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On some investigations of alpha-conformable Ostrowski–Trapezoid–Grüss dynamic inequalities on time scales

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Abstract

We prove new Ostrowski-type α-conformable dynamic inequalities and its companion inequalities on time scales by using the integration-by-parts formula on time scales associated with two parameters for functions with bounded second delta derivatives. When $\alpha = 1$, we obtain some well-known time-scale inequalities due to Ostrowski. As particular cases, we obtain new continuous and discrete inequalities.

MSC: 26D15; 26E70

Keywords: Montgomery identity; Ostrowski-type inequality; Trapezoid-type inequality; Grüss-type inequality; Time scales

1 Introduction

The theory of time scales has become a trend and is now part of the mathematics subject classification: see 26E70 for "Real analysis on time scales"; 34K42 for "Functional-differential equations on time scales"; 34N05 for "Dynamic equations on time scales"; and 35R07 for "PDEs on time scales". The subject has began with the PhD thesis of Hilger to get continuous and discrete results together [20, 21]. In books [8, 9], Bohner and Peterson introduce most basic concepts and definitions related with the theory of time scales. In [3, 10, 14, 22], several mathematicians investigate new forms of dynamic inequalities.

For instance, Bohner and Matthews [6] seem to be the first mathematicians to introduce the dynamic Ostrowski inequality on time scales as follows.

Theorem 1.1 *Let $\varrho, \varsigma, \omega, \tau \in \mathbb{T}$, $\varrho < \varsigma$, and let $\Theta : [\varrho, \varsigma]_{\mathbb{T}} \to \mathbb{R}$ be a delta differentiable function. Then for all $\omega \in [\varrho, \varsigma]_{\mathbb{T}}$, we have*

$$\left| \Theta(\omega) - \frac{1}{\varsigma - \varrho} \int_{\varrho}^{\varsigma} \Theta^{\sigma}(\tau) \Delta\tau \right| \leq \frac{M}{\varsigma - \varrho} \bigl(h_2(\omega, \varrho) + h_2(\omega, \varsigma) \bigr), \tag{1.1}$$

where $h_2(\omega, \tau) = \int_{\tau}^{\omega} (s - \tau) \Delta s$ and $M = \sup_{\varrho < \tau < \varsigma} |\Theta^{\Delta}(\tau)| < \infty$. Inequality (1.1) is sharp in the sense that the right-hand side cannot be replaced by a smaller one.

Also, Bohner and Matthews [5] are the first mathematicians to introduce the dynamic Grüss inequality on time scales as follows.

Theorem 1.2 *Let* $\Theta, \phi \in C_{rd}([\varrho,\varsigma]_{\mathbb{T}},\mathbb{R})$ *with*

$$m_1 \le \Theta(\tau) \le M_1 \quad \textit{and} \quad m_2 \le \phi(\tau) \le M_2 \quad \textit{for all } \tau \in [\varrho,\varsigma].$$

Then we have

$$\left| \frac{1}{\varsigma-\varrho}\int_{\varrho}^{\varsigma} \Theta^{\sigma}(\tau)\phi^{\sigma}(\tau)\Delta\tau - \frac{1}{(\varsigma-\varrho)^2}\int_{\varrho}^{\varsigma}\Theta^{\sigma}(\tau)\Delta\tau\int_{\varrho}^{\varsigma}\phi^{\sigma}(\tau)\Delta\tau \right| \le \frac{1}{4}(M_1-m_1)(M_2-m_2).$$

Ostrowski's inequality has a significant importance in many fields, particularly in numerical analysis. One of its applications is the estimation of the error in the approximation of integrals. Many generalizations and refinements of the Ostrowski inequality and its companion inequalities were done during the past several decades; we refer the reader to the papers [1, 3, 10, 12–14, 19, 22, 24, 25, 28, 29], the books [2, 26, 27], and the references cited therein.

Some various generalizations and extensions of the dynamic Ostrowski inequality and its companion inequalities can be found in [7, 11, 16–18, 23, 30].

Here we prove new dynamic Ostrowski-type dynamic inequalities via the α-conformable calculus on time scales or functions with bounded second delta derivatives. Then we prove new generalized dynamic trapezoid- and Grüss-type inequalities on time scales. Our inequalities have a completely new form. As particular cases, we obtain some new continuous and discrete inequalities of Ostrowski type generalizing those obtained in the literature. The paper is organized as follows. In Sect. 2, we briefly recall necessary results and notions. Then we give and prove the original results in Sect. 3. We end with Sect. 4 of conclusion.

2 Time scales preliminaries

This section is devoted to the presentation of some preliminaries about fractional conformable derivatives developed in [4].

Now, let us take a journey to the center of the time scales calculus. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. Throughout the paper, we assume that $\mathbb{T}$ has the topology inherited from the standard topology on $\mathbb{R}$. We define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ for any $\tau \in \mathbb{T}$ by

$$\sigma(\tau) := \inf\{s \in \mathbb{T} : s > \tau\}$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ for any $\tau \in \mathbb{T}$ by

$$\rho(\tau) := \sup\{s \in \mathbb{T} : s < \tau\}.$$

In the preceding two definitions, we set $\inf\emptyset = \sup\mathbb{T}$ (i.e., if τ is the maximum of $\mathbb{T}$, then $\sigma(\tau) = \tau$) and $\sup\emptyset = \inf\mathbb{T}$ (i.e., if τ is the minimum of $\mathbb{T}$, then $\rho(\tau) = \tau$), where $\emptyset$ denotes the empty set.

Definition 2.1 Let $\xi : \mathbb{T} \to \mathbb{R}$, $\tau \in \mathbb{T}^k$, and $\alpha \in (0,1]$. For $\tau > 0$, we define $T_\alpha^\Delta(\xi)(\tau)$ to be the number (provided that it exists) such that, given any $\epsilon > 0$, there is a δ-neighborhood $U_\tau \subset \mathbb{T}$ of τ, $\delta > 0$, such that

$$\left|\left[\xi\left(\sigma(\tau)\right) - \xi(s)\right]\tau^{1-\alpha} - T_\alpha^\Delta(\xi)(\tau)\left[\sigma(\tau) - s\right]\right| \le \varepsilon\left|\sigma(\tau) - s\right|$$

for all $s \in U_\tau$. We call $T_\alpha^\Delta(\xi)(\tau)$ the conformable derivative of ξ of order α at τ, and we define the conformable derivative on $\mathbb{T}$ at 0 as $T_\alpha^\Delta(\xi)(0) = \lim_{\tau \longrightarrow 0+} T_\alpha^\Delta(\xi)(\tau)$.

Remark 2.2 If $\alpha = 1$, then from Definition 2.1 we obtain the delta derivative of time scales. The conformable derivative of order zero is defined as the identity operator, $T_0^\Delta(\xi) = \xi$.

Remark 2.3 Along the work, we also use the notation $(\xi)^{\Delta_\alpha}(\tau) = T_\alpha^\Delta(\xi)(\tau)$.

Theorem 2.4 *Let $\alpha \in (0,1]$, and let $\mathbb{T}$ be a time scale. Let $\xi : \mathbb{T} \to \mathbb{R}$ and $\tau \in \mathbb{T}^k$. Then:*

(i) *If ξ is conformal differentiable of order α at $\tau > 0$, then ξ is continuous at τ;*

(ii) *If ξ is continuous at τ and τ is right-scattered, then ξ is conformable differentiable of order α at τ with*

$$T_\alpha^\Delta(\xi)(\tau) = \frac{\xi(\sigma(\tau)) - \xi(\tau)}{\mu(\tau)}\tau^{1-\alpha};$$

(iii) *If τ is right-dense, then ξ is conformable differentiable of order α at τ if and only if there exists the finite limit $T_\alpha^\Delta(\xi)(\tau) := \lim_{s \longrightarrow \tau} \frac{\xi(\tau)-\xi(s)}{\tau - s}\tau^{1-\alpha}$;*

(iv) *If ξ is differentiable of order α at τ, then*

$$\xi\left(\sigma(\tau)\right) = \xi(\tau) + \mu(\tau)\tau^{\alpha-1}T_\alpha^\Delta(\xi)(\tau).$$

Theorem 2.5 *Let $\xi, \varpi : \mathbb{T} \longrightarrow \mathbb{R}$ be conformable differentiable of order $\alpha \in (0,1]$. Then:*

(i) *The sum $\xi + \varpi : \mathbb{T} \longrightarrow \mathbb{R}$ is conformable differentiable with*

$$T_\alpha^\Delta(\xi + \varpi) = T_\alpha^\Delta(\xi) + T_\alpha^\Delta(\varpi);$$

(ii) *For any $k \in \mathbb{R}$, $k\xi : \mathbb{T} \longrightarrow \mathbb{R}$ is conformable differentiable with*

$$T_\alpha^\Delta(k\xi) = kT_\alpha^\Delta(\xi);$$

(iii) *If ξ and ϖ are continuous, then the product $\xi\varpi : \mathbb{T} \longrightarrow \mathbb{R}$ is conformable differentiable with*

$$T_\alpha^\Delta(\xi\varpi) = T_\alpha^\Delta(\xi)\varpi + \xi^\sigma T_\alpha^\Delta(\varpi) = T_\alpha^\Delta(\xi)\varpi^\sigma + \xi T_\alpha^\Delta(\varpi);$$

(iv) *If ξ is continuous, then $1/\xi$ is conformable differentiable with*

$$T_\alpha^\Delta\left(\frac{1}{\xi}\right) = \frac{-T_\alpha^\Delta(\xi)}{\xi(\xi \circ \sigma)}$$

at all points $\tau \in \mathbb{T}^k$ for which $\xi(\xi \circ \sigma) \neq 0$;

(iv) *If ξ and ϖ are continuous, then ξ/ϖ is conformable differentiable with*

$$T_{\alpha}^{\Delta}\left(\frac{\xi}{\varpi}\right)=\frac{T_{\alpha}^{\Delta}(\xi)\varpi-\xi T_{\alpha}^{\Delta}(\varpi)}{\varpi\varpi^{\sigma}}$$

for all $\tau\in\mathbb{T}^{k}$ for which $\varpi\varpi^{\sigma}\neq 0$.

Definition 2.6 Let $\xi:\mathbb{T}\to\mathbb{R}$ be a regulated function. Then for $0<\alpha\leq 1$, the α-conformable integral of ξ is defined by

$$\int\xi(\tau)\Delta_{\alpha}\tau=\int\xi(\tau)\tau^{\alpha-1}\Delta\tau.$$

Definition 2.7 Let $\xi:\mathbb{T}\to\mathbb{R}$ be a regulated function. The indefinite α-conformable integral of ξ of order $\alpha\in(0,1]$ is defined as $F_{\alpha}(\tau)=\int\xi(\tau)\Delta_{\alpha}\tau$. Then, for all $a,b\in\mathbb{T}$, we define the Cauchy α-conformable integral by

$$\int_{a}^{b}\xi(\tau)\Delta_{\alpha}\tau=F_{\alpha}(b)-F_{\alpha}(a).$$

Theorem 2.8 *Let $\alpha\in(0,1]$. Then for any rd-continuous function $\xi:\mathbb{T}\to\mathbb{R}$, there exists a function $F_{\alpha}:\mathbb{T}\to\mathbb{R}$ such that $T_{\alpha}^{\Delta}(F_{\alpha})(\tau)=\xi(\tau)$ for all $\tau\in\mathbb{T}^{k}$. The function F_{α} is said to be an α-antiderivative of ξ.*

The conformable integral satisfies the following properties.

Theorem 2.9 *Let $\alpha\in(0,1]$, a, b, $c\in\mathbb{T}$, and $\omega\in\mathbb{R}$, and let ξ, ϖ be two rd-continuous functions. Then*:

(i) $\int_{a}^{b}[\xi(\tau)+\varpi(\tau)]\Delta_{\alpha}\tau=\int_{a}^{b}\xi(\tau)\Delta_{\alpha}\tau+\int_{a}^{b}\varpi(\tau)\Delta_{\alpha}\tau$;
(ii) $\int_{a}^{b}\omega\xi(\tau)\Delta_{\alpha}\tau=\omega\int_{a}^{b}\xi(\tau)\Delta_{\alpha}\tau$;
(iii) $\int_{a}^{b}\xi(\tau)\Delta_{\alpha}\tau=-\int_{b}^{a}\xi(\tau)\Delta_{\alpha}\tau$;
(iv) $\int_{a}^{b}\xi(\tau)\Delta_{\alpha}\tau=\int_{a}^{c}\xi(\tau)\Delta_{\alpha}\tau+\int_{c}^{b}\xi(\tau)\Delta_{\alpha}\tau$;
(v) $\int_{a}^{a}\xi(\tau)\Delta_{\alpha}\tau=0$;
(vi) *if there exists $\xi:\mathbb{T}\to\mathbb{R}$ with $|\zeta(\tau)|\leq\xi(\tau$ for all $\tau\in[a,b]$, then* $|\int_{a}^{b}\zeta(\tau)\Delta_{\alpha}\tau|\leq\int_{a}^{b}\xi(\tau)\Delta_{\alpha}\tau$;
(vii) *if $\xi>0$ for all $\tau\in[a,b]$, then* $\int_{a}^{b}\xi(\tau)\Delta_{\alpha}\tau\geq 0$.

The α-conformable integration-by-parts formula on time scales is given in the following lemma.

Lemma 2.10 ([31, Theorem 4.3(v)]) *Let a, $b\in\mathbb{T}$ with $b>a$. If ξ, ϖ are conformable α-fractional differentiable and $\alpha\in(0,1]$, then*

$$\int_{a}^{b}\xi(\tau)T_{\alpha}^{\Delta}\varpi(\tau)\Delta_{\alpha}\tau=\left[\xi(\tau)\varpi(\tau)\right]_{a}^{b}-\int_{a}^{b}T_{\alpha}^{\Delta}\xi(\tau)\varpi^{\sigma}(\tau)\Delta_{\alpha}\tau. \tag{2.1}$$

We use the following crucial relations between calculus on time scales $\mathbb{T}$, differential calculus on $\mathbb{R}$, and difference calculus on $\mathbb{Z}$. Note that:

(i) For any time scales $\mathbb{T}$, we have

$$(\xi)^{\Delta_\alpha}(\tau) = (\xi)^{\Delta}(\tau)\tau^{1-\alpha}, \qquad \int_a^b \xi(\tau)\Delta_\alpha\tau = \int_a^b \xi(\tau)\tau^{\alpha-1}\Delta\tau;$$

(ii) If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(\tau) = \tau, \qquad \mu(\tau) = 0, \qquad f^{\Delta}(\tau) = f'(\tau), \qquad \int_a^b f(\tau)\Delta\tau = \int_a^b f(\tau)\,d\tau; \quad (2.2)$$

(iii) If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{aligned} &\sigma(\tau) = \tau + 1, \qquad \mu(\tau) = 1, \\ &f^{\Delta}(\tau) = \Delta f(\tau), \qquad \int_a^b f(\tau)\Delta\tau = \sum_{\tau=a}^{b-1} f(\tau). \end{aligned} \quad (2.3)$$

3 Main results

3.1 An Ostrowski-type inequality on time scales

Theorem 3.1 *Let $\mathbb{T}$ be a time scale with $\varrho, \varsigma, \omega, \tau \in \mathbb{T}$ and $\varrho < \varsigma$. Further, assume that $\Theta : [\varrho, \varsigma]_{\mathbb{T}} \to \mathbb{T}$ is a twice delta-alpha differentiable function. Then, for all $\omega \in [\varrho, \varsigma]_{\mathbb{T}}$ and $\theta, \vartheta \in \mathbb{R}$, we have*

$$\begin{aligned} &\Bigg|\Theta(\omega) - \frac{1}{\theta + \vartheta}\left[\frac{\theta - \alpha + 1}{\omega - \varrho}\int_\varrho^\omega \Theta^\sigma(\tau)\Delta_\alpha\tau + \frac{\vartheta + \alpha - 1}{\varsigma - \omega}\int_\omega^\varsigma \Theta^\sigma(\tau)\Delta_\alpha\tau\right] \\ &\quad - \frac{1}{\theta + \vartheta}\left[\int_\varrho^\varsigma \int_\varrho^\tau \frac{\theta - \alpha + 1}{\tau - \varrho}\Lambda(\omega, \tau)\Theta^{\Delta_\alpha}\big(\sigma(s)\big)\Delta_\alpha s\Delta_\alpha\tau \right. \\ &\quad \left. + \int_\varrho^\varsigma \int_\tau^\varsigma \frac{\vartheta + \alpha - 1}{\varsigma - \omega}\Lambda(\omega, \tau)\Theta^{\Delta_\alpha}\big(\sigma(s)\big)\Delta_\alpha s\Delta_\alpha\tau\right]\Bigg| \\ &\quad \le \frac{K}{(\theta + \vartheta)^2}\int_\varrho^\varsigma \int_\varrho^\varsigma \Lambda(\omega, \tau)\Lambda(\tau, s)\Delta_\alpha s\Delta_\alpha\tau, \end{aligned} \quad (3.1)$$

where

$$\Lambda(\omega, \tau) = \begin{cases} \frac{\theta - \alpha + 1}{\theta + \vartheta}(\frac{\tau - \varrho}{\omega - \varrho}), & \varrho \le \tau < \omega, \\ \frac{1 - (\vartheta + \alpha)}{\theta + \vartheta}(\frac{\varsigma - \tau}{\varsigma - \omega}), & \omega \le \tau \le \varsigma, \end{cases}$$

and

$$K = \sup_{\varrho < \tau < \varsigma}\left|\Theta^{\Delta_\alpha\Delta_\alpha}(\tau)\right| < \infty.$$

Proof Using the integration-by-parts formula on time scales (2.1), we have

$$\begin{aligned} &\int_\varrho^\omega \frac{\theta - \alpha + 1}{\theta + \vartheta}\left(\frac{\tau - \varrho}{\omega - \varrho}\right)\Theta^{\Delta_\alpha}(\tau)\Delta_\alpha\tau \\ &\quad = \frac{\theta - \alpha + 1}{\theta + \vartheta}\Theta(\omega) - \frac{\theta - \alpha + 1}{(\theta + \vartheta)(\omega - \varrho)}\int_\varrho^\omega \Theta^\sigma(\tau)\Delta_\alpha\tau \end{aligned} \quad (3.2)$$

and

$$
\begin{aligned}
&\int_{\omega}^{\varsigma} \frac{1-(\vartheta+\alpha)}{\theta+\vartheta}\left(\frac{\varsigma-\tau}{\varsigma-\omega}\right)\Theta^{\Delta_\alpha}(\tau)\Delta_\alpha\tau \\
&\quad = \frac{\vartheta+\alpha-1}{\theta+\vartheta}\Theta(\omega)-\frac{\vartheta+\alpha-1}{(\theta+\vartheta)(\varsigma-\omega)}\int_{\omega}^{\varsigma}\Theta^{\sigma}(\tau)\Delta_\alpha\tau. \qquad (3.3)
\end{aligned}
$$

Adding (3.2) and (3.3), we get

$$
\begin{aligned}
\int_{\varrho}^{\varsigma}\Lambda(\omega,\tau)\Theta^{\Delta_\alpha}(\tau)\Delta_\alpha\tau &= \Theta(\omega)-\frac{1}{\theta+\vartheta}\left[\frac{\theta-\alpha+1}{\omega-\varrho}\int_{\varrho}^{\omega}\Theta^{\sigma}(\tau)\Delta_\alpha\tau\right. \\
&\quad \left.+\frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_{\omega}^{\varsigma}\Theta^{\sigma}(\tau)\Delta_\alpha\tau\right]. \qquad (3.4)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\int_{\varrho}^{\varsigma}\Lambda(\tau,s)\Theta^{\Delta_\alpha\Delta_\alpha}(s)\Delta_\alpha s &= \Theta^{\Delta_\alpha}(\tau)-\frac{1}{\theta+\vartheta}\left[\frac{\theta-\alpha+1}{\tau-\varrho}\int_{\varrho}^{\tau}\Theta^{\Delta_\alpha}\big(\sigma(s)\big)\Delta_\alpha s\right. \\
&\quad \left.+\frac{\vartheta+\alpha-1}{\varsigma-\tau}\int_{\tau}^{\varsigma}\Theta^{\Delta_\alpha}\big(\sigma(s)\big)\Delta_\alpha s\right]. \qquad (3.5)
\end{aligned}
$$

Substituting (3.5) into (3.4) leads to

$$
\begin{aligned}
&\int_{\varrho}^{\varsigma}\int_{\varrho}^{\varsigma}\Lambda(\omega,\tau)\Lambda(\tau,s)\Theta^{\Delta_\alpha\Delta_\alpha}(s)\Delta_\alpha s\Delta_\alpha\tau \\
&\quad +\frac{1}{\theta+\vartheta}\left[\int_{\varrho}^{\varsigma}\int_{\varrho}^{\tau}\frac{\theta-\alpha+1}{\tau-\varrho}\Lambda(\omega,\tau)\Theta^{\Delta_\alpha}\big(\sigma(s)\big)\Delta_\alpha s\Delta_\alpha\tau\right. \qquad (3.6) \\
&\quad \left.+\int_{\varrho}^{\varsigma}\int_{\tau}^{\varsigma}\frac{\vartheta+\alpha-1}{\varsigma-\tau}\Lambda(\omega,\tau)\Theta^{\Delta_\alpha}\big(\sigma(s)\big)\Delta_\alpha s\Delta_\alpha\tau\right] \\
&\quad = \Theta(\omega)-\frac{1}{\theta+\vartheta}\left[\frac{\theta-\alpha+1}{\omega-\varrho}\int_{\varrho}^{\omega}\Theta^{\sigma}(\tau)\Delta_\alpha\tau+\frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_{\omega}^{\varsigma}\Theta^{\sigma}(\tau)\Delta_\alpha\tau\right].
\end{aligned}
$$

Inequality (3.1) follows directly from (3.6) and the properties of modulus. This completes the proof. □

Remark 3.2 Taking $\alpha = 1$ in Theorem 3.1, we get Theorem 3.1 in [15].

Corollary 3.3 *If we take* $\mathbb{T} = \mathbb{R}$ *in Theorem* 3.1, *then by relation* (2.2) *inequality* (3.1) *becomes*

$$
\begin{aligned}
&\left|\Theta(\omega)-\frac{1}{\theta+\vartheta}\left[\frac{\theta-\alpha+1}{\omega-\varrho}\int_{\varrho}^{\omega}\Theta(\tau)\,d_\alpha\tau+\frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_{\omega}^{\varsigma}\Theta(\tau)\,d_\alpha\tau\right]\right. \\
&\quad -\frac{1}{\theta+\vartheta}\left[\int_{\varrho}^{\varsigma}\int_{\varrho}^{\tau}\frac{\theta-\alpha+1}{\tau-\varrho}\Lambda(\omega,\tau)\Theta'(s)\,d_\alpha s d_\alpha\tau\right. \\
&\quad \left.\left.+\int_{\varrho}^{\varsigma}\int_{\tau}^{\varsigma}\frac{\vartheta+\alpha-1}{\varsigma-\omega}\Lambda(\omega,\tau)\Theta'(s)\,d_\alpha s d_\alpha\tau\right]\right| \\
&\quad \le \frac{K}{(\theta+\vartheta)^2}\int_{\varrho}^{\varsigma}\int_{\varrho}^{\varsigma}\Lambda(\omega,\tau)\Lambda(\tau,s)\,d-\alpha s d_\alpha\tau,
\end{aligned}
$$

where

$$\Lambda(\omega,\tau)=\begin{cases}\frac{\theta-\alpha+1}{\theta+\vartheta}(\frac{\tau-\varrho}{\omega-\varrho}), & \varrho\le\tau<\omega,\\ \frac{1-(\vartheta+\alpha)}{\theta+\vartheta}(\frac{\varsigma-\tau}{\varsigma-\omega}), & \omega\le\tau\le\varsigma,\end{cases}$$

and

$$K=\sup_{\varrho<\tau<\varsigma}\left|\Theta''(\tau)\right|<\infty.$$

Corollary 3.4 *If we take $\mathbb{T}=\mathbb{Z}$ in Theorem 3.1, then by relation (2.3) inequality (3.1) becomes*

$$\begin{aligned}&\Bigg|\Theta(\omega)-\frac{1}{\theta+\vartheta}\left[\frac{\theta}{\omega-\varrho}\sum_{\tau=\varrho}^{\omega-1}\Theta(\tau+1)\tau^{\alpha-1}+\frac{\vartheta}{\varsigma-\omega}\sum_{\tau=\omega}^{\varsigma-1}\Theta(\tau+1)\tau^{\alpha-1}\right]\\&\quad-\frac{1}{\theta+\vartheta}\Bigg[\sum_{\tau=\varrho}^{\varsigma-1}\sum_{s=\varrho}^{\tau-1}\frac{\theta}{\tau-\varrho}\Lambda(\omega,\tau)\Delta_\alpha\Theta(s+1)\tau^{\alpha-1}s^{\alpha-1}\\&\quad+\sum_{\tau=\varrho}^{\varsigma-1}\sum_{s=\tau}^{\varsigma-1}\frac{\vartheta}{\varsigma-\omega}\Lambda(\omega,\tau)\Delta_\alpha\Theta(s+1)\tau^{\alpha-1}s^{\alpha-1}\Bigg]\Bigg|\\&\quad\le\frac{K}{(\theta+\vartheta)^2}\sum_{\tau=\varrho}^{\varsigma-1}\sum_{s=\tau}^{\varsigma-1}\Lambda(\omega,\tau)\Lambda(\tau,s)\tau^{\alpha-1}s^{\alpha-1},\end{aligned}$$

where

$$\Lambda(\omega,\tau)=\begin{cases}\frac{\theta-\alpha+1}{\theta+\vartheta}(\frac{\tau-\varrho}{\omega-\varrho}), & \tau=\varrho,\dots,\omega-1,\\ \frac{1-(\vartheta+\alpha)}{\theta+\vartheta}(\frac{\varsigma-\tau}{\varsigma-\omega}), & \tau=\omega,\dots,\varsigma,\end{cases}$$

and

$$K=\max_{\varrho<\tau<\varsigma}\left|\Delta_\alpha^2\Theta(\tau)\right|<\infty.$$

3.2 A trapezoid-type inequality on time scales

Theorem 3.5 *Under the assumptions of Theorem 3.1, we have*

$$\begin{aligned}&\Bigg|\Theta^2(\varsigma)-\Theta^2(\varrho)-\frac{1}{\theta+\vartheta}\int_\varrho^\varsigma\Bigg[\frac{\theta-\alpha+1}{\omega-\varrho}\int_\varrho^\omega\left[\Theta^\sigma(\tau)+\Theta^{\sigma^2}(\tau)\right]\Delta_\alpha\tau\\&\quad+\frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_\omega^\varsigma\left[\Theta^\sigma(\tau)+\Theta^{\sigma^2}(\tau)\right]\Delta_\alpha\tau\Bigg]\Delta_\alpha\omega\Bigg|\\&\quad\le M(M+P)\int_\varrho^\varsigma\int_\varrho^\varsigma\left|\Lambda(\omega,\tau)\right|\Delta_\alpha\tau\,\Delta_\alpha\omega,\end{aligned}\tag{3.7}$$

where

$$\Lambda(\omega,\tau)=\begin{cases}\frac{\theta-\alpha+1}{\theta+\vartheta}(\frac{\tau-\varrho}{\omega-\varrho}), & \varrho\le\tau<\omega,\\ \frac{1-(\vartheta+\alpha)}{\theta+\vartheta}(\frac{\varsigma-\tau}{\varsigma-\omega}), & \omega\le\tau\le\varsigma,\end{cases}$$

and

$$M = \sup_{\varrho<\tau<\varsigma} \left|\Theta^{\Delta_\alpha}(\tau)\right| \quad \textit{and} \quad P = \sup_{\varrho<\tau<\varsigma} \left|\left(\Theta^\sigma\right)^{\Delta_\alpha}(\tau)\right|.$$

Proof From (3.4) we have

$$\Theta(\omega) = \int_\varrho^\varsigma \Lambda(\omega,\tau)\Theta^{\Delta_\alpha}(\tau)\Delta_\alpha\tau + \frac{1}{\theta+\vartheta}\left[\frac{\theta-\alpha+1}{\omega-\varrho}\int_\varrho^\omega \Theta^\sigma(\tau)\Delta_\alpha\tau + \frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_\omega^\varsigma \Theta^\sigma(\tau)\Delta_\alpha\tau\right] \tag{3.8}$$

and, similarly,

$$\Theta^\sigma(\omega) = \int_\varrho^\varsigma \Lambda(\omega,\tau)\left(\Theta^\sigma\right)^{\Delta_\alpha}(\tau)\Delta_\alpha\tau + \frac{1}{\theta+\vartheta}\left[\frac{\theta-\alpha+1}{\omega-\varrho}\int_\varrho^\omega \Theta^{\sigma^2}(\tau)\Delta_\alpha\tau]] + \frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_\omega^\varsigma \Theta^{\sigma^2}(\tau)\Delta_\alpha\tau\right]. \tag{3.9}$$

Now adding (3.8) and (3.9) produces

$$\begin{aligned}\Theta(\omega)+\Theta^\sigma(\omega) = {}& \int_\varrho^\varsigma \Lambda(\omega,\tau)\left[\Theta^{\Delta_\alpha}(\tau)+\left(\Theta^\sigma\right)^{\Delta_\alpha}(\tau)\right]\Delta_\alpha\tau \\ &+ \frac{1}{\theta+\vartheta}\left[\frac{\theta-\alpha+1}{\omega-\varrho}\int_\varrho^\omega \left[\Theta^\sigma(\tau)+\Theta^{\sigma^2}(\tau)\right]\Delta_\alpha\tau \right.\\ &\left.+ \frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_\omega^\varsigma \left[\Theta^\sigma(\tau)+\Theta^{\sigma^2}(\tau)\right]\Delta_\alpha\tau\right].\end{aligned}$$

Multiplying the last identity by $\Theta^{\Delta_\alpha}(\omega)$, using (2.3), and integrating the resulting identity with respect to ω from ϱ to ς yield

$$\begin{aligned}\Theta^2(\varsigma)-\Theta^2(\varrho) = {}& \int_\varrho^\varsigma\int_\varrho^\varsigma \Theta^{\Delta_\alpha}(\omega)\Lambda(\omega,\tau)\left[\Theta^{\Delta_\alpha}(\tau)+\left(\Theta^\sigma\right)^{\Delta_\alpha}(\tau)\right]\Delta_\alpha\tau\Delta_\alpha\omega \\ &+ \frac{1}{\theta+\vartheta}\int_\varrho^\varsigma \Theta^{\Delta_\alpha}(\omega)\left[\frac{\theta-\alpha+1}{\omega-\varrho}\int_\varrho^\omega \left[\Theta^\sigma(\tau)+\Theta^{\sigma^2}(\tau)\right]\Delta_\alpha\tau \right.\\ &\left.+ \frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_\omega^\varsigma \left[\Theta^\sigma(\tau)+\Theta^{\sigma^2}(\tau)\right]\Delta_\alpha\tau\right]\Delta_\alpha\omega.\end{aligned}$$

Equivalently,

$$\begin{aligned}&\Theta^2(\varsigma)-\Theta^2(\varrho) - \frac{1}{\theta+\vartheta}\int_\varrho^\varsigma \Theta^{\Delta_\alpha}(\omega)\left[\frac{\theta-\alpha+1}{\omega-\varrho}\int_\varrho^\omega \left[\Theta^\sigma(\tau)+\Theta^{\sigma^2}(\tau)\right]\Delta_\alpha\tau \right.\\ &\quad\left.+ \frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_\omega^\varsigma \left[\Theta^\sigma(\tau)+\Theta^{\sigma^2}(\tau)\right]\Delta_\alpha\tau\right]\Delta_\alpha\omega \\ &\quad= \int_\varrho^\varsigma\int_\varrho^\varsigma \Theta^{\Delta_\alpha}(\omega)\Lambda(\omega,\tau)\left[\Theta^{\Delta_\alpha}(\tau)+\left(\Theta^\sigma\right)^{\Delta_\alpha}(\tau)\right]\Delta_\alpha\tau\Delta_\alpha\omega.\end{aligned}$$

Taking the absolute values on both sides, we get

$$\begin{aligned}
&\Bigg|\Theta^2(\varsigma) - \Theta^2(\varrho) - \frac{1}{\theta+\vartheta}\int_\varrho^\varsigma \Theta^{\Delta_\alpha}(\omega)\bigg[\frac{\theta-\alpha+1}{\omega-\varrho}\int_\varrho^\omega \big[\Theta^\sigma(\tau)+\Theta^{\sigma^2}(\tau)\big]\Delta_\alpha\tau \\
&\quad + \frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_\omega^\varsigma \big[\Theta^\sigma(\tau)+\Theta^{\sigma^2}(\tau)\big]\Delta_\alpha\tau\bigg]\Delta_\alpha\omega\Bigg| \\
&\quad = \bigg|\int_\varrho^\varsigma\int_\varrho^\varsigma \Theta^{\Delta_\alpha}(\omega)\Lambda(\omega,\tau)\big[\Theta^{\Delta_\alpha}(\tau)+\big(\Theta^\sigma\big)^{\Delta_\alpha}(\tau)\big]\Delta_\alpha\tau\,\Delta_\alpha\omega\bigg| \\
&\quad \le \int_\varrho^\varsigma\int_\varrho^\varsigma \big|\Theta^{\Delta_\alpha}(\omega)\big|\big|\Lambda(\omega,\tau)\big|\big[\big|\Theta^{\Delta_\alpha}(\tau)\big| + \big|\big(\Theta^\sigma\big)^{\Delta_\alpha}(\tau)\big|\big]\Delta_\alpha\tau\,\Delta_\alpha\omega \\
&\quad \le M(M+P)\int_\varrho^\varsigma\int_\varrho^\varsigma \big|\Lambda(\omega,\tau)\big|\Delta_\alpha\tau\,\Delta_\alpha\omega.
\end{aligned}$$

This shows (3.7). □

Remark 3.6 Taking $\alpha = 1$ in Theorem 3.5, we get Theorem 3.4 in [15].

Corollary 3.7 *If we take $\mathbb{T} = \mathbb{R}$ in Theorem 3.5, then by relation (2.2) inequality (3.7) becomes*

$$\begin{aligned}
&\bigg|\frac{\Theta^2(\varsigma) - \Theta^2(\varrho)}{2} - \frac{1}{\theta+\vartheta}\int_\varrho^\varsigma \Theta'(\omega)\bigg[\frac{\theta}{\omega-\varrho}\int_\varrho^\omega \Theta(\tau)\,d_\alpha\tau + \frac{\vartheta}{\varsigma-\omega}\int_\omega^\varsigma \Theta(\tau)\,d_\alpha\tau\bigg]d_\alpha\omega\bigg| \\
&\quad \le M^2\int_\varrho^\varsigma\int_\varrho^\varsigma \big|\Lambda(\omega,\tau)\big|\,d_\alpha t d_\alpha x,
\end{aligned}$$

where

$$\Lambda(\omega,\tau) = \begin{cases} \frac{\theta-\alpha+1}{\theta+\vartheta}\big(\frac{\tau-\varrho}{\omega-\varrho}\big), & \varrho \le \tau < \omega, \\ \frac{1-(\vartheta+\alpha)}{\theta+\vartheta}\big(\frac{\varsigma-\tau}{\varsigma-\omega}\big), & \omega \le \tau \le \varsigma, \end{cases}$$

and

$$M = \sup_{\varrho<\tau<\varsigma}\big|\Theta'(\tau)\big|.$$

Corollary 3.8 *If we take $\mathbb{T} = \mathbb{Z}$ in Theorem 3.5, then by relation (2.3) inequality (3.7) becomes*

$$\begin{aligned}
&\Bigg|\Theta^2(\varsigma) - \Theta^2(\varrho) - \frac{1}{\theta+\vartheta}\sum_{\omega=\varrho}^{\varsigma-1}\Delta_\alpha\Theta(\omega)\Bigg[\frac{\theta}{\omega-\varrho}\sum_{\tau=\varrho}^{\omega-1}\big[\Theta(\tau+1)+\Theta(\tau+2)\big]\tau^{\alpha-1} \\
&\quad + \frac{\vartheta}{\varsigma-\omega}\sum_{\tau=\omega}^{\varsigma-1}\big[\Theta(\tau+1)+\Theta(\tau+2)\big]\tau^{\alpha-1}\Bigg]\omega^{\alpha-1}\Bigg| \\
&\quad \le M(M+N)\sum_{\omega=\varrho}^{\varsigma-1}\sum_{\tau=\varrho}^{\varsigma-1}\big|\Lambda(\omega,\tau)\big|\omega^{\alpha-1}\tau^{\alpha-1},
\end{aligned}$$

where

$$\Lambda(\omega,\tau)=\begin{cases}\frac{\theta+\alpha-1}{\theta+\vartheta}(\frac{\tau-\varrho}{\omega-\varrho}), & \tau=\varrho,\dots,\omega-1,\\ \frac{1-(\vartheta+\alpha)}{\theta+\vartheta}(\frac{\varsigma-\tau}{\varsigma-\omega}), & \tau=\omega,\dots,\varsigma,\end{cases}$$

and

$$M=\max_{\varrho<\tau<\varsigma}\left|\Delta_\alpha\Theta(\tau)\right| \quad \textit{and} \quad P=\max_{\varrho<\tau<\varsigma}\left|\Delta_\alpha\Theta(\tau+1)\right|.$$

3.3 A Grüss-type inequality on time scales

Theorem 3.9 *Let $\mathbb{T}$ be a time scale with ϱ, ς, ω, $\tau\in\mathbb{T}$ and $\varrho<\varsigma$. Moreover, let Θ, ϕ : $[\varrho,\varsigma]_{\mathbb{T}}\to\mathbb{R}$ be delta-alpha differentiable functions. Then for all $\omega\in[\varrho,\varsigma]_{\mathbb{T}}$ and $\theta,\vartheta\in\mathbb{R}$, we have*

$$\begin{aligned}&\Bigg|2\int_\varrho^\varsigma\Theta(\omega)\phi(\omega)\Delta_\alpha\omega-\frac{1}{\theta+\vartheta}\Bigg[\frac{\theta-\alpha+1}{\omega-\varrho}\int_\varrho^\varsigma\int_\varrho^\omega\big(\Theta^\sigma(\tau)\phi(\omega)+\phi^\sigma(\tau)\Theta(\omega)\big)\Delta_\alpha\tau\,\Delta_\alpha\omega\\&\qquad+\frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_\varrho^\varsigma\int_\omega^\varsigma\big(\Theta^\sigma(\tau)\phi(\omega)+\phi^\sigma(\tau)\Theta(\omega)\big)\Delta_\alpha\tau\,\Delta_\alpha\omega\Bigg]\Bigg|\\&\quad\le\int_\varrho^\varsigma\int_\varrho^\varsigma\left|\Lambda(\omega,\tau)\right|\big[M\left|\phi(\omega)\right|+N\left|\Theta(\omega)\right|\big]\Delta_\alpha\tau\,\Delta_\alpha\omega,\end{aligned}\tag{3.10}$$

where

$$\Lambda(\omega,\tau)=\begin{cases}\frac{\theta-\alpha+1}{\theta+\vartheta}(\frac{\tau-\varrho}{\omega-\varrho}), & \varrho\le\tau<\omega,\\ \frac{1-(\vartheta+\alpha)}{\theta+\vartheta}(\frac{\varsigma-\tau}{\varsigma-\omega}), & \omega\le\tau\le\varsigma,\end{cases}$$

and

$$M=\sup_{\varrho<\tau<\varsigma}\left|\Theta^{\Delta_\alpha}(\tau)\right|<\infty \quad \textit{and} \quad N=\sup_{\varrho<\tau<\varsigma}\left|\phi^{\Delta_\alpha}(\tau)\right|<\infty.$$

Proof From (3.4) we have

$$\begin{aligned}\Theta(\omega)={}&\int_\varrho^\varsigma\Lambda(\omega,\tau)\Theta^{\Delta_\alpha}(\tau)\Delta_\alpha\tau+\frac{1}{\theta+\vartheta}\Bigg[\frac{\theta-\alpha+1}{\omega-\varrho}\int_\varrho^\omega\Theta^\sigma(\tau)\Delta_\alpha\tau\\&+\frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_\omega^\varsigma\Theta^\sigma(\tau)\Delta_\alpha\tau\Bigg]\end{aligned}\tag{3.11}$$

and, similarly,

$$\begin{aligned}\phi(\omega)={}&\int_\varrho^\varsigma\Lambda(\omega,\tau)\phi^{\Delta_\alpha}(\tau)\Delta_\alpha\tau+\frac{1}{\theta+\vartheta}\Bigg[\frac{\theta-\alpha+1}{\omega-\varrho}\int_\varrho^\omega\phi^\sigma(\tau)\Delta_\alpha\tau\\&+\frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_\omega^\varsigma\phi^\sigma(\tau)\Delta_\alpha\tau\Bigg].\end{aligned}\tag{3.12}$$

Multiplying (3.11) by $\phi(\omega)$ and (3.12) by $\Theta(\omega)$, adding them, and integrating the resulting identity with respect to ω from ϱ to ς yield

$$2\int_\varrho^\varsigma\Theta(\omega)\phi(\omega)\Delta_\alpha\omega$$

$$
\begin{aligned}
&= \int_{\varrho}^{\varsigma}\int_{\varrho}^{\varsigma}\Lambda(\omega,\tau)\big[\Theta^{\Delta_\alpha}(\tau)\phi(\omega)+\phi^{\Delta_\alpha}(\tau)\Theta(\omega)\big]\Delta_\alpha\tau\,\Delta_\alpha\omega \\
&\quad + \frac{1}{\theta+\vartheta}\Bigg[\frac{\theta-\alpha+1}{\omega-\varrho}\int_{\varrho}^{\varsigma}\int_{\varrho}^{\omega}\big(\Theta^{\sigma}(\tau)\phi(\omega)+\phi^{\sigma}(\tau)\Theta(\omega)\big)\Delta_\alpha\tau\,\Delta_\alpha\omega \\
&\quad + \frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_{\varrho}^{\varsigma}\int_{\omega}^{\varsigma}\big(\Theta^{\sigma}(\tau)\phi(\omega)+\phi^{\sigma}(\tau)\Theta(\omega)\big)\Delta_\alpha\tau\,\Delta_\alpha\omega\Bigg].
\end{aligned}
$$

By using the properties of modulus we obtain

$$
\begin{aligned}
&\Bigg|2\int_{\varrho}^{\varsigma}\Theta(\omega)\phi(\omega)\Delta_\alpha\omega - \frac{1}{\theta+\vartheta}\Bigg[\frac{\theta-\alpha+1}{\omega-\varrho}\int_{\varrho}^{\varsigma}\int_{\varrho}^{\omega}\big(\Theta^{\sigma}(\tau)\phi(\omega)+\phi^{\sigma}(\tau)\Theta(\omega)\big)\Delta_\alpha\tau\,\Delta_\alpha\omega \\
&\quad + \frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_{\varrho}^{\varsigma}\int_{\omega}^{\varsigma}\big(\Theta^{\sigma}(\tau)\phi(\omega)+\phi^{\sigma}(\tau)\Theta(\omega)\big)\Delta_\alpha\tau\,\Delta_\alpha\omega\Bigg]\Bigg| \\
&= \left|\int_{\varrho}^{\varsigma}\int_{\varrho}^{\varsigma}\Lambda(\omega,\tau)\big[\Theta^{\Delta_\alpha}(\tau)\phi(\omega)+\phi^{\Delta_\alpha}(\tau)\Theta(\omega)\big]\Delta_\alpha\tau\,\Delta_\alpha\omega\right| \\
&\le \int_{\varrho}^{\varsigma}\int_{\varrho}^{\varsigma}\big|\Lambda(\omega,\tau)\big|\big[\big|\Theta^{\Delta_\alpha}(\tau)\big|\big|\phi(\omega)\big|+\big|\phi^{\Delta_\alpha}(\tau)\big|\big|\Theta(\omega)\big|\big]\Delta_\alpha\tau\,\Delta_\alpha\omega \\
&\le \int_{\varrho}^{\varsigma}\int_{\varrho}^{\varsigma}\big|\Lambda(\omega,\tau)\big|\big[M\big|\phi(\omega)\big|+N\big|\Theta(\omega)\big|\big]\Delta_\alpha\tau\,\Delta_\alpha\omega.
\end{aligned}
$$

This concludes the proof. □

Remark 3.10 Taking $\alpha = 1$ in Theorem 3.9, we get Theorem 3.7 in [15].

Corollary 3.11 *If we take $\mathbb{T} = \mathbb{R}$ in Theorem 3.9, then by relation (2.2) inequality (3.10) becomes*

$$
\begin{aligned}
&\Bigg|2\int_{\varrho}^{\varsigma}\Theta(\omega)\phi(\omega)\,d\omega - \frac{1}{\theta+\vartheta}\Bigg[\frac{\theta-\alpha+1}{\omega-\varrho}\int_{\varrho}^{\varsigma}\int_{\varrho}^{\omega}\big(\Theta(\tau)\phi(\omega)+\phi(\tau)\Theta(\omega)\big)\,d\tau\,d\omega \\
&\quad + \frac{\vartheta+\alpha-1}{\varsigma-\omega}\int_{\varrho}^{\varsigma}\int_{\omega}^{\varsigma}\big(\Theta(\tau)\phi(\omega)+\phi(\tau)\Theta(\omega)\big)\,d\tau\,d\omega\Bigg]\Bigg| \\
&\le \int_{\varrho}^{\varsigma}\int_{\varrho}^{\varsigma}\big|\Lambda(\omega,\tau)\big|\big[M\big|\phi(\omega)\big|+N\big|\Theta(\omega)\big|\big]\,d\tau\,d\omega,
\end{aligned}
$$

where

$$
\Lambda(\omega,\tau) = \begin{cases} \frac{\theta-\alpha+1}{\theta+\vartheta}\left(\frac{\tau-\varrho}{\omega-\varrho}\right), & \varrho \le \tau < \omega, \\ \frac{1-(\vartheta+\alpha)}{\theta+\vartheta}\left(\frac{\varsigma-\tau}{\varsigma-\omega}\right), & \omega \le \tau \le \varsigma, \end{cases}
$$

and

$$
M = \sup_{\varrho<\tau<\varsigma}\big|\Theta'(\tau)\big| < \infty \quad \textit{and} \quad N = \sup_{\varrho<\tau<\varsigma}\big|\phi'(\tau)\big| < \infty.
$$

Corollary 3.12 *If we take $\mathbb{T}=\mathbb{Z}$ in Theorem 3.9, then by relation (2.3) inequality (3.10) becomes*

$$\begin{aligned}&\Bigg|2\sum_{\omega=\varrho}^{\varsigma-1}\Theta(\omega)\phi(\omega)\omega^{\alpha-1}-\frac{1}{\theta+\vartheta}\Bigg[\frac{\theta}{\omega-\varrho}\sum_{\omega=\varrho}^{\varsigma-1}\sum_{\tau=\varrho}^{\omega-1}\big(\Theta(\tau+1)\phi(\omega)+\phi(\tau+1)\Theta(\omega)\big)\omega^{\alpha-1}\tau^{\alpha-1}\\&\quad+\frac{\vartheta}{\varsigma-\omega}\sum_{\omega=\varrho}^{\varsigma-1}\sum_{\tau=\omega}^{\varsigma-1}\big(\Theta(\tau+1)\phi(\omega)+\phi(\tau+1)\Theta(\omega)\big)\omega^{\alpha-1}\tau^{\alpha-1}\Bigg]\Bigg|\\&\quad\leq\sum_{\omega=\varrho}^{\varsigma-1}\sum_{\tau=\varrho}^{\varsigma-1}\big|\Lambda(\omega,\tau)\big|\big[M\big|\phi(\omega)\big|+N\big|\Theta(\omega)\big|\big]\omega^{\alpha-1}\tau^{\alpha-1},\end{aligned}$$

where

$$\Lambda(\omega,\tau)=\begin{cases}\frac{\theta-\alpha+1}{\theta+\vartheta}\big(\frac{\tau-\varrho}{\omega-\varrho}\big), & \tau=\varrho,\dots,\omega-1,\\ \frac{1-(\vartheta+\alpha)}{\theta+\vartheta}\big(\frac{\varsigma-\tau}{\varsigma-\omega}\big), & \tau=\omega,\dots,\varsigma,\end{cases}$$

and

$$M=\max_{\varrho<\tau<\varsigma}\big|\Delta_\alpha\Theta(\tau)\big|<\infty\quad\textit{and}\quad N=\max_{\varrho<\tau<\varsigma}\big|\Delta_\alpha\phi(\tau)\big|<\infty.$$

4 Conclusions

The Ostrowski inequality and its companion inequalities have many applications and are subject to strong research: see the books [2, 26, 27] and recent publications [1, 12, 13, 25]. In this paper, by employing the α-conformable fractional calculus on time scales of Benkhettou et al. [4], we prove several new Ostrowski-type inequalities by using two parameters. These inequalities have certain conditions that have not been studied before. The results extend several dynamic inequalities known in the literature, which are new even in the discrete and continuous settings.

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