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On a Duffing-type oscillator differential equation on the transition to chaos with fractional q -derivatives

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Abstract

In this paper, by applying fractional quantum calculus, we study a nonlinear Duffing-type equation with three sequential fractional q -derivatives. We prove the existence and uniqueness results by using standard fixed-point theorems (Banach and Schaefer fixed-point theorems). We also discuss the Ulam–Hyers and the Ulam–Hyers–Rassias stabilities of the mentioned Duffing problem. Finally, we present an illustrative example and nice application; a Duffing-type oscillator equation with regard to our outcomes.

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1 Introduction

It is known that the difference equations involving quantum calculus play an important role in modeling many problems in engineering, physics, and mathematics, to obtain further information the reader can address the following works [1–3]. In recent years, differential equations with fractional quantum calculus have been extensively studied by several scientific researchers, see, for instance, [4–8]. In this sense, several interesting topics concerning research for differential equations involving fractional quantum calculus are devoted to the existence and Ulam–Hyers stability of the solutions [9]. Recently, many interesting results concerning the existence and Ulam-type stability of solutions for differential equations with fractional q -calculus were obtained, see [10–16] and the references therein. In [17–19], the existence and uniqueness of solutions were investigated for sequential differential equations with q -fractional calculus.

We have already seen that chaotic behavior can emerge in a system as simple as a logistic map. In that case the “route to chaos” is called period doubling. In practice, one would like to understand the route to chaos in systems described by partial differential equations, such as flow in a randomly stirred fluid. This is, however, very complicated and difficult to treat either analytically or numerically. Here, we consider an intermediate situation where the dynamics is described by a single ordinary differential equation, called the Duffing

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equation. The Duffing equation is considered with an important type of differential problem that has many applications in chaotic phenomena, for more details, we refer to the articles [20–25]. The classical form of the Duffing problem ($D\mathbb{P}$) [20], can be displayed by

$$D^2w(\tau) + \delta D^1w(\tau) = \varphi(\tau) - g(\tau, w(\tau)), \quad \tau \in \Omega := [0, 1], \delta > 0,$$

under initial values $w(0) = d_1, D^1w(0) = d_2, d_1, d_2 \in \mathbb{R}$, where φ and g are given continuous functions. Recently, the fractional-type Duffing equation ($D\mathbb{E}$) [26] of the form

$${}^C D^\zeta w(\tau) + \delta {}^C D^\zeta w(\tau) = c \sin(\varrho \tau) - bw(\tau) - aw^3(\tau), \quad \tau \in \Omega,$$

with initial values $w(0) = d_1, D^\zeta w(0) = d_2, d_1, d_2 \in \mathbb{R}^*$, for $\delta > 0, 1 < \zeta < 2, 0 < \zeta < 1, a, b, c, \varrho > 0$, where ${}^C D^\zeta, \zeta \in \{\varsigma, \zeta\}$ is the Caputo fractional derivative, has received considerable interest among scientific researchers, see [27–30] and references therein. In [25], the authors studied the existence and the uniqueness of solutions for sequential $\mathbb{FD}\mathbb{P}$ with Caputo-type fractional derivatives of different orders:

$$\begin{cases} {}^C D^\zeta [{}^C D^\zeta [{}^C D^\varepsilon w(\tau)]] \\ = \varphi(\tau) - \delta \psi(\tau, {}^C D^\varepsilon w(\tau)) \\ - g(\tau, w(\tau), {}^C D^\mu w(\tau)) - h(\tau, w(\tau), {}_{RL}I^\eta w(\tau)), \quad \delta > 0, \tau \in \Omega, \\ w(0) = d_1, \quad {}^C D^\varepsilon w(0) = d_2, \quad {}_{RL}I^\varepsilon w(1) = d_3, \quad d_1, d_2, d_3 \in \mathbb{R}, \end{cases}$$

for $0 \leq \mu < \varepsilon \leq 1, 0 \leq \varsigma, \zeta \leq 1, 1 < \varepsilon + \zeta \leq 2, 1 < \zeta + \varsigma \leq 2$, where ${}^C D^\zeta, \zeta \in \{\varsigma, \zeta, \varepsilon\}$ is the derivative in the sense of Caputo and ${}_{RL}I^\eta$ is the Riemann–Liouville integral of order $\eta \geq 0$, φ, ψ, g , and h are given continuous functions. Also, the authors discussed the fractional order $D\mathbb{P}$ of the form

$${}^C D^\zeta w(\tau) + \delta {}^C D^\zeta w(\tau) = \varphi(\tau) - g(\tau, w(\tau)), \quad \tau \in \Omega, \delta > 0,$$

via initial values $w(\tau_0) = w_0, D^1w(\tau_0) = w_1$, where ${}^C D^\zeta$ is the Caputo fractional derivative of order $\zeta \in \{\varsigma, \zeta\}, 1 < \zeta < 2, 0 < \varsigma < 1$, and τ_0 is an initial value in Ω [29]. Riaz and Zada studied coupled \mathbb{FDE} s with the help of a Laplace-transform method

$$\begin{cases} {}^C D^{\theta_1} w_1(\tau) - \alpha_1 {}^C D^{\vartheta_1} w_1(\tau) = F_1(\tau, {}^C D^{1\lambda_1} w_1(\tau), {}^C D^{1\lambda_2} w_2(\tau)), \\ {}^C D^{\theta_2} w_2(\tau) - \alpha_2 {}^C D^{\vartheta_2} w_2(\tau) = F_2(\tau, {}^C D^{2\lambda_1} w_1(\tau), {}^C D^{2\lambda_2} w_2(\tau)), \\ {}^C D^\kappa w_1(0) = {}_1 w_\kappa^*, \quad {}^C D^\kappa w_2(0) = {}_2 w_\kappa^*, \quad \kappa = 0, 1, \dots, p-1, \end{cases}$$

where $\alpha_i \in \mathbb{R}, p-1 < \theta_i < p, q-1 < \vartheta_i \leq q, p, q \in \mathbb{Z}^+, q \leq p, 0 < {}_1\lambda_i, {}_2\lambda_i \leq 1, F_i : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$, and ${}_i w_\kappa^* \in \mathbb{R}, i = 1, 2$ [31]. They studied the following class of implicit \mathbb{FDE} with implicit integral boundary condition:

$$\begin{cases} {}^C D^\zeta w(\tau) = g(\tau, w(\tau), {}^C D^\zeta w(\tau)) + \int_0^\tau \frac{(\tau-\tilde{\tau})^{\zeta-1}}{\Gamma(\zeta)} F(\tilde{\tau}, w(\tilde{\tau}), {}^C D^\zeta w(\tilde{\tau})) d\tilde{\tau}, \\ w(0) = \int_0^{\tilde{\tau}} \frac{(\tilde{\tau}-\tilde{\tau})^{\varsigma-1}}{\Gamma(\varsigma)} g(\tilde{\tau}, w(\tilde{\tau}), {}^C D^\zeta w(\tilde{\tau})) d\tilde{\tau}, \end{cases}$$

for $\tau \in [0, \tilde{\tau}], \tilde{\tau} > 0$, where the notation ${}^C D^\zeta$ is used for the Caputo fractional derivative of order $0 < \zeta \leq 1, g, g, F : [0, \tilde{\tau}] \times \mathbb{R}^2 \rightarrow \mathbb{R}, \delta, \sigma$ are real constants greater than zero [32].

Guo *et al.* investigated the existence, uniqueness, and at least one solution of the coupled system of FDEs in the sense of Hadamard derivatives:

$$\begin{cases} {}_{\text{H}}D^{\varsigma_1}w_1(\tau) + g(\tau, w_1(\tau), {}_{\text{H}}D^{\varsigma_1}w_2(\tau)) = 0, & 2 < \varsigma_1 \leq 3, \\ {}_{\text{H}}D^{\varsigma_2}w_2(\tau) + g(\tau, w_2(\tau), {}_{\text{H}}D^{\varsigma_2}w_1(\tau)) = 0, & 2 < \varsigma_2 \leq 3, \end{cases}$$

for $1 \leq \tau \leq \tilde{r}$ under the generalized Hadamard fractional integrodifferential boundary conditions [33].

In this work, we discuss the existence, uniqueness, and Ulam–Hyers and Ulam–Hyers–Rassias stability (UHS & UHRS) of solutions for a sequential fractional Duffing q -differential equation ($\text{FD}_q - \text{DE}$) involving Riemann–Liouville-type and Caputo-type fractional q -derivatives given by

$$\begin{cases} {}^{\text{R.L}}D_q^\theta [{}^C D_q^\vartheta [{}^C D_q^\lambda w(\tau)]] \\ \quad = \varphi(\tau) - \delta g(\tau, w(\tau), {}^C D_q^\mu w(\tau)) - h(\tau, w(\tau), {}^{\text{R.L}}I_q^\eta w(\tau)), & \delta > 0, \tau \in \Omega, \\ w(0) = 0, \quad {}^C D_q^\vartheta [{}^C D_q^\lambda w(1)] = 0, \\ \beta {}^C D_q^\lambda w(0) = \alpha_1 {}^C D_q^\lambda w(\gamma) + \alpha_2 {}^C D_q^\lambda w(1), \end{cases} \quad (1)$$

for $0 < \theta, \vartheta, \lambda, \mu, \gamma, q < 1, \eta \geq 0, \beta, \alpha_i \in \mathbb{R}$, with $\beta \neq \sum_{i=1}^2 \alpha_i$, where ${}^{\text{R.L}}D_q^\theta$ and ${}^C D_q^\nu$, $\nu \in \{\vartheta, \lambda\}$ are the Riemann–Liouville and Caputo fractional q -derivatives, respectively, ${}^{\text{R.L}}I_q^\eta$ is the fractional q -derivative of the Riemann–Liouville type, $\varphi : \Omega \rightarrow \mathbb{R}$, and $g, h : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are given continuous functions.

In Sect. 2, we recall some essential definitions of fractional quantum calculus. Sections 3 and 4 contain our main results about the existence, uniqueness, UHS, and Ulam–Hyers–Rassias stability UHRS of the $\text{FD}_q - \text{DE}$ (1), respectively. An application and illustrative example with some needed algorithms for the problem are given in Sect. 5. In Sect. 6, some conclusions are presented.

2 Fractional quantum calculus

We consider the fractional q -calculus on the specific time scale $\mathbb{T}_{\tau_0} = \{0\} \cup \{\tau : \tau = \tau_0 q^n\}$, for $n \in \mathbb{N}$, $\tau_0 \in \mathbb{R}$, $0 < q < 1$, and in short we shall denote \mathbb{T}_{τ_0} by \mathbb{T} . Let $\kappa \in \mathbb{R}$. Define $[\kappa]_q = \frac{1-q^\kappa}{1-q}$ and the q -factorial function $(\tau_1 - q\tau_2)^{(n)}$ by

$$(\kappa_1 - q\kappa_2)^{(n)} = \begin{cases} 1, & n = 0, \\ \prod_{l=0}^{n-1} (\kappa_1 - q^l \kappa_2), & n \in \mathbb{N}, \end{cases} \quad (2)$$

where $\kappa_1, \kappa_2 \in \mathbb{R}$ [34, 35]. In particular, $\kappa_1 = 1, \kappa_2 = q$, we obtain

$$(1 - q)^{(n)} = \prod_{l=0}^{n-1} (1 - q^{l+1}), \quad n \in \mathbb{N}.$$

Algorithm 1 in [36] shows MATLAB lines to obtain the q -gamma function. Also, in [37], one can find that

$$(\tau_1 - q\tau_2)^{(\sigma)} = \tau_1^\sigma \prod_{l=0}^{\infty} \frac{\tau_1 - q^l \tau_2}{\tau_1 - q^{\sigma+l} \tau_2}, \quad \sigma \in \mathbb{R}^{\geq 0} \setminus \mathbb{N}_0, \tau_1, \tau_2 \in \mathbb{T}.$$

Obviously, if $\tau_2 = 0$, then $(\tau_1)^{(\sigma)} = \tau_1^\sigma$. The q -Gamma function is expressed by $\Gamma_q(\kappa) = \frac{(1-q)^{(\kappa-1)}}{(1-q)^{\kappa-1}}$, for $\kappa \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, and satisfies $\Gamma_q(\kappa + 1) = [\kappa]_q \Gamma_q(\kappa)$ [34, 37]. The operator ${}^{RL}D_q^\theta$ is the fractional q -derivative of Riemann–Liouville type [38, 39], defined by

$$\begin{cases} {}^{RL}D_q^\theta[w(\tau)] = D_q^{[\theta]}[{}^{RL}I_q^{[\theta]-\theta}[w(\tau)]], & \theta > 0, \\ {}^{RL}D_q^0[w(\tau)] = w(\tau), \end{cases}$$

where $[\theta]$ is the smallest integer greater than or equal to θ . The fractional q -derivative of the Caputo type of order v is given by

$$\begin{cases} {}^C D_q^v[w(\tau)] = {}^{RL}I_q^{[v]-v}[D_q^{[v]}[w(\tau)]], & v > 0, \\ {}^C D_q^0[w(\tau)] = w(\tau), \end{cases}$$

while the fractional q -integral of Riemann–Liouville type [38, 39] is defined by

$$\begin{cases} {}^{RL}I_q^\kappa[w(\tau)] = \int_0^\tau (\tau - q\xi)^{(\kappa-1)} \frac{w(\xi)}{\Gamma_q(\kappa)} d_q\xi, & \kappa > 0, \\ {}^{RL}I_q^0[w(\tau)] = w(\tau). \end{cases}$$

We recall the following lemmas [38, 39].

Lemma 2.1 Let $\kappa, \rho \geq 0$ and z be a function defined in $[0, 1]$. Then,

$${}^{RL}I_q^\kappa[{}^{RL}I_q^\rho[z]] = {}^{RL}I_q^{\kappa+\rho}[z], \quad D_q^\kappa {}^{RL}I_q^\kappa[z] = z.$$

Lemma 2.2 Suppose that $\kappa > 0$, and ϵ is a positive integer. Then, we have

$${}^{RL}I_q^\kappa[D_q^\epsilon[z]] = D_q^\epsilon[{}^{RL}I_q^\kappa[z]] - \sum_{l=0}^{\epsilon-1} \frac{\tau^{\kappa-\epsilon+l}}{\Gamma_q(\kappa+l-\epsilon+1)} D_q^l z(0).$$

Lemma 2.3 Let $\kappa \in \mathbb{R}^+ \setminus \mathbb{N}$. Then, the following equality is valid

$${}^{RL}I_q^\kappa[{}^C D_q^\kappa[z]] = z(l) - \sum_{l=0}^{n-1} \frac{\tau^l}{\Gamma_q(l+1)} D_q^l z(0),$$

such that n is the smallest integer greater than or equal to κ .

In discrete fractional q -calculus, the fractional Riemann–Liouville-type q -integral of the function w is obtained by [36]

$$\int_0^\tau (\tau - q\eta)^{(\sigma-1)} \frac{w(\eta)}{\Gamma_q(\sigma)} d_q\eta = \frac{\tau^\sigma (1-q)}{\Gamma_q(\sigma)} \sum_{k=0}^{\infty} q^k \prod_{i=0}^{\infty} \frac{1 - q^{k+i}}{1 - q^{\sigma+k+i-1}} w(\tau q^k).$$

By using [36, Algorithm 2], one can calculate this type of q -integral.

Lemma 2.4 For $\kappa \in \mathbb{R}_+$ and $\rho > -1$, we have ${}^{RL}I_q^\kappa[\tau^{(\rho)}] = \frac{\Gamma_q(\rho+1)}{\Gamma_q(\kappa+\rho+1)} \tau^{(\kappa+\rho)}$. If $\rho = 0$, we can obtain ${}^{RL}I_q^\kappa[1] = \frac{1}{\Gamma_q(\kappa+1)} \tau^{(\kappa)}$.

In order to study the problem (1), we need the following space $B = \{w : w \& {}^C D_q^\mu w \in C(\Omega)\}$, endowed with the norm

$$\|w\|_B = \|w\| + \|{}^C D_q^\mu w\| = \sup_{\tau \in \Omega} |w(\tau)| + \sup_{\tau \in \Omega} |{}^C D_q^\mu w(\tau)|.$$

Then, it is well known that $(B, \|\cdot\|_B)$ is a Banach space. Now, we consider the Ulam-stability type for the sequential $\text{FD}_q - \text{DP}$ (1).

Definition 2.5 The $\text{FD}_q - \text{DE}$ (1) is stable in the

- UH sense if there exists a real number $\Sigma_{g^*, h^*} > 0$ such that for each $\omega > 0$ and for each solution \tilde{w} of the inequality

$$|^{\text{R.L}} D_q^\theta [{}^C D_q^\vartheta [{}^C D_q^\lambda \tilde{w}(\tau)]] - [\varphi(\tau) - \delta g_{\tilde{w}}^*(\tau) - h_{\tilde{w}}^*(\tau)]| \leq \omega, \quad \tau \in \Omega, \quad (3)$$

there exists a solution w of the $\text{FD}_q - \text{DE}$ (1) with $\|\tilde{w} - w\|_B \leq \Sigma_{g^*, h^*} \omega$, where

$g_{\tilde{w}}^*(\tau) = g(\tau, \tilde{w}(\tau), {}^C D_q^\mu \tilde{w}(\tau))$ and $h_{\tilde{w}}^*(\tau) = h(\tau, \tilde{w}(\tau), {}^{\text{R.L}} I_q^\eta \tilde{w}(\tau))$;

- UHR sense with respect to $p \in C(\Omega, \mathbb{R}_+)$ if there exists a real number $\Sigma_{g^*, h^*} > 0$ such that for each $\omega > 0$ and for each solution \tilde{w} of the inequality

$$|^{\text{R.L}} D_q^\theta [{}^C D_q^\vartheta [{}^C D_q^\lambda \tilde{w}(\tau)]] - [\varphi(\tau) - \delta g_{\tilde{w}}^*(\tau) - h_{\tilde{w}}^*(\tau)]| \leq \Sigma_{g^*, h^*} \omega p(\tau), \quad (4)$$

for $\tau \in \Omega$, there exists a solution w of the $\text{FD}_q - \text{DE}$ (1) with

$$\|\tilde{w} - w\|_B \leq \Sigma_{g^*, h^*} \omega p(\tau).$$

Remark 2.1 A function $\tilde{w} \in C(\Omega)$ is a solution of the inequality (3) iff there exists a function $p : \Omega \rightarrow \mathbb{R}$ (which depends on \tilde{w}) such that $|p(\tau)| \leq \omega$, for each $\tau \in \Omega$, and

$${}^{\text{R.L}} D_q^\theta [{}^C D_q^\vartheta [{}^C D_q^\lambda \tilde{w}(\tau)]] = \delta g_{\tilde{w}}^*(\tau) + h_{\tilde{w}}^*(\tau) - \varphi(\tau) + p(\tau), \quad \tau \in \Omega.$$

3 Existence of solution for $\text{FD}_q - \text{DP}$

In this section, we prove the existence and uniqueness of a solution of problem (1). First, we state the following key lemma.

Lemma 3.1 Suppose that $v \in C(\Omega)$. Then, the $\text{FD}_q - \text{DP}$

$$\begin{cases} {}^{\text{R.L}} D_q^\theta [{}^C D_q^\vartheta [{}^C D_q^\lambda w(\tau)]] = v(\tau), & \tau \in \Omega, \\ w(0) = 0, \quad {}^C D_q^\vartheta [{}^C D_q^\lambda w(1)] = 0, \\ \beta {}^C D_q^\lambda w(0) = \alpha_1 {}^C D_q^\lambda w(\gamma) + \alpha_2 {}^C D_q^\lambda w(1), \end{cases} \quad (5)$$

for $0 < \theta, \vartheta, \lambda, q < 1$, admits the following solution

$$\begin{aligned} w(\tau) &= \frac{1}{\Gamma_q(\theta + \vartheta + \lambda)} \int_0^\tau (\tau - q\eta)^{(\theta+\vartheta+\lambda-1)} v(\eta) d_q \eta \\ &\quad - \frac{\tau^{\theta+\vartheta+\lambda-1}}{\Gamma_q(\vartheta + \theta + \lambda)} \int_0^1 (1 - q\eta)^{(\theta-1)} v(\eta) d_q \eta \end{aligned} \quad (6)$$

$$\begin{aligned}
& + \frac{\alpha_2^\tau}{(\beta - \sum_{i=1}^2 \alpha_i) \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)} \int_0^\gamma (\gamma - q\eta)^{(\theta+\vartheta-1)} v(\eta) d_q \eta \\
& + \frac{\alpha_3^\tau}{(\beta - \sum_{i=1}^2 \alpha_i) \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)} \int_0^1 (1 - q\eta)^{(\theta+\vartheta-1)} v(\eta) d_q \eta \\
& - \frac{(\alpha_2 \gamma^{\theta+\vartheta-1} + \alpha_3) \tau^\lambda}{\Gamma_q(\theta+\vartheta) (\beta - \sum_{i=1}^2 \alpha_i) \Gamma_q(\lambda+1)} \int_0^1 (1 - q\eta)^{(\theta-1)} v(\eta) d_q \eta,
\end{aligned}$$

such that $\beta \neq \alpha_1 + \alpha_2$.

Proof Using Lemma 2.2, we can write

$$[{}^C D_q^\vartheta [{}^C D_q^\lambda w(\tau)]] = {}_{RL} I_q^\theta v(\tau) + b_1^{\theta-1} \tau, \quad b_1 \in \mathbb{R}. \quad (7)$$

Next, applying Lemma 2.3, we obtain

$${}^C D_q^\lambda w(\tau) = {}_{RL} I_q^{\theta+\vartheta} v(\tau) + b_1 \frac{\Gamma_q(\theta)}{\Gamma_q(\theta+\vartheta)} \tau^{\theta+\vartheta-1} + b_2, \quad b_i \in \mathbb{R}, i = 1, 2. \quad (8)$$

Now, applying the operator ${}_{RL} I_q^\lambda$ to both sides of equation (8), we obtain

$$w(\tau) = {}_{RL} I_q^{\theta+\vartheta+\lambda} v(\tau) + \frac{b_1 \Gamma_q(\theta) \tau^{\theta+\vartheta+\lambda-1}}{\Gamma_q(\vartheta+\theta+\lambda)} + \frac{b_2^\tau}{\Gamma_q(\lambda+1)} + b_3, \quad (9)$$

here $b_i \in \mathbb{R}$, $i = 1, 2, 3$. By using the boundary conditions $w(0) = 0$ and $[{}^C D_q^\vartheta [{}^C D_q^\lambda w(1)]] = 0$, we obtain

$$b_3 = 0, \quad \& \quad b_1 = -{}_{RL} I_q^\theta v(1). \quad (10)$$

Now, by applying conditions (3) and (10), we have

$$\begin{aligned}
b_2 &= \frac{1}{(\beta - \sum_{i=1}^2 \alpha_i)} \left[\alpha_1 ({}_{RL} I_q^{\theta+\vartheta} v(\gamma)) + \alpha_2 ({}_{RL} I_q^{\theta+\vartheta} v(1)) \right. \\
&\quad \left. - \frac{\Gamma_q(\theta)}{\Gamma_q(\theta+\vartheta)} (\alpha_1 \gamma^{\theta+\vartheta-1} + \alpha_2) {}_{RL} I_q^\theta v(1) \right].
\end{aligned}$$

Hence, we obtain Eq. (6). \square

Using Lemma 3.1, we introduce an operator $\mathcal{Z} : B \rightarrow B$ as follows

$$\begin{aligned}
\mathcal{Z} w(\tau) &= \frac{1}{\Gamma_q(\theta+\vartheta+\lambda)} \int_0^\tau (\tau - q\eta)^{(\theta+\vartheta+\lambda-1)} (\varphi(\eta) - \delta g_w^*(\eta) - h_w^*(\eta)) d_q \eta \\
&\quad - \frac{\tau^{\theta+\vartheta+\lambda-1}}{\Gamma_q(\vartheta+\theta+\lambda)} \int_0^1 (1 - q\eta)^{(\theta-1)} (\varphi(\eta) - \delta g_w^*(\eta) - h_w^*(\eta)) d_q \eta \\
&\quad + \frac{\alpha_2^\tau}{(\beta - \sum_{i=1}^2 \alpha_i) \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)} \\
&\quad \times \int_0^\gamma (\gamma - q\eta)^{(\theta+\vartheta-1)} (\varphi(\eta) - \delta g_w^*(\eta) - h_w^*(\eta)) d_q \eta
\end{aligned}$$

$$\begin{aligned}
& + \frac{\alpha_3^\tau}{(\beta - \sum_{i=1}^2 \alpha_i) \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta)} \\
& \times \int_0^1 (1 - q\eta)^{(\theta+\vartheta-1)} (\varphi(\eta) - \delta g_w^*(\eta) - h_w^*(\eta)) d_q \eta \\
& - \frac{(\alpha_2 \gamma^{\theta+\vartheta-1} + \alpha_3) \tau^\lambda}{(\beta - \sum_{i=1}^2 \alpha_i) \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta)} \\
& \times \int_0^1 (1 - q\eta)^{(\theta-1)} (\varphi(\eta) - \delta g_w^*(\eta) - h_w^*(\eta)) d_q \eta.
\end{aligned}$$

Before stating and proving the main results, we impose the following hypotheses:

- (H₁) The functions g and h are continuous over $\Omega \times \mathbb{R}^2$ and φ is continuous over Ω .
- (H₂) There exist constant $\varpi_i > 0$ such that for all $\tau \in \Omega$ and $w_i, v_i \in \mathbb{R}^2$, $i = 1, 2$, we have

$$|g(\tau, w_1, w_2) - g(\tau, v_1, v_2)| \leq \varpi_1 (|w_1 - v_1| + |w_2 - v_2|) \quad (11)$$

and

$$|h(\tau, w_1, w_2) - h(\tau, v_1, v_2)| \leq \varpi_2 (|w_1 - v_1| + |w_2 - v_2|). \quad (12)$$

- (H₃) There exist positive constants N_i , $i = 1, 2, 3$ such that for all $\tau \in \Omega$ and $w, v \in \mathbb{R}$,

$$|g(\tau, w, v)| \leq N_1, \quad |h(\tau, w, v)| \leq N_2, \quad |\varphi(\tau)| \leq N_3.$$

For the sake of convenience, we introduce the following quantities:

$$\begin{aligned}
\aleph_1 &:= \frac{1}{\Gamma_q(\theta + \vartheta + \lambda + 1)} + \frac{1}{\Gamma_q(\theta + \vartheta + \lambda) [\theta]_q} \\
&+ \frac{|\alpha_2| \gamma^{\theta+\vartheta} + |\alpha_3|}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta + 1)} \\
&+ \frac{|\alpha_1| \gamma^{\theta+\vartheta-1} + |\alpha_2|}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta) [\theta]_q}, \\
\aleph_2 &:= \frac{1}{\Gamma_q(\theta + \vartheta + \lambda - \mu + 1)} + \frac{1}{\Gamma_q(\theta + \vartheta + \lambda - \mu) \Gamma_q(\vartheta + \theta + \lambda) [\theta]_q} \\
&+ \frac{|\alpha_1| \gamma^{\theta+\vartheta} + |\alpha_2|}{\Gamma_q(\lambda - \mu + 1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta + 1)} \\
&+ \frac{|\alpha_1| \gamma^{\theta+\vartheta-1} + |\alpha_2|}{\Gamma_q(\lambda - \mu + 1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta) [\theta]_q}.
\end{aligned} \quad (13)$$

Theorem 3.2 Assume that (H₁) and (H₂) hold and that

$$\Delta := \frac{(\delta \varpi_1 + \varpi_2) \Gamma_q(\eta + 1) + \varpi_2}{\Gamma_q(\eta + 1)} < \frac{1}{\aleph_1 + \aleph_2}, \quad (14)$$

where \aleph_i , $i = 1, 2$, are given by (13), then the $\mathbb{FD}q - \mathbb{DP}$ (1) has a unique solution.

Proof Let us define $\Lambda = \max\{\Lambda_i : i = 1, 2, 3\}$, where Λ_i are finite numbers given by

$$\Lambda_1 = \sup_{\tau \in \Omega} |g(\tau, 0, 0)|, \quad \Lambda_2 = \sup_{\tau \in \Omega} |h(\tau, 0, 0)|, \quad \Lambda_3 = \sup_{\tau \in \Omega} |\varphi(\tau)|.$$

Setting real number r such that

$$r \geq (\aleph_1 + \aleph_2)\Lambda(\theta + 2) \left[1 - (\aleph_1 + \aleph_2) \left(\delta \varpi_1 + \varpi_2 + \frac{\varpi_2}{\Gamma_q(\eta + 1)} \right) \right]^{-1}, \quad (15)$$

we show that $\mathcal{Z}B_r \subset B_r$, where $B_r = \{w \in B : \|w\|_B \leq r\}$. For $w \in B_r$ and by (H_1) , we obtain

$$\begin{aligned} |g_w^*(\tau)| &= |g(\tau, w(\tau), {}^C D_q^\mu w(\tau))| \\ &\leq |g(\tau, w(\tau), {}^C D_q^\mu w(\tau)) - g(\tau, 0, 0)| + |g(\tau, 0, 0)| \\ &\leq \varpi_1 (|w(\tau)| + |{}^C D_q^\mu w(\tau)|) + \Lambda_1 \leq \varpi_1 \|w\|_B + \Lambda \leq \varpi_1 r + \Lambda \end{aligned} \quad (16)$$

and

$$\begin{aligned} |h_w^*(\tau)| &= |h(\tau, w(\tau), {}_{RL}I_q^\eta w(\tau))| \\ &\leq |h(\tau, w(\tau), {}_{RL}I_q^\eta z(\tau)) - h(\tau, 0, 0)| + |h(\tau, 0, 0)| \\ &\leq \varpi_2 (|w(\tau)| + |{}_{RL}I_q^\eta w(\tau)|) + \Lambda_2 \\ &\leq \varpi_2 \left(\|w\|_B + \frac{1}{\Gamma_q(\eta + 1)} \|w\| \right) + \Lambda_2 \\ &\leq \varpi_2 \left[1 + \frac{1}{\Gamma_q(\eta + 1)} \right] \|w\|_B + \Lambda_2 \leq \varpi_2 \left[1 + \frac{1}{\Gamma_q(\eta + 1)} \right] r + \Lambda. \end{aligned} \quad (17)$$

By Eqs. (16) and (17), we obtain

$$\begin{aligned} |\mathcal{Z}w(\tau)| &\leq \sup_{\tau \in \Omega} \left\{ \frac{1}{\Gamma_q(\theta + \vartheta + \lambda)} \right. \\ &\quad \times \int_0^\tau (\tau - q\xi)^{(\theta+\vartheta+\lambda-1)} |(\varphi(\xi) - \delta g_w^*(\xi) - h_w^*(\xi))| d_q\xi \\ &\quad + \frac{\tau^{\theta+\vartheta+\lambda-1}}{\Gamma_q(\vartheta + \theta + \lambda)} \\ &\quad \times \int_0^1 (1 - q\xi)^{(\theta-1)} |(\varphi(\xi) - \delta g_w^*(\xi) - h_w^*(\xi))| d_q\xi \\ &\quad + \frac{|\alpha_2| \tau^\lambda}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta)} \\ &\quad \times \int_0^\gamma (\gamma - q\xi)^{(\theta+\vartheta-1)} |(\varphi(\xi) - \delta g_w^*(\xi) - h_w^*(\xi))| d_q\xi \\ &\quad + \frac{|\alpha_3| \tau^\lambda}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta)} \\ &\quad \times \int_0^1 (1 - q\xi)^{(\theta+\vartheta-1)} |(\varphi(\xi) - \delta g_w^*(\xi) - h_w^*(\xi))| d_q\xi \end{aligned}$$

$$\begin{aligned}
& + \frac{(\alpha_2 \gamma^{\theta+\vartheta-1} + \alpha_3) \tau^\lambda}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)} \\
& \times \int_0^1 (1-q\xi)^{(\theta-1)} |(\varphi(\xi) - \delta g_w^*(\xi) - h_w^*(\xi))| d_q \xi \Big\},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|\mathcal{Z}w\| & \leq \left(\frac{1}{\Gamma_q(\theta+\vartheta+\lambda+1)} + \frac{1}{\Gamma_q(\theta+\vartheta+\lambda)[\theta]_q} \right. \\
& + \frac{|\alpha_2| \gamma^{(\theta+\vartheta)} + |\alpha_3|}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta+1)} \\
& \left. + \frac{|\alpha_2| \gamma^{\theta+\vartheta-1} + |\alpha_3|}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)[\theta]_q} \right) \\
& \times \left[\left(\delta \varpi_1 + \varpi_2 + \frac{\varpi_2}{\Gamma_q(\eta+1)} \right) r + \Lambda(\theta+2) \right] \\
& = \aleph_1 \left[\left(\delta \varpi_1 + \varpi_2 + \frac{\varpi_2}{\Gamma_q(\eta+1)} \right) r + \Lambda(\theta+2) \right].
\end{aligned}$$

Also, we have

$$\begin{aligned}
|{}^C D_q^\mu \mathcal{Z}w(\tau)| & \leq \sup_{\tau \in \Omega} \left\{ \frac{1}{\Gamma_q(\theta+\vartheta+\lambda-\mu)} \right. \\
& \times \int_0^\tau (\tau-q\xi)^{(\theta+\vartheta+\lambda-\mu-1)} |(\varphi(\xi) - \delta g_w^*(\xi) - h_w^*(\xi))| d_q \xi \\
& + \frac{\tau^{\theta+\vartheta+\lambda-\mu-1}}{\Gamma_q(\theta+\vartheta+\lambda-\mu) \Gamma_q(\vartheta+\theta+\lambda)} \\
& \times \int_0^1 (1-q\xi)^{(\theta-1)} |(\varphi(\xi) - \delta g_w^*(\xi) - h_w^*(\xi))| d_q \xi \\
& + \frac{|\alpha_2| \tau^{\lambda-\mu}}{\Gamma_q(\lambda-\mu+1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)} \\
& \times \int_0^\gamma (\gamma-q\xi)^{(\theta+\vartheta-1)} (\varphi(\xi) - \delta g_w^*(\xi) - h_w^*(\xi)) d_q \xi \\
& + \frac{|\alpha_3| \tau^{\lambda-\mu}}{\Gamma_q(\lambda-\mu+1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)} \\
& \times \int_0^1 (1-q\xi)^{(\theta+\vartheta-1)} |(\varphi(\xi) - \delta h_w^*(\xi) - g_w^*(\xi))| d_q \xi \\
& + \frac{(|\alpha_2| \gamma^{\theta+\vartheta-1} + |\alpha_3|) \tau^{\lambda-\mu}}{\Gamma_q(\lambda-\mu+1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)} \\
& \times \int_0^1 (1-q\xi)^{(\theta-1)} |(\varphi(\xi) - \delta g_w^*(\xi) - h_w^*(\xi))| d_q \xi \Big\}.
\end{aligned}$$

This implies that

$$\|{}^C D_q^\mu \mathcal{Z}(w)\| \leq \left(\frac{1}{\Gamma_q(\theta+\vartheta+\lambda-\mu+1)} \right.$$

$$\begin{aligned}
& + \frac{1}{\Gamma_q(\theta + \vartheta + \lambda - \mu)\Gamma_q(\vartheta + \theta + \lambda)[\theta]_q} \\
& + \frac{|\alpha_2|\gamma^{\theta+\vartheta} + |\alpha_3|}{\Gamma_q(\lambda - \mu + 1)|\beta - \sum_{i=1}^2 \alpha_i|\Gamma_q(\lambda + 1)\Gamma_q(\theta + \vartheta + 1)} \\
& + \frac{|\alpha_2|\gamma^{\theta+\vartheta-1} + |\alpha_3|}{\Gamma_q(\lambda - \mu + 1)|\beta - \sum_{i=1}^2 \alpha_i|\Gamma_q(\lambda + 1)\Gamma_q(\theta + \vartheta)[\theta]_q} \\
& \times \left[\left(\delta\varpi_1 + \varpi_2 + \frac{\varpi_2}{\Gamma_q(\eta + 1)} \right) r + \Lambda(\theta + 2) \right] \\
& = \aleph_2 \left[\left(\delta\varpi_1 + \varpi_2 + \frac{\varpi_2}{\Gamma_q(\eta + 1)} \right) r + \Lambda(\theta + 2) \right].
\end{aligned}$$

In consequence, we obtain

$$\begin{aligned}
\|\mathcal{Z}(w)\|_B &= \|\mathcal{Z}(w)\| + \|D_q^\mu \mathcal{Z}(w)\| \\
&\leq (\aleph_1 + \aleph_2) \left(\delta\varpi_1 + \varpi_2 + \frac{\varpi_2}{\Gamma_q(\eta + 1)} \right) r + (\aleph_1 + \aleph_2)\Lambda(\theta + 2) \leq r,
\end{aligned}$$

which means that $\mathcal{Z}B_r \subset B_r$. For $w, v \in B_r$ and for each $\tau \in \Omega$, we have

$$\begin{aligned}
|\mathcal{Z}w(\tau) - \mathcal{Z}v(\tau)| &\leq \sup_{\tau \in \Omega} \left\{ \frac{1}{\Gamma_q(\theta + \vartheta + \lambda)} \right. \\
&\quad \times \int_0^\tau (\tau - q\xi)^{(\theta+\vartheta+\lambda-1)} (|\delta g_w^*(\xi) - g_v^*(\xi)| + |h_w^*(\xi) - h_v^*(\xi)|) d_q\xi \\
&\quad + \frac{\tau^{\theta+\vartheta+\lambda-\mu-1}}{\Gamma_q(\theta + \vartheta + \lambda - \mu)\Gamma_q(\vartheta + \theta + \lambda)} \\
&\quad \times \int_0^1 (1 - q\xi)^{(\theta-1)} (\delta|g_w^*(\xi) - g_v^*(\xi)| + |h_w^*(\xi) - h_v^*(\xi)|) d_q\xi \\
&\quad + \frac{|\alpha_2|\tau^\lambda}{|\beta - \sum_{i=1}^2 \alpha_i|\Gamma_q(\lambda + 1)\Gamma_q(\theta + \vartheta)} \\
&\quad \times \int_0^\gamma (\gamma - q\xi)^{(\theta+\vartheta-1)} (\delta|g_w^*(\xi) - g_v^*(\xi)| + |h_w^*(\xi) - h_v^*(\xi)|) d_q\xi \\
&\quad + \frac{|\alpha_3|\tau^\lambda}{|\beta - \sum_{i=1}^2 \alpha_i|\Gamma_q(\lambda + 1)\Gamma_q(\theta + \vartheta)} \\
&\quad \times \int_0^1 (1 - q\xi)^{(\theta+\vartheta-1)} (\delta|g_w^*(\xi) - g_v^*(\xi)| + |h_w^*(\xi) - h_v^*(\xi)|) d_q\xi \\
&\quad + \frac{(|\alpha_2|\gamma^{\theta+\vartheta-1} + |\alpha_3|)\tau^{\lambda-\mu}}{\Gamma_q(\lambda - \mu + 1)(\beta - \sum_{i=1}^2 \alpha_i)\Gamma_q(\lambda + 1)\Gamma_q(\theta + \vartheta)} \\
&\quad \times \left. \int_0^1 (1 - q\xi)^{(\theta-1)} (\delta|g_w^*(\xi) - g_v^*(\xi)| + |h_w^*(\xi) - h_v^*(\xi)|) d_q\xi \right\}.
\end{aligned}$$

Using (H_1) , we obtain

$$\|\mathcal{Z}(w) - \mathcal{Z}(v)\| \leq \left(\frac{1}{\Gamma_q(\theta + \vartheta + \lambda + 1)} + \frac{1}{\Gamma_q(\theta + \vartheta + \lambda)[\theta]_q} \right)$$

$$\begin{aligned}
& + \frac{|\alpha_2| \gamma^{(\theta+\vartheta)} + |\alpha_3|}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta+1)} \\
& + \frac{|\alpha_2| \gamma^{\theta+\vartheta-1} + |\alpha_3|}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)[\theta]_q} \\
& \times \left[\frac{(\delta \varpi_1 + \varpi_2) \Gamma_q(\eta+1) + \varpi_2}{\Gamma_q(\eta+1)} \right] \|w-v\|_B \\
& = \aleph_1 \frac{(\delta \varpi_1 + \varpi_2) \Gamma_q(\eta+1) + \varpi_2}{\Gamma_q(\eta+1)} \|w-v\|_B.
\end{aligned}$$

Hence,

$$\|\mathcal{Z}(w) - \mathcal{Z}(v)\| \leq \aleph_1 \frac{(\delta \varpi_1 + \varpi_2) \Gamma_q(\eta+1) + \varpi_2}{\Gamma_q(\eta+1)} \|w-v\|_B. \quad (18)$$

On the other hand, for each $\tau \in \Omega$, we have

$$\begin{aligned}
& |{}^C D_q^\mu \mathcal{Z}w(\tau) - {}^C D_q^\mu \mathcal{Z}v(\tau)| \\
& \leq \sup_{\tau \in \Omega} \left\{ \frac{1}{\Gamma_q(\theta+\vartheta+\lambda-\mu)} \right. \\
& \times \int_0^\tau (\tau - q\xi)^{(\theta+\vartheta+\lambda-\mu-1)} (\delta |g_w^*(\xi) - g_v^*(\xi)| \\
& + |h_w^*(\xi) - h_v^*(\xi)|) d_q\xi + \frac{\tau^{\theta+\vartheta+\lambda-\mu-1}}{\Gamma_q(\theta+\vartheta+\lambda-\mu) \Gamma_q(\vartheta+\theta+\lambda)} \\
& \times \int_0^1 (1 - q\xi)^{(\theta-1)} (\delta |g_w^*(\xi) - g_v^*(\xi)| + |h_w^*(\xi) - h_v^*(\xi)|) d_q\xi \\
& + \frac{|\alpha_2| \tau^{\lambda-\mu}}{\Gamma_q(\lambda-\mu+1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)} \\
& \times \int_0^\gamma (\gamma - q\xi)^{(\theta+\vartheta-1)} (\delta |g_w^*(\xi) - g_v^*(\xi)| + |h_w^*(\xi) - h_v^*(\xi)|) d_q\xi \\
& + \frac{|\alpha_3| \tau^{\lambda-\mu}}{\Gamma_q(\lambda-\mu+1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)} \\
& \times \int_0^1 (1 - q\xi)^{(\theta+\vartheta-1)} (\delta |g_w^*(\xi) - g_v^*(\xi)| + |h_w^*(\xi) - h_v^*(\xi)|) d_q\xi \\
& + \frac{(|\alpha_2| \gamma^{\theta+\vartheta-1} + |\alpha_3|) \tau^{\lambda-\mu}}{\Gamma_q(\lambda-\mu+1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda+1) \Gamma_q(\theta+\vartheta)} \\
& \times \left. \int_0^1 (1 - q\xi)^{(\theta-1)} (\delta |g_w^*(\xi) - g_v^*(\xi)| + |h_w^*(\xi) - h_v^*(\xi)|) d_q\xi \right\}.
\end{aligned}$$

Thanks to (H_1) , we have

$$\begin{aligned}
\|{}^C D_q^\mu \mathcal{Z}(w) - {}^C D_q^\mu \mathcal{Z}(v)\| & \leq \left(\frac{1}{\Gamma_q(\theta+\vartheta+\lambda-\mu+1)} \right. \\
& \left. + \frac{1}{\Gamma_q(\theta+\vartheta+\lambda-\mu) \Gamma_q(\vartheta+\theta+\lambda)[\theta]_q} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{|\alpha_2| \gamma^{\theta+\vartheta} + |\alpha_3|}{\Gamma_q(\lambda - \mu + 1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta + 1)} \\
& + \frac{|\alpha_2| \gamma^{\theta+\vartheta-1} + |\alpha_3|}{\Gamma_q(\lambda - \mu + 1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta) [\theta]_q} \Big) \\
& \times \frac{(\delta \varpi_1 + \varpi_2) \Gamma_q(\eta + 1) + \varpi_2}{\Gamma_q(\eta + 1)} \|w - v\|_B \\
& = \aleph_2 \frac{(\delta \varpi_1 + \varpi_2) \Gamma_q(\eta + 1) + \varpi_2}{\Gamma_q(\eta + 1)} \|w - v\|_B.
\end{aligned}$$

Therefore,

$$\|{}^C D_q^\mu \mathcal{Z}(w) - {}^C D_q^\mu \mathcal{Z}(v)\| \leq \aleph_2 \frac{(\delta \varpi_1 + \varpi_2) \Gamma_q(\eta + 1) + \varpi_2}{\Gamma_q(\eta + 1)} \|w - v\|_B. \quad (19)$$

Then, thanks to Eqs. (18) and (19), we conclude that

$$\|\mathcal{Z}(w) - \mathcal{Z}(v)\|_B \leq (\aleph_1 + \aleph_2) \frac{(\delta \varpi_1 + \varpi_2) \Gamma_q(\eta + 1) + \varpi_2}{\Gamma_q(\eta + 1)} \|w - v\|_B.$$

By (14), we see that \mathcal{Z} is a contractive operator. Consequently, by the Banach fixed-point theorem, \mathcal{Z} has a fixed point that is a solution of (1). \square

Now, we prove the existence of a solution for the FD q -DP (1) by applying the following lemma.

Lemma 3.3 ([40]) *Let S be a Banach space. Assume that $\mathcal{Z} : S \rightarrow S$ is a completely continuous operator and that the set $B = \{w \in S : w = \zeta \mathcal{Z}(w), 0 < \zeta < 1\}$, is bounded. Then, \mathcal{Z} has a fixed point in S .*

Theorem 3.4 *If conditions (H_1) and (H_3) are satisfied, then the FD q -DP (1) has at least one solution.*

Proof By the continuity of functions g , h , and φ , the operator \mathcal{Z} is continuous. Now, we show that the operator \mathcal{Z} is completely continuous. First, we show that \mathcal{Z} maps bounded sets of B into bounded sets of B . Let us take $\sigma > 0$ and $B_\sigma = \{w \in B : \|w\|_B \leq \sigma\}$. Then, for all $w \in B_\sigma$, we have

$$\begin{aligned}
\|\mathcal{Z}w\| & \leq \left(\frac{1}{\Gamma_q(\theta + \vartheta + \lambda + 1)} + \frac{1}{\Gamma_q(\theta + \vartheta + \lambda) [\theta]_q} \right. \\
& + \frac{|\alpha_2| \gamma^{(\theta+\vartheta)} + |\alpha_3|}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta + 1)} \\
& + \left. \frac{|\alpha_2| \gamma^{\theta+\vartheta-1} + |\alpha_3|}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta) [\theta]_q} \right) (\Lambda N_1 + N_2 + N_3) \\
& = \aleph_1 (\Lambda N_1 + N_2 + N_3)
\end{aligned}$$

and

$$\|{}^C D_q^\mu \mathcal{Z}(w)\| \leq \left(\frac{1}{\Gamma_q(\theta + \vartheta + \lambda - \mu + 1)} \right.$$

$$\begin{aligned}
& + \frac{1}{\Gamma_q(\theta + \vartheta + \lambda - \mu)\Gamma_q(\vartheta + \theta + \lambda)[\theta]_q} \\
& + \frac{|\alpha_2|\gamma^{\theta+\vartheta} + |\alpha_3|}{\Gamma_q(\lambda - \mu + 1)|\beta - \sum_{i=1}^2 \alpha_i|\Gamma_q(\lambda + 1)\Gamma_q(\theta + \vartheta + 1)} \\
& + \frac{|\alpha_2|\gamma^{\theta+\vartheta-1} + |\alpha_3|}{\Gamma_q(\lambda - \mu + 1)|\beta - \sum_{i=1}^2 \alpha_i|\Gamma_q(\lambda + 1)\Gamma_q(\theta + \vartheta)[\theta]_q} \\
& \times (\Lambda N_1 + N_2 + N_3) = \aleph_2(\Lambda N_1 + N_2 + N_3).
\end{aligned}$$

From the above inequalities, it follows that the operator \mathcal{Z} is uniformly bounded. Next, we show that \mathcal{Z} is equicontinuous. Let $w \in B_\sigma$ and $\tau_1, \tau_2 \in \Omega$, with $\tau_1 < \tau_2$, we have

$$\begin{aligned}
|\mathcal{Z}w(\tau_2) - \mathcal{Z}w(\tau_1)| & \leq \left(\frac{1}{\Gamma_q(\theta + \vartheta + \lambda + 1)} [(\tau_2 - \tau_1)^{\theta+\vartheta+\lambda} \right. \\
& + |\tau_2^{\theta+\vartheta+\lambda} - \tau_1^{\theta+\vartheta+\lambda}|] + \frac{|\tau_1^{\theta+\vartheta+\lambda-1} - \tau_2^{\theta+\vartheta+\lambda-1}|}{\Gamma_q(\vartheta + \theta + \lambda)[\theta]_q} \\
& + \frac{(|\alpha_2|\tau^\lambda\gamma^{\theta+\vartheta} + |\alpha_3|)|\tau_2^\lambda - \tau_1^\lambda|}{|\beta - \sum_{i=1}^2 \alpha_i|\Gamma_q(\lambda + 1)\Gamma_q(\theta + \vartheta + 1)} \\
& \left. + \frac{(|\alpha_2|\gamma^{\theta+\vartheta-1} + |\alpha_3|)|\tau_1^\lambda - \tau_2^\lambda|}{|\beta - \sum_{i=1}^2 \alpha_i|\Gamma_q(\lambda + 1)\Gamma_q(\theta + \vartheta)[\theta]_q} \right) (\Lambda N_1 + N_2 + N_3)
\end{aligned}$$

and

$$\begin{aligned}
& |{}^C D_q^\mu \mathcal{Z}w(\tau_2) - {}^C D_q^\mu \mathcal{Z}w(\tau_1)| \\
& \leq \left(\frac{[(\tau_2 - \tau_1)^{\theta+\vartheta+\lambda-\mu} + |\tau_2^{\theta+\vartheta+\lambda-\mu} - \tau_1^{\theta+\vartheta+\lambda-\mu}|]}{\Gamma_q(\theta + \vartheta + \lambda - \mu + 1)} \right. \\
& + \frac{|\tau_1^{\theta+\vartheta+\lambda-\mu-1} - \tau_2^{\theta+\vartheta+\lambda-\mu-1}|}{\Gamma_q(\theta + \vartheta + \lambda - \mu)\Gamma_q(\vartheta + \theta + \lambda)[\theta]_q} \\
& + \frac{(|\alpha_2|\gamma^{\theta+\vartheta} + |\alpha_3|)|\tau_2^{\lambda-\mu} - \tau_1^{\lambda-\mu}|}{\Gamma_q(\lambda - \mu + 1)|\beta - \sum_{i=1}^2 \alpha_i|\Gamma_q(\lambda + 1)\Gamma_q(\theta + \vartheta + 1)} \\
& \left. + \frac{(|\alpha_2|\gamma^{\theta+\vartheta-1} + |\alpha_3|)|\tau_1^{\lambda-\mu} - \tau_2^{\lambda-\mu}|}{\Gamma_q(\lambda - \mu + 1)|\beta - \sum_{i=1}^2 \alpha_i|\Gamma_q(\lambda + 1)\Gamma_q(\theta + \vartheta)[\theta]_q} \right) \\
& \times (\Lambda N_1 + N_2 + N_3),
\end{aligned}$$

which imply that $\|\mathcal{Z}w(\tau_2) - \mathcal{Z}w(\tau_1)\|_B \rightarrow 0$, as $\tau_2 \rightarrow \tau_1$. Combining these results and using the Arzelà–Ascoli theorem, we conclude that \mathcal{Z} is a completely continuous operator. Finally, it will be verified that the set Π , defined by $\Pi = \{w \in B : w = \xi \mathcal{Z}(w), 0 < \xi < 1\}$, is bounded. Let $w \in \Pi$, then $w = \xi \mathcal{Z}(w)$. For all $\tau \in \Omega$, we have $w(\tau) = \xi \mathcal{Z}w(\tau)$. Then, by (H_3) , we obtain

$$\|w\| \leq \xi \aleph_1(\Lambda N_1 + N_2 + N_3), \quad \|{}^C D_q^\mu w\| \leq \xi \aleph_2(\Lambda N_1 + N_2 + N_3). \quad (20)$$

It follows from Inequality (20) that

$$\|w\|_B = \|w\| + \|{}^C D_q^\mu w\|$$

$$\begin{aligned} &\leq \xi(\aleph_1 + \aleph_2)(\Lambda N_1 + N_2 + N_3) \\ &\leq (\aleph_1 + \aleph_2)(\Lambda N_1 + N_2 + N_3). \end{aligned}$$

Consequently, $\|w\|_B < \infty$. This shows that the set Π is bounded. Thanks to previous results, and by Lemma 3.3, we deduce that \mathcal{Z} has at least one fixed point, which is a solution of problem (1). \square

4 Ulam-stability of $\text{FD}_q - \text{DP}$

In this part, the Ulam stability of the $\text{FD}_q - \text{DP}$ (1) will be discussed.

Theorem 4.1 Suppose that conditions (H_1) and (H_2) are valid. In addition, it is assumed that

$$\tilde{\Delta} := (\delta \varpi_1 + \varpi_2) \Gamma_q(\eta + 1) + \varpi_2 < (*): \Gamma_q(\theta + \vartheta + \lambda + 1) \Gamma_q(\eta + 1), \quad (21)$$

then the $\text{FD}_q - \text{DP}$ (1) is UH stable.

Proof Suppose that $\tilde{w} \in B$ is a solution (3) and $w \in B$ is a unique solution of the problem

$$\begin{cases} {}^{\text{R,L}}\text{D}_q^\theta [{}^C\text{D}_q^\vartheta [{}^C\text{D}_q^\lambda w(\tau)]] + \delta g_w^*(\tau) + h_w^*(\tau) - \varphi(\tau) = 0, & \tau \in \Omega, \\ w(0) = \tilde{w}(0), \quad {}^C\text{D}_q^\vartheta [{}^C\text{D}_q^\lambda w(1)] = {}^C\text{D}_q^\vartheta [{}^C\text{D}_q^\lambda \tilde{w}(1)], \\ {}^C\text{D}_q^\lambda w(0) = {}^C\text{D}_q^\lambda \tilde{w}(0), \quad {}^C\text{D}_q^\lambda w(\gamma) = {}^C\text{D}_q^\lambda \tilde{w}(\gamma), \quad {}^C\text{D}_q^\lambda w(1) = {}^C\text{D}_q^\lambda \tilde{w}(1). \end{cases} \quad (22)$$

Thanks to Lemma 3.1, we have

$$w(\tau) = {}_{\text{R,L}}\text{I}_q^{\theta+\vartheta+\lambda} v_w(\tau) + \frac{\Gamma_q(\theta)\tau^{\theta+\vartheta+\lambda-1} b_1}{\Gamma_q(\vartheta+\theta+\lambda)} + \frac{\tau^\lambda b_2}{\Gamma_q(\lambda+1)} + b_3,$$

for $b_i \in \mathbb{R}$, $i = 1, 2, 3$, where $v_w(\tau) = \varphi(\tau) - \delta g_w^*(\tau) - h_w^*(\tau)$, for $\tau \in \Omega$. By integration of the inequality (3), we obtain

$$\begin{aligned} &\left| \tilde{w}(\tau) - {}_{\text{R,L}}\text{I}_q^{\theta+\vartheta+\lambda} v_{\tilde{w}}(\tau) - \frac{\Gamma_q(\theta)\tau^{\theta+\vartheta+\lambda-1}}{\Gamma_q(\vartheta+\theta+\lambda)} b'_1 - \frac{\tau^\lambda}{\Gamma_q(\lambda+1)} b'_2 - b'_3 \right| \\ &\leq \frac{\omega}{\Gamma_q(\theta+\vartheta+\lambda+1)} \tau^{\theta+\vartheta+\lambda} \leq \frac{\omega}{\Gamma_q(\theta+\vartheta+\lambda+1)}. \end{aligned}$$

Then, for any $\tau \in \Omega$, we obtain

$$\begin{aligned} |\tilde{w}(\tau) - w(\tau)| &= \left| \tilde{w}(\tau) - {}_{\text{R,L}}\text{I}_q^{\theta+\vartheta+\lambda} v_{\tilde{w}}(\tau) - \frac{\Gamma_q(\theta)\tau^{\theta+\vartheta+\lambda-1}}{\Gamma_q(\vartheta+\theta+\lambda)} b'_1 \right. \\ &\quad \left. - \frac{\tau^\lambda}{\Gamma_q(\lambda+1)} b'_2 - b'_3 + {}_{\text{R,L}}\text{I}_q^{\theta+\vartheta+\lambda} [v_{\tilde{w}}(\tau) - v_w(\tau)] \right| \\ &\leq \left| \tilde{w}(\tau) - {}_{\text{R,L}}\text{I}_q^{\theta+\vartheta+\lambda} v_{\tilde{w}}(\tau) - \frac{\Gamma_q(\theta)\tau^{\theta+\vartheta+\lambda-1}}{\Gamma_q(\vartheta+\theta+\lambda)} b'_1 \right. \\ &\quad \left. - \frac{\tau^\lambda}{\Gamma_q(\lambda+1)} b'_2 - b'_3 \right| + \left| {}_{\text{R,L}}\text{I}_q^{\theta+\vartheta+\lambda} [v_{\tilde{w}}(\tau) - v_w(\tau)] \right| \end{aligned}$$

$$\begin{aligned} &\leq \frac{\omega}{\Gamma_q(\theta + \vartheta + \lambda + 1)} \\ &+ \frac{\delta}{\Gamma_q(\theta + \vartheta + \lambda)} \int_0^\tau (\tau - q\xi)^{(\theta+\vartheta+\lambda-1)} |g_w^*(\xi) - g_w^*(\xi)| d_q\xi \\ &+ \frac{1}{\Gamma_q(\theta + \vartheta + \lambda)} \int_0^\tau (\tau - q\xi)^{(\theta+\vartheta+\lambda-1)} |h_w^*(\xi) - h_w^*(\xi)| d_q\xi. \end{aligned}$$

Now, using (H_2) , we obtain

$$\|\tilde{w} - w\|_B \leq \frac{\omega}{\Gamma_q(\theta + \vartheta + \lambda + 1)} + \frac{(\delta\varpi_1 + \varpi_2)\Gamma_q(\eta + 1) + \varpi_2}{\Gamma_q(\eta + 1)\Gamma_q(\theta + \vartheta + \lambda + 1)} \|\tilde{w} - w\|_B.$$

Then, we have

$$\|\tilde{w} - w\|_B \leq \left[\Gamma_q(\theta + \vartheta + \lambda + 1) - \frac{(\delta\varpi_1 + \varpi_2)\Gamma_q(\eta + 1) + \varpi_2}{\Gamma_q(\eta + 1)} \right]^{-1} \omega = \Sigma_{g^*, h^*} \omega.$$

Hence, the $\mathbb{FD}q - \mathbb{DP}$ (1) is stable in the UH sense. \square

Theorem 4.2 Suppose that conditions (H_1) , (H_2) , and Eq. (21) are valid. Suppose there exists $\chi_p > 0$ such that

$$\frac{1}{\Gamma_q(\theta + \vartheta + \lambda)} \int_0^\tau (\tau - q\xi)^{(\theta+\vartheta+\lambda-1)} p(\xi) d_q\xi \leq \chi_p p(\tau), \quad (23)$$

where $p \in C(\Omega, \mathbb{R}_+)$ is nondecreasing. Then, the $\mathbb{FD}q - \mathbb{DP}$ (1) is UHR stable.

Proof Let $\tilde{w} \in B$ be a solution of the inequality (4). Applying Remark 2.1, we can obtain

$$\begin{aligned} &\left| \tilde{w}(\tau) - {}_{\text{R.L}}I_q^{\theta+\vartheta+\lambda} v_{\tilde{w}}(\tau) - \frac{\Gamma_q(\theta)\tau^{\theta+\vartheta+\lambda-1}}{\Gamma_q(\vartheta + \theta + \lambda)} b'_1 - \frac{\tau^\lambda}{\Gamma_q(\lambda + 1)} b'_2 - b'_3 \right| \\ &\leq \frac{\omega}{\Gamma_q(\theta + \vartheta + \lambda)} \int_0^\tau (\tau - q\xi)^{(\theta+\vartheta+\lambda-1)} p(\xi) d_q\xi. \end{aligned}$$

Let us denote by $w \in B$ the unique solution of the problem (22). Then, we have

$$w(\tau) = {}_{\text{R.L}}I_q^{\theta+\vartheta+\lambda} m_z(\tau) + \frac{\Gamma_q(\theta)\tau^{\theta+\vartheta+\lambda-1} b_1}{\Gamma_q(\vartheta + \theta + \lambda)} + \frac{\tau^\lambda b_2}{\Gamma_q(\lambda + 1)} + b_3,$$

here $b_i \in \mathbb{R}$, $i = 1, 2, 3$. Then,

$$\begin{aligned} |\tilde{w}(\tau) - w(\tau)| &\leq \left| \tilde{w}(\tau) - {}_{\text{R.L}}I_q^{\theta+\vartheta+\lambda} v_{\tilde{w}}(\tau) \right. \\ &\quad \left. - \frac{\Gamma_q(\theta)\tau^{\theta+\vartheta+\lambda-1}}{\Gamma_q(\vartheta + \theta + \lambda)} b'_1 - \frac{\tau^\lambda}{\Gamma_q(\lambda + 1)} b'_2 - b'_3 \right| \\ &\quad + \frac{1}{\Gamma_q(\theta + \vartheta + \lambda)} \int_0^\tau (\tau - q\xi)^{(\theta+\vartheta+\lambda-1)} |v_{\tilde{w}}(\tau) - v_w(\tau)| d_q\xi \\ &\leq \frac{\omega}{\Gamma_q(\theta + \vartheta + \lambda)} \int_0^\tau (\tau - q\xi)^{(\theta+\vartheta+\lambda-1)} p(\xi) d_q\xi \end{aligned}$$

$$\begin{aligned}
& + \frac{\delta}{\Gamma_q(\theta + \vartheta + \lambda)} \int_0^\tau (\tau - q\xi)^{(\theta + \vartheta + \lambda - 1)} |g_w^*(\xi) - g_w^*(\xi)| d_q \xi \\
& + \frac{1}{\Gamma_q(\theta + \vartheta + \lambda)} \int_0^\tau (\tau - q\xi)^{(\theta + \vartheta + \lambda - 1)} |h_w^*(\xi) - h_w^*(\xi)| d_q \xi.
\end{aligned}$$

Hence, by (H_2) and (23), we obtain

$$|\dot{w}(\tau) - w(\tau)| \leq \omega \chi_p p(\tau) + \frac{(\delta \varpi_1 + \varpi_2) \Gamma_q(\eta + 1) + \varpi_2}{\Gamma_q(\eta + 1) \Gamma_q(\theta + \vartheta + \lambda)} \| \dot{w} - w \|_B.$$

Then, we have

$$\| \dot{w} - w \|_B \leq \chi_p \left[1 - \frac{(\delta \varpi_1 + \varpi_2) \Gamma_q(\eta + 1) + \varpi_2}{\Gamma_q(\eta + 1) \Gamma_q(\theta + \vartheta + \lambda)} \right]^{-1} \omega p(\tau) = \Sigma_{g^*, h^*} \omega p(\tau),$$

for $\tau \in \Omega$. Hence, the $\text{FD}_q - \mathbb{D}\mathbb{P}$ (1) is stable in the UHR sense. \square

5 Numerical results

5.1 An illustrative example

Example 5.1 Based on the problem (1), we consider the following $\text{FD}_q - \mathbb{D}\mathbb{P}$

$$\begin{cases} {}_{\text{R.L}}\text{D}_q^{\frac{\exp(1)}{2}} [{}^C\text{D}_q^{\frac{\sqrt{11}}{6}} [{}^C\text{D}_q^{\frac{3}{5}} w(\tau)]] + \frac{\sqrt{\exp(1)}}{25^2} \\ \times \left(\frac{|w(\tau)|}{50\sqrt{\pi(2+\tau^2)(1+|w(\tau)|)}} + \frac{\arctan({}^C\text{D}_q^{\frac{1}{4}} w(\tau))}{50\sqrt{\pi(2+\tau^2)}} \right) \\ + \frac{|w(\tau)|}{25(\exp(1))^{\tau^2+2}(|w(\tau)|+2)} \\ + \frac{\arctan({}_{\text{R.L}}\text{I}_q^{\frac{5}{4}} w(\tau))}{25(\exp(1))^{\tau^2+2}} - \frac{\ln \tau + 2}{3+\tau^2} = 0, \\ w(0) = 0, \quad {}^C\text{D}_q^{\frac{\sqrt{11}}{6}} [{}^C\text{D}_q^{\frac{3}{5}} w(1)] = 0, \\ \frac{2\exp(1)}{13} {}^C\text{D}_q^{\frac{3}{5}} w(0) = \frac{\sqrt{7}}{3} {}^C\text{D}_q^{\frac{3}{5}} w\left(\frac{2}{5}\right) + \frac{\sin 7}{5} {}^C\text{D}_q^{\frac{3}{5}} w(1), \end{cases} \quad (24)$$

for $\tau \in \Omega = [0, 1]$, $q \in \{\frac{3}{10}, \frac{1}{2}, \frac{8}{9}\} \in (0, 1)$, and the following q -fractional inequalities

$$\begin{aligned}
& \left| {}_{\text{R.L}}\text{D}_q^{\frac{\exp(1)}{2}} [{}^C\text{D}_q^{\frac{\sqrt{11}}{6}} [{}^C\text{D}_q^{\frac{3}{5}} w(\tau)]] \right. \\
& \left. - \left(\frac{\ln \tau + 2}{3 + \tau^2} - \frac{\sqrt{\exp(1)}}{25^2} g_w^*(\tau) - h_w^*(\tau) \right) \right| \leq \omega
\end{aligned} \quad (25)$$

and

$$\begin{aligned}
& \left| {}_{\text{R.L}}\text{D}_q^{\frac{\exp(1)}{2}} [{}^C\text{D}_q^{\frac{\sqrt{11}}{6}} [{}^C\text{D}_q^{\frac{3}{5}} w(\tau)]] \right. \\
& \left. - \left(\frac{\ln \tau + 2}{3 + \tau^2} - \frac{\sqrt{\exp(1)}}{25^2} g_w^*(\tau) - h_w^*(\tau) \right) \right| \leq \omega p(\tau),
\end{aligned} \quad (26)$$

where

$$g_w^*(\tau) = \frac{|w(\tau)|}{50\sqrt{\pi(2+\tau^2)(1+|w(\tau)|)}} + \frac{\arctan({}^C\text{D}_q^{\frac{1}{4}} w(\tau))}{50\sqrt{\pi(2+\tau^2)}},$$

$$h_w^*(\tau) = \frac{|w(\tau)|}{45(\exp(1))^{\tau^2+2}(|w(\tau)| + 2)} + \frac{\arctan({}_{\text{R.L}} I_q^{\frac{5}{4}} w(\tau))}{45(\exp(1))^{\tau^2+2}}.$$

Clearly $\theta = \frac{1}{2}\exp(1) \in (0, 1)$, $\vartheta = \frac{\sqrt{11}}{6} \in (0, 1)$, $\lambda = \frac{3}{5} \in (0, 1)$, $\delta = \frac{1}{25^2}\sqrt{\exp(1)} > 0$, $\varphi(\tau) = \frac{1}{3+\tau^2}(\ln \tau + 2)$, $\tau \in \Omega$, $\mu = \frac{1}{4} \in (0, 1)$, $\eta = \frac{5}{4} \geq 0$, and $\gamma = \frac{2}{5} \in (0, 1)$, $\beta = \frac{2\exp(1)}{13} \in \mathbb{R}$, $\alpha_1 = \frac{\sqrt{7}}{3} \in \mathbb{R}$, $\alpha_2 = \frac{\sin 7}{5} \in \mathbb{R}$, with $\beta \neq \alpha_1 + \alpha_2$. Condition (H_1) holds because g, h are continuous over $\Omega \times \mathbb{R}^2$ and φ is continuous over Ω . Also, for $\tau \in \Omega$ and $w_i, v_i \in \mathbb{R}^2$, $i = 1, 2$, we have

$$\begin{aligned} & |g(\tau, w_1, w_2) - g(\tau, v_1, v_2)| \\ &= \left| \frac{|w_1(\tau)|}{50\sqrt{\pi(2+\tau^2)}(1+|w_1(\tau)|)} + \frac{\arctan(w_2(\tau))}{50\sqrt{\pi(2+\tau^2)}} \right. \\ &\quad \left. - \left(\frac{|v_1(\tau)|}{50\sqrt{\pi(2+\tau^2)}(1+|v_1(\tau)|)} + \frac{\arctan(v_2(\tau))}{50\sqrt{\pi(2+\tau^2)}} \right) \right| \\ &\leq \frac{1}{50\sqrt{\pi(2+\tau^2)}} \left(\left| \frac{|w_1(\tau)|}{1+|w_1(\tau)|} - \frac{|v_1(\tau)|}{1+|v_1(\tau)|} \right| \right. \\ &\quad \left. + |\arctan(w_2(\tau)) - \arctan(v_2(\tau))| \right) \\ &\leq \frac{1}{50\sqrt{\pi(2+\tau^2)}} (||w_1(\tau)| - |v_1(\tau)|| + |w_2(\tau) - v_2(\tau)|) \\ &\leq \frac{1}{50\sqrt{\pi(2+\tau^2)}} (|w_1 - v_1| + |w_2 - v_2|) \end{aligned}$$

and

$$\begin{aligned} & |h(\tau, w_1, w_2) - h(\tau, v_1, v_2)| \\ &= \left| \frac{|w_1(\tau)|}{25(\exp(1))^{\tau^2+2}(|w_1(\tau)| + 2)} + \frac{\arctan(w_2(\tau))}{25(\exp(1))^{\tau^2+2}} \right. \\ &\quad \left. - \left(\frac{|v_1(\tau)|}{25(\exp(1))^{\tau^2+2}(|v_1(\tau)| + 2)} + \frac{\arctan(v_2(\tau))}{25(\exp(1))^{\tau^2+2}} \right) \right| \\ &= \left| \frac{1}{25(\exp(1))^{\tau^2+2}} \left| \frac{|w_1(\tau)|}{|w_1(\tau)| + 2} + \arctan(w_2(\tau)) \right| \right. \\ &\quad \left. - \left(\frac{|v_1(\tau)|}{|v_1(\tau)| + 2} + \arctan(v_2(\tau)) \right) \right| \\ &\leq \left(\left| \frac{|w_1(\tau)|}{|w_1(\tau)| + 2} - \frac{|v_1(\tau)|}{|v_1(\tau)| + 2} \right| \right. \\ &\quad \left. + |\arctan(w_2(\tau)) - \arctan(v_2(\tau))| \right) \\ &\leq \frac{1}{25(\exp(1))^{\tau^2+2}} (|w_1 - v_1| + |w_2 - v_2|). \end{aligned}$$

Hence, by choosing $\varpi_1 = \frac{1}{50\sqrt{2\pi}}$, $\varpi_2 = \frac{1}{25(\exp(1))^2}$, condition (H_2) , inequalities (11) and (12) hold. Now, we consider three cases for $q \in \{\frac{3}{10}, \frac{1}{2}, \frac{8}{9}\}$, in $\mathbb{FD}q - \mathbb{DP}$ (24). By employing (13),

we find that

$$\begin{aligned}
 \aleph_1 &= \frac{1}{\Gamma_q(\theta + \vartheta + \lambda + 1)} + \frac{1}{\Gamma_q(\theta + \vartheta + \lambda)[\theta]_q} \\
 &\quad + \frac{|\alpha_1| \gamma^{\theta+\vartheta} + |\alpha_2|}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta + 1)} \\
 &\quad + \frac{|\alpha_1| \gamma^{\theta+\vartheta-1} + |\alpha_2|}{|\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta)[\theta]_q}, \\
 &= \frac{1}{\Gamma_q(\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6} + \frac{3}{5} + 1)} + \frac{1}{\Gamma_q(\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6} + \frac{3}{5})[\frac{\exp(1)}{2}]_q} \\
 &\quad + \frac{|\frac{\sqrt{7}}{3}|(\frac{2}{5})^{\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6}} + |\frac{\sin 7}{4}|}{|\frac{2\exp(1)}{13} - \frac{5\sqrt{7}+3\sin 7}{15}| \Gamma_q(\frac{8}{5}) \Gamma_q(\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6} + 1)} \\
 &\quad + \frac{|\frac{\sqrt{7}}{3}|(\frac{2}{5})^{\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6}-1} + |\frac{\sin 7}{5}|}{|\frac{2\exp(1)}{13} - \frac{5\sqrt{7}+3\sin 7}{15}| \Gamma_q(\frac{8}{5}) \Gamma_q(\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6})[\frac{\exp(1)}{2}]_q} \\
 &\simeq \begin{cases} 1.9774, & q = \frac{3}{10}, \\ 1.1928, & q = \frac{1}{2}, \\ 0.1926, & q = \frac{8}{9} \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 \aleph_2 &= \frac{1}{\Gamma_q(\theta + \vartheta + \lambda - \mu + 1)} \\
 &\quad + \frac{1}{\Gamma_q(\theta + \vartheta + \lambda - \mu) \Gamma_q(\vartheta + \theta + \lambda)[\theta]_q} \\
 &\quad + \frac{|\alpha_1| \gamma^{\theta+\vartheta} + |\alpha_2|}{\Gamma_q(\lambda - \mu + 1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta + 1)} \\
 &\quad + \frac{|\alpha_1| \gamma^{\theta+\vartheta-1} + |\alpha_2|}{\Gamma_q(\lambda - \mu + 1) |\beta - \sum_{i=1}^2 \alpha_i| \Gamma_q(\lambda + 1) \Gamma_q(\theta + \vartheta)[\theta]_q} \\
 &= \frac{1}{\Gamma_q(\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6} + \frac{3}{5} - \frac{1}{4} + 1)} \\
 &\quad + \frac{1}{\Gamma_q(\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6} + \frac{3}{5} - \frac{1}{4}) \Gamma_q(\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6} + \frac{3}{5})[\frac{\exp(1)}{2}]_q} \\
 &\quad + \frac{|\frac{\sqrt{7}}{3}|(\frac{2}{5})^{\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6}} + |\frac{\sin 7}{5}|}{\Gamma_q(\frac{27}{20}) |\frac{2\exp(1)}{13} - \frac{5\sqrt{7}+3\sin 7}{15}| \Gamma_q(\frac{8}{5}) \Gamma_q(\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6} + 1)} \\
 &\quad + \frac{|\frac{\sqrt{7}}{3}|(\frac{2}{5})^{\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6}-1} + |\frac{\sin 7}{5}|}{\Gamma_q(\frac{27}{20}) |\frac{2\exp(1)}{13} - \frac{5\sqrt{7}+3\sin 7}{15}| \Gamma_q(\frac{8}{5}) \Gamma_q(\frac{\exp(1)}{2} + \frac{\sqrt{11}}{6})[\frac{\exp(1)}{2}]_q} \\
 &\simeq \begin{cases} 1.3979, & q = \frac{3}{10}, \\ 0.7761, & q = \frac{1}{2}, \\ 0.1104, & q = \frac{8}{9}. \end{cases}
 \end{aligned}$$

Table 1 Numerical results of $\Gamma_q(\theta + \vartheta + \lambda + 1)$, $\Gamma_q(\theta + \vartheta + \lambda)$, $\Gamma_q(\lambda + 1)$, $\Gamma_q(\theta + \vartheta + 1)$, $\Gamma_q(\theta + \vartheta)$, $\Gamma_q(\theta + \vartheta + \lambda - \mu + 1)$, $\Gamma_q(\theta + \vartheta + \lambda - \mu)$, $\Gamma_q(\theta - \mu + 1)$, and \aleph_1 , \aleph_2 of FDq – DDP (24) with $q = \frac{3}{10}$

n	$q = \frac{3}{10}$	$\Gamma_q(\theta + \vartheta + \lambda + 1)$	$\Gamma_q(\theta + \vartheta + \lambda)$	$\Gamma_q(\lambda + 1)$	$\Gamma_q(\theta + \vartheta + 1)$	$\Gamma_q(\theta + \vartheta)$	\aleph_1
1	1.6644	1.3700	1.7954	1.4432	1.4704	1.9579	
2	1.6265	1.3527	1.8267	1.4170	1.4750	1.9714	
3	1.6155	1.3477	1.8360	1.4093	1.4763	1.9756	
4	1.6122	1.3461	1.8388	1.4070	1.4767	1.9768	
5	1.6112	1.3457	1.8396	1.4064	1.4768	1.9772	
6	1.6109	1.3456	1.8398	1.4062	1.4769	1.9773	
7	<u>1.6108</u>	<u>1.3455</u>	1.8399	<u>1.4061</u>	1.4769	1.9773	
8	1.6108	1.3455	1.8399	1.4061	1.4769	1.9774	
9	1.6108	1.3455	<u>1.8400</u>	1.4061	1.4769	1.9774	
10	1.6108	1.3455	1.8400	1.4061	1.4769	1.9774	

n	$\Gamma_q(\theta + \vartheta + \lambda - \mu + 1)$	$\Gamma_q(\theta + \vartheta + \lambda - \mu)$	$\Gamma_q(\theta - \mu + 1)$	\aleph_2
1	1.5583	1.3689	2.6131	1.3817
2	1.5252	1.3587	2.7022	1.3928
3	1.5156	1.3557	2.7286	1.3964
4	1.5127	1.3548	2.7365	1.3974
5	1.5118	1.3546	2.7389	1.3978
6	1.5116	<u>1.3545</u>	2.7396	1.3979
7	<u>1.5115</u>	1.3545	2.7398	1.3979
8	1.5115	1.3545	<u>2.7399</u>	1.3979
9	1.5115	1.3545	2.7399	1.3979
10	1.5115	1.3545	2.7399	1.3979

One can see these results in Tables 1, 2, 3, and 4. We can see graphical representation of \aleph_1 and \aleph_2 for three cases of q in Fig. 1. Curves 1a and 1b show well that as q approaches 1, the values of \aleph_1 and \aleph_2 decrease harmoniously. Similarly, Tables 1, 2, and 3 show the values of \aleph_1 and \aleph_2 for three cases of $q = \frac{3}{10}, \frac{1}{2}, \frac{8}{9}$, respectively, and the interesting thing is that when q approaches 1, we reach a constant value up to 4 decimal places in the number of steps higher than n .

It is found that $\Lambda_1 = \sup_{\tau \in \Omega} |g(\tau, 0, 0)| = 0$, $\Lambda_2 = \sup_{\tau \in \Omega} |h(\tau, 0, 0)| = 0$,

$$\Lambda_3 = \sup_{\tau \in \Omega} |\varphi(\tau)| = \sup_{\tau \in \Omega} \frac{\ln \tau + 2}{3 + \tau^2} = \frac{1}{2}$$

and so $\Lambda = \max\{\Lambda_i : i = 1, 2, 3\} = \frac{1}{2}$. From inequality (14), we remark that

$$\begin{aligned} \Delta &= \frac{(\delta \varpi_1 + \varpi_2) \Gamma_q(\eta + 1) + \varpi_2}{\Gamma_q(\eta + 1)} \\ &\simeq \left\{ \begin{array}{ll} 0.0094, & q = \frac{3}{10}, \\ 0.0083, & q = \frac{1}{2}, \\ 0.0061, & q = \frac{8}{9}. \end{array} \right\} < \left\{ \begin{array}{ll} 0.2963, & q = \frac{3}{10}, \\ 0.5079, & q = \frac{1}{2}, \\ 3.3008, & q = \frac{8}{9}. \end{array} \right\} = \frac{1}{\aleph_1 + \aleph_2}. \end{aligned}$$

According to Table 4, the calculations performed are consistent with our analysis and analytical proofs. Figure 2 shows this for the FDq – DDP (24) with three cases of q . Curves 2a, 2b, and 2c show well that as q approaches 1, the value of Δ , Eq. (14) holds, but its rate of change increases with the increase of q . Similarly, Table 4 shows the values of Δ for three

Table 2 Numerical results of $\Gamma_q(\theta + \vartheta + \lambda + 1)$, $\Gamma_q(\theta + \vartheta + \lambda)$, $\Gamma_q(\lambda + 1)$, $\Gamma_q(\theta + \vartheta + 1)$, $\Gamma_q(\theta + \vartheta)$, $\Gamma_q(\theta + \vartheta + \lambda - \mu + 1)$, $\Gamma_q(\theta + \vartheta + \lambda - \mu)$, $\Gamma_q(\theta - \mu + 1)$, and \aleph_1, \aleph_2 of $\text{FD}q - \text{DP}$ (24) with $q = \frac{1}{2}$

n	$q = \frac{1}{2}$					
	$\Gamma_q(\theta + \vartheta + \lambda + 1)$	$\Gamma_q(\theta + \vartheta + \lambda)$	$\Gamma_q(\lambda + 1)$	$\Gamma_q(\theta + \vartheta + 1)$	$\Gamma_q(\theta + \vartheta)$	\aleph_1
1	2.8428	1.9971	2.4929	2.2164	2.0510	1.1087
2	2.6015	1.9154	2.6121	2.0773	2.0696	1.1497
3	2.4936	1.8780	2.6690	2.0144	2.0783	1.1710
:	:	:	:	:	:	:
9	2.3945	1.8431	2.7233	1.9562	2.0866	1.1924
10	2.3938	1.8429	2.7237	1.9558	2.0867	1.1926
11	2.3934	1.8427	2.7239	1.9556	2.0867	1.1927
12	2.3932	1.8427	2.7240	1.9554	2.0867	1.1927
13	2.3931	<u>1.8426</u>	<u>2.7241</u>	1.9554	2.0867	1.1927
14	<u>2.3930</u>	1.8426	2.7241	1.9554	2.0867	1.1928
15	2.3930	1.8426	2.7241	1.9554	2.0867	1.1928
16	2.3930	1.8426	2.7241	<u>1.9553</u>	2.0867	1.1928
17	2.3930	1.8426	2.7241	1.9553	2.0867	1.1928

n	$\Gamma_q(\theta + \vartheta + \lambda - \mu + 1)$	$\Gamma_q(\theta + \vartheta + \lambda - \mu)$	$\Gamma_q(\theta - \mu + 1)$	\aleph_2
1	2.5368	1.9488	3.6511	0.6997
2	2.3418	1.9037	3.9742	0.7367
3	2.2542	1.8829	4.1309	0.7561
:	:	:	:	:
11	2.1726	1.8631	4.2842	0.7760
12	2.1725	1.8631	4.2845	0.7761
13	2.1724	1.8631	4.2846	0.7761
14	2.1724	<u>1.8630</u>	4.2847	0.7761
15	2.1724	1.8630	4.2847	0.7761
16	<u>2.1723</u>	1.8630	<u>4.2848</u>	0.7761
17	2.1723	1.8630	4.2848	0.7761

cases of $q = \frac{3}{10}, \frac{1}{2}, \frac{8}{9}$, respectively, and the interesting thing is that when q approaches 1, we reach a constant value up to 4 decimal places in the number of steps higher than n . Setting real number r such that $r \geq 5.8553, 3.3620, 0.5098$, whenever $q = \frac{3}{10}, \frac{1}{2}, \frac{8}{9}$, respectively. Therefore, the $\text{FD}q - \text{DP}$ (24) satisfies all conditions of Theorem 3.2 and so problem (24) has a unique solution. In the other hand, condition (H_3) holds, because there exists $N_i \geq 0$, $i = 1, 2, 3$ such that

$$\begin{aligned}
|g(\tau, w_1, w_2)| &= \left| \frac{|w_1(\tau)|}{50\sqrt{\pi(2+\tau^2)}(1+|w_1(\tau)|)} + \frac{\arctan(w_2(\tau))}{50\sqrt{\pi(2+\tau^2)}} \right| \\
&\leq \frac{1}{50\sqrt{\pi(2+\tau^2)}} \left(\left| \frac{|w_1(\tau)|}{1+|w_1(\tau)|} \right| + |\arctan(w_2(\tau))| \right) \\
&\leq \frac{2\pi+1}{50\sqrt{\pi(2+\tau^2)}} =: N_1, \\
|h(\tau, w_1, w_2)| &= \left| \frac{|w_1(\tau)|}{25(\exp(1))^{\tau^2+2}(|w_1(\tau)|+2)} + \frac{\arctan(w_2(\tau))}{25(\exp(1))^{\tau^2+2}} \right| \\
&\leq \frac{1}{25(\exp(1))^{\tau^2+2}} \left(\left| \frac{|w_1(\tau)|}{|w_1(\tau)|+2} \right| + |\arctan(w_2(\tau))| \right) \\
&\leq \frac{2\pi+1}{25(\exp(1))^{\tau^2+2}} =: N_2
\end{aligned}$$

Table 3 Numerical results of $\Gamma_q(\theta + \vartheta + \lambda + 1)$, $\Gamma_q(\theta + \vartheta + \lambda)$, $\Gamma_q(\lambda + 1)$, $\Gamma_q(\theta + \vartheta + 1)$, $\Gamma_q(\theta + \vartheta)$, $\Gamma_q(\theta + \vartheta + \lambda - \mu + 1)$, $\Gamma_q(\theta + \vartheta + \lambda - \mu)$, $\Gamma_q(\theta - \mu + 1)$, and \aleph_1, \aleph_2 of $\mathbb{FD}q - \mathbb{DP}$ (24) with $q = \frac{8}{9}$

n	$q = \frac{8}{9}$					
	$\Gamma_q(\theta + \vartheta + \lambda + 1)$	$\Gamma_q(\theta + \vartheta + \lambda)$	$\Gamma_q(\lambda + 1)$	$\Gamma_q(\theta + \vartheta + 1)$	$\Gamma_q(\theta + \vartheta)$	\aleph_1
1	67.0460	15.4708	7.4364	26.5870	8.4220	0.0832
2	48.4129	13.5941	8.3916	21.4361	8.6345	0.0938
3	38.0888	12.3894	9.1229	18.3336	8.7868	0.1038
:	:	:	:	:	:	:
71	11.8035	8.0386	13.1855	8.7147	9.5062	0.1925
72	11.8032	8.0385	13.1856	8.7146	9.5063	0.1926
73	11.8028	8.0384	13.1857	8.7144	9.5063	0.1926
:	:	:	:	:	:	:
79	11.8016	8.0381	13.1861	8.7138	9.5063	0.1926
80	11.8014	8.0381	13.1862	8.7138	<u>9.5064</u>	0.1926
81	11.8013	8.0380	13.1862	8.7137	9.5064	0.1926
:	:	:	:	:	:	:
93	11.8006	8.0379	13.1864	8.7134	9.5064	0.1926
94	11.8005	8.0379	<u>13.1865</u>	<u>8.7133</u>	9.5064	0.1926
95	<u>11.8005</u>	<u>8.0378</u>	13.1865	8.7133	9.5064	0.1926
96	11.8005	8.0378	13.1865	8.7133	9.5064	0.1926

n	$\Gamma_q(\theta + \vartheta + \lambda - \mu + 1)$	$\Gamma_q(\theta + \vartheta + \lambda - \mu)$	$\Gamma_q(\theta - \mu + 1)$	\aleph_2
1	45.0140	11.5522	8.4748	0.0287
2	33.9505	10.7796	10.4335	0.0363
3	27.6133	10.2616	12.0231	0.0435
:	:	:	:	:
65	10.1749	8.1827	22.2055	0.1103
66	10.1744	8.1826	22.2062	0.1104
67	10.1739	8.1825	22.2067	0.1104
:	:	:	:	:
82	10.1709	8.1820	22.2106	0.1104
83	10.1708	<u>8.1819</u>	22.2107	0.1104
:	:	:	:	:
94	10.1704	8.1819	22.2112	0.1104
95	10.1704	8.1819	<u>22.2113</u>	0.1104
96	<u>10.1703</u>	8.1819	22.2113	0.1104
97	10.1703	8.1819	22.2113	0.1104

and $|\varphi(\tau)| = |\frac{\ln \tau + 2}{3 + \tau^2}| \leq \frac{1}{2} =: N_3$. Hence Theorem 3.4 implies that the $\mathbb{FD}q - \mathbb{DP}$ (24) has at least one solution. Also, according to the numerical results in Table 5, we have

$$\begin{aligned} \Delta &= (\delta\varpi_1 + \varpi_2)\Gamma_q(\eta + 1) + \varpi_2 \\ &\simeq \left\{ \begin{array}{ll} 0.0128, & q = \frac{3}{10}, \\ 0.0156, & q = \frac{1}{2}, \\ 0.0500, & q = \frac{8}{9}, \end{array} \right\} \\ &< \left\{ \begin{array}{ll} 2.1845, & q = \frac{3}{10}, \\ 4.4660, & q = \frac{1}{2}, \\ 96.7859, & q = \frac{8}{9}, \end{array} \right\} \simeq \Gamma_q(\theta + \vartheta + \lambda + 1)\Gamma_q(\eta + 1). \end{aligned}$$

Table 4 Numerical results of $\frac{1}{\aleph_1 + \aleph_2}$, $\Gamma_q(\eta + 1)$, Δ , and suitable r of $\text{FD}q - \text{DP}$ (24) with $q = \frac{3}{10}$ in Example 5.1

n	$\frac{1}{\aleph_1 + \aleph_2}$	$\Gamma_q(\eta + 1)$	Δ	$r \geq$
$q = \frac{3}{10}$				
1	0.2994	1.3701	0.0094	5.7907
2	0.2972	1.3602	0.0094	5.8352
3	0.2966	1.3574	0.0094	5.8492
4	0.2964	1.3565	0.0094	5.8534
5	0.2963	1.3562	0.0094	5.8547
6	0.2963	1.3562	0.0094	5.8551
7	0.2963	1.3561	0.0094	5.8552
8	0.2963	1.3561	0.0094	5.8553
9	0.2963	1.3561	0.0094	5.8553
$q = \frac{1}{2}$				
1	0.5530	1.9486	0.0082	3.0831
2	0.5301	1.9053	0.0083	3.2186
3	0.5189	1.8853	0.0083	3.2894
⋮	⋮	⋮	⋮	⋮
10	0.5080	1.8664	0.0083	3.3614
11	0.5080	1.8664	0.0083	3.3617
12	0.5079	1.8663	0.0083	3.3619
13	0.5079	1.8663	0.0083	3.3619
14	0.5079	1.8663	0.0083	3.3620
15	0.5079	1.8663	0.0083	3.3620
$q = \frac{8}{9}$				
1	8.9368	11.4057	0.0059	0.1881
2	7.6865	10.6748	0.0059	0.2187
3	6.7896	10.1838	0.0060	0.2476
⋮	⋮	⋮	⋮	⋮
10	4.3214	8.8257	0.0060	0.3892
11	4.1818	8.7441	0.0061	0.4022
12	4.0643	8.6747	0.0061	0.4139
⋮	⋮	⋮	⋮	⋮
75	3.3010	8.2022	0.0061	0.5097
76	3.3009	8.2022	0.0061	0.5098
77	3.3009	8.2022	0.0061	0.5098
78	3.3009	8.2021	0.0061	0.5098
79	3.3008	8.2021	0.0061	0.5098

Hence, the inequality (21) in Theorem 4.1 holds for all $q \in (0, 1)$ and so Theorem 4.1 implies that the $\text{FD}q - \text{DP}$ (24) is UH stable with

$$\|\hat{\mathbf{w}} - \mathbf{w}\|_B \leq \begin{cases} 0.6245\omega, & q = \frac{3}{10}, \\ 0.4193\omega, & q = \frac{1}{2}, \\ 0.0848\omega, & q = \frac{8}{9}, \end{cases} \simeq \Sigma_{g^*, h^*} \omega, \quad (\omega > 0).$$

We can see graphical representations of UH stable Σ_{g^*, h^*} with three cases of q in Fig. 3.
Let $p(\tau) = \tau^{\frac{\sqrt{3}}{2}}$, then

$$\frac{1}{\Gamma_q(\frac{15\exp(1)+5\sqrt{11}-12}{30})} \int_0^\tau (\tau - q\xi)^{(\frac{15\exp(1)+5\sqrt{11}-12}{30})} \xi^{\frac{\sqrt{3}}{2}} d_q \xi$$

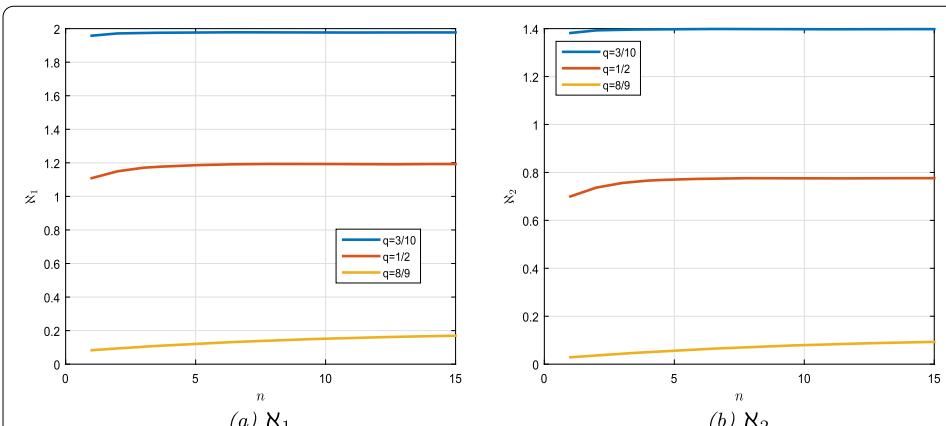


Figure 1 Graphical representation of N_1 and N_2 for $q \in \{\frac{3}{10}, \frac{1}{2}, \frac{8}{9}\}$ in Example 5.1

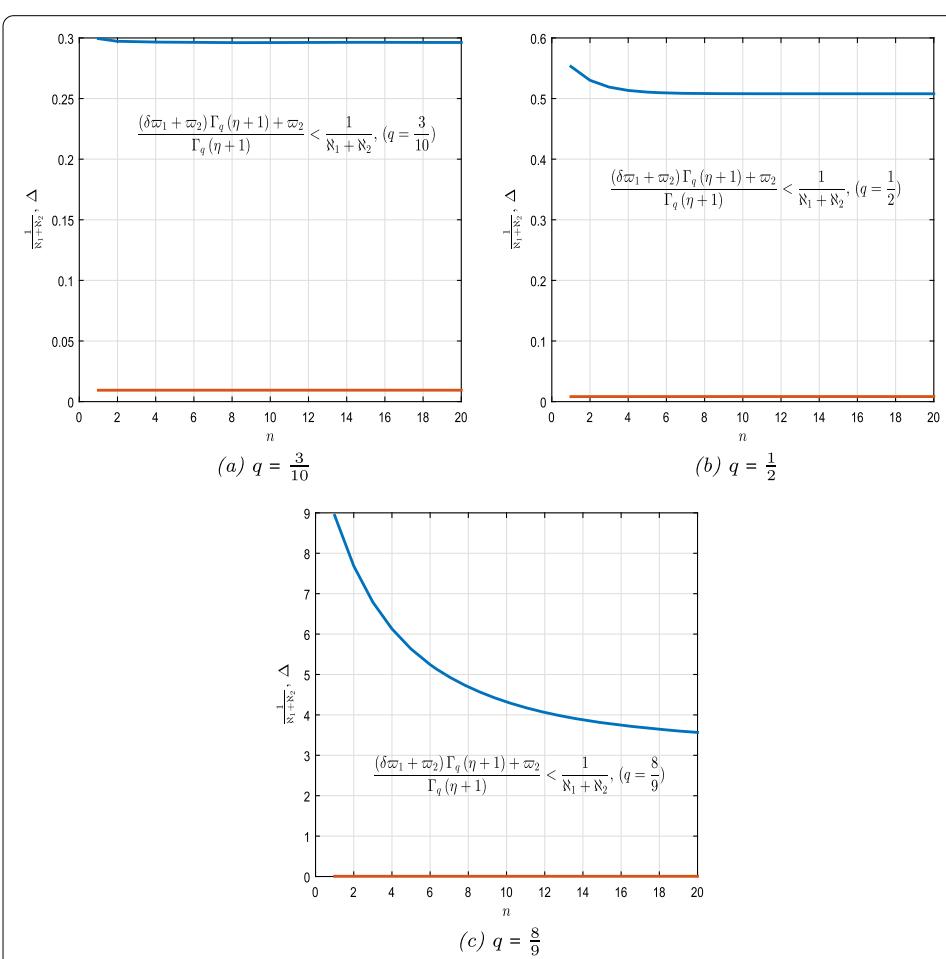


Figure 2 Graphical representation of Δ and $\frac{1}{N_1+N_2}$ for $q \in \{\frac{3}{10}, \frac{1}{2}, \frac{8}{9}\}$ in Example 5.1

$$\leq \begin{cases} 0.9366p(\tau), & q = \frac{3}{10}, \\ 0.8957p(\tau), & q = \frac{1}{2}, \\ 0.8334p(\tau), & q = \frac{8}{9}, \end{cases}$$

Table 5 Numerical results of $\tilde{\Delta} = (\delta\varpi_1 + \varpi_2)\Gamma_q(\eta + 1) + \varpi_2, (*) : \Gamma_q(\theta + \vartheta + \lambda + 1)\Gamma_q(\eta + 1)$, and UH stable Σ_{g^*, h^*} of FDq - DIP (24) with $q \in \{\frac{3}{10}, \frac{1}{2}, \frac{8}{9}\}$ in Example 5.1

n	$q = \frac{3}{10}$			$q = \frac{1}{2}$			$q = \frac{8}{9}$		
	$\tilde{\Delta}$	(*)	Σ_{g^*, h^*}	$\tilde{\Delta}$	(*)	Σ_{g^*, h^*}	$\tilde{\Delta}$	(*)	Σ_{g^*, h^*}
1	0.0129	2.2803	0.6042	0.0160	5.5396	0.3528	0.0674	764.7056	0.0149
2	<u>0.0128</u>	2.2125	0.6184	0.0158	4.9566	0.3856	0.0634	516.8007	0.0207
3	0.0128	2.1928	0.6226	0.0157	4.7011	0.4024	0.0608	387.8897	0.0263
4	0.0128	2.1870	0.6239	<u>0.0156</u>	4.5811	0.4108	0.0588	311.0775	0.0316
5	0.0128	2.1852	0.6243	0.0156	4.5230	0.4151	0.0574	261.0760	0.0366
6	0.0128	2.1847	0.6244	0.0156	4.4943	0.4172	0.0562	226.4491	0.0413
7	0.0128	<u>2.1845</u>	0.6245	0.0156	4.4801	0.4183	0.0553	201.3523	0.0456
8	0.0128	2.1845	0.6245	0.0156	4.4731	0.4188	0.0545	182.5200	0.0495
:	:	:	:	:	:	:	:	:	:
10	0.0128	2.1845	0.6245	0.0156	4.4678	0.4192	0.0534	156.5566	0.0564
11	0.0128	2.1845	0.6245	0.0156	4.4669	0.4193	0.0529	147.3789	0.0594
12	0.0128	2.1845	0.6245	0.0156	4.4664	0.4193	0.0526	139.9099	0.0620
13	0.0128	2.1845	0.6245	0.0156	4.4662	0.4193	0.0522	133.7573	0.0644
14	0.0128	2.1845	0.6245	0.0156	4.4661	0.4193	0.0520	128.6372	0.0666
15	0.0128	2.1845	0.6245	0.0156	<u>4.4660</u>	0.4193	0.0517	124.3393	0.0685
16	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	0.0515	120.7051	0.0703
:	:	:	:	:	:	:	:	:	:
42	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	0.0501	97.7327	0.0841
43	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	<u>0.0500</u>	97.6267	0.0842
44	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	0.0500	97.5327	0.0842
45	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	0.0500	97.4493	0.0843
:	:	:	:	:	:	:	:	:	:
65	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	0.0500	96.8484	0.0847
66	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	0.0500	96.8415	0.0847
67	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	0.0500	96.8353	0.0848
68	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	0.0500	96.8298	0.0848
:	:	:	:	:	:	:	:	:	:
117	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	0.0500	96.7860	0.0848
118	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	0.0500	<u>96.7859</u>	0.0848
119	0.0128	2.1845	0.6245	0.0156	4.4660	0.4193	0.0500	96.7859	0.0848

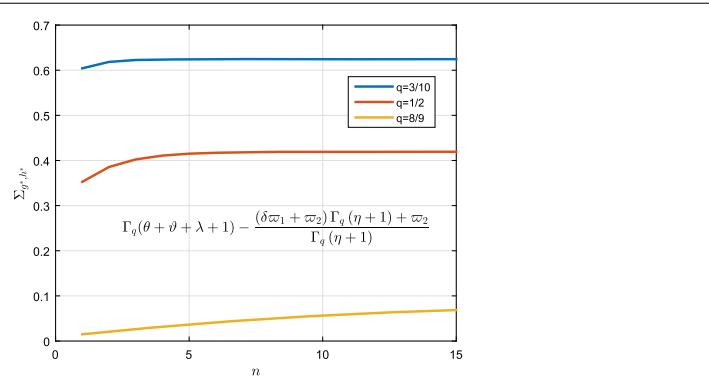


Figure 3 Graphical representation of UH stable Σ_{g^*, h^*} for $q \in \{\frac{3}{10}, \frac{1}{2}, \frac{8}{9}\}$ in Example 5.1

$$\simeq \frac{\Gamma_q(\frac{\sqrt{3}+2}{2})}{\Gamma_q(\frac{15\exp(1)+5\sqrt{11}+15\sqrt{3}+48}{30})} \tau^{\frac{\sqrt{3}}{2}} = \chi_p p(\tau).$$

Table 6 Numerical results of $\Gamma_q(\rho + 1)$, here $p(\tau) = \tau^{(\rho)}$, χ_p and UHR stable Σ_{g^*, h^*} of $\text{FD}q - \text{DP}$ (24) with $q \in \{\frac{3}{10}, \frac{1}{2}, \frac{8}{9}\}$ in Example 5.1

n	$q = \frac{3}{10}$			$q = \frac{1}{2}$			$q = \frac{8}{9}$		
	$\Gamma_q(\rho + 1)$	χ_p	Σ_{g^*, h^*}	$\Gamma_q(\rho + 1)$	χ_p	Σ_{g^*, h^*}	$\Gamma_q(\rho + 1)$	χ_p	Σ_{g^*, h^*}
1	1.4983	0.9002	0.9064	2.0868	0.7341	0.7371	8.1718	0.1219	0.1219
2	1.5056	0.9257	0.9321	2.1162	0.8135	0.8170	8.4905	0.1754	0.1755
3	1.5078	0.9333	0.9399	2.1301	0.8542	0.8580	8.7209	0.2290	0.2291
4	1.5084	0.9356	0.9422	2.1368	0.8749	0.8788	8.8974	0.2811	0.2813
5	1.5086	0.9363	0.9429	2.1401	0.8852	0.8892	9.0373	0.3309	0.3310
6	1.5087	0.9365	0.9431	2.1417	0.8905	0.8945	9.1510	0.3777	0.3779
7	1.5087	0.9366	0.9432	2.1425	0.8931	0.8971	9.2449	0.4213	0.4216
8	1.5087	0.9366	0.9432	2.1429	0.8944	0.8984	9.3235	0.4617	0.4619
9	1.5087	0.9366	0.9432	2.1431	0.8950	0.8991	9.3899	0.4987	0.4990
10	1.5087	0.9366	0.9432	2.1432	0.8953	0.8994	9.4466	0.5325	0.5329
11	1.5087	0.9366	0.9432	2.1433	0.8955	0.8996	9.4951	0.5634	0.5637
12	1.5087	0.9366	0.9432	2.1433	0.8956	0.8997	9.5370	0.5913	0.5917
13	1.5087	0.9366	0.9432	2.1433	0.8956	0.8997	9.5732	0.6166	0.6170
:	:	:	:	:	:	:	:	:	:
66	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8348	0.8330	0.8337
67	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8349	0.8331	0.8337
68	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8349	0.8331	0.8338
69	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8350	0.8332	0.8338
70	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8350	0.8332	0.8338
71	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8350	0.8332	0.8339
72	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8350	0.8333	0.8339
73	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8351	0.8333	0.8339
74	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8351	0.8333	0.8339
75	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8351	0.8333	0.8340
76	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8351	0.8333	0.8340
77	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8352	0.8334	0.8340
78	1.5087	0.9366	0.9432	2.1433	0.8957	0.8997	9.8352	0.8334	0.8340

Table 6 shows these results. Then, the condition (23) is satisfied with $p(\tau) = \tau^{\frac{\sqrt{3}}{2}}$ and

$$\chi_p = \Gamma_q\left(\frac{\sqrt{3}+2}{2}\right)\left[\Gamma_q\left(\frac{15\exp(1)+5\sqrt{11}+15\sqrt{3}+48}{30}\right)\right]^{-1}.$$

It follows from Theorem 4.2 that $\text{FD}q - \text{DP}$ (24) is UHR stable with

$$\|\tilde{w} - w\|_B \leq \begin{cases} 0.9432, & q = \frac{3}{10}, \\ 0.8997, & q = \frac{1}{2}, \\ 0.8347, & q = \frac{8}{9}, \end{cases} \times \omega \tau^{\frac{\sqrt{3}}{2}}, \quad (\omega > 0), \tau \in \Omega.$$

One can see graphical representations of UHR stable Σ_{g^*, h^*} with three cases of q in Fig. 4.

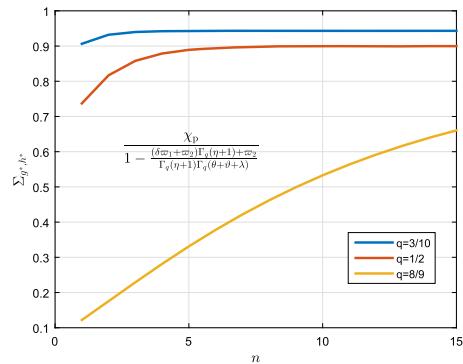
5.2 An application: a Duffing-type oscillator

As is known, the equation of motion of a Duffing oscillator under initial conditions is normally written

$$w''(\tau) + \alpha_1 w'(\tau) + \alpha_2 w(\tau) + \alpha_3 w^3(\tau) = F_0 \cos \omega \tau, \quad (27)$$

$w(0) = r_1$, $w'(0) = r_2$, where α_i , r_1 , r_2 are real constants, $w'' + \alpha_2 w$ is a simple harmonic oscillator with angular frequency $\sqrt{\alpha_2}$, $\alpha_1 w'$ is a small damping, $\alpha_3 w^3$ is a small nonlinearity,

Figure 4 Graphical representation of UHR stable Σ_{g^*, h^*} for $q \in \{\frac{3}{10}, \frac{1}{2}, \frac{8}{9}\}$ in Example 5.1



and $F_0 \cos \omega\tau$ is a small periodic forcing term with angular frequency ω . In this work, the differential equation (27) is rearranged to $\text{FD}q - \text{DP}$ as follows:

$$\begin{aligned} & {}^{\text{RL}}D_q^\theta [{}^C D_q^\vartheta [{}^C D_q^\lambda w(\tau)]] \\ &= \varphi(\tau) - \delta g(\tau, w(\tau), {}^C D_q^\mu w(\tau)) - h(\tau, w(\tau), {}_{\text{RL}}I_q^\eta w(\tau)) \\ &= F_0 \cos \omega\tau - \alpha_1 w'(\tau) - \alpha_2 w(\tau) - \alpha_3 w^3(\tau), \end{aligned}$$

for $\delta > 0$, $\tau \in \Omega := [0, 2\pi]$. Hence, $\varphi(\tau) = F_0 \cos \omega\tau$,

$$\begin{aligned} \delta g(\tau, w(\tau), {}^C D_q^\mu w(\tau)) &= \alpha_1 w'(\tau), \\ h(\tau, w(\tau), {}_{\text{RL}}I_q^\eta w(\tau)) &= \alpha_2 w(\tau) + \alpha_3 w^3(\tau), \end{aligned}$$

with $\mu = 1$, $\eta = 0$, $\delta = \alpha_1$, and $\alpha_2, \alpha_3 \neq 0$. It is obvious that Theorems 3.2, 3.4, and 4.1 confirm the existence of the solution and its stability.

6 Conclusion

The $\text{FD}q - \text{DP}$ has been investigated in this work in detail. The investigation of this particular equation provides us with a powerful tool in modeling most scientific phenomena without the need to remove most parameters that have an essential role in the physical interpretation of the studied phenomena. $\text{FD}q - \text{DP}$ (1) has been studied on a time scale under some BCs. An application that describes the motion of a particle in the plane has been provided to support our results' validity and applicability in fields of physics and engineering.

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Data availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Competing interests

The authors declare no competing interests.

Author contributions

MH: Actualization, methodology, formal analysis, validation, investigation and initial draft. MES: Methodology, formal analysis, validation, actualization, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. SSS: Formal analysis, methodology, validation, investigation and initial draft. JA: Methodology, formal analysis, validation and investigation. All authors read and approved the final manuscript.

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References

1. Finkelstein, R., Marcus, E.: Transformation theory of the q -oscillator. *J. Math. Phys.* **36**, 2652–2672 (1995). <https://doi.org/10.1063/1.531057>
2. Floreanini, R., Vinet, L.: Symmetries of the q -difference heat equation. *Lett. Math. Phys.* **32**, 37–44 (1994). <https://doi.org/10.1007/BF00761122>
3. Han, G.N., Zeng, J.: On a q -sequence that generalizes the median Genocchi numbers. *Ann. Math. Qué.* **23**, 63–72 (1999)
4. Tajadodi, H.: Variable-order Mittag-Leffler fractional operator and application to mobile-immobile advection-dispersion model. *Alex. Eng. J.* **61**(5), 3719–3728 (2022). <https://doi.org/10.1016/j.aej.2021.09.007>
5. Abdeljawad, T., Samei, M.E.: Applying quantum calculus for the existence of solution of q -integro-differential equations with three criteria. *Discrete Contin. Dyn. Syst., Ser. S* **14**(10), 3351–3386 (2021). <https://doi.org/10.3934/dcdss.2020440>
6. Guo, C., Guo, J., Gao, Y., Kang, S.: Existence of positive solutions for two-point boundary value problems of nonlinear fractional q -difference equation. *Adv. Differ. Equ.* **2018**, 180 (2018). <https://doi.org/10.1186/s13662-018-1578-y>
7. Liang, S., Samei, M.E.: New approach to solutions of a class of singular fractional q -differential problem via quantum calculus. *Adv. Differ. Equ.* **2020**, 14 (2020). <https://doi.org/10.1186/s13662-019-2489-2>
8. Sheng, Y., Zhang, T.: Some results on the q -calculus and fractional q -differential equations. *Mathematics* **10**(1), 64 (2022). <https://doi.org/10.3390/math10010064>
9. Shah, D., Riaz, U., Zada, A.: Exponential and Hyers-Ulam stability of impulsive linear system of first order. *Differential Equations & Applications* **15**(1), 317–326 (2023). <https://doi.org/10.7153/dea-2023-15-01>
10. Houas, M., Samei, M.E.: Existence and Mittag-Leffler-Ulam-stability results for Duffing type problem involving sequential fractional derivatives. *Int. J. Appl. Comput. Math.* **8**, 185 (2022). <https://doi.org/10.1007/s40819-022-01398-y>
11. Abbas, S., Benchohra, M., Laledj, N., Zhou, Y.: Existence and Ulam stability for implicit fractional q -difference equations. *Adv. Differ. Equ.* **2019**, 48 (2019). <https://doi.org/10.1186/s13662-019-2411-y>
12. Afshari, H., Shojaat, H., Moradi, S.: Existence of the positive solutions for a tripled system of fractional differential equations via integral boundary conditions. *Results Nonlinear Anal.* **4**(3), 186–199 (2021). <https://doi.org/10.53006/rna.938851>
13. Jiang, M., Huang, R.: Existence and stability results for impulsive fractional q -difference equation. *J. Appl. Math. Phys.* **8**(7), 1413–1423 (2020). <https://doi.org/10.4236/jamp.2020.87107>
14. Kalvandi, V., Samei, M.E.: New stability results for a sum-type fractional q -integro-differential equation. *J. Adv. Math. Stud.* **12**(2), 201–209 (2019)
15. Waheed, H., Zada, A., Rizwan, R., Popa, I.L.: Hyers-Ulam stability for a coupled system of fractional differential equation with p -Laplacian operator having integral boundary conditions. *Qual. Theory Dyn. Syst.* **21**, 92 (2022). <https://doi.org/10.1007/s12346-022-00624-8>
16. Riaz, U., Zada, A.: Hyers-Ulam types stability of nonlinear summation equations with delay. *Int. J. Nonlinear Anal. Appl.* **12**(2), 317–326 (2021). <https://doi.org/10.22075/ijnaa.2019.18218.1995>
17. Agarwal, R.P., Ahmad, B., Alsaedi, A., Al-Hutami, H.: Existence theory for q -antiperiodic boundary value problems of sequential q -fractional integrodifferential equations. *Abstr. Appl. Anal.* **2014**, Article ID 207547 (2014). <https://doi.org/10.1186/s13662-019-2411-y>
18. Etemad, S., Ntouyas, S., Ahmad, B.: Existence theory for a fractional q -integro-difference equation with q -integral boundary conditions of different orders. *Mathematics* **7**(8), 659 (2019). <https://doi.org/10.3390/math7080659>
19. Phuong, N.D., Etemad, S., Rezapour, S.: On two structures of the fractional q -sequential integro-differential boundary value problems. *Math. Methods Appl. Sci.* **45**(2), 618–639 (2022). <https://doi.org/10.1002/mma.7800>
20. Chandrasekhar, S.: An introduction to the study of stellar structure. *Ciel Terre* **55**, 412–415 (1939)
21. Afshari, H.: Solution of fractional differential equations in quasi- b -metric and b -metric-like spaces. *Adv. Differ. Equ.* **2019**, 285 (2019). <https://doi.org/10.1186/s13662-019-2227-9>
22. Chen, H., Li, Y.: Rate of decay of stable periodic solutions of Duffing equations. *J. Differ. Equ.* **236**, 493–503 (2007). <https://doi.org/10.1016/j.jde.2007.01.023>
23. Duffing, G.: Forced Oscillations with Variable Natural Frequency and Their Technical Significance. Vieweg, Braunschweig (1918)

24. Lazer, A.C., McKenna, P.J.: On the existence of stable periodic solutions of differential equations of Duffing type. Proc. Am. Math. Soc. **110**(1), 125–133 (1990). <https://doi.org/10.2307/2048251>
25. Sunday, J.: The Duffing oscillator: applications and computational simulations. Asian Res. J. Math. **2**(3), 1–13 (2017). <https://doi.org/10.9734/ARJOM/2017/31199>
26. Ejikeme, C.L., Oyesanya, M.O., Agbebaku, D.F., Okofu, M.B.: Solution to nonlinear Duffing oscillator with fractional derivatives using Homotopy Analysis Method (HAM). Glob. J. Pure Appl. Math. **14**(10), 1363–1383 (2018)
27. Gouari, Y., Dahmani, Z., Jebril, I.: Application of fractional calculus on a new differential problem of Duffing type. Adv. Math. Sci. J. **9**(12), 10989–11002 (2020). <https://doi.org/10.37418/amsj.9.12.82>
28. Niu, J., Liu, R., Shen, Y., Yang, S.: Chaos detection of Duffing system with fractional order derivative by Melnikov method. Chaos, Interdiscip. J. Nonlinear Sci. **29**, 123106 (2019). <https://doi.org/10.1063/1.5124367>
29. Pirmohabbati, P., Refahi Sheikhan, A.H., Saberi Najaf, H., Abdolahzadeh Ziabari, A.: Numerical solution of full fractional Duffing equations with Cubic-Quintic-Heptic nonlinearities. AIMS Math. **5**(2), 1621–1641 (2020). <https://doi.org/10.3934/math.2020110>
30. Tablennehas, K., Zoubir, D.: A three sequential fractional differential problem of Duffing type. Appl. Math. E-Notes **21**, 587–598 (2021)
31. Riaz, U., Zada, A.: Analysis of (α, β) -order coupled implicit Caputo fractional differential equations using topological degree method. Fractal Fract. **22**(7–8), 897–915 (2021). <https://doi.org/10.1515/ijsns-2020-0082>
32. Zada, A., Waheed, H.: Stability analysis of implicit fractional differential equation with anti-periodic integral boundary value problem. Ann. Univ. Paedagog. Crac. Stud. Math. **19**(1), 5–25 (2020). <https://doi.org/10.2478/aupcsm-2020-0001>
33. Guo, L., Riaz, U., Zada, A., Alam, M.: On implicit coupled Hadamard fractional differential equations with generalized Hadamard fractional integro-differential boundary conditions. Fractal Fract. **7**(1), 13 (2023). <https://doi.org/10.3390/fractfract7010013>
34. Jackson, F.H.: q -Difference equations. Am. J. Math. **32**, 305–314 (1910). <https://doi.org/10.2307/2370183>
35. Adams, C.R.: The general theory of a class of linear partial q -difference equations. Trans. Am. Math. Soc. **26**, 283–312 (1924)
36. Samei, M.E., Zanganeh, H., Aydogan, S.M.: Investigation of a class of the singular fractional integro-differential quantum equations with multi-step methods. J. Math. Ext. **15**, 1–45 (2021). <https://doi.org/10.30495/JMSE.2021.2070>
37. Georgiev, S.G.: Fractional Dynamic Calculus and Fractional Dynamic Equations on Time Scales. Springer, Switzerland (2012). <https://doi.org/10.1007/978-3-319-73954-0>
38. Annaby, M.H., Mansour, Z.S.: q -Fractional Calculus and Equations. Springer, Cambridge (2012). <https://doi.org/10.1007/978-3-642-30898-7>
39. Rajković, P.M., Marinković, S.D., Stanković, M.S.: On q -analogues of Caputo derivative and Mittag-Leffler function. Fract. Calc. Appl. Anal. **10**(4), 359–373 (2007)
40. Smart, D.R.: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)

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