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# Sharp coefficient inequalities of starlike functions connected with secant hyperbolic function

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**Abstract**

This article comprises the study of class $\mathcal{S}_E^*$ that represents the class of normalized analytic functions f satisfying $\varsigma f'(z)/f(\varsigma) \prec \sec h(\varsigma)$. The geometry of functions of class $\mathcal{S}_E^*$ is star-shaped, which is confined in the symmetric domain of a secant hyperbolic function. We find sharp coefficient results and sharp Hankel determinants of order two and three for functions in the class $\mathcal{S}_E^*$. We also investigate the same sharp results for inverse coefficients.

**Mathematics Subject Classification:** 30C45; 30C50

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## 1 Introduction and preliminaries

Denote by $\mathcal{A}$ a class of functions f that contains analytic functions in $\mathbb{D} = \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$ and having the Maclaurin series expansion of the form

$$f(\varsigma) = \varsigma + \sum_{m=2}^{\infty} a_m \varsigma^m, \quad \varsigma \in \mathbb{D}. \tag{1.1}$$

A subclass of $\mathcal{A}$ that contains univalent functions in $\mathbb{D}$ is denoted by $\mathcal{S}$. The classes of starlike and convex functions are, respectively, represented by $\mathcal{S}^*$ and $\mathcal{C}$ in $\mathbb{D}$. These are analytically defined for the functions fby the relations $Re(\varsigma f'(\varsigma)/f(\varsigma)) > 0$ and $Re(1 + \varsigma f''(\varsigma)/f'(\varsigma)) > 0$ in $\mathbb{D}$, respectively. Let $\mathcal{B}$ denote the family of functions $w$, which are analytic (holomorphic) in $\mathbb{D}$ such that $w$ maps zero onto zero and the disk to itself that is mathematically written as $w(0) = 0$ and $|w| < 1$ for $\varsigma \in \mathbb{D}$. The said function is called a Schwarz function. Considering functions $f$ and $g$ are both analytic (holomorphic) in $\mathbb{D}$, then we write mathematically as $f \prec g$ and read as $f$ is subordinated to $g$ if there exists a function $w$ that is Schwarz in $\mathbb{D}$ such that $f(\varsigma) = g(w(\varsigma))$ for $\varsigma \in \mathbb{D}$. When $g$ is univalent (one-to-one) and $f(0) = g(0)$, then $f(\mathbb{D}) \subset g(\mathbb{D})$.

Ma and Minda [12] introduced generalized subclasses of $\mathcal{S}^*$ and $\mathcal{C}$ denoted by $\mathcal{S}^*(\Psi)$ and $\mathcal{C}(\Psi)$, respectively, which are defined as follows:

$$\mathcal{S}^*(\Psi) := \left\{ \mathsf{f} \in \mathcal{A} : \frac{\varsigma \mathsf{f}'(\varsigma)}{\mathsf{f}(\varsigma)} \prec \Psi(\varsigma) \right\}$$

and

$$\mathcal{C}(\Psi) := \left\{ \mathsf{f} \in \mathcal{A} : 1 + \frac{\varsigma \mathsf{f}''(\varsigma)}{\mathsf{f}'(\varsigma)} \prec \Psi(\varsigma) \right\}.$$

The analytic and univalent function $\Psi$ satisfies the conditions $\Psi(0) = 1$ and $Re\{\Psi'(\varsigma)\} > 0$ in $\mathbb{D}$ and $\Psi(\mathbb{D})$ is convex.

Certain generating functions related with well-known numbers have attracted various authors. These functions are connected with starlike functions. For instance, Sokół [17, 18] used Fibonacci numbers to introduce a subclass that was strongly starlike. This concept was further utilized by Dziok et al [7, 8] to study various class of analytic functions. Deniz [6] studied a class of starlike functions related with Telephone numbers, also see [14]. The geometry of generating a function for Bell numbers was explored in [4, 9] and a subclass of starlike functions was introduced. Similar work for Bernoulli numbers was done by Raza et al. [16].

Recently, Bano et al. [2] introduced the class $\mathcal{S}_E^*$ related with Euler numbers defined as

$$\mathcal{S}_E^* := \left\{ \mathsf{f} \in \mathcal{A} : \frac{\varsigma \mathsf{f}'(\varsigma)}{\mathsf{f}(\varsigma)} \prec \sec h(\varsigma) = \frac{2}{e^{\varsigma} + e^{-\varsigma}} = \sum_{m=0}^{\infty} \frac{\mathbf{E}_m}{m!} \varsigma^m \right\}.$$

The constants $\mathbf{E}_m$ are known as Eulers numbers. These numbers hold the relation $\mathbf{E}_{2m+1} = 0$, $m = 0, 1, 2, \ldots$. The first few Euler numbers are given as $\mathbf{E}_0 = 1$, $\mathbf{E}_2 = -1$, $\mathbf{E}_4 = 5$, and $\mathbf{E}_6 = -61$. These numbers have very close association with Bernoulli, Genocchi, tangent, and Stirling numbers of two kinds. These are also related with the Euler polynomials and Riemann zeta function. Due to this relation, Euler numbers are very useful in combinatorics and number theory, see [1, 10, 13, 21] and references therein.

## 2 Coefficient bounds and Hankel determinants

The $q$th Hankel determinant for analytic functions was first given by Pommerenke [15] and is defined by

$$H_{q,m}(\mathsf{f}) := \begin{vmatrix} a_m & a_{m+1} & \ldots & a_{m+q-1} \\ a_{m+1} & a_{m+2} & \ldots & a_{m+q} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m+q-1} & a_{m+q} & \ldots & a_{m+2q-2} \end{vmatrix}, \tag{2.1}$$

where $m \geq 1$ and $q \geq 1$. We see that

$$H_{2,1}(\mathsf{f}) = a_3 - a_2^2, \qquad H_{2,2}(\mathsf{f}) = a_2 a_4 - a_3^2,$$
$$H_{3,1}(\mathsf{f}) = 2a_2a_3a_4 - a_4^2 - a_3^3 - a_2^2 a_5 + a_3 a_5 \tag{2.2}$$

and

$$H_{2,3}(\mathsf{f}) = a_3 a_5 - a_4^2. \tag{2.3}$$

The Hankel determinant $H_{2,1}(\mathsf{f})$ is known as a Fekete–Szegö functional that is generalized as a $|a_3 - \mu a_2^2|$ for $\mu \in \mathbb{C}$. For some recent work on it, we refer the readers to [3, 19, 20].

The class $\mathcal{P}$ has great importance in function theory as most of the functions in various classes can be related with this class. The main tool in this study is to convert the coefficients for functions in class $\mathcal{S}_E^*$ to the class $\mathcal{P}$. We define it with some of its useful results for our investigation as follows.

Let $\mathcal{P}$ represent the class of analytic functions $\mathsf{p}$ of the form

$$\mathsf{p}(\varsigma) = 1 + \sum_{m=1}^{\infty} q_m \varsigma^m. \tag{2.4}$$

**Lemma 2.1** [11] *Let* $\mathsf{p} \in \mathcal{P}$ *and be of the form* (2.4). *Then,*

$$2q_2 = q_1^2 + \rho\left(4 - q_1^2\right),$$
$$4q_3 = q_1^3 + 2q_1\left(4 - q_1^2\right)\rho - q_1\left(4 - q_1^2\right)\rho^2 + 2\left(4 - q_1^2\right)\left(1 - \left(|\rho|\right)^2\right)\eta$$

*for some* $\rho$ *and* $\eta$ *such that* $|\rho| \leq 1$ *and* $|\eta| \leq 1$.

**Lemma 2.2** [12] *Let* $\mathsf{p} \in \mathcal{P}$ *be given by* (2.4). *Then,*

$$\left|q_2 - \upsilon q_1^2\right| \leq 2 - \upsilon\left|q_1^2\right|$$

*for* $0 < \upsilon \leq \frac{1}{2}$.

**Lemma 2.3** [5]. *Let* $\overline{\mathbb{D}} := \{\rho \in \mathbb{C} : |\rho| \leq 1\}$, *and for* $J, K, L \in \mathbb{R}$, *let*

$$Y(J,K,L) := \max\left\{\left|J + K\rho + L\rho^2\right| + 1 - |\rho|^2 : \rho \in \overline{\mathbb{D}}\right\}. \tag{2.5}$$

*If* $JL \geq 0$, *then*

$$Y(J,K,L) = \begin{cases} |J| + |K| + |L|, & |K| \geq 2(1 - |L|), \\ 1 + |J| + \frac{K^2}{4(1-|L|)}, & |K| < 2(1 - |L|). \end{cases}$$

*If* $JL < 0$, *then*

$$Y(J,K,L) = \begin{cases} 1 - |J| + \frac{K^2}{4(1-|L|)}, & -4JL(L^{-2} - 1) \leq K^2 \wedge |K| < 2(1 - |L|), \\ 1 + |J| + \frac{K^2}{4(1+|L|)}, & K^2 < \min\{4(1 + |L|)^2, -4JL(L^{-2} - 1)\}. \\ R(J,K,L), & \textit{otherwise}, \end{cases}$$

*where*

$$R(J,K,L)=\begin{cases}|J|+|K|-|L|, & |L|(|K|+4|J|)\le|JK|,\\ 1+|J|+\frac{K^2}{4(1+|L|)}, & |JK|<|L|(|K|-4|J|),\\ |L|+|J|\sqrt{1-\frac{K^2}{4JL}}, & \textit{otherwise}.\end{cases}$$

**Theorem 2.4** *Let* $\mathsf{f}\in\mathcal{S}_E^*$ *and be of the form* (1.1). *Then,*

$$a_2=0,\qquad |a_3|\le\frac{1}{4},\qquad |a_4|\le\frac{2}{27}\sqrt{3},\qquad |a_5|\le\frac{1}{8}.$$

*These bounds are sharp.*

*Proof* Let $\mathsf{f}\in\mathcal{S}_E^*$. Then, by using the definition of subordination, we can write

$$\frac{\varsigma\mathsf{f}'(\varsigma)}{\mathsf{f}(\varsigma)}=\sec h\bigl(w(z)\bigr), \tag{2.6}$$

where $w\in\mathcal{B}$. Let $p\in\mathcal{P}$. Then, we can write

$$p(z)=\frac{1+w(z)}{1-w(z)}=1+q_1z+q_2z^2+\cdots. \tag{2.7}$$

Now, using (2.6) and (2.7) and comparing the coefficients, we obtain

$$a_2=0,\qquad a_3=-\frac{1}{16}q_1^2,\qquad a_4=-\frac{1}{12}q_1q_2+\frac{1}{24}q_1^3 \tag{2.8}$$

and

$$a_5=\frac{-7}{384}q_1^4-\frac{1}{32}q_2^2+\frac{3}{32}q_2^2q_1^2-\frac{1}{16}q_1q_3. \tag{2.9}$$

(i) The bound for $a_3$ follows by applying well-known coefficient bounds for class $\mathcal{P}$. This bound is sharp for the function

$$\mathsf{f}_1(\varsigma)=\varsigma\exp\left(\int_0^{\varsigma}\frac{\sec h(t)-1}{t}\,dt\right)=\varsigma-\frac{1}{4}\varsigma^3+\frac{1}{12}\varsigma^5+\cdots. \tag{2.10}$$

(ii) For the bound on $a_4$, we write

$$a_4=-\frac{q_1}{12}\left[q_2-\frac{q_1^2}{2}\right].$$

Using Lemma 2.2, we obtain

$$|a_4|\le\frac{q_1}{12}\left[2-\frac{q_1^2}{2}\right].$$

Since the class $\mathcal{S}_E^*$ is rotationally invariant, we take $q:=q_1$, so that $0\le q\le 2$. Hence,

$$|a_4|\le\frac{q}{12}\left[2-\frac{q^2}{2}\right].$$

Let

$$\Psi(q) = \frac{q}{12}\left[2 - \frac{q^2}{2}\right].$$

Then, it is easy to see that $\Psi(q) \le \frac{2}{27}\sqrt{3}$. For the sharpness, consider $t_0 = \frac{2}{3}\sqrt{3}$ and $q_0(\varsigma) = \frac{1-\varsigma^2}{1-t_0\varsigma+\varsigma^2}$ such that

$$w_0(\varsigma) = \frac{q_0(\varsigma) - 1}{q_0(\varsigma) + 1} = \frac{\varsigma(3\varsigma - \sqrt{3})}{-3 + \sqrt{3}\varsigma}.$$

It is easy to see that $w_0(0) = 0$ and $|w_0(\varsigma)| < 1$. Hence, the function

$$\mathsf{f}_0(\varsigma) = \varsigma \exp\left(\int_0^{\varsigma} \frac{\sec h(w_0(t)) - 1}{t}\, dt\right) = \varsigma - \frac{1}{12}\varsigma^3 + \frac{2}{27}\sqrt{3}\varsigma^4 + \cdots \tag{2.11}$$

is in class $\mathcal{S}^*_E$. Here, $a_4 = \frac{2}{27}\sqrt{3}$.
(iii) For the bound on $a_5$, using the inequalities from Lemma 2.1, we can write

$$a_5 = \frac{1}{384}\left[\upsilon_1(q,\rho) + \upsilon_2(q,\rho)\eta\right],$$

where $\rho, \eta \in \overline{\mathbb{D}}$, and

$$\begin{aligned}
&\upsilon_1(q,\rho) := 2q^4 + 6q^2\left(4-q^2\right)\rho^2 - 3\rho^2\left(4-q^2\right)^2,\\
&\upsilon_2(q,\rho) := -12q\left(4-q^2\right)\left(1-|\rho|^2\right).
\end{aligned}$$

Letting $t := |\rho|$, and $u := |\eta|$, we have

$$\begin{aligned}
|a_5| &\le \frac{1}{384}\left(\left|\upsilon_1(q,\rho)\right| + \left|\upsilon_2(q,\rho)\right|u\right)\\
&\le J(q,t,u),
\end{aligned}$$

where

$$J(q,t,u) := \frac{1}{384}\left(j_1(q,t) + j_2(q,t)u\right), \tag{2.12}$$

with

$$\begin{aligned}
&j_1(q,t) := 2q^4 + 6q^2\left(4-q^2\right)t^2 + 3t^2\left(4-q^2\right)^2,\\
&j_2(q,t) := 12q\left(4-q^2\right)\left(1-t^2\right).
\end{aligned}$$

Now, we have to find the maximum value in $S : [0,2] \times [0,1] \times [0,1]$. We determine the maximum value of the function $J(q,t,u)$ in the interior of $S$ on the edges and vertices of $S$.

I. First, we consider the interior points of $S$. Differentiating (2.12) with respect to $u$ and after some simplifications, we obtain

$$\frac{\partial J(q,t,u)}{\partial u} = \frac{q(4-q^2)(1-t^2)}{32}. \tag{2.13}$$

Since equation (2.13) is independent of $u$, we have no maximum value in $(0,2) \times (0,1) \times (0,1)$.

II. Next, we discuss the optimum value in the interior of six faces of $S$.

On the face $q = 0$, $J(q,t,u)$ reduces to

$$\kappa_1(t,u) := J(0,t,u) = \frac{t^2}{8}, \quad t,u \in (0,1). \tag{2.14}$$

The function $\kappa_1$ has no maxima in $(0,1) \times (0,1)$ since

$$\frac{\partial \kappa_1}{\partial t} = \frac{t}{4} \neq 0, \quad t \in (0,1).$$

On $q = 2$, $J(q,t,u)$ takes the form

$$J(2,t,u) = \frac{1}{12}, \quad t,\ u \in (0,1). \tag{2.15}$$

On $t = 0$, $J(q,t,u)$ can be written as

$$J(q,0,u) = \kappa_2(q,u) := \frac{[2q^4 + 12uq(4-q^2)]}{384}, \quad q \in (0,2), u \in (0,1). \tag{2.16}$$

This implies function $\kappa_2$ has maxima $(0,2) \times (0,1)$, since

$$\frac{\partial \kappa_2}{\partial u} = \frac{q(4-q^2)}{32} \neq 0, \quad q \in (0,2).$$

On $t = 1$, $J(q,t,u)$ reduces to

$$\kappa_3(q,u) := J(q,1,u) = \frac{-q^4 + 48}{384}, \quad q \in (0,2). \tag{2.17}$$

There is no critical point for the function $\kappa_3$ in $(0,2) \times (0,1)$, since

$$\frac{\partial \kappa_3}{\partial q} = -\frac{q^3}{96} \neq 0, \quad q \in (0,2).$$

On $u = 0$, $J(q,t,u)$ reduces to

$$\kappa_4(q,t) = \frac{1}{384}\left[2q^4 + 6q^2\left(4-q^2\right)t^2 + 3t^2\left(4-q^2\right)^2\right].$$

Therefore,

$$\frac{\partial \kappa_4}{\partial q} = \frac{12q^2(4-q^2)t + 6t(4-q^2)^2}{384}$$

and

$$\frac{\partial \kappa_4}{\partial t} = \frac{2q^3 - 3q^3t^2}{96}.$$

By implementing numerical methods, we see that the system $\frac{\partial \kappa_4}{\partial q} = 0$ and $\frac{\partial \kappa_4}{\partial t} = 0$ has no solution in $(0,2) \times (0,1)$.

On $u = 1$, the function $J(q,t,u)$ takes the form

$$\kappa_5(q,t) = \frac{1}{384}\left[3t^2\left(4-q^2\right)^2 + \left(6q^2t^2 + 12q - 12qt^2\right)\left(4-q^2\right) + 2q^4\right].$$

Similarly, on the face $u = 0$, the system of equations $\frac{\partial \kappa_5}{\partial q} = 0$ and $\frac{\partial \kappa_5}{\partial t} = 0$ has no solution in $(0,2) \times (0,1)$.

III. Now, we investigate the maximum of $J(q,t,u)$ on the edges of $S$. From (2.16), we obtain $J(q,0,0) =: s_1(q) = q^4/192$. Since $s_1'(q) > 0$ for $[0,2]$. Therefore, $s_1$ is increasing in $[0,2]$ and hence a maximum is achieved at $q = 2$. Therefore,

$$J(q,0,0) \leq \frac{1}{12} \approx 0.08333.$$

Also, from (2.16) at $u = 1$, we write $J(q,0,1) := s_2(q) = (q^4 - 6q^3 + 24q)/192$, since $s_2' = 0$ for $q = 1.388684545$ in $[0,2]$. Thus,

$$J(q,0,1) \leq 0.1092672897.$$

Putting $q = 0$ in (2.16), we have

$$J(0,0,u) = 0.$$

Since (2.17) is independent of $t$, we obtain $J(q,1,1) = J(q,1,0) = s_3(q) := \frac{-q^4+48}{384}$. Since $s_3'(q) < 0$ for $[0,2]$, $s_3(q)$ is decreasing in $[0,2]$ and hence a maximum is achieved at $q = 0$. Thus,

$$J(q,1,1) = J(q,1,0) \leq \frac{1}{8} \approx 0.1250.$$

Putting $q = 0$ in (2.17), we obtain

$$J(0,1,u) \leq \frac{1}{8} \approx 0.1250.$$

Equation (2.15) is independent of variables $u$ and $t$. Thus,

$$J(2,t,0) = J(2,t,1) = J(2,0,u) = J(2,1,u) = \frac{1}{12} \approx 0.0833, \quad t,u \in [0,1].$$

As equation (2.14) is independent of the variable $u$, we have $J(0,t,0) = J(0,t,1) = s_4(t) := t^2/8$. Since $s_4'(t) > 0$ for $[0,1]$, $s_4(t)$ is increasing in $[0,1]$ and hence a maximum is achieved at $t = 1$. Thus,

$$J(0,t,0) = J(0,t,1) \leq \frac{1}{8}.$$

The bound for $|a_5|$ is sharp for the function

$$\mathrm{f}_2(\varsigma) = \varsigma \exp\left(\int_0^{\varsigma} \frac{\sec h(t^2) - 1}{t}\, dt\right) = \varsigma - \frac{1}{8}\varsigma^5 + \cdots. \qquad (2.18)$$

This completes the result. □

**Theorem 2.5** *Let* $\mathsf{f} \in \mathcal{S}_E^*$ *be given by* (1.1). *Then,*

$$\left|H_{2,2}(\mathsf{f})\right| \le \frac{1}{16}.$$

*The result is sharp.*

*Proof* Since

$$H_{2,2}(\mathsf{f}) = a_2 a_4 - a_3^2,$$

now using (2.8), we obtain

$$\left|H_{2,2}(\mathsf{f})\right| = \frac{1}{256}|q_1|^4.$$

Now, using well-known coefficient bounds for class $\mathcal{P}$, we obtain the required result. The function $\mathsf{f}_1$ given in (2.10) that gives a sharp result. □

**Theorem 2.6** *Let* $\mathsf{f} \in \mathcal{S}_E^*$ *be given by* (1.1). *Then,*

$$\left|H_{3,1}(\mathsf{f})\right| \le \frac{-3115}{164{,}268} + \frac{671}{657{,}072}\sqrt{1342}.$$

*This inequality is sharp.*

*Proof* Using (2.8)–(2.9) in (2.2), we obtain

$$H_{3,1}(\mathsf{f}) = a_3 a_5 - a_4^2 - a_3^3 = \frac{1}{36{,}864}\psi, \tag{2.19}$$

where

$$\psi = -13q_1^6 + 40q_1^4 q_2 + 144q_1^3 q_3 - 184q_1^2 q_2^2.$$

As we see that the functional $H_{3,1}(\mathsf{f})$ and the class $\mathcal{S}_E^*$ are rotationally invariant, we may take $q := q_1$, such that $q \in [0,2]$. Then, by using Lemma 2.1 and after some computations we may write

$$\psi = -3q^6 - q^2\left(4-q^2\right)\left(184-10q^2\right)\rho^2 + 72q^3\left(4-q^2\right)\left(1-|\rho|^2\right)\eta,$$

where $\rho$ and $\eta$ satisfy the relation $|\rho| \le 1$, $|\eta| \le 1$.

First, we consider the case when $q = 2$. Then, $|\psi| = 192$, therefore from (2.19) we obtain $|H_{3,1}(\mathsf{f})| = \frac{1}{192}$. Next, we assume that $q = 0$, then $|\psi| = 0$. Now, suppose that $q \in (0,2)$ and using the well-known triangle inequality, we obtain

$$|\psi| \le 72q^3\left(4-q^2\right)\Phi(J,K,L),$$

where

$$\Phi(J,K,L) = \left|J + K\rho + L\rho^2\right| + 1 - |\rho|^2, \quad \rho \in \overline{\mathbb{D}},$$

with $J = \frac{-q^3}{24(4-q^2)}$, $K = 0$ and $L = \frac{-(184-10q^2)}{72q}$, then clearly

$$JL = \frac{q^2(184-10q^2)}{1728(4-q^2)} \geq 0, \quad \text{for } q \in (0,2).$$

Also,

$$K - 2\bigl(1-|L|\bigr) = \frac{(2-q)(5q+46)}{18q} > 0 \quad q \in (0,2),$$

so that $K > 2(1-|L|)$, and applying Lemma 2.3, we can have

$$|\psi| \leq 72q^3\bigl(4-q^2\bigr)\bigl(|J|+|K|+|L|\bigr) = g(q),$$

where

$$g(q) = 13q^6 - 224q^4 + 736q^2. \tag{2.20}$$

Clearly, $g'(q) = 0$ has roots at $q_1 = 0$ and $q_2 = \frac{2}{39}\sqrt{2184 - 39\sqrt{1342}}$. Also, $g''(q_2) \approx -2328.5$, and so

$$\max \psi(q) = \psi(q_2) = \frac{-3{,}189{,}760}{4563} + \frac{171{,}776}{4563}\sqrt{1342},$$

which from (2.19) gives the required result.

For the sharpness, consider $t_0 = \frac{2}{39}\sqrt{2184 - 39\sqrt{1342}}$ and $q_0(\varsigma) = \frac{1+t_0\varsigma+\varsigma^2}{1-\varsigma^2}$ such that

$$w_0(\varsigma) = \frac{q_0(\varsigma)-1}{q_0(\varsigma)+1} = \frac{\varsigma(\sqrt{2184-39\sqrt{1342}} + 39\varsigma)}{39 + \sqrt{2184-39\sqrt{1342}}\varsigma}.$$

It is easy to see that $w_0(0) = 0$ and $|w_0(\varsigma)| < 1$. Hence, the function

$$\begin{aligned} \mathsf{f}_0(\varsigma) = \varsigma \exp\left(\int_0^{\varsigma} \frac{\sec h(w_0(t)) - 1}{t}\,dt\right) &= \varsigma + \left(\frac{-14}{39} + \frac{1}{156}\sqrt{1342}\right)\varsigma^3 \\ &\quad - \frac{1}{4563}(-17 + \sqrt{1342})\sqrt{2184 - 39\sqrt{1342}}\varsigma^4 \\ &\quad + \left(\frac{-9701}{36{,}504} + \frac{79}{9126}\sqrt{1342}\right)\varsigma^5 + \cdots \end{aligned}$$

is in class $\mathcal{S}_E^*$. Now, after simplifications, we see that

$$H_{3,1}(\mathsf{f}) = \frac{3115}{164{,}268} - \frac{671}{657{,}072}\sqrt{1342}.$$

Hence, the required result is obtained. □

**Theorem 2.7** *Let* $\mathsf{f} \in \mathcal{S}_E^*$ *and be of the form* (1.1). *Then,*

$$\bigl|H_{2,3}(\mathsf{f})\bigr| \leq \frac{1}{48}.$$

*The result is sharp.*

*Proof* Using (2.8)–(2.9) in (2.3), we obtain

$$H_{2,3}(\mathsf{f}) = a_3a_5 - a_4^2 = \frac{1}{18{,}432}\psi, \tag{2.21}$$

where

$$\psi = -11q_1^6 + 20q_1^4q_2 + 72q_1^3q_3 - 92q_1^2q_2^2.$$

As we see that the functional $H_{2,3}(\mathsf{f})$ and the class $\mathcal{S}_E^*$ are rotationally invariant, we may take $q := q_1$, such that $q \in [0,2]$. Then, by using Lemma 2.1 and after some computations we may write

$$\psi = -6q^6 - q^2\left(4-q^2\right)\left(92-5q^2\right)\rho^2 + 36q^3\left(4-q^2\right)\left(1-|\rho|^2\right)\eta,$$

where $\rho$ and $\eta$ satisfy the relation $|\rho| \leq 1$, $|\eta| \leq 1$.

First, we consider the case when $q = 0$. Then, $|\psi| = 0$, therefore from (2.21) we obtain $|H_{2,3}(\mathsf{f})| = 0$. Next, we assume that $q = 2$, then $|\psi| = 384$ and $|H_{2,3}(\mathsf{f})| = \frac{1}{48}$. Now, suppose that $q \in (0,2)$ and using the well-known triangle inequality, we obtain

$$|\psi| \leq 36q^3\left(4-q^2\right)\Phi(J,K,L),$$

where

$$\Phi(J,K,L) = \left|J + K\rho + L\rho^2\right| + 1 - |\rho|^2, \quad \rho \in \overline{\mathbb{D}},$$

with $J = \frac{-q^3}{6(4-q^2)}$, $K = 0$ and $L = \frac{-(92-5q^2)}{36q}$, then clearly

$$JL = \frac{q^2(92-5q^2)}{216(4-q^2)} > 0, \quad \text{for } q \in (0,2).$$

Now,

$$36q\left(1-|L|\right) = -(5q+46)(2-q) < 0 \quad q \in (0,2).$$

Thus, we conclude that $|K| > 2(1-|L|)$ and applying Lemma 2.3, we can have

$$|\psi| \leq 36q^3\left(4-q^2\right)\left(|J| + |K| + |L|\right) = g(q),$$

where

$$g(q) = q^2\left(11q^4 - 112q^2 + 368\right).$$

Clearly, $g'(q) = 2q(4-q^2)(92-33q^2) > 0$ for $q \in (0,2)$, therefore

$$\max g(q) = g(2) = 384,$$

which from (2.21) gives the result. It is sharp for $\mathsf{f}_1$ given in (2.10). Hence, the required result is obtained. □

## 3 Inverse coefficients

For a function $f$ that is univalent, its inverse function $f^{-1}$ is defined on some disc $|w| \leq 1/4 \leq r_0(f)$, having a series expansion of the form

$$f^{-1}(w) = w + B_2 w^2 + B_3 w^3 + \cdots . \tag{3.1}$$

Next, note that since $f(f^{-1}(w)) = w$, using (3.1) it follows that

$$B_2 = -a_2, \tag{3.2}$$

$$B_3 = 2a_2^2 - a_3, \tag{3.3}$$

$$B_4 = 5a_2a_3 - 5a_2^3 - a_4, \tag{3.4}$$

$$B_5 = 14a_2^4 - 21a_3a_2^2 + 6a_2a_4 + 3a_3^2 - a_5. \tag{3.5}$$

Now, using (2.8) in the above relations, we can write

$$B_2 = 0, \qquad B_3 = \frac{1}{16}q_1^2, \qquad B_4 = \frac{1}{12}q_1q_2 - \frac{1}{24}q_1^3, \tag{3.6}$$

$$B_5 = \frac{23}{768}q_1^4 + \frac{1}{32}q_2^2 - \frac{3}{32}q_2q_1^2 + \frac{1}{16}q_1q_3. \tag{3.7}$$

**Theorem 3.1** *Let* $\mathsf{f} \in \mathcal{S}_E^*$ *and be of the form* (1.1). *Then,*

$$|B_2| = 0, \qquad |B_3| \leq \frac{1}{4}, \qquad B_4 = \frac{1}{12}q_1q_2 - \frac{1}{24}q_1^3.$$

*Proof* (i) The bound for $B_3$ follows by applying well-known coefficient bounds for class $\mathcal{P}$. Sharpness can be found by $\mathsf{f}_1$ given in (2.10). Here, $a_2 = 0$, $a_3 = \frac{-1}{4}$. Now, $B_3 = 2a_2^2 - a_3$ implies $B_3 = \frac{1}{4}$.

(ii) For the bound on $B_4$, we write

$$B_4 = -\frac{q_1}{12}\left[q_2 - \frac{q_1^2}{2}\right].$$

Using Lemma 2.2, we obtain

$$|B_4| \leq \frac{q_1}{12}\left[2 - \frac{q_1^2}{2}\right].$$

Since the class $\mathcal{S}_E^*$ is rotationally invariant, we take $q := q_1$, so that $0 \leq q \leq 2$. Hence,

$$|B_4| \leq \frac{q}{12}\left[2 - \frac{q^2}{2}\right].$$

Let

$$f(q) = \frac{q}{12}\left[2 - \frac{q^2}{2}\right].$$

Then, $f(q) \leq \frac{2}{27}\sqrt{3}$. For the sharpness, consider the function $\mathsf{f}_0$ given by (2.11). Here, $a_2 = 0$, $a_3 = -\frac{1}{12}$, $a_4 = \frac{2}{27}\sqrt{3}$. Since $B_4 = 5a_2a_3 - 5a_2^3 - a_4$, $B_4 = -\frac{2}{27}\sqrt{3}$.

(iii) Using Lemma 2.1 in (3.7), we obtain

$$B_5 = \frac{23}{768}q_1^4 + \frac{1}{32}q_2^2 - \frac{3}{32}q_2q_1^2 + \frac{1}{16}q_1q_3 = \frac{1}{768}\psi, \tag{3.8}$$

where

$$\psi = 23q_1^4 + 24q_2^2 - 72q_2q_1^2 + 48q_1q_3.$$

As we see that the functional $B_5$ and the class $\mathcal{S}_E^*$ are rotationally invariant, we may take $q := q_1$, such that $q \in [0,2]$. Then, by using Lemma 2.1 and after some computations we may write

$$\psi = 5q^4 + 6(4-q^2)(4-3q^2)\rho^2 + 24q(4-q^2)(1-|\rho|^2)\eta,$$

where $\rho$ and $\eta$ satisfy the relation $|\rho| \leq 1$, $|\eta| \leq 1$.

First, we consider the case when $q = 0$. Then, $|\psi| = 96|\rho^2|$, and from (3.8) we obtain $|B_5| \leq 1/8$. Next, we assume that $q = 2$, then $|\psi| = 80$ and $|B_5| = \frac{5}{48}$. Now, suppose that $q \in (0,2)$ and using the well-known triangle inequality, we obtain

$$|\psi| \leq 24q(4-q^2)\Phi(J,K,L),$$

where

$$\Phi(J,K,L) = \left|J + K\rho + L\rho^2\right| + 1 - |\rho|^2, \quad \rho \in \overline{\mathbb{D}},$$

with $J = \frac{5q^3}{24(4-q^2)}$, $K = 0$ and $L = \frac{(4-3q^2)}{4q}$, then clearly

$$JL = \frac{5q^2(4-3q^2)}{96(4-q^2)} \geq 0, \quad \text{for } q \in \left(0, \frac{2}{\sqrt{3}}\right].$$

Now,

$$4q(1-|L|) = (q+2)(3q-2).$$

Here, we have two cases: $4q(1-|L|) \leq 0$ for $q \in (0, \frac{2}{3}]$ and $4q(1-|L|) > 0$ for $(\frac{2}{3}, \frac{2}{\sqrt{3}}]$. Thus, we conclude that $|K| \geq 2(1-|L|)$ for $q \in (0, \frac{2}{3}]$ and applying Lemma 2.3, we can have

$$|\psi| \leq 24q(4-q^2)(|J| + |K| + |L|) = g_1(q),$$

where

$$g_1(q) = 23q^4 - 96q^2 + 96).$$

Clearly, $g_1'(q) = 4q(23q^2 - 48) < 0$ for $(0, \frac{2}{3}]$, therefore,

$$\max g_1(q) = g_1(0) = 96,$$

which from (3.8) gives the required result.

For the case $(\frac{2}{3}, \frac{2}{\sqrt{3}}]$, we see that $4q(1 - |L|) > 0$ and $|K| > 2(1 - |L|)$ and applying Lemma 2.3, we can have

$$|\psi| \leq 24q(4-q^2)\left(1 + |J| + \frac{K^2}{4(1-|L|)}\right) = g_2(q),$$

where

$$g_2(q) = 5q^4 - 24q^3 + 96q). \tag{3.9}$$

We see that $g_2'(q) = 20q^3 - 72q^2 + 96 > 0$, therefore,

$$\max g_2(q) = g_2\left(\frac{2}{\sqrt{3}}\right) = \frac{80}{9} + \frac{128\sqrt{3}}{3} \approx 82.789 < 96.$$

Lastly, we consider the case when $q \in (\frac{2}{\sqrt{3}}, 2)$. Then, $JL < 0$. Now,

$$-4JL(L^{-2} - 1) = \frac{5q^2(3q-2)(3q+2)}{24(3q^2-4)} > 0, \quad q \in \left(\frac{2}{\sqrt{3}}, 2\right).$$

Also,

$$4(1 + |L|)^2 = \frac{(3q-2)^2(q+2)}{q^2} > 0, \quad q \in \left(\frac{2}{\sqrt{3}}, 2\right).$$

This shows that $K^2 < \min\{4(1+|L|)^2, -4JL(L^{-2}-1)\}$. Now, applying Lemma 2.3, we can have

$$|\psi| \leq 24q(4-q^2)\left(1 + |J| + \frac{K^2}{4(1+|L|)}\right) = g_2(q),$$

where $g_2$ is given by (3.9). The result is sharp for the function $\mathsf{f}_2$ given in (2.18). We see that $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $a_5 = -\frac{1}{8}$. Now, using (3.5), we have $|B_5| = \frac{1}{8}$. Hence, the required result is obtained. □

**Theorem 3.2** *Let* $\mathsf{f} \in \mathcal{S}_E^*$ *be given by* (1.1). *Then,*

$$|H_{2,2}(\mathsf{f}^{-1})| \leq \frac{1}{16}.$$

*This result is sharp.*

*Proof* Since

$$H_{2,2}(\mathsf{f}^{-1}) = B_2B_4 - B_3^2.$$

Now, using (3.6), we obtain

$$|H_{2,2}(\mathsf{f}^{-1})| = \frac{1}{256}|q_1|^4.$$

Now, using well-known coefficient bounds for class $\mathcal{P}$, we obtain the required result. This result is sharp for the function $\mathsf{f}_1$ given in (2.10). □

**Theorem 3.3** *Let* $\mathsf{f} \in \mathcal{S}_E^*$ *be given by* (1.1). *Then,*

$$\left|H_{3,1}\left(\mathsf{f}^{-1}\right)\right| \leq \frac{1}{15{,}552}(77 + 29\sqrt{58}).$$

*Proof* Using (3.6)–(3.7) in (2.2), we obtain

$$H_{3,1}\left(\mathsf{f}^{-1}\right) = B_3 B_5 - B_4^2 - B_3^3 = \frac{1}{9216}\psi, \tag{3.10}$$

where

$$\psi = -q_1^6 + 10q_1^4 q_2 + 36q_1^3 q_3 - 46q_1^2 q_2^2.$$

As we see that the functional $H_{3,1}(\mathsf{f}^{-1})$ and the class $\mathcal{S}_E^*$ are rotationally invariant, we may take $q := q_1$, such that $q \in [0,2]$. Then, by using Lemma 2.1 and after some computations we may write

$$\psi = \frac{3}{2}q^6 - \frac{q^2}{2}\left(4 - q^2\right)\left(92 - 5q^2\right)\rho^2 + 18q^3\left(4 - q^2\right)\left(1 - |\rho|^2\right)\eta,$$

where $\rho$ and $\eta$ satisfy the relation $|\rho| \leq 1$, $|\eta| \leq 1$.

First, we consider the case when $q = 0$. Then, $|\psi| = 0$. Next, we assume that $q = 2$, then $|\psi| = 96$, therefore, from (3.10) we obtain $|H_{3,1}(\mathsf{f}^{-1})| = \frac{1}{96}$. Now, suppose that $q \in (0,2)$ and using the well-known triangle inequality, we obtain

$$|\psi| \leq 18q^3\left(4 - q^2\right)\Phi(J,K,L),$$

where

$$\Phi(J,K,L) = \left|J + K\rho + L\rho^2\right| + 1 - |\rho|^2, \quad \rho \in \overline{\mathbb{D}},$$

with $J = \frac{q^3}{12(4-q^2)}$, $K = 0$ and $L = \frac{-(92-5q^2)}{36q}$, then clearly,

$$JL = -\frac{q^2(92 - 5q^2)}{432(4 - q^2)} < 0, \quad \text{for } q \in (0,2).$$

Now,

$$-4JL\left(L^{-2} - 1\right) = -\frac{1}{108}\frac{q^2(46 - 5q)(46 + 5q)}{92 - 5q^2} < 0, \quad q \in (0,2)$$

and

$$2\left(1 - |C|\right) = \frac{(5q + 46)(q - 2)}{18q} < 0, \quad q \in (0,2).$$

This shows that $-4JL(L^{-2}-1) \leq K^2 \wedge |K| < 2(1-|L|)$ and $K^2 < \min\{4(1+|L|)^2, -4JL(L^{-2}-1)\}$ do not hold. We also see that

$$|L|\big(|K| + 4|J|\big) = \frac{q^2(92-5q^2)}{9(48-12q^2)} > 0 \quad q \in (0,2)$$

and

$$|L|\big(|K| - 4|J|\big) = -\frac{q^2(92-5q^2)}{9(48-12q^2)} < 0 \quad q \in (0,2).$$

We conclude that $|L|(|K|+4|J|) \nleq |JK|$ and $|JK| \nleq |L|(|K|-4|J|)$. Applying Lemma 2.3, we can have

$$|\psi| \leq 18q^3\big(4-q^2\big)\left(|L| + |J|\sqrt{1-\frac{K^2}{4JL}}\right) = g_3(q),$$

where

$$g_3(q) = 4q^2\big(q^4 - 14q^2 + 46\big).$$

Clearly, $g_3'(q) = 0$ has roots at $q_1 = 0$ and $q_2 = \frac{1}{3}\sqrt{42-3\sqrt{58}}$. Also, $g_3''(q_2) \approx -518.6$, and so

$$\max \psi(q) = \psi(q_2) = \frac{1232}{27} + \frac{464}{27}\sqrt{58},$$

which from (3.10) gives the required result. □

**Theorem 3.4** *Let* $\mathsf{f} \in \mathcal{S}_E^*$ *and be of the form* (1.1). *Then,*

$$\big|H_{2,3}\big(\mathsf{f}^{-1}\big)\big| \leq \frac{5}{192}.$$

*This inequality is sharp.*

*Proof* Using (3.6)–(3.7) in (2.3), we obtain

$$H_{2,3}\big(\mathsf{f}^{-1}\big) = B_3B_5 - B_4^2 = \frac{1}{36{,}864}\psi, \tag{3.11}$$

where

$$\psi = 5q_1^6 + 40q_1^4q_2 + 144q_1^3q_3 - 184q_1^2q_2^2.$$

As we see that the functional $H_{2,3}(\mathsf{f}^{-1})$ and the class $\mathcal{S}_E^*$ is rotationally invariant, we may take $q := q_1$, such that $q \in [0,2]$. Then, by using Lemma 2.1 and after some computations we may write

$$\psi = 15q^6 - 2q^2\big(4-q^2\big)\big(92-5q^2\big)\rho^2 + 72q^3\big(4-q^2\big)\big(1-|\rho|^2\big)\eta,$$

where $\rho$ and $\eta$ satisfy the relation $|\rho| \leq 1$, $|\eta| \leq 1$.

First, we consider the case when $q = 0$. Then, $|\psi| = 0$. Next, we assume that $q = 2$, then $|\psi| = 960$, therefore, from (3.11) we obtain $|H_{2,3}(\mathsf{f}^{-1})| = \frac{5}{192}$. Now, suppose that $q \in (0,2)$ and using the well-known triangle inequality, we obtain

$$|\psi| \leq 72q^3\left(4-q^2\right)\Phi(J,K,L),$$

where

$$\Phi(J,K,L) = \left|J + K\rho + L\rho^2\right| + 1 - |\rho|^2, \quad \rho \in \overline{\mathbb{D}},$$

with $J = \frac{15q^3}{72(4-q^2)}$, $K = 0$ and $L = \frac{-(92-5q^2)}{36q}$, then clearly

$$JL = \frac{-5q^2(92-5q^2)}{864(4-q^2)} < 0, \quad \text{for } q \in (0,2).$$

Now,

$$-4JL\left(L^{-2}-1\right) = -\frac{5q(46+5q)(46-5q)}{216(92-5q^2)} < 0 \quad q \in (0,2)$$

and

$$2\left(1-|L|\right) = -\frac{5q(46+5q)(2-q)}{q}0 \quad q \in (0,2).$$

Thus, we see that $|K| \nleq 2(1-|L|)$. We also note that $K^2 \nless \min\{4(1+|L|)^2, -4JL(L^{-2}-1)\}$. Also,

$$|L|\left(|K| + 4|J|\right) = \frac{5q^2(92-5q^2)}{216(4-q^2)} > 0, \quad q \in (0,2)$$

and

$$|L|\left(|K| - 4|J|\right) = -\frac{5q^2(92-5q^2)}{216(4-q^2)} < 0, \quad q \in (0,2).$$

Thus, $|L|(|K|+4|J|) \nleq |JK|$ and $|JK| \nleq |L|(|K|-4|J|)$. Now, applying Lemma 2.3, we can have

$$|\psi| \leq 72q^3\left(4-q^2\right)\left(|L| + |J|\sqrt{1-\frac{K^2}{4JL}}\right) = g_4(q),$$

where

$$g_4(q) = q^2\left(11q^4 - 224q^2 + 736\right).$$

Clearly, $g_4'(q) = 2q75q^4 - 448q^2 + 736) > 0$ for $q \in (0,2)$, therefore

$$\max g_4(q) = g_4(2) = 960,$$

which from (3.11) gives the result. The result is sharp for $\mathsf{f}_1$ given in (2.10). Here, $a_2 = 0$, $a_3 = -\frac{1}{4}$, $a_4 = 0$, and $a_5 = \frac{1}{12}$. Now, using (3.2)–(3.5), we have $B_2 = 0$, $B_3 = \frac{1}{4}$, $B_4 = 0$, and $B_5 = \frac{5}{48}$. Lastly, by using (3.11), we have the required result. □

## 4 Conclusion

We have found sharp coefficient results and sharp Hankel determinants of orders two and three for starlike functions related with Euler numbers. We have obtained the fifth coefficient by using the results of Libera and Zlotkiewicz [11]. This approach can be utilized for the investigations of fifth coefficients bound for various classes of univalent functions. Furthermore, we have obtained the bounds on $H_{1,3}(\mathsf{f})$ and $H_{2,3}(\mathsf{f})$ by using the results of Choi et al [5]. This approach is useful for obtaining these kind of results for functions whose second coefficients are zero.

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## Publisher's Note