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Analytical and geometrical approach to the generalized Bessel function

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Abstract

In continuation of Zayed and Bulboacă work in (J. Inequal. Appl. 2022:158, 2022), this paper discusses the geometric characterization of the normalized form of the generalized Bessel function defined by

$$\mathrm{V}_{\rho,r}(z) := z + \sum_{k=1}^{\infty} \frac{(-r)^k}{4^k (1)_k (\rho)_k} z^{k+1}, \quad z \in \mathbb{U},$$

for $\rho, r \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Precisely, we will use a sharp estimate for the Pochhammer symbol, that is, $\Gamma(a+n)/\Gamma(a+1) > (a+\alpha)^{n-1}$, or equivalently $(a)_n > a(a+\alpha)^{n-1}$, that was firstly proved by Baricz and Ponnusamy for $n \in \mathbb{N} \setminus \{1,2\}$, $a > 0$ and $\alpha \in [0, 1.302775637\ldots]$ in (Integral Transforms Spec. Funct. 21(9):641–653, 2010), and then proved in our paper by another method to improve it using the partial derivatives and the two-variable functions' extremum technique for $n \in \mathbb{N} \setminus \{1,2\}$, $a > 0$ and $0 \le \alpha \le \sqrt{2}$, and used to investigate the orders of starlikeness and convexity. We provide the reader with some examples to illustrate the efficiency of our theory. Our results improve, complement, and generalize some well-known (nonsharp) estimates, as seen in the *Concluding Remarks and Outlook* section.

Mathematics Subject Classification: 30C45; 30C50

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1 Introduction and preliminary results

It is well known that classical special functions theory has found applications in different branches of mathematics, physics, engineering, and other sciences more than ever. Indeed, these functions were discovered through the study of physical problems involving vibrations, heat flow, equilibrium, and so on. This branch of mathematics has a respectable history with great names like Gauss, Bessel, Fourier, Euler, Legendre, Riemann, etc. The majority of special functions are solutions of certain second-order linear differential equations, and the associated equations are partial differential equations of second order.

We will restrict our present study to the generalized Bessel function. Bessel functions appear in several problems such as heat conduction in a cylindrical object, electromagnetic waves in a cylindrical waveguide, probability density function of a product of two normally distributed random variables, solutions to the radial Schrödinger equation for a free particle, etc.

Complex analysis constitutes a well-developed and active subject in mathematics to solve some physical problems. One of the important branches of complex analysis is the geometric function theory, which studies the geometric properties of the analytic and harmonic functions. Naturally, the foundation of geometric function theory was supplied by Cauchy, Weierstrass, and Riemann in their pioneering work. Then, in 1851, Riemann replaced an orbital domain with the open disc centered at the origin, and this made this direction of mathematics exceptional. He was the leader in investigating the analyticity and univalence of complex variable functions inside the open unit disc, which are in fact necessary and fundamental thoughts in this field, and led to the well-know Riemann mapping theorem, which is considered one of the most useful theorems in complex analysis and states that if $\Lambda \subset \mathbb{C}$ is a simply connected set and $\Lambda \neq \mathbb{C}$, then there exists a univalent mapping f that maps Λ onto the disk $|z| < 1$. This function is known as the Riemann mapping. Nevertheless, his proof was incomplete, and the full proof was given by Carathéodory only in 1912 using Riemann surfaces. It was simplified by Koebe two years later in a way that did not require these preliminaries (see, for example, [1, 6, 7, 11, 14]).

The hypergeometric functions of one complex variable have long been successfully used in different fields of pure and applied mathematics, as well as physics, due to their importance in proving the well-known Bieberbach conjecture for the coefficients of the normalized univalent functions (see De Branges [5]). There is a comprehensive literature dealing with the geometric properties of several kinds of hypergeometric functions and other functions such as the Bessel function, the generalized Struve function, the Lommel function, the generalized Lommel–Wright function, and the Fox–Wright function. For more extensive information in this direction, see the previous studies [2, 4, 9, 10, 12, 15–17, 20–25] and the references therein.

The contents of the paper is summarized as follows: we first outline that $\Gamma(a+n)/\Gamma(a+1) > (a+\alpha)^{n-1}$, or equivalently $(a)_n > a(a+\alpha)^{n-1}$, which was firstly proved by [4] for $n \in \mathbb{N} \setminus \{1,2\}$, $a > 0$ and $0 \leq \alpha \leq 1.302775637\ldots$, and then proved in our paper by another method to improve it using the partial derivatives and the two-variable functions' extremum technique for $n \in \mathbb{N} \setminus \{1,2\}$, $a > 0$ and $0 \leq \alpha \leq \sqrt{2}$. Then, the previous results were used to determine the orders of starlikeness and convexity of the normalized form of the generalized Bessel function that will be defined later. One can show by elementary reasoning that our results complement, improve, and generalize some familiar (nonsharp) estimates.

Let $\mathcal{S}$ denote the class of univalent functions f in the open unit disc $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$, satisfying the conditions $f(0) = 0 = f'(0) - 1$. Such functions have a power series expansion of the form

$$f(z) = \sum_{k=1}^{\infty} f_k z^k, \quad z \in \mathbb{U}, \text{ with } f_1 = 1. \tag{1.1}$$

It is worth noting that the class of functions that have the expansion (1.1) for some function f that is analytic will be denoted by $\mathcal{A}$.

The class $\mathcal{S}$ has an important property that it is compact (i.e., locally bounded and closed), and the proof of this fact is made by using of the growth and distortion theorems, which control, as their names indicate, the distortion of any function of the class

$\mathcal{S}$ and its derivative by sharp bounds. The most classic example of a function in $\mathcal{S}$ is the so-called Koebe function, that is,

$$k(z) = z(1-z)^{-2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \sum_{k=1}^{\infty} kz^k.$$

Looking at the second expression, one can see that it involves the square of the Cayley transform $z \mapsto (1+z)/(1-z)$, and belongs to $\mathcal{S}$ since it is normalized. It maps $\mathbb{U}$ onto $\mathbb{C}$ slit along the negative half-line of the real axis connecting $-1/4$ to $-\infty$, i.e., $k(\mathbb{U}) = \mathbb{C} \setminus (-\infty, -1/4]$. This function has an essential role since it is extremal in many problems in the theory of univalent functions.

At this stage, if $h \in \mathcal{A}$, then h has the expansion $h(z) = \sum_{k=1}^{\infty} h_k z^k$, $z \in \mathbb{U}$, with $h_1 = 1$ and the *convolution* of f and h is given by $(f * h)(z) := \sum_{k=1}^{\infty} f_k h_k z^k$, $z \in \mathbb{U}$. One can show that the convolution definition follows from (see [6])

$$(f * h)\left(r^2 e^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} f\left(re^{i(\theta-t)}\right) h\left(re^{it}\right) dt, \quad r < 1.$$

We are now in a position to consider the families of starlike and convex functions, and that is why we shall start with the starlike domain with respect to a point and the convex domain, that have particular interest since the image domain of $\mathbb{U}$ under a univalent function has interesting geometric properties. If $f(\mathbb{U})$ is a starlike domain with respect to the origin, then $f \in \mathcal{S}$ is called *starlike with respect to the origin (or briefly, starlike)*; the set of such functions is denoted by $\mathcal{S}^*$. Hereby, a domain $\Lambda \subset \mathbb{C}$ is *starlike with respect to an interior point* $z_0 \in \Lambda$ if the line segment that joins z_0 to any other point of Λ lies totally in Λ. Particularly, if $z_0 = 0$, then Λ is called a *starlike domain*. A function $f \in \mathcal{S}^*$ if and only if $f \in \mathcal{A}$ and $\operatorname{Re}(zf'(z)/f(z)) > 0$, $z \in \mathbb{U}$. The Koebe function and its rotations provide an example of starlike functions and this function is extremal for the class $\mathcal{S}^*$.

Moreover, if $f(\mathbb{U})$ is a convex domain, then $f \in \mathcal{S}$ is called *convex*; the collection of such sets is denoted by $\mathcal{K}$. It is well-known that a domain $\Lambda \subset \mathbb{C}$ is *convex* if the line segment joining any two points of Λ lies totally in Λ. Analytically, convex functions $f \in \mathcal{A}$ can be characterized as satisfying $\operatorname{Re}(zf''(z)/f'(z)) + 1 > 0$, $z \in \mathbb{U}$. The main branch of the function $f(z) = -\log(1-z) \in \mathcal{K}$ since $1 + \operatorname{Re}(zf''(z)/f'(z)) = 1 + \operatorname{Re}(z/(1-z)) > 1/2 > 0$ for all $z \in \mathbb{U}$.

Additionally, $f \in \mathcal{A}$ is *starlike of order* α, $0 \le \alpha \le 1$, with the set of such functions denoted by $\mathcal{S}^*(\alpha)$, if and only if $\operatorname{Re}(zf'(z)/f(z)) > \alpha$, $z \in \mathbb{U}$. Also, it belongs to the class of *convex functions of order* α, denoted by $\mathcal{K}(\alpha)$, if and only if $\operatorname{Re}(zf''(z)/f'(z)) + 1 > \alpha$, $z \in \mathbb{U}$. It is well-known that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) =: \mathcal{S}^*$, $\mathcal{K}(\alpha) \subset \mathcal{K}(0) =: \mathcal{K}$, and $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$.

It is obvious that if $0 \le \alpha < 1$, then $\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(0) =: \mathcal{S}^* \subsetneq \mathcal{S}$. For $\alpha < 0$, we have $\mathcal{S}^*(\alpha) \not\subset \mathcal{S}$, and for proving this fact let us consider the function $\widehat{f}(z) = z + az^2$ with $a = 0.6$. A simple computation show that

$$\operatorname{Re}\frac{z\widehat{f}'(z)}{\widehat{f}(z)} > -\frac{1}{2}, \quad z \in \mathbb{U},$$

that is, $\widehat{f} \in \mathcal{S}^*(-1/2)$, but $\widehat{f}$ is not univalent in $\mathbb{U}$ because $\widehat{f}'(z) = 0$ for $z = -1/1.2 \in \mathbb{U}$.

The Marx–Strohhäcker theorem [8, 19] shows that $\mathcal{K} \subset \mathcal{S}^*(1/2)$, but $\mathcal{K}(\alpha) \not\subset \mathcal{S}^*$ for $\alpha < 0$. Thus, in [13] the authors show that for each $\alpha \in [-1/2, 0)$ there exists a function $\widetilde{f}_\alpha \in \mathcal{A}$ such that $\widetilde{f}_\alpha \in \mathcal{K}(\alpha)$ but $\widetilde{f}_\alpha \notin \mathcal{S}^*$.

A well-known homogeneous differential equation of the second-order is given explicitly by (see, for details, [3])

$$z^2\zeta''(z) + qz\zeta'(z) + \left[rz^2 - p^2 + (1-q)p\right]\zeta(z) = 0, \tag{1.2}$$

whose solutions are generalized Bessel functions, with $r \in \mathbb{C}$ and $p, q \in \mathbb{R}$. A particular solution of (1.2) is the generalized Bessel function of order p that has the series expansion

$$\zeta_{p,q,r}(z) = \sum_{k=0}^{\infty} \frac{(-r)^k}{\Gamma(p+k+\frac{q+1}{2})\Gamma(k+1)} \left(\frac{z}{2}\right)^{2k+p}. \tag{1.3}$$

It is worth of note that (1.2) has a particular interest because it enables us to know extensive information concerning the Bessel, spherical Bessel, and modified Bessel functions. Additionally, the series (1.3) converges everywhere, while it is not univalent in $\mathbb{U}$. Bear in mind that special values of p, q, and r will give us the famous Bessel, spherical Bessel, and modified Bessel functions. For example, setting $q = r = 1$, we get the Bessel function, which can be defined as

$$\mathrm{J}_p(z) := \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+p+1)\Gamma(k+1)} \left(\frac{z}{2}\right)^{p+2k}, \quad z \in \mathbb{C}.$$

For $q = 1 = -r$, the modified Bessel function will follow that can be expanded as

$$\mathrm{I}_p(z) := \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+p+1)\Gamma(k+1)} \left(\frac{z}{2}\right)^{p+2k}, \quad z \in \mathbb{C},$$

whilst for $r = -1$ and $q = 2$, we obtain the spherical Bessel function given by

$$\mathrm{S}_p(z) := \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+p+\frac{3}{2})\Gamma(k+1)} \left(\frac{z}{2}\right)^{p+2k}, \quad z \in \mathbb{C}.$$

According to the series expansion $\zeta_{p,q,r}$, we have $\zeta_{p,q,r} \notin \mathcal{S}$, therefore we consider the following transformation:

$$\mathrm{v}_{p,q,r}(z) := 2^p \Gamma\left(p + \frac{q+2}{2}\right) z^{-\frac{p}{2}} \zeta_{p,q,r}(\sqrt{z}). \tag{1.4}$$

From (1.4), the series of $\mathrm{v}_{p,q,r}$ has the representation

$$\mathrm{v}_{p,q,r}(z) = \sum_{k=0}^{\infty} \frac{(-r)^k}{4^k (1)_k (p + \frac{q+2}{2})_k} z^k,$$

where $p + (q+2)/2 \notin \{0, -1, -2, \dots\}$ and $(\sigma)_k$ stands for the *Pochhammer symbol* defined by

$$(\sigma)_k := \begin{cases} 1 & \text{if } k = 0, \\ \sigma(\sigma+1)(\sigma+2)\cdots(\sigma+k-1) & \text{if } k \in \mathbb{N}. \end{cases}$$

Having in mind the prior representations, it can be easily seen the following definition:

Definition 1.1 For $p, q \in \mathbb{C}$ and $r \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, the normalization of the of $\mathrm{v}_{p,q,r}$ is given explicitly by

$$\mathrm{V}_{\rho,r}(z) := z \cdot \mathrm{v}_{p,q,r}(z) = z + \sum_{k=1}^{\infty} \frac{(-r)^k}{4^k (1)_k (\rho)_k} z^{k+1}, \quad z \in \mathbb{U}, \tag{1.5}$$

where $\rho := p + (q+2)/2 \in (0, +\infty)$.

We will require the next technical lemma in our investigation. This lemma was first proved in [4] for $n \in \mathbb{N} \setminus \{1, 2\}$, $a > 0$, and $0 \le \alpha \le \alpha_0$, where $\alpha_0 \simeq 1.302775637\ldots$ is the greatest root of the equation $\alpha^2 + \alpha - 3 = 0$. Here, we employ another method to improve it using the partial derivatives and the two-variable functions' extremum technique. It is shown that for $n \in \mathbb{N} \setminus \{1, 2\}$, $a > 0$, and $0 \le \alpha \le \sqrt{2}$, the following inequality is satisfied.

Lemma 1.1 *If* $a > 0$, $0 \le \alpha \le \alpha_0 = \sqrt{2} \simeq 1.4142\ldots$, *and* $n \in \mathbb{N} \setminus \{1, 2\}$, *then the following sharp result holds*:

$$(a)_n > a(a+\alpha)^{n-1}. \tag{1.6}$$

Proof Let $f : (0, +\infty) \times [3, +\infty) \to \mathbb{R}$ be defined by

$$f(a, n) = \frac{\Gamma(a+n)}{\Gamma(a+1)} (a+\alpha)^{1-n} - 1, \tag{1.7}$$

where $0 \le \alpha \le 2$ is a given number. It is easy to check that

$$\frac{\partial}{\partial n} f(a, n) = \frac{(a+\alpha)^{1-n} \Gamma(a+n)(\Psi(a+n) - \ln(a+\alpha))}{\Gamma(a+1)},$$
$$(a, n) \in (0, +\infty) \times [3, +\infty). \tag{1.8}$$

Using that

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt, \quad \operatorname{Re} z > 0,$$

it follows that $\Gamma(a+1) > 0$, $\Gamma(a+n) > 0$ for all $(a, n) \in (0, +\infty) \times [3, +\infty)$. Therefore, since $\alpha \ge 0$, according to (1.8), the sign of $\frac{\partial}{\partial n} f(a, n)$ will match the sign of

$$G(a, n) := \Psi(a+n) - \ln(a+\alpha).$$

Since it is well-known that

$$\ln x - \frac{1}{x} \le \psi(x) \le \ln x - \frac{1}{2x}, \quad x > 0, \tag{1.9}$$

from the above definition we get

$$G(a, n) \ge \ln(a+n) - \frac{1}{a+n} - \ln(a+\alpha) = \ln \frac{a+n}{a+\alpha} - \frac{1}{a+n}$$

$$\geq \ln\frac{a+3}{a+\alpha}-\frac{1}{a+3}=: H(a), \quad a>0,\ n\geq 3. \tag{1.10}$$

Hence

$$H'(a)=\frac{1}{a+3}\left(\frac{\alpha-3}{a+\alpha}+\frac{1}{a+3}\right), \quad a>0,\ 0\leq\alpha\leq 2,$$

and we have the following equivalences:

$$H'(a)<0 \quad\Longleftrightarrow\quad \frac{\alpha-3}{a+\alpha}<-\frac{1}{a+3} \quad\Longleftrightarrow\quad \alpha<2+\frac{1}{a+4}, \quad a>0,\ \ 0\leq\alpha\leq 2.$$

Because

$$\inf\left\{2+\frac{1}{a+4}:a>0\right\}=2,$$

if $\alpha\leq 2$, the last inequality above is satisfied, i.e., $H'(a)<0$ for all $a>0$ whenever $0\leq\alpha\leq 2$. Thus, the function H is strictly decreasing on $(0,+\infty)$, which implies

$$H(a)>\lim_{x\to+\infty}H(x)=0, \quad a>0.$$

This inequality, combined with (1.8) and (1.10), implies that

$$\frac{\partial}{\partial n}f(a,n)>0, \quad (a,n)\in(0,+\infty)\times[3,+\infty),$$

hence $f(a,n)$ is a strictly increasing function on n if $n\in[3,+\infty)$ for all $a>0$, that is,

$$f(a,n)\geq f(a,3)=\frac{\Gamma(a+3)}{(a+\alpha)^2\Gamma(a+1)}-1=\frac{(a+1)(a+2)}{(a+\alpha)^2}-1, \quad a>0. \tag{1.11}$$

Using the facts that $a>0$ and $\alpha\geq 0$, we obtain

$$\begin{aligned}\frac{(a+1)(a+2)}{(a+\alpha)^2}-1>0 \quad&\Longleftrightarrow\quad |a+\alpha|<\sqrt{(a+1)(a+2)}\\ &\Longleftrightarrow\quad \alpha<\sqrt{(a+1)(a+2)}-a=: h(a),\end{aligned} \tag{1.12}$$

and it is easy to check the inequality

$$\begin{aligned}h'(a)&=\frac{2a+3}{2\sqrt{(a+1)(a+2)}}-1=\frac{2a+3-2\sqrt{(a+1)(a+2)}}{2\sqrt{(a+1)(a+2)}}\\ &=\frac{1}{2\sqrt{(a+1)(a+2)}(2a+3+2\sqrt{(a+1)(a+2)})}>0, \quad a>0.\end{aligned}$$

This implies that h is strictly increasing on $(0,+\infty)$, hence

$$\lim_{x\to 0^+}h(x)=\sqrt{2}<h(a)<\lim_{x\to+\infty}h(x)=\frac{3}{2}, \quad a>0. \tag{1.13}$$

Consequently, if $\alpha \leq \sqrt{2}$, by using the inequalities (1.11) and (1.12), it follows that

$$f(a,n) > 0, \quad a > 0,\ n \geq 3. \tag{1.14}$$

The left-hand side of the latter inequality could not be improved because

$$\inf\left\{\psi(x) - \ln x + \frac{1}{x} : x > 0\right\} = 0,$$

where we used the fact that $x \mapsto \psi(x) - \ln x + \frac{1}{x}$ is strictly decreasing and positive on $(0, +\infty)$ and, using the inequality (1.9), we have

$$\lim_{x\to+\infty}\left(\psi(x) - \ln x + \frac{1}{x}\right) = 0.$$

From here, using the inequalities (1.10), (1.11), and (1.13), the value $\sqrt{2}$ is the maximum possible such that the inequality (1.14) holds, hence our result is sharp (i.e., the best possible) and the proof is complete. □

2 Sufficient conditions for starlikeness and convexity of order β

The first two theorems of the present section have a particular interest. They provide the reader with the orders of starlikeness and convexity of $\mathrm{V}_{\rho,r}$ that slightly improve the results given in [23], as seen in Sect. 5, items 2 and 3.

Theorem 2.1 *Assume that* $\rho > 0$, *and let* $r \in \mathbb{C}^*$ *with*

$$0 < |r| < \frac{4\rho}{1+\rho} =: r_*. \tag{2.1}$$

If

$$\beta \leq 1 - \frac{|r|}{\rho(4-|r|) - |r|} =: \beta_*, \tag{2.2}$$

then $\mathrm{V}_{\rho,r} \in \mathcal{S}^*(\beta)$.

Proof To prove that $\mathrm{V}_{\rho,r} \in \mathcal{S}^*(\beta)$, it is sufficient to show that

$$\left|\frac{z(\mathrm{V}_{\rho,r}(z))'}{\mathrm{V}_{\rho,r}(z)} - 1\right| < 1 - \beta, \quad z \in \mathbb{U}, \tag{2.3}$$

where $\beta \leq 1$. From the familiar triangle inequality and the maximum modulus theorem for an analytic function, we get

$$\begin{aligned}\left|\left(\mathrm{V}_{\rho,r}(z)\right)' - \frac{\mathrm{V}_{\rho,r}(z)}{z}\right| &= \left|\sum_{k=1}^{\infty}\frac{k(-r)^k}{4^k k!(\rho)_k}z^k\right| < \sup_{\theta\in[0,2\pi]}\left|\sum_{k=1}^{\infty}\frac{k(-r)^k}{4^k k!(\rho)_k}e^{ik\theta}\right| \\ &\leq \frac{\Gamma(\rho+1)}{\rho}\sum_{k=1}^{\infty}\frac{k|r|^k}{4^k\Gamma(k+\rho)\Gamma(k+1)}, \quad z \in \mathbb{U},\end{aligned} \tag{2.4}$$

where $\rho > 0$.

Letting the function $\phi : [1, +\infty) \to \mathbb{R}$ be defined by

$$\phi(s) := \frac{s}{\Gamma(s+\rho)\Gamma(s+1)},$$

a simple computation shows that

$$\phi(k+1) - \phi(k) = \frac{1 - k^2 - k\rho}{\Gamma(k+1+\rho)\Gamma(k+1)} < 0, \quad k \in \mathbb{N},$$

for all $\rho > 0$. Consequently, ϕ is a strictly decreasing function on $\mathbb{N}$, hence

$$\frac{s}{\Gamma(s+\rho)\Gamma(s+1)} \leq \phi(1) = \frac{1}{\Gamma(\rho+1)}, \quad s \in \mathbb{N},$$

and, using this inequality, from (2.4), we get

$$\left| \left(\mathrm{V}_{\rho,r}(z)\right)' - \frac{\mathrm{V}_{\rho,r}(z)}{z} \right| < \frac{1}{\rho} \sum_{k=1}^{\infty} \left(\frac{|r|}{4} \right)^k = \frac{|r|}{\rho(4-|r|)}, \quad z \in \mathbb{U}. \tag{2.5}$$

Also, we should assume for the above result that $|r|/(\rho(4-|r|)) > 0$, equivalent to $0 < |r| < 4$, which holds because of (2.9) and the assumption $0 < |r| < 4\rho/(1+\rho)$. We mention that the case $|r| = 0$ is trivial because $\mathrm{V}_{\rho,0}(z) = z$ is the identity function.

On the other hand, by using the theorem of *the maximum of the modulus for an analytic function*, we have

$$\begin{aligned} \left| \frac{\mathrm{V}_{\rho,r}(z)}{z} \right| &= \left| 1 + \sum_{k=1}^{\infty} \frac{(-r)^k}{4^k k! (\rho)_k} z^k \right| > 1 - \sup_{\theta \in [0,2\pi]} \left| \sum_{k=1}^{\infty} \frac{(-r)^k}{4^k k! (\rho)_k} e^{ik\theta} \right| \\ &\geq 1 - \frac{\Gamma(\rho+1)}{\rho} \sum_{k=1}^{\infty} \frac{|r|^k}{4^k \Gamma(k+1)\Gamma(k+\rho)}, \quad z \in \mathbb{U}, \end{aligned} \tag{2.6}$$

where $\theta \in \mathbb{R}$, $\rho > 0$. The function $1/(\Gamma(k+1)\Gamma(k+\rho))$ is strictly decreasing for $k \in \mathbb{N}$, which leads to

$$\left| \frac{\mathrm{V}_{\rho,r}(z)}{z} \right| > 1 - \frac{\Gamma(\rho+1)}{\rho} \sum_{k=1}^{\infty} \left(\frac{|r|}{4} \right)^k \cdot \frac{1}{\Gamma(\rho+1)\Gamma(2)} = \frac{\rho(4-|r|) - |r|}{\rho(4-|r|)}, \quad z \in \mathbb{U}, \tag{2.7}$$

where

$$\frac{\rho(4-|r|) - |r|}{\rho(4-|r|)} > 0. \tag{2.8}$$

Note that (2.8) holds because $\rho > 0$ and

$$|r| < \min\left\{ 4; \frac{4\rho}{1+\rho} \right\} = \frac{4\rho}{1+\rho} \tag{2.9}$$

for all $\rho \in \mathbb{C}$, hence this inequality holds by the assumptions of the theorem. Since

$$\left| \frac{z\mathrm{V}'_{\rho,r}(z)}{\mathrm{V}_{\rho,r}(z)} - 1 \right| = \left| \mathrm{V}'_{\rho,r}(z) - \frac{\mathrm{V}_{\rho,r}(z)}{z} \right| \cdot \left| \frac{z}{\mathrm{V}_{\rho,r}(z)} \right|, \quad z \in \mathbb{U},$$

from (2.5) and (2.7), according to the assumption (2.2), we deduce that

$$\left|\frac{z\mathrm{V}'_{\rho,r}(z)}{\mathrm{V}_{\rho,r}(z)}-1\right|<\frac{|r|}{\rho(4-|r|)-|r|}\leq 1-\beta,\quad z\in\mathbb{U}.$$

Finally, from (2.3), it follows that $\mathrm{V}_{\rho,r}\in\mathcal{S}^*(\beta)$. □

Theorem 2.2 *Assume that* $\rho\geq 1/2$ *and let* $r\in\mathbb{C}^*$ *with*

$$0<|r|<\frac{4\rho}{2+\rho}=:r_c. \tag{2.10}$$

If

$$\beta\leq 1-\frac{2|r|}{\rho(4-|r|)-2|r|}=:\beta_c, \tag{2.11}$$

then $\mathrm{V}_{\rho,r}\in\mathcal{K}(\beta)$.

Proof We could check immediately that the condition

$$\left|\frac{z\mathrm{V}''_{\rho,r}(z)}{\mathrm{V}'_{\rho,r}(z)}\right|<1-\beta,\quad z\in\mathbb{U}, \tag{2.12}$$

implies $\mathrm{V}_{\rho,r}\in\mathcal{K}(\beta)$, where $0\leq\beta<1$.

Using the triangle inequality and the theorem of the maximum of the modulus for an analytic function, we get

$$\begin{aligned}\left|z\mathrm{V}''_{\rho,r}(z)\right|&=\left|\sum_{k=1}^{\infty}\frac{k(k+1)(-r)^k}{4^k k!(\rho)_k}z^k\right|<\sup_{\theta\in[0,2\pi]}\left|\sum_{k=1}^{\infty}\frac{k(k+1)(-r)^k}{4^k k!(\rho)_k}e^{ik\theta}\right|\\&\leq\frac{\Gamma(\rho+1)}{\rho}\sum_{k=1}^{\infty}\frac{k(k+1)|r|^k}{4^k\Gamma(k+\rho)\Gamma(k+1)},\quad z\in\mathbb{U},\end{aligned} \tag{2.13}$$

where $\rho>0$.

Defining the function $\omega:[1,+\infty)\to\mathbb{R}$ by

$$\omega(s):=\frac{s(s+1)}{\Gamma(s+\rho)\Gamma(s+1)},$$

we get

$$\omega(k+1)-\omega(k)=-\frac{\varphi(k)}{\Gamma(k+1+\rho)\Gamma(k+1)},\quad k\in\mathbb{N}, \tag{2.14}$$

where

$$\varphi(k):=k^3+(\rho+1)k^2+(\rho-1)k-2.$$

Since it is easy to check that if $\rho>0$,

$$\varphi'(s)=3s^2+2(\rho+1)s+\rho-1>0,\quad s\in[1,+\infty),$$

it follows that

$$\min\{\varphi(s): s \geq 1\} = \varphi(1) = 2\rho - 1 \geq 0$$

wherever $\rho \geq 1/2$. Therefore, according to (2.14), we get

$$\omega(k+1) - \omega(k) \leq 0, \quad k \in \mathbb{N},$$

hence ω is a decreasing function on $\mathbb{N}$ and

$$\frac{k(k+1)}{\Gamma(\rho+k)\Gamma(k+1)} \leq \omega(1) = \frac{2}{\Gamma(\rho+1)}, \quad k \in \mathbb{N}.$$

Using this inequality, from (2.13) we get

$$\left|z\mathrm{V}''_{\rho,r}(z)\right| < \frac{2}{\rho}\sum_{k=1}^{\infty}\left(\frac{|r|}{4}\right)^k = \frac{2|r|}{\rho(4-|r|)}, \quad z \in \mathbb{U}. \tag{2.15}$$

In the above inequality, we have $2|r|/(\rho(4-|r|)) > 0$, equivalent to $0 < |r| < 4$, which holds because of our assumptions.

Furthermore, from the theorem of the maximum of the modulus for an analytic function, we have

$$\begin{aligned}\left|\mathrm{V}'_{\rho,r}(z)\right| &= \left|1 + \sum_{k=1}^{\infty}\frac{(k+1)(-r)^k}{4^k(1)_k(\rho)_k}z^k\right| > 1 - \sup_{\theta\in[0,2\pi]}\left|\sum_{k=1}^{\infty}\frac{(k+1)(-r)^k}{4^k(1)_k(\rho)_k}e^{ik\theta}\right| \\ &\geq 1 - \frac{\Gamma(\rho+1)}{\rho}\sum_{k=1}^{\infty}\frac{(k+1)|r|^k}{4^k\Gamma(k+1)\Gamma(k+\rho)}, \quad z \in \mathbb{U},\end{aligned}$$

where $\theta \in \mathbb{R}$. Since the function $k(k+1)/(\Gamma(k+1)\Gamma(k+\rho))$ is strictly decreasing for $k \in \mathbb{N}$, it follows that the function $(k+1)/(\Gamma(k+1)\Gamma(k+\rho))$ is also strictly decreasing for $k \in \mathbb{N}$, thus

$$\left|\mathrm{V}'_{\rho,r}(z)\right| > 1 - \frac{\Gamma(\rho+1)}{\rho}\sum_{k=1}^{\infty}\left(\frac{|r|}{4}\right)^k \cdot \frac{2}{\Gamma(\rho+1)\Gamma(2)} = \frac{\rho(4-|r|)-2|r|}{\rho(4-|r|)}, \quad z \in \mathbb{U}, \tag{2.16}$$

where

$$\frac{\rho(4-|r|)-2|r|}{\rho(4-|r|)} > 0. \tag{2.17}$$

It is worth noting that (2.17) holds because $\rho > 0$ and

$$|r| < \min\left\{4; \frac{4\rho}{2+\rho}\right\} = \frac{4\rho}{2+\rho}, \tag{2.18}$$

for all $\rho > 0$, hence this inequality holds by the assumptions of the theorem. Since

$$\left|\frac{z\mathrm{V}''_{\rho,r}(z)}{\mathrm{V}'_{\rho,r}(z)}\right| = \left|z\mathrm{V}''_{\rho,r}(z)\right| \cdot \left|\frac{1}{\mathrm{V}'_{\rho,r}(z)}\right|, \quad z \in \mathbb{U},$$

from (2.15) and (2.16), by using the assumption (2.11), we deduce that

$$\left|\frac{z\mathrm{V}''_{\rho,r}(z)}{\mathrm{V}'_{\rho,r}(z)}\right| < \frac{2|r|}{\rho(4-|r|)-2|r|} \leq 1-\beta, \quad z \in \mathbb{U},$$

and, according to (2.12), it follows that $\mathrm{V}_{\rho,r} \in \mathcal{K}(\beta)$. □

3 Special cases and examples

Remark 3.1 The above two theorems could be summarized as follows:

(i) Suppose that $\rho > 0$ and let $r \in \mathbb{C}^*$ be such that the conditions (2.1) and (2.2) are satisfied. Then, $\mathrm{V}_{\rho,r} \in \mathcal{S}^*(\beta)$.

(ii) Suppose that $\rho \in \mathbb{C}$ with $\rho \geq 1/2$, and let $r \in \mathbb{C}^*$ be such that the conditions (2.10) and (2.11) are satisfied. Then, $\mathrm{V}_{\rho,r} \in \mathcal{K}(\beta)$.

Remark 3.2 We could make the following remarks regarding the restrictions for the parameters r and ρ in both theorems:

(i) The restriction $\rho \geq 1/2$ in Theorem 2.2 is stronger than the condition $\rho > 0$ in Theorem 2.1.

(ii) Regarding the assumptions (2.1) and (2.10) for the upper bound of the parameter $r \in \mathbb{C}^*$, for all $\rho > 0$ we have

$$0 < r_c < r_*.$$

(iii) According to (2.9) and (2.18), we get

$$\max\left\{4; \frac{4\rho}{1+\rho}; \frac{4\rho}{2+\rho}\right\} = 4,$$

hence each assumption (2.1) or (2.10) implies $|r| < 4$. Consequently, for all $r, \rho > 0$, we obtain that

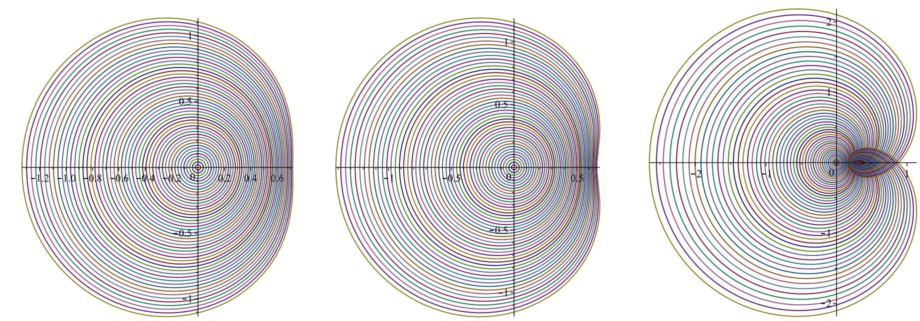
$$0 < \beta_c < \beta_*.$$

Next we will show a few particular cases for Theorems 2.1 and 2.2 obtained for different choices of the parameters r, ρ, and β.

Example 3.1 (i) Taking $\rho = 1.2591$, $r = 1.5453$, and $\beta = 0$ in Theorem 2.1, both assumptions (2.1) and (2.2) are satisfied, hence $\mathrm{V}_{1.2591,1.5453} \in \mathcal{S}^*(0) =: \mathcal{S}^*$, and the image of the open unit disc $\mathbb{U}$ is shown in Fig. 1(A).

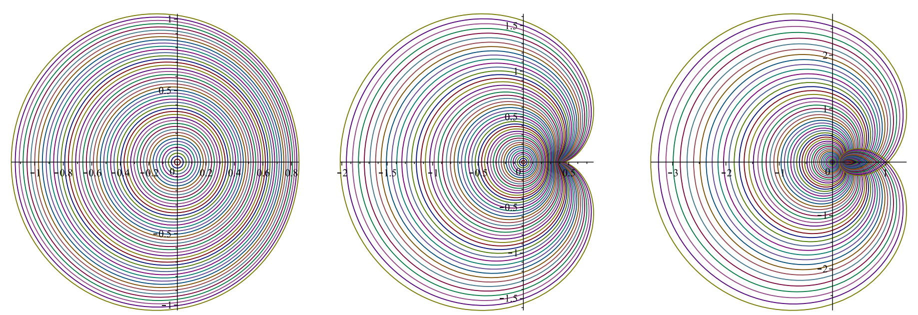
(ii) For the values $\rho = 1.259$, $r = 1.9$, and $\beta = -1.555$, all assumptions of Theorem 2.1 are satisfied, therefore $\mathrm{V}_{1.259,1.9} \in \mathcal{S}^*(-1.555)$. As seen in Fig. 1(B), the function $\mathrm{V}_{1.259,1.9}$ is not convex, but, according to this theorem, it is starlike of order $\beta = -1.555$.

(iii) If we take $\rho = 1.259$, $r = 6$, and $\beta = 0.9$ in Theorem 2.1, then it is easy to check that the assumption (2.1) is not satisfied. As seen in Fig. 1(C), the function $\mathrm{V}_{1.259,6} \notin \mathcal{S}^*(0.9) \subset \mathcal{S}$ because it is not univalent in $\mathbb{U}$. Thus, we could see from the mentioned figure that there exists a subset of $\mathbb{C}$ that it is twice overlapped by $\mathrm{V}_{1.259,6}(\mathbb{U})$, therefore form here and item (ii), we see the assumption (2.1) is essential for the validity of this theorem.



(A) The image of $V_{1.2591,1.5453}(\mathbb{U})$ (B) The image of $V_{1.259,1.9}(\mathbb{U})$ (C) The image of $V_{1.259,6}(\mathbb{U})$

Figure 1 Illustrations for Example 3.1



(A) The image of $V_{2.424,1.509}(\mathbb{U})$ (B) The image of $V_{2.424,7.5}(\mathbb{U})$ (C) The image of $V_{2.424,14}(\mathbb{U})$

Figure 2 Illustrations for Example 3.2

Example 3.2 (i) For $\rho = 2.424$, $r = 1.509$, and $\beta = 0$ in Theorem 2.2, the assumptions (2.10) and (2.11) hold for these values, hence $V_{2.424,1.509} \in \mathcal{K}(0) =: \mathcal{K}$, while the image of $V_{2.424,1.509}(\mathbb{U})$ is shown in Fig. 2(A).

(ii) Considering the values $\rho = 2.424$, $r = 7.5$, and $\beta = 0.9$, it is easy to check that the assumption (2.10) is not satisfied. Therefore, in this case we cannot use the result of Theorem 2.2. Moreover, we see from Fig. 2(B) that $V_{2.424,7.5} \notin \mathcal{K}(0) \subset \mathcal{K}(0.9)$ because the domain $V_{2.424,7.5}(\mathbb{U})$ is not convex.

(iii) If $\rho = 2.424$, $r = 14$, and $\beta = 0.9$, then we may see that the assumption (2.10) is not satisfied, hence for these values we cannot use the conclusion of Theorem 2.2. We may see in Fig. 2(C) that the function $V_{2.424,14} \notin \mathcal{K}(0.9) \subset \mathcal{K}$ because it is not univalent in $\mathbb{U}$. From this figure we see that there exists a subset of $\mathbb{C}$ that it is twice overlapped by $V_{2.424,14}(\mathbb{U})$, thus form here and item (ii) we see the assumption (2.10) is essential for Theorem 2.2.

4 Sufficient conditions for starlikeness and convexity of order β using Silverman's theorem

The theorems of this section allow us to obtain sufficient conditions on the parameters ρ and r such that $V_{\rho,r}$ is in the classes of starlike and convex functions of order β by making use of Lemma 1.1 and the well-known result of H. Silverman [18, Theorem 1].

Theorem 4.1 *Let*

$$\begin{aligned} W_{\rho,r}(\beta) := \Bigg(1 - \frac{|r|}{4\rho} - \frac{|r|^2}{32\rho(\rho+1)} \\ - \frac{|r|^3}{16\rho(1+\sqrt{2})(\rho+\sqrt{2})(8+4\sqrt{2}+4\rho+4\rho\sqrt{2}-|r|)}\Bigg)\beta \\ + \frac{|r|}{2\rho} + \frac{3|r|^2}{32\rho(\rho+1)} + \frac{3|r|^3}{64\rho(1+\sqrt{2})(\rho+\sqrt{2})(2+\sqrt{2}+\rho+\rho\sqrt{2}-|r|)} \\ + \frac{|r|^3}{16\rho(1+\sqrt{2})(\rho+\sqrt{2})(8+4\sqrt{2}+4\rho+4\rho\sqrt{2}-|r|)} - 1, \end{aligned}$$

where $\rho > 0$ *and* $r \in \mathbb{C}^*$. *If there exists a* $\beta < 1$ *such that*

$$W_{\rho,r}(\beta) \le 0, \tag{4.1}$$

then $\mathrm{V}_{\rho,r} \in \mathcal{S}^*(\beta)$.

Proof A well-known result from [18, Theorem 1] shows that if f has the form (1.1) and satisfies $\sum_{k=2}^{\infty}(k-\beta)|f_k| \le 1-\beta$, then $f \in \mathcal{S}^*(\beta)$. Thus, according to (1.5), it sufficient to show that

$$\mathrm{A}_1 := \sum_{k=2}^{\infty}(k-\beta)\left|\frac{(-r)^{k-1}}{4^{k-1}(1)_{k-1}(\rho)_{k-1}}\right| \le 1-\beta.$$

Since $\rho > 0$ and $r \in \mathbb{C}^*$, we have

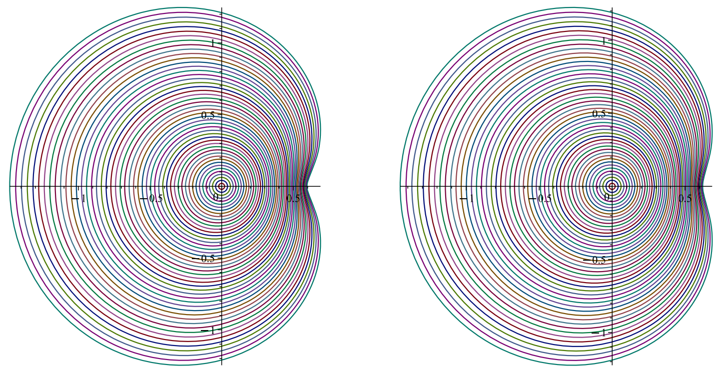
$$\begin{aligned} \mathrm{A}_1 &= \sum_{k=2}^{\infty}(k-\beta)\frac{|r|^{k-1}}{4^{k-1}(1)_{k-1}(\rho)_{k-1}} = \sum_{k=1}^{\infty}(k+1-\beta)\frac{|r|^k}{4^k(1)_k(\rho)_k} \\ &= \frac{(2-\beta)|r|}{4\rho} + \frac{(3-\beta)|r|^2}{32\rho(\rho+1)} + \sum_{k=3}^{\infty}\frac{k|r|^k}{4^k(1)_k(\rho)_k} + (1-\beta)\sum_{k=3}^{\infty}\frac{|r|^k}{4^k(1)_k(\rho)_k}. \end{aligned}$$

It is easy to show by induction that $k \le (3/64)4^k$ for all $k \in \mathbb{N}\setminus\{1,2\}$. Moreover, using Lemma 1.1, we have $(1)_k > (1+\alpha)^{k-1}$, $k \in \mathbb{N}\setminus\{1,2\}$, and $0 \le \alpha \le \sqrt{2}$. Since

$$\max\left\{(1+\alpha)^{k-1} : 0 \le \alpha \le \sqrt{2}\right\} = (1+\sqrt{2})^{k-1},$$

it follows that $(1)_k \ge (1+\sqrt{2})^{k-1}$, $\mathbb{N}\setminus\{1,2\}$. Similarly, $(\rho)_k \ge \rho(\rho+\sqrt{2})^{k-1}$, $\rho > 0$, and $k \in \mathbb{N}\setminus\{1,2\}$, hence from the assumption (4.1), it follows that

$$\begin{aligned} \mathrm{A}_1 &\le \frac{(2-\beta)|r|}{4\rho} + \frac{(3-\beta)|r|^2}{32\rho(\rho+1)} + \frac{3|r|}{64\rho}\sum_{k=3}^{\infty}\frac{|r|^{k-1}}{(1+\sqrt{2})^{k-1}(\rho+\sqrt{2})^{k-1}} \\ &\quad + \frac{(1-\beta)|r|}{4\rho}\sum_{k=3}^{\infty}\frac{|r|^{k-1}}{4^{k-1}(1+\sqrt{2})^{k-1}(\rho+\sqrt{2})^{k-1}} \\ &= \frac{(2-\beta)|r|}{4\rho} + \frac{(3-\beta)|r|^2}{32\rho(\rho+1)} + \frac{3|r|^3}{64\rho(1+\sqrt{2})(\rho+\sqrt{2})(2+\sqrt{2}+\rho+\rho\sqrt{2}-|r|)} \end{aligned}$$



(A) The image of $\mathrm{V}_{0.6240754208,1.1}(\mathbb{U})$ (B) The image of $\mathrm{V}_{0.7229723004,1.2}(\mathbb{U})$

Figure 3 Illustrations for Example 4.1

$$+\frac{|r|^3(1-\beta)}{16\rho(1+\sqrt{2})(\rho+\sqrt{2})(8+4\sqrt{2}+4\rho+4\rho\sqrt{2}-|r|)}\leq 1-\beta,$$

and thus the proof is complete. □

Example 4.1 The next special cases of the above theorem represent two situations that could treated using the above theorem.

(i) For the case $\beta = 0$ and $r = 1.1$, the inequality (4.1) gives $\rho \geq 0.6240754208\ldots$, hence we could formulate the following result:

If $r = 1.1$ and $\rho \geq 0.6240754208\ldots$, then $\mathrm{V}_{\rho,1.1} \in \mathcal{S}^*(0) = \mathcal{S}^*$.

(ii) A similar case obtained for the case $\beta = 0.1$ and $r = 1.2$ is the following:

If $r = 1.2$ and $\rho \geq 0.7229723004\ldots$, then $\mathrm{V}_{\rho,1.2} \in \mathcal{S}^*(0.1)$.

As seen in Fig. 3, in both on the above cases the functions $\mathrm{V}_{\rho,r}$ are starlike but not convex in $\mathbb{U}$.

Theorem 4.2 *Let*

$$\begin{aligned}T_{\rho,r}(\beta) := \biggl(1-\frac{|r|}{2\rho}-\frac{3|r|^2}{32\rho(\rho+1)}-\frac{3|r|^3}{64\rho(1+\sqrt{2})(\rho+\sqrt{2})(2+\sqrt{2}+\rho+\rho\sqrt{2}-|r|)}\\ -\frac{|r|^3}{16\rho(1+\sqrt{2})(\rho+\sqrt{2})(8+4\sqrt{2}+4\rho+4\rho\sqrt{2}-|r|)}\biggr)\beta\\ +\frac{|r|}{\rho}+\frac{9|r|^2}{32\rho(\rho+1)}+\frac{15|r|^3}{64\rho(1+\sqrt{2})(\rho+\sqrt{2})(2+\sqrt{2}+\rho+\rho\sqrt{2}-|r|)}\\ +\frac{|r|^3}{16\rho(1+\sqrt{2})(\rho+\sqrt{2})(8+4\sqrt{2}+4\rho+4\rho\sqrt{2}-|r|)}-1,\end{aligned}$$

where $\rho > 0$ *and* $r \in \mathbb{C}^*$. *If there exists a* $\beta < 1$ *such that*

$$T_{\rho,r}(\beta) \leq 0, \tag{4.2}$$

then $\mathrm{V}_{\rho,r} \in \mathcal{K}(\beta)$.

Proof As stated above, if f is of the form (1.1) and satisfies $\sum_{k=2}^{\infty}(k-\beta)|f_k| \leq 1-\beta$, then $f \in \mathcal{S}^*(\beta)$. From the Alexander duality relation, i.e., for a function f of the form (1.1), we have $f \in \mathcal{K}(\beta)$ if and only if $zf'(z) \in \mathcal{S}^*(\beta)$, according to [18, Theorem 1], if a function f of the form (1.1), satisfies $\sum_{k=2}^{\infty} k(k-\beta)|f_k| \leq 1-\beta$, then $f \in \mathcal{K}(\beta)$. Therefore, it is enough to prove that

$$A_2 := \sum_{k=2}^{\infty} k(k-\beta)\left|\frac{(-r)^{k-1}}{4^{k-1}(1)_{k-1}(\rho)_{k-1}}\right| \leq 1-\beta. \tag{4.3}$$

Since $\rho > 0$ and $r \in \mathbb{C}^*$, we have

$$\begin{aligned}
\mathrm{A}_2 :=& \sum_{k=1}^{\infty} \frac{(k+1)(k+1-\beta)|r|^k}{4^k(1)_k(\rho)_k} \\
=& \frac{(2-\beta)|r|}{2\rho} + \frac{3(3-\beta)|r|^2}{32\rho(\rho+1)} + \sum_{k=3}^{\infty} \frac{(k+1)(k+1-\beta)|r|^k}{4^k(1)_k(\rho)_k} \\
=& \frac{(2-\beta)|r|}{2\rho} + \frac{3(3-\beta)|r|^2}{32\rho(\rho+1)} + \sum_{k=3}^{\infty} \frac{k^2|r|^k}{4^k(1)_k(\rho)_k} \\
&+ (2-\beta)\sum_{k=3}^{\infty} \frac{k|r|^k}{4^k(1)_k(\rho)_k} + (1-\beta)\sum_{k=3}^{\infty} \frac{|r|^k}{4^k(1)_k(\rho)_k}.
\end{aligned}$$

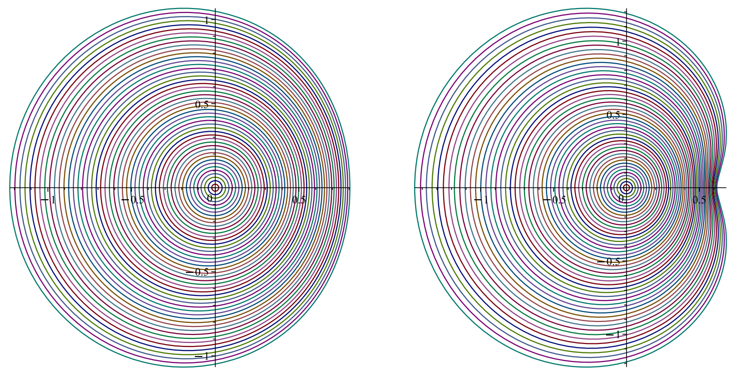
It is worth noting that by mathematical induction we have $k^2 \leq (9/64)4^k$ and $k \leq (3/64)4^k$, $k \in \mathbb{N} \setminus \{1,2\}$. We shall now proceed to combine the latter inequalities again with the estimates $(1)_k \geq (1+\sqrt{2})^{k-1}$, $(\rho)_k \geq \rho(\rho+\sqrt{2})^{k-1}$, $k \in \mathbb{N} \setminus \{1,2\}$, which follow from Lemma 1.1, to get

$$\begin{aligned}
\mathrm{A}_2 \leq& \frac{(2-\beta)|r|}{2\rho} + \frac{3(3-\beta)|r|^2}{32\rho(\rho+1)} + \frac{9+3(2-\beta)}{64\rho} \cdot |r| \sum_{k=3}^{\infty} \frac{|r|^{k-1}}{(1+\sqrt{2})^{k-1}(\rho+\sqrt{2})^{k-1}} \\
&+ \frac{(1-\beta)|r|}{4\rho} \sum_{k=3}^{\infty} \frac{|r|^{k-1}}{4^{k-1}(1+\sqrt{2})^{k-1}(\rho+\sqrt{2})^{k-1}} \\
=& \frac{(2-\beta)|r|}{2\rho} + \frac{3(3-\beta)|r|^2}{32\rho(\rho+1)} \\
&+ \frac{3(5-\beta)}{64\rho} \cdot \frac{|r|^3}{(1+\sqrt{2})(\rho+\sqrt{2})(2+\sqrt{2}+\rho+\rho\sqrt{2}-|r|)} \\
&+ \frac{(1-\beta)|r|^3}{16\rho(1+\sqrt{2})(\rho+\sqrt{2})(8+4\sqrt{2}+4\rho+4\rho\sqrt{2}-|r|)} \leq 1-\beta,
\end{aligned}$$

where the last inequality holds under the assumption (4.2). Thus, the inequality (4.3) is satisfied, hence $\mathrm{V}_{\rho,r} \in \mathcal{K}(\beta)$. □

Example 4.2 The next particular cases of Theorem 4.2 represent two situations obtained for special cases of the parameters.

(i) For the case $\beta = 0$ and $r = 2.1$, the inequality (4.2) leads to $\rho \geq 2.488707149\ldots$, thus the following result gives us a better result for ρ than that of Theorem 2.2 but assuming that $\rho > 0$:



(A) The image of $\mathrm{V}_{2.488707149,2.1}(\mathbb{U})$ (B) The image of $\mathrm{V}_{1.400245864,2.1}(\mathbb{U})$

Figure 4 Illustrations for Example 4.2

If $r = 2.1$ and $\rho \geq 2.488707149\ldots$, then $\mathrm{V}_{\rho,2.1} \in \mathcal{K}(0) = \mathcal{K}$.

(ii) A similar case obtained for the case $\beta = 0.1$ and $r = 1.2$ is the following:

If $r = 2.1$ and $\rho \geq 1.400245864\ldots$, then $\mathrm{V}_{\rho,1.2} \in \mathcal{K}(-8)$.

As seen in Fig. 4, in the first case, the image $\mathrm{V}_{2.488707149,2.1}(\mathbb{U})$ of the first function is a convex domain, while in the second case, the image $\mathrm{V}_{1.400245864,2.1}(\mathbb{U})$ is not a convex domain, and it seems to be starlike. This fact is expected because $\mathrm{V}_{1.400245864,1.2} \in \mathcal{K}(-8) \not\subset \mathcal{K}$.

5 Concluding remarks and outlook

The highlights of the present paper are given below:

1. At first, Lemma 1.1 is an improvement of Lemma 2 of [4], showing that $(a)_n > a(a+\alpha)^{n-1}$ for $n \in \mathbb{N} \setminus \{1,2\}$, $a > 0$, and $0 \leq \alpha \leq \sqrt{2}$;

2. In Theorem 2.1, we have proved that if $\rho > 0$, $r \in \mathbb{C}^*$ are such that the conditions (2.1) and (2.2) are satisfied, then $\mathrm{V}_{\rho,r} \in \mathcal{S}^*(\beta)$. This theorem slightly improves Theorem 2.1 of [23]. Thus, for the values $\rho = 2$ and $r = 0.05$, the conditions of both of these two theorems are satisfied. Therefore, Theorem 2.1 of [23] implies $V_{\rho,r} \in \mathcal{S}^*(0.9915966387)$ while the present Theorem 2.1 gives a better result $V_{\rho,r} \in \mathcal{S}^*(0.9936305732)$. Also, for $\rho = 0.5$ and $r = 0.09$, the conditions of both of these two theorems are satisfied, and Theorem 2.1 of [23] gives us $V_{\rho,r} \in \mathcal{S}^*(0.9361702128)$ while Theorem 2.1 implies a better result $V_{\rho,r} \in \mathcal{S}^*(0.9517426274)$.

3. In Theorem 2.2, we proved that if $\rho \geq 1/2$ and $r \in \mathbb{C}^*$ are such that the conditions (2.10) and (2.11) are satisfied, then we have$\mathrm{V}_{\rho,r} \in \mathcal{K}(\beta)$. Like in the above item, Theorem 2.2 slightly improves Theorem 2.2 of [23]. For example, taking the values $\rho = 1$ and $r = 0.05$, the assumptions of both of these theorems hold. Thus, Theorem 2.2 of [23] implies $V_{\rho,r} \in \mathcal{K}(0.9655172414)$ while Theorem 2.1 actually gives a better result $V_{\rho,r} \in \mathcal{K}(0.9740259740)$. In addition, for $\rho = 0.5$ and $r = 0.09$, since all the conditions of these theorems are satisfied, Theorem 2.2 of [23] implies $V_{\rho,r} \in \mathcal{K}(0.8636363636)$ while Theorem 2.1 gives us a better result $V_{\rho,r} \in \mathcal{K}(0.8985915493)$.

4. In Theorems 4.1 and 4.2, and with the help of Lemma 1.1, we have obtained sufficient conditions on ρ, r, with $\rho > 0$ and $r \in \mathbb{C}^*$, such that $\mathrm{V}_{\rho,r}$ belongs to the sets $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$. The proof uses our new Lemma 1.1 that improves Lemma 2 of [4].

5. We provided the reader with some examples to illustrate the efficiency of our approach.

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Author contributions
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References
1. Ahlfors, L.V.: Complex Analysis. McGraw-Hill, New York (1953)
2. Baricz, Á.: Bessel transforms and Hardy space of generalized Bessel functions. Mathematica **48**(71), 127–136 (2006)
3. Baricz, Á.: Geometric properties of generalized Bessel functions. Publ. Math. (Debr.) **73**(1–2), 15–178 (2008)
4. Baricz, Á., Ponnusamy, S.: Starlikeness and convexity of generalized Bessel functions. Integral Transforms Spec. Funct. **21**(9), 641–653 (2010)
5. De Branges, L.: A proof of the Bieberbach conjecture. Acta Math. **154**, 137–152 (1985)
6. Duren, P.L.: Univalent Functions. Springer, New York (1983)
7. Graham, I., Kohr, G.: Geometric Function Theory in One and Higher Dimensions. Dekker, New York (2003)
8. Marx, A.: Untersuchungen über schlichte Abbildungen. Math. Ann. **107**, 40–67 (1932–1933)
9. Mehrez, K.: Some geometric properties of a class of functions related to the Fox–Wright functions. Banach J. Math. Anal. **14**, 1222–1240 (2020)
10. Miller, S.S., Mocanu, P.T.: Univalence of Gaussian and confluent hypergeometric functions. Proc. Am. Math. Soc. **110**(2), 333–342 (1990)
11. Nehari, Z.: Conformal Mapping. McGraw-Hill, New York (1952)
12. Orhan, H., Yağmur, N.: Geometric properties of generalized Struve functions. An. Ştiinţ. Univ. 'Al.I. Cuza' Iaşi, Mat. **63**(2), 229–244 (2017)
13. Pfaltzgraff, J.A., Reade, M.O., Umezawa, T.: Sufficient conditions for univalence. Ann. Fac. Sci. de Kinshasa, Zaïre, Sect. Math. Phys. **2**, 211–218 (1976)
14. Pommerenke, C.: Univalent Functions. Vandenhoeck & Ruprecht, Gottingen (1975)
15. Ponnusamy, S.: The Hardy space of hypergeometric functions. Complex Var. Theory Appl. **29**(1), 83–96 (1996)
16. Ponnusamy, S., Rønning, F.: Geometric properties for convolutions of hypergeometric functions and functions with the derivative in a halfplane. Integral Transforms Spec. Funct. **8**(1–2), 121–138 (1999)
17. Ponnusamy, S., Vuorinen, M.: Univalence and convexity properties for confluent hypergeometric functions. Complex Var. Theory Appl. **36**(1), 73–97 (1998). https://doi.org/10.1080/17476939808815101
18. Silverman, H.: Univalent functions with negative coefficients. Proc. Am. Math. Soc. **51**(1), 109–116 (1975)
19. Strohhäcker, E.: Beiträge zur Theorie der schlichten Functionen. Math. Z. **37**, 356–380 (1933)
20. Yağmur, N.: Hardy space of Lommel functions. Bull. Korean Math. Soc. **52**(3), 1035–1046 (2015)
21. Yağmur, N., Orhan, H.: Hardy space of generalized Struve functions. Complex Var. Elliptic Equ. **59**(7), 929–936 (2014)
22. Zayed, H.M., Bulboacă, T.: On some geometric properties for the combination of generalized Lommel–Wright function. J. Inequal. Appl. **2021**, Article ID 158 (2021). https://doi.org/10.1186/s13660-021-02690-z
23. Zayed, H.M., Bulboacă, T.: Normalized generalized Bessel function and its geometric properties. J. Inequal. Appl. **2022**, Article ID 158 (2022) https://doi.org/10.1186/s13660-022-02891-0
24. Zayed, H.M., Bulboacă, T., Morais, J.: The geometric characterizations for a combination of generalized Struve functions. Comput. Methods Funct. Theory (2021). https://doi.org/10.1007/s40315-021-00421-5
25. Zayed, H.M., Mehrez, K.: Generalized Lommel–Wright function and its geometric properties. J. Inequal. Appl. **2022**, Article ID 115 (2022)

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