p+t\theta_2^p}{2}\right]^{\frac{1}{p}}\right)\\
&\quad= \ln\varphi\left(\left[\frac{t\theta_1^p+(1-t)\theta_2^p}{2}+\frac{(1-t)\theta_1^p+t\theta_2^p}{2}\right]^{\frac{1}{p}}\right)\\
&\quad\le \ln\Big[\big(\varphi\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)^{\frac{1}{2}}\big(\varphi\big((1-t)\theta_1^p+t\theta_2^p\big)^{\frac{1}{p}}\big)^{\frac{1}{2}}\Big]\\
&\quad= \frac{1}{2}\ln\varphi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)+\frac{1}{2}\ln\varphi\big(\big((1-t)\theta_1^p+t\theta_2^p\big)^{\frac{1}{p}}\big).
\end{aligned}$$

Integrating the above inequality with respect to t on $[0,1]$, we have

$$\begin{aligned}
&\ln\varphi\left(\left[\frac{\theta_1^p+\theta_2^p}{2}\right]^{\frac{1}{p}}\right)\\
&\quad\le \frac{1}{2}\int_0^1 \ln\varphi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\,dt+\frac{1}{2}\int_0^1 \ln\varphi\big(\big((1-t)\theta_1^p+t\theta_2^p\big)^{\frac{1}{p}}\big)\,dt\\
&\quad= \frac{p}{\theta_1^p-\theta_2^p}\int_{\theta_1}^{\theta_2}\ln\left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)d\varkappa,
\end{aligned}$$

which implies that

$$\ln\varphi\left(\left[\frac{\theta_1^p+\theta_2^p}{2}\right]^{\frac{1}{p}}\right) \le \frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\ln\left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)d\varkappa.$$

Thus, we have

$$\varphi\left(\left[\frac{\theta_1^p+\theta_2^p}{2}\right]^{\frac{1}{p}}\right) \le e^{\frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\ln(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa} = \left(\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}},$$

which gives the first inequality. Now, consider the second inequality

$$\begin{aligned}
&\left(\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}}\\
&= e^{(\int_{\theta_1}^{\theta_2}\ln(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa)^{\frac{p}{\theta_2^p-\theta_1^p}}}\\
&= e^{\frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\ln(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa}\\
&= e^{\int_0^1\ln\varphi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}})\,dt}\\
&\le e^{\int_0^1\ln((\varphi(\theta_1))^t(\varphi(\theta_2))^{(1-t)})\,dt}\\
&= e^{\int_0^1(t\ln\varphi(\theta_1)+(1-t)\ln\varphi(\theta_2))\,dt}\\
&= e^{(\ln(\varphi(\theta_1)\varphi(\theta_2))\int_0^1 t\,dt)}\\
&= G\big(\varphi(\theta_1),\varphi(\theta_2)\big).
\end{aligned}$$

So, the proof is completed. □

Remark 3.1 Choosing $p=1$ in Theorem 3.1, we get Theorem 5 in [12].

Corollary 3.1 *Let φ and ψ be multiplicatively p-convex functions on* $[\theta_1,\theta_2]$. *Then*

$$\begin{aligned}
&\varphi\left(\left[\frac{\theta_1^p+\theta_2^p}{2}\right]^{\frac{1}{p}}\right)\psi\left(\left[\frac{\theta_1^p+\theta_2^p}{2}\right]^{\frac{1}{p}}\right)\\
&\le \left(\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\int_{\theta_1}^{\theta_2}\left(\frac{\psi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}}\\
&\le G\big(\varphi(\theta_1),\varphi(\theta_2)\big).G\big(\psi(\theta_1),\psi(\theta_2)\big).
\end{aligned}$$

Proof Since φ and ψ are multiplicatively p-convex functions, then $\varphi\psi$ is a multiplicatively p-convex function. Thus, if we apply Theorem 3.1 to the function $\varphi\psi$, we obtain the desired result. □

Remark 3.2 Choosing $p=1$ in Corollary 3.1, we get Theorem 7 in [12].

Corollary 3.2 *Let φ and ψ be multiplicatively p-convex functions on* $[\theta_1,\theta_2]$. *Then*

$$\frac{\varphi([\frac{\theta_1^p+\theta_2^p}{2}]^{\frac{1}{p}})}{\psi([\frac{\theta_1^p+\theta_2^p}{2}]^{\frac{1}{p}})} \le \left(\frac{\int_{\theta_1}^{\theta_2}(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})^{d\varkappa}}{\int_{\theta_1}^{\theta_2}(\frac{\psi(\varkappa)}{\varkappa^{1-p}})^{d\varkappa}}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \le \frac{G(\varphi(\theta_1),\varphi(\theta_2))}{G(\psi(\theta_1),\psi(\theta_2))}.$$

Proof Since φ and ψ are multiplicatively p-convex functions, then $\frac{\varphi}{\psi}$ is a multiplicatively p-convex function. Thus, if we apply Theorem 3.1 to the function $\frac{\varphi}{\psi}$, we obtain the required result. □

Remark 3.3 Choosing $p = 1$ in Corollary 3.2, we get Theorem 9 in [12].

Theorem 3.2 *Let φ be a convex function and ψ be a multiplicatively p-convex function. Then*

$$\left(\frac{\int_{\theta_1}^{\theta_2}(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})^{d\varkappa}}{\int_{\theta_1}^{\theta_2}(\frac{\psi(\varkappa)}{\varkappa^{1-p}})^{d\varkappa}}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \leq \frac{(\frac{(\varphi(\theta_2))^{\varphi(\theta_2)}}{(\varphi(\theta_1))^{\varphi(\theta_1)}})^{\frac{1}{\varphi(\theta_2)-\varphi(\theta_1)}}}{G(\psi(\theta_1),\psi(\theta_2)).e}.$$

Proof Note that

$$\begin{aligned}
&\left(\frac{\int_{\theta_1}^{\theta_2}(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})^{d\varkappa}}{\int_{\theta_1}^{\theta_2}(\frac{\psi(\varkappa)}{\varkappa^{1-p}})^{d\varkappa}}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \\
&\quad= \left(\frac{e^{\int_{\theta_1}^{\theta_2}\ln(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa}}{e^{\int_{\theta_1}^{\theta_2}\ln(\frac{\psi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa}}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \\
&\quad= e^{(\int_{\theta_1}^{\theta_2}\ln(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa-\int_{\theta_1}^{\theta_2}\ln(\frac{\psi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa)^{\frac{p}{\theta_2^p-\theta_1^p}}} \\
&\quad= e^{(\int_0^1\ln\varphi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}})\,dt-\int_0^1\ln\psi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}})\,dt)} \\
&\quad\leq e^{(\int_0^1\ln(\varphi(\theta_2)+t(\varphi(\theta_1)-\varphi(\theta_2)))\,dt-\int_0^1\ln((\psi(\theta_1))^t(\psi(\theta_2))^{(1-t)})\,dt)} \\
&\quad= e^{\ln((\frac{(\varphi(\theta_2))^{\varphi(\theta_2)}}{(\varphi(\theta_1))^{\varphi(\theta_1)}})^{\frac{1}{\varphi(\theta_2)-\varphi(\theta_1)}})-1-\ln(\psi(\theta_1)\psi(\theta_2))^{\int_0^1t\,dt}} \\
&\quad= \frac{(\frac{(\varphi(\theta_2))^{\varphi(\theta_2)}}{(\varphi(\theta_1))^{\varphi(\theta_1)}})^{\frac{1}{\varphi(\theta_2)-\varphi(\theta_1)}}}{G(\psi(\theta_1),\psi(\theta_2)).e}.
\end{aligned}$$

So, the proof is completed. □

Remark 3.4 Choosing $p = 1$ in Theorem 3.2, we get Theorem 11 in [12].

Theorem 3.3 *Let φ be a multiplicatively p-convex function and ψ be a convex function. Then*

$$\left(\frac{\int_{\theta_1}^{\theta_2}(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})^{d\varkappa}}{\int_{\theta_1}^{\theta_2}(\frac{\psi(\varkappa)}{\varkappa^{1-p}})^{d\varkappa}}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \leq \frac{G(\varphi(\theta_1),\varphi(\theta_2)).e}{(\frac{(\psi(\theta_2))^{\psi(\theta_2)}}{(\psi(\theta_1))^{\psi(\theta_1)}})^{\frac{1}{\psi(\theta_2)-\psi(\theta_1)}}}.$$

Proof Note that

$$\begin{aligned}
&\left(\frac{\int_{\theta_1}^{\theta_2}(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})^{d\varkappa}}{\int_{\theta_1}^{\theta_2}(\frac{\psi(\varkappa)}{\varkappa^{1-p}})^{d\varkappa}}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \\
&\quad= \left(\frac{e^{\int_{\theta_1}^{\theta_2}\ln(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa}}{e^{\int_{\theta_1}^{\theta_2}\ln(\frac{\psi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa}}\right)^{\frac{p}{\theta_2^p-\theta_1^p}}
\end{aligned}$$

$$
\begin{aligned}
&= e^{\left(\int_{\theta_1}^{\theta_2} \ln\left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right) d\varkappa - \int_{\theta_1}^{\theta_2} \ln\left(\frac{\psi(\varkappa)}{\varkappa^{1-p}}\right) d\varkappa\right)^{\frac{p}{\theta_2^p-\theta_1^p}}} \\
&= e^{\left(\int_0^1 \ln \varphi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}}) dt - \int_0^1 \ln \psi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}}) dt\right)} \\
&\leq e^{\left(\int_0^1 \ln((\varphi(\theta_1))^t(\varphi(\theta_2))^{(1-t)}) dt - \int_0^1 \ln(\psi(\theta_2)+t(\psi(\theta_1)-\psi(\theta_2))) dt\right)} \\
&= e^{\ln(\varphi(\theta_1)\varphi(\theta_2))^{\int_0^1 t\,dt} - \ln\left(\left(\frac{(\varphi(\theta_2))^{\varphi(\theta_2)}}{(\varphi(\theta_1))^{\varphi(\theta_1)}}\right)^{\frac{1}{\varphi(\theta_2)-\varphi(\theta_1)}}\right)+1} \\
&= \frac{G(\varphi(\theta_1),\varphi(\theta_2)).e}{\left(\frac{(\psi(\theta_2))^{\psi(\theta_2)}}{(\psi(\theta_1))^{\psi(\theta_1)}}\right)^{\frac{1}{\psi(\theta_2)-\psi(\theta_1)}}}.
\end{aligned}
$$

Hence, the proof is completed. □

Remark 3.5 Choosing $p = 1$ in Theorem 3.3, we get Theorem 12 in [12].

Theorem 3.4 *Let φ be a convex function and ψ be a multiplicatively p-convex function. Then*

$$
\left(\int_{\theta_1}^{\theta_2} \left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa} . \int_{\theta_1}^{\theta_2} \left(\frac{\psi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \leq \frac{\left(\frac{(\varphi(\theta_2))^{\varphi(\theta_2)}}{(\varphi(\theta_1))^{\varphi(\theta_1)}}\right)^{\frac{1}{\varphi(\theta_2)-\varphi(\theta_1)}}.G(\psi(\theta_1),\psi(\theta_2))}{e}.
$$

Proof Note that

$$
\begin{aligned}
&\left(\int_{\theta_1}^{\theta_2} \left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa} . \int_{\theta_1}^{\theta_2} \left(\frac{\psi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \\
&\quad= e^{\left(\int_{\theta_1}^{\theta_2} \ln\left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right) d\varkappa + \int_{\theta_1}^{\theta_2} \ln\left(\frac{\psi(\varkappa)}{\varkappa^{1-p}}\right) d\varkappa\right)^{\frac{p}{\theta_2^p-\theta_1^p}}} \\
&\quad= e^{\left(\int_0^1 \ln \varphi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}}) dt + \int_0^1 \ln \psi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}}) dt\right)} \\
&\quad\leq e^{\left(\int_0^1 \ln(\varphi(\theta_2)+t(\varphi(\theta_1)-\varphi(\theta_2))) dt + \int_0^1 \ln((\psi(\theta_1))^t(\psi(\theta_2))^{(1-t)}) dt\right)} \\
&\quad= e^{\ln\left(\left(\frac{(\varphi(\theta_2))^{\varphi(\theta_2)}}{(\varphi(\theta_1))^{\varphi(\theta_1)}}\right)^{\frac{1}{\varphi(\theta_2)-\varphi(\theta_1)}}\right)-1+\ln(\psi(\theta_1)\psi(\theta_2))^{\int_0^1 t\,dt}} \\
&\quad= \frac{\left(\frac{(\varphi(\theta_2))^{\varphi(\theta_2)}}{(\varphi(\theta_1))^{\varphi(\theta_1)}}\right)^{\frac{1}{\varphi(\theta_2)-\varphi(\theta_1)}}.G(\psi(\theta_1),\psi(\theta_2))}{e}.
\end{aligned}
$$

This completes the proof. □

Remark 3.6 Choosing $p = 1$ in Theorem 3.4, we get Theorem 13 in [12].

Theorem 3.5 *Let $\varphi : I \to \mathbb{R}$ be multiplicatively p-convex function, where $\theta_1, \theta_2 \in I$ and $\theta_1 < \theta_2$. Then*

$$
\left(\int_{\theta_1}^{\theta_2} \left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \leq \frac{\varphi(\theta_1)+\varphi(\theta_2)}{2}.
$$

Proof Note that

$$\begin{aligned}
&\left(\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \\
&\quad= e^{(\int_{\theta_1}^{\theta_2}\ln(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa)^{\frac{p}{\theta_2^p-\theta_1^p}}} \\
&\quad= e^{\frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\ln(\frac{\varphi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa} \\
&\quad= e^{\int_0^1\ln\varphi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}})\,dt} \\
&\quad\leq \int_0^1 e^{\ln\varphi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}})}\,dt \\
&\quad= \int_0^1 \varphi\left(\left(t\theta_1^p+(1-t)\theta_2^p\right)^{\frac{1}{p}}\right)dt \\
&\quad\leq \int_0^1 \left[\left(\varphi(\theta_1)\right)^t\left(\varphi(\theta_2)\right)^{(1-t)}\right]dt \\
&\quad= \varphi(\theta_2)\int_0^1\left(\frac{\varphi(\theta_1)}{\varphi(\theta_2)}\right)^t dt \\
&\quad= \frac{\varphi(\theta_1)-\varphi(\theta_2)}{\log\varphi(\theta_1)-\log\varphi(\theta_2)} \\
&\quad\leq \frac{\varphi(\theta_1)+\varphi(\theta_2)}{2}.
\end{aligned}$$

This completes the proof. □

Theorem 3.6 *Let $\varphi,\psi: I\to\mathbb{R}$ be multiplicatively p-convex functions, where $\theta_1,\theta_2\in I$ and $\theta_1<\theta_2$. Then*

$$\left(\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)\psi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \leq \frac{1}{4}\phi(\theta_1,\theta_2)$$

where

$$\phi(\theta_1,\theta_2) = \left(\varphi(\theta_1)\right)^2+\left(\varphi(\theta_2)\right)^2+\left(\psi(\theta_1)\right)^2+\left(\psi(\theta_2)\right)^2.$$

Proof Note that

$$\begin{aligned}
&\left(\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)\psi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \\
&\quad= e^{(\int_{\theta_1}^{\theta_2}\ln(\frac{\varphi(\varkappa)\psi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa)^{\frac{p}{\theta_2^p-\theta_1^p}}} \\
&\quad= e^{\frac{p}{\theta_2^p-\theta_1^p}\int_{\theta_1}^{\theta_2}\ln(\frac{\varphi(\varkappa)\psi(\varkappa)}{\varkappa^{1-p}})\,d\varkappa} \\
&\quad= e^{\int_0^1\ln[\varphi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}})\psi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}})]\,dt} \\
&\quad\leq \int_0^1 e^{\ln[\varphi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}})\psi((t\theta_1^p+(1-t)\theta_2^p)^{\frac{1}{p}})]}\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \left[\varphi\left(\left(t\theta_1^p + (1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\psi\left(\left(t\theta_1^p + (1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\right] dt \\
&\le \int_0^1 \left[\left(\varphi(\theta_1)\right)^t\left(\varphi(\theta_2)\right)^{(1-t)}\left(\psi(\theta_1)\right)^t\left(\psi(\theta_2)\right)^{(1-t)}\right] dt \\
&= \varphi(\theta_1)\varphi(\theta_2)\int_0^1 \left(\frac{\varphi(\theta_1)\psi(\theta_1)}{\varphi(\theta_2)\psi(\theta_2)}\right)^t dt \\
&= \frac{\varphi(\theta_1)\psi(\theta_1) - \varphi(\theta_2)\psi(\theta_2)}{\log(\varphi(\theta_1)\psi(\theta_1)) - \log(\varphi(\theta_2)\psi(\theta_2))} \\
&\le \frac{\varphi(\theta_1)\psi(\theta_1) + \varphi(\theta_2)\psi(\theta_2)}{2} \\
&\le \frac{1}{2}\int_0^1 \left[\left(\varphi\left(\left(t\theta_1^p + (1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\right)^2 + \left(\psi\left(\left(t\theta_1^p + (1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\right)^2\right] dt \\
&\le \frac{1}{2}\int_0^1 \left[\left(\left(\varphi(\theta_1)\right)^t\left(\varphi(\theta_2)\right)^{(1-t)}\right)^2 + \left(\left(\psi(\theta_1)\right)^t\left(\psi(\theta_2)\right)^{(1-t)}\right)^2\right] dt \\
&= \frac{(\varphi(\theta_2))^2}{2}\int_0^1 \left(\frac{\varphi(\theta_1)}{\varphi(\theta_2)}\right)^{2t} dt + \frac{(\psi(\theta_2))^2}{2}\int_0^1 \left(\frac{\psi(\theta_1)}{\psi(\theta_2)}\right)^{2t} dt \\
&= \frac{1}{4}\frac{(\varphi(\theta_1))^2 - (\varphi(\theta_2))^2}{\log\varphi(\theta_1) - \log\varphi(\theta_1)} + \frac{1}{4}\frac{(\psi(\theta_1))^2 - (\psi(\theta_2))^2}{\log\psi(\theta_1) - \log\psi(\theta_1)} \\
&\le \frac{1}{2}\frac{\varphi(\theta_1) + \varphi(\theta_2)}{2}\frac{\varphi(\theta_1) - \varphi(\theta_2)}{\log\varphi(\theta_1) - \log\varphi(\theta_2)} + \frac{1}{2}\frac{\psi(\theta_1) + \psi(\theta_2)}{2}\frac{\psi(\theta_1) - \psi(\theta_2)}{\log\psi(\theta_1) - \log\psi(\theta_2)} \\
&\le \frac{1}{4}\left[\left(\varphi(\theta_1)\right)^2 + \left(\varphi(\theta_2)\right)^2 + \left(\psi(\theta_1)\right)^2 + \left(\psi(\theta_2)\right)^2\right].
\end{aligned}
$$

This completes the proof. □

Theorem 3.7 *Let $\varphi, \psi : I \to \mathbb{R}$ be multiplicatively p-convex functions, where $\theta_1, \theta_2 \in I$ and $\theta_1 < \theta_2$. Then*

$$
\left(\int_{\theta_1}^{\theta_2} \left(\frac{\varphi(\varkappa)\psi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p - \theta_1^p}} \le \frac{1}{8}\phi(\theta_1, \theta_2) + \frac{1}{4}\rho(\theta_1, \theta_2)
$$

where

$$
\phi(\theta_1, \theta_2) = \left(\varphi(\theta_1)\right)^2 + \left(\varphi(\theta_2)\right)^2 + \left(\psi(\theta_1)\right)^2 + \left(\psi(\theta_2)\right)^2
$$

and

$$
\rho(\theta_1, \theta_2) = \varphi(\theta_1)\psi(\theta_1) + \varphi(\theta_2)\psi(\theta_2).
$$

Proof Let φ and ψ be multiplicatively p-convex functions. Using the inequality

$$
\theta_1\theta_2 \le \frac{1}{4}(\theta_1 + \theta_2)^2, \quad \forall \theta_1, \theta_2 \in \mathbb{R},
$$

we have

$$
\begin{aligned}
&\left(\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)\psi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \\
&\quad\le \int_0^1\left[\varphi\left(\left(t\theta_1^p+(1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\psi\left(\left(t\theta_1^p+(1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\right]dt \\
&\quad\le \frac{1}{4}\int_0^1\left[\varphi\left(\left(t\theta_1^p+(1-t)\theta_2^p\right)^{\frac{1}{p}}\right)+\psi\left(\left(t\theta_1^p+(1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\right]^2dt \\
&\quad\le \frac{1}{4}\int_0^1\left[\left(\varphi(\theta_1)\right)^t\left(\varphi(\theta_2)\right)^{(1-t)}+\left(\psi(\theta_1)\right)^t\left(\psi(\theta_2)\right)^{(1-t)}\right]dt \\
&\quad= \frac{1}{4}\int_0^1\left[\varphi(\theta_2)\left(\frac{\varphi(\theta_1)}{\varphi(\theta_2)}\right)^t+\psi(\theta_2)\left(\frac{\psi(\theta_1)}{\psi(\theta_2)}\right)^t\right]^2dt \\
&\quad= \frac{(\varphi(\theta_2))^2}{8}\int_0^2\left(\frac{\varphi(\theta_1)}{\varphi(\theta_2)}\right)^{\lambda}d\lambda+\frac{(\psi(\theta_2))^2}{8}\int_0^2\left(\frac{\psi(\theta_1)}{\psi(\theta_2)}\right)^{\lambda}d\lambda \\
&\qquad+\frac{\varphi(\theta_2)\psi(\theta_2)}{2}\int_0^1\left(\frac{\varphi(\theta_1)\psi(\theta_1)}{\varphi(\theta_2)\psi(\theta_2)}\right)^t dt \\
&\quad= \frac{1}{8}\frac{(\varphi(\theta_1))^2-(\varphi(\theta_2))^2}{\log\varphi(\theta_1)-\log\varphi(\theta_2)}+\frac{1}{8}\frac{(\psi(\theta_1))^2-(\psi(\theta_2))^2}{\log\psi(\theta_1)-\log\psi(\theta_2)} \\
&\qquad+\frac{1}{2}\frac{\varphi(\theta_1)\psi(\theta_1)-\varphi(\theta_2)\psi(\theta_2)}{\log(\varphi(\theta_1)\psi(\theta_1))-\log(\varphi(\theta_2)\psi(\theta_2))} \\
&\quad= \frac{1}{4}\frac{\varphi(\theta_1)+\varphi(\theta_2)}{2}\frac{\varphi(\theta_1)-\varphi(\theta_2)}{\log\varphi(\theta_1)-\log\varphi(\theta_2)}+\frac{1}{4}\frac{\psi(\theta_1)+\psi(\theta_2)}{2}\frac{\psi(\theta_1)-\psi(\theta_2)}{\log\psi(\theta_1)-\log\psi(\theta_2)} \\
&\qquad+\frac{1}{2}\frac{\varphi(\theta_1)\psi(\theta_1)-\varphi(\theta_2)\psi(\theta_2)}{\log(\varphi(\theta_1)\psi(\theta_1))-\log(\varphi(\theta_2)\psi(\theta_2))} \\
&\quad\le \frac{1}{8}\left[\left(\varphi(\theta_1)\right)^2+\left(\varphi(\theta_2)\right)^2+\left(\psi(\theta_1)\right)^2+\left(\psi(\theta_2)\right)^2\right]+\frac{1}{4}\left[\varphi(\theta_1)\psi(\theta_1)+\varphi(\theta_2)\psi(\theta_2)\right].
\end{aligned}
$$

So, the proof is completed. □

Theorem 3.8 *Let $\varphi,\psi: I\to\mathbb{R}$ be multiplicatively p-convex functions, where $\theta_1,\theta_2\in I$ and $\theta_1<\theta_2$. Then*

$$
\begin{aligned}
&\left(\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)\psi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}} \\
&\quad\le \alpha\frac{\varphi(\theta_1)+\varphi(\theta_2)}{2}\left[L_{\frac{1}{\alpha}-1}\left(\varphi(\theta_1),\varphi(\theta_2)\right)\right]^{\frac{1}{\alpha}-1} \\
&\qquad+\beta\frac{\psi(\theta_1)+\psi(\theta_2)}{2}\left[L_{\frac{1}{\beta}-1}\left(\psi(\theta_1),\psi(\theta_2)\right)\right]^{\frac{1}{\beta}-1}.
\end{aligned}
$$

Proof Let φ and ψ be multiplicatively p-convex functions. Then using the inequality

$$
\theta_1\theta_2\le\alpha\theta_1^{\frac{1}{\alpha}}+\beta\theta_2^{\frac{1}{\beta}},\quad \alpha,\beta>0,\alpha+\beta=1,
$$

we have

$$\begin{aligned}
&\left(\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)\psi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p}{\theta_2^p-\theta_1^p}}\\
&\quad\le\int_0^1\left[\varphi\left(\left(t\theta_1^p+(1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\psi\left(\left(t\theta_1^p+(1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\right]dt\\
&\quad\le\int_0^1\left[\alpha\left\{\varphi\left(\left(t\theta_1^p+(1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\right\}^{\frac{1}{\alpha}}+\beta\left\{\psi\left(\left(t\theta_1^p+(1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\right\}^{\frac{1}{\beta}}\right]dt\\
&\quad\le\int_0^1\left\{\alpha\left[\left(\varphi(\theta_1)\right)^t\left(\varphi(\theta_2)\right)^{(1-t)}\right]^{\frac{1}{\alpha}}+\beta\left[\left(\psi(\theta_1)\right)^t\left(\psi(\theta_2)\right)^{(1-t)}\right]^{\frac{1}{\beta}}\right\}dt\\
&\quad=\alpha\left(\varphi(\theta_2)\right)^{\frac{1}{\alpha}}\int_0^1\left(\frac{\varphi(\theta_1)}{\varphi(\theta_2)}\right)^{\frac{t}{\alpha}}dt+\beta\left(\psi(\theta_2)\right)^{\frac{1}{\beta}}\int_0^1\left(\frac{\psi(\theta_1)}{\psi(\theta_2)}\right)^{\frac{t}{\beta}}dt\\
&\quad=\alpha^2\left(\varphi(\theta_2)\right)^{\frac{1}{\alpha}}\int_0^{\frac{1}{\alpha}}\left(\frac{\varphi(\theta_1)}{\varphi(\theta_2)}\right)^{\lambda}d\lambda+\beta^2\left(\psi(\theta_2)\right)^{\frac{1}{\beta}}\int_0^{\frac{1}{\beta}}\left(\frac{\psi(\theta_1)}{\psi(\theta_2)}\right)^{\lambda}d\lambda\\
&\quad=\alpha^2\frac{(\varphi(\theta_1))^{\frac{1}{\alpha}}-(\varphi(\theta_2))^{\frac{1}{\alpha}}}{\log\varphi(\theta_1)-\log\varphi(\theta_2)}+\beta^2\frac{(\psi(\theta_1))^{\frac{1}{\beta}}-(\psi(\theta_2))^{\frac{1}{\beta}}}{\log\psi(\theta_1)-\log\psi(\theta_2)}\\
&\quad=\alpha^2\frac{(\varphi(\theta_1))^{\frac{1}{\alpha}}-(\varphi(\theta_2))^{\frac{1}{\alpha}}}{\varphi(\theta_1)-\varphi(\theta_2)}L\left[\varphi(\theta_1),\varphi(\theta_2)\right]\\
&\qquad+\beta^2\frac{(\psi(\theta_1))^{\frac{1}{\beta}}-(\psi(\theta_2))^{\frac{1}{\beta}}}{\psi(\theta_1)-\psi(\theta_2)}L\left[\psi(\theta_1),\psi(\theta_2)\right]\\
&\quad=\alpha\left[L_{\frac{1}{\alpha}-1}\left(\varphi(\theta_1),\varphi(\theta_2)\right)\right]^{\frac{1}{\alpha}-1}L\left[\varphi(\theta_1),\varphi(\theta_2)\right]\\
&\qquad+\beta\left[L_{\frac{1}{\beta}-1}\left(\varphi(\theta_1),\varphi(\theta_2)\right)\right]^{\frac{1}{\beta}-1}L\left[\psi(\theta_1),\psi(\theta_2)\right]\\
&\quad\le\alpha\frac{\varphi(\theta_1)+\varphi(\theta_2)}{2}\left[L_{\frac{1}{\alpha}-1}\left(\varphi(\theta_1),\varphi(\theta_2)\right)\right]^{\frac{1}{\alpha}-1}\\
&\qquad+\beta\frac{\psi(\theta_1)+\psi(\theta_2)}{2}\left[L_{\frac{1}{\beta}-1}\left(\psi(\theta_1),\psi(\theta_2)\right)\right]^{\frac{1}{\beta}-1}.
\end{aligned}$$

This completes the proof. □

Theorem 3.9 *Let $\varphi,\psi:I\to\mathbb{R}$ be increasing multiplicatively p-convex functions, where $\theta_1,\theta_2\in I$ and $\theta_1<\theta_2$. Then*

$$\begin{aligned}
&\left(\int_{\theta_1}^{\theta_2}\left(\frac{\varphi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p.\ln G(\psi(\theta_1),\psi(\theta_2))}{\theta_2^p-\theta_1^p}}.\left(\int_{\theta_1}^{\theta_2}\left(\frac{\psi(\varkappa)}{\varkappa^{1-p}}\right)^{d\varkappa}\right)^{\frac{p.\ln G(\varphi(\theta_1),\varphi(\theta_2))}{\theta_2^p-\theta_1^p}}\\
&\quad\le 2L\left[\varphi(\theta_1)\psi(\theta_2),\varphi(\theta_2)\psi(\theta_1)\right].
\end{aligned}$$

Proof Let φ and ψ be increasing multiplicatively p-convex functions. Then

$$\begin{aligned}
&\varphi\left(\left(t\theta_1^p+(1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\le\left(\varphi(\theta_1)\right)^t\left(\varphi(\theta_2)\right)^{(1-t)},\\
&\psi\left(\left(t\theta_1^p+(1-t)\theta_2^p\right)^{\frac{1}{p}}\right)\le\left(\psi(\theta_1)\right)^t\left(\psi(\theta_2)\right)^{(1-t)}.
\end{aligned}$$

Using $\langle\lambda_1-\lambda_2,\lambda_3-\lambda_4\rangle\geq 0$, $\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{R}$ and $\lambda_1<\lambda_2<\lambda_3<\lambda_4$, we have

$$\begin{aligned}&\varphi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\big(\psi(\theta_1)\big)^t\big(\psi(\theta_2)\big)^{(1-t)}\\&\quad+\psi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\big(\varphi(\theta_1)\big)^t\big(\varphi(\theta_2)\big)^{(1-t)}\\&\quad\leq\varphi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\psi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\\&\quad+\big(\varphi(\theta_1)\big)^t\big(\varphi(\theta_2)\big)^{(1-t)}\big(\psi(\theta_1)\big)^t\big(\psi(\theta_2)\big)^{(1-t)}.\end{aligned}$$

Taking logarithm and integrating above inequalities with respect to t on $[0,1]$, we have

$$\begin{aligned}&\int_0^1\ln\big[\varphi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\big(\psi(\theta_1)\big)^t\big(\psi(\theta_2)\big)^{(1-t)}\big]\,dt\\&\quad+\int_0^1\ln\big[\psi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\big(\varphi(\theta_1)\big)^t\big(\varphi(\theta_2)\big)^{(1-t)}\big]\,dt\\&\quad\leq\int_0^1\ln\big[\varphi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\psi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\\&\quad+\big(\varphi(\theta_1)\big)^t\big(\varphi(\theta_2)\big)^{(1-t)}\big(\psi(\theta_1)\big)^t\big(\psi(\theta_2)\big)^{(1-t)}\big]\,dt.\end{aligned}$$

Since φ and ψ are increasing, we have

$$\begin{aligned}&\int_0^1\ln\varphi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\,dt\int_0^1\ln\big[\big(\psi(\theta_1)\big)^t\big(\psi(\theta_2)\big)^{(1-t)}\big]\,dt\\&\quad+\int_0^1\ln\psi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\,dt\int_0^1\ln\big[\big(\varphi(\theta_1)\big)^t\big(\varphi(\theta_2)\big)^{(1-t)}\big]\,dt\\&\quad\leq\int_0^1\ln\big[\varphi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\psi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\\&\quad+\big(\varphi(\theta_1)\big)^t\big(\varphi(\theta_2)\big)^{(1-t)}\big(\psi(\theta_1)\big)^t\big(\psi(\theta_2)\big)^{(1-t)}\big]\,dt.\end{aligned}$$

which means

$$\begin{aligned}&\ln G\big(\psi(\theta_1),\psi(\theta_2)\big)\int_0^1\ln\varphi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\,dt\\&\quad+\ln G\big(\varphi(\theta_1),\varphi(\theta_2)\big)\int_0^1\ln\psi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\,dt\\&\quad\leq\int_0^1\ln\big[\varphi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\psi\big(\big(t\theta_1^p+(1-t)\theta_2^p\big)^{\frac{1}{p}}\big)\\&\quad+\big(\varphi(\theta_1)\big)^t\big(\varphi(\theta_2)\big)^{(1-t)}\big(\psi(\theta_1)\big)^t\big(\psi(\theta_2)\big)^{(1-t)}\big]\,dt.\end{aligned}$$

Now, taking exponential on both sides, we get the required result

$$\begin{aligned}&\bigg(\int_{\theta_1}^{\theta_2}\bigg(\frac{\varphi(\varkappa)}{x\varkappa^{1-p}}\bigg)^{d\varkappa}\bigg)^{\frac{p.\ln G(\psi(\theta_1),\psi(\theta_2))}{\theta_2^p-\theta_1^p}}.\bigg(\int_{\theta_1}^{\theta_2}\bigg(\frac{\psi(\varkappa)}{\varkappa^{1-p}}\bigg)^{d\varkappa}\bigg)^{\frac{p.\ln G(\varphi(\theta_1),\varphi(\theta_2))}{\theta_2^p-\theta_1^p}}\\&\quad\leq 2L\big[\varphi(\theta_1)\psi(\theta_2),\varphi(\theta_2)\psi(\theta_1)\big].\end{aligned}\tag{2}$$

□

4 Concluding remarks

In this paper, we defined and examined the class of multiplicatively p-convex functions. We established a novel form of Hermite-Hadamard type inequality utilizing multiplicative calculus for convex and multiplicatively p-convex functions. Additionally, we derived several integral inequalities of Hermite-Hadamard type for the product and quotient of convex and multiplicatively p-convex functions. Furthermore, we provided upper bounds for the product of two multiplicatively p-convex functions. As a result, several new integral inequalities of the Hermite-Hadamard type are established. The findings we proved are generalizations of the results in the literature. In recent years, Hermite-Hadamard inequalities played a crucial role in mathematical analysis, probability theory, optimization, and other branches of mathematics. So, several studies have been devoted to introducing novel dimensions to the theory of inequalities. We presume that our newly created class of functions will be the focus of much research in this fascinating field of inequalities and analysis.

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