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Approximation by multivariate Baskakov–Durrmeyer operators in Orlicz spaces

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Abstract

Employing some properties of multivariate Baskakov–Durrmeyer operators and utilizing modified *K*-functional and a decomposition technique, the authors obtain the direct theorem and weak type inverse theorem in the Orlicz spaces.

Mathematics Subject Classification: Primary 41A17; secondary 41A35

Keywords: Direct theorem; Weak type inverse theorem; Orlicz space; Jensen's inequality; *K*-functional; Multivariate Baskakov–Durrmeyer operator

1 Preliminaries

For proceeding smoothly, we recall from [27] some definitions and related results.

A continuous convex function $\Phi(t)$ on $[0,\infty)$ is called a Young function if it satisfies

$$\lim_{t\to 0^+}\frac{\Phi(t)}{t}=0 \quad\text{and}\quad \lim_{t\to\infty}\frac{\Phi(t)}{t}=\infty.$$

For a Young function $\Phi(t)$, its complementary Young function is denoted by $\Psi(t)$.

A function $\varphi:[0,\infty)\to\mathbb{R}$ is said to be star-shaped if $\varphi(\nu t)\le\nu\varphi(t)$ for all $\nu\in[0,1]$ and $t\ge 0$. A real function φ defined on a set $S\subset\mathbb{R}^n$ is said to be super-additive if $s,t\in S$ implies $s+t\in S$ and $\varphi(s+t)\ge\varphi(s)+\varphi(t)$. See [21, Chap. 16] and [23, Sect. 3.4]. Among convex functions, star-shaped functions, and super-additive functions, the following relations hold true:

1) If φ is convex on $[0,\infty)$ with $\varphi(0)\le 0$, then φ is star-shaped;
2) If $\varphi:[0,\infty)\to\mathbb{R}$ is star-shaped, then φ is super-additive.

See [21, pp. 650–651, Section B.9], [24, p. 706], [25, pp. 616–617], or [26, Lemma 2.2]. Therefore, a Young function $\Phi(t)$ is both star-shaped and super-additive.

A Young function $\Phi(t)$ is said to satisfy the Δ_2-condition, denoted by $\Phi\in\Delta_2$, if there exist $t_0\ge 0$ and $C\ge 1$ such that $\Phi(2t)\le C\Phi(t)$ for $t\ge t_0$.

Throughout the paper we shall use the following standard notations:

$$\mathbb{N}=\{1,2,\dots\},\qquad \mathbb{N}_0=\{0,1,2,\dots\},\qquad \binom{n}{\boldsymbol{k}}=\frac{n!}{\boldsymbol{k}!(n-|\boldsymbol{k}|)!},$$

$$\boldsymbol{k} = (k_1, k_2, \dots, k_m) \in \mathbb{N}_0^m, \qquad \boldsymbol{k}! = k_1! k_2! \cdots k_m!, \qquad |\boldsymbol{k}| = \sum_{i=1}^{m} k_i,$$

$$\boldsymbol{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m, \qquad \boldsymbol{x}^{\boldsymbol{k}} = x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}, \qquad |\boldsymbol{x}| = \sum_{i=1}^{m} x_i,$$

$$\sum_{\boldsymbol{k}=0}^{\infty} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_m=0}^{\infty}, \qquad D^{\boldsymbol{k}} = D_1^{k_1} D_2^{k_2} \cdots D_m^{k_m}, \qquad D_i^r = \frac{\partial^r}{\partial x_i^r},$$

and

$$\mathbb{R}_0^m = \{\boldsymbol{x} = (x_1, x_2, \dots, x_m) \in \mathbb{R}^m : 0 \le x_i < \infty, 1 \le i \le m\}$$

for $m \in \mathbb{N}$ and $r \in \mathbb{N}$.

Let $\Phi(t)$ be a Young function. We define the Orlicz class $L_\Phi(\mathbb{R}_0^m)$ as the collection of all Lebesgue measurable functions $f(\boldsymbol{x})$ on $\mathbb{R}_0^m$ for which

$$\rho(f, \Phi) = \int_{\mathbb{R}_0^m} \Phi(|f(\boldsymbol{x})|) \, \mathrm{d}\boldsymbol{x} < \infty.$$

We also define the Orlicz space $L_\Phi^*(\mathbb{R}_0^m)$ as the set of all Lebesgue measurable functions $f(\boldsymbol{x})$ on $\mathbb{R}_0^m$, such that $\int_{\mathbb{R}_0^m} \Phi(|\alpha f(\boldsymbol{x})|) \, \mathrm{d}\boldsymbol{x} < \infty$ for some $\alpha > 0$. The Orlicz space is a Banach space under the Luxemburg norm

$$\|f\|_{(\Phi)} = \inf_{\lambda > 0} \left\{ \lambda : \rho\left(\frac{f}{\lambda}, \Phi\right) \le 1 \right\}.$$

The Orlicz norm, an equivalence of the Luxemburg norm on $L_\Phi^*(\mathbb{R}_0^m)$, is given by

$$\|f\|_\Phi = \sup_{\rho(g, \Psi) \le 1} \left| \int_{\mathbb{R}_0^m} f(\boldsymbol{x}) g(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right|$$

and satisfies

$$\|f\|_{(\Phi)} \le \|f\|_\Phi \le 2\|f\|_{(\Phi)}. \tag{1}$$

If $\Phi(u) = \frac{u^p}{p}$ for $1 < p < \infty$, then the complementary function becomes $\Psi(u) = \frac{|u|^q}{q}$ with $\frac{1}{p} + \frac{1}{q} = 1$, and then $L_\Phi^*(\mathbb{R}_0^m) = L_p(\mathbb{R}_0^m)$. So the Orlicz spaces $L_\Phi^*(\mathbb{R}_0^m)$ are more general than the classical $L_p(\mathbb{R}_0^m)$ spaces which are composed of measurable functions $f(\boldsymbol{x})$ such that $|f(\boldsymbol{x})|^p$ are integrable.

Throughout this paper we use C to denote a constant independent of n and $\boldsymbol{x}$, which may be not necessarily the same in different cases.

For $\boldsymbol{x} \in \mathbb{R}_0^m$, we introduce weight functions

$$\varphi(x) = \sqrt{x(1+x)}$$

for $m = 1$ and

$$\varphi_i(\boldsymbol{x}) = \sqrt{x_i(1 + |\boldsymbol{x}|)}$$

for $m > 1$ and $1 \leq i \leq m$. We also define the weighted Sobolev space

$$W_{\varphi}^{r,\Phi}(\mathbb{R}_0^m) = \{f \in L_{\Phi}^*(\mathbb{R}_0^m) : D^{\boldsymbol{k}} f \in A.C.loc(\mathring{\mathbb{R}}_0^m), \varphi_i^r D_i^r f \in L_{\Phi}^*(\mathbb{R}_0^m)\},$$

where $|\boldsymbol{k}| \leq r$ and $\mathring{\mathbb{R}}_0^m$ is the interior of $\mathbb{R}_0^m$.

The modified Peetre K-functionals are defined by

$$\bar{K}_{r,\varphi}(f,t^r)_{\Phi} = \inf\left\{\|f-g\|_{\Phi} + t^r \sum_{i=1}^{m} \|\varphi_i^r D_i^r g\|_{\Phi} + t^{2r} \sum_{i=1}^{m} \|D_i^r g\|_{\Phi} : g \in W_{\varphi}^{r,\Phi}(\mathbb{R}_0^m)\right\}$$

and

$$\tilde{K}_{r,\varphi}(f,t^r)_{\Phi} = \inf\left\{\|f-g\|_{\Phi} + t^r \max_{1\leq i\leq m} \|\varphi_i^r D_i^r g\|_{\Phi} : g \in W_{\varphi}^{r,\Phi}(\mathbb{R}_0^m)\right\}$$

for $t > 0$.

For any vector $\boldsymbol{e} \in \mathbb{R}^m$, we write

$$\Delta_{h\boldsymbol{e}}^r f(\boldsymbol{x}) = \begin{cases} \sum_{i=0}^{r} \binom{r}{i}(-1)^i f(\boldsymbol{x} + ih\boldsymbol{e}), & \boldsymbol{x}, \boldsymbol{x} + rh\boldsymbol{e} \in \mathbb{R}_0^m, \\ 0, & \text{otherwise} \end{cases}$$

for the rth forward difference of a function f in the direction of $\boldsymbol{e}$. We define the modulus of smoothness of $f \in L_{\Phi}^*(\mathbb{R}_0^m)$ as

$$\omega_{r,\varphi}(f,t)_{\Phi} = \sup_{0<h\leq t} \sum_{i=1}^{m} \|\Delta_{h\varphi_i \boldsymbol{e}_i}^r f\|_{\Phi}.$$

2 Motivations and main results

Between the modulus of smoothness and the K-functional there exists the following equivalent theorems.

Theorem A ([13]) *Let $f \in L_{\Phi}^*(\mathbb{R}_0^m)$ and $r \in \mathbb{N}$. Then there exist some constants C and t_0 such that*

$$\frac{\omega_{r,\varphi}(f,t)_{\Phi}}{C} \leq \bar{K}_{r,\varphi}(f,t^r)_{\Phi} \leq C\omega_{r,\varphi}(f,t)_{\Phi}, \quad 0 < t \leq t_0. \tag{2}$$

Theorem B ([31]) *Let $f \in L_{\Phi}^*(\mathbb{R}_0^m)$ and $r \in \mathbb{N}$. Then there exist some constants C and t_0 such that*

$$\frac{\omega_{r,\varphi}(f,t)_{\Phi}}{C} \leq \tilde{K}_{r,\varphi}(f,t^r)_{\Phi} \leq C\omega_{r,\varphi}(f,t)_{\Phi}, \quad 0 < t \leq t_0. \tag{3}$$

Let

$$p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, \quad x \in [0,\infty), n \in \mathbb{N}.$$

The well-known Baskakov operators were defined in [2] as

$$B_n(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right).$$

These operators can be used to approximate any function f defined on $[0,\infty)$. For $f \in L_p[0,\infty)$ and $1 \le p < \infty$, the Baskakov–Durrmeyer operators were defined in [17] as

$$V_{n,1}(f,x) = \sum_{k=0}^{\infty} p_{n,k}(x)(n-1)\int_0^{\infty} p_{n,k}(t)f(t)\,\mathrm{d}t, \quad x \in [0,\infty).$$

For a function f defined on $\mathbb{R}_0^m$, the multivariate Baskakov operators were defined in [5] as

$$B_{n,m}(f,\boldsymbol{x}) = \sum_{\boldsymbol{k}=0}^{\infty} f\left(\frac{\boldsymbol{k}}{n}\right) p_{n,\boldsymbol{k}}(\boldsymbol{x}),$$

where

$$p_{n,\boldsymbol{k}}(\boldsymbol{x}) = \binom{n+|\boldsymbol{k}|-1}{\boldsymbol{k}} \frac{\boldsymbol{x}^k}{(1+|\boldsymbol{x}|)^{n+|\boldsymbol{k}|}}.$$

The multivariate Baskakov–Durrmeyer operators were defined in [4] as

$$V_{n,m}(f,\boldsymbol{x}) = \sum_{\boldsymbol{k}=0}^{\infty} p_{n,\boldsymbol{k}}(\boldsymbol{x}) Q_{n,\boldsymbol{k},m}(f), \quad f \in L_p(\mathbb{R}_0^m),$$

where

$$Q_{n,\boldsymbol{k},m}(f) = \frac{\int_{\mathbb{R}_0^m} p_{n,\boldsymbol{k}}(\boldsymbol{u}) f(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}}{\int_{\mathbb{R}_0^m} p_{n,\boldsymbol{k}}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}} = \prod_{k=1}^{m}(n-k)\int_{\mathbb{R}_0^m} p_{n,\boldsymbol{k}}(\boldsymbol{u}) f(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u}.$$

There are many approximation results about one variable operator of the Baskakov type in $C[0,\infty)$ or $L_p[0,\infty)$, see [1, 2, 7–9, 15, 17–19, 29, 30]. But there are few approximation results about multivariate Baskakov type operators (see [4, 5, 13, 22]) or multivariate Durrmeyer type operators (see [3, 20]).

In the paper [4], Cao and An obtained the strong direct inequality

$$\left\| V_{n,m}(f) - f \right\|_p \le C\left(\omega_{2,\varphi}\left(f, \frac{1}{\sqrt{n}}\right)_p + \frac{1}{n}\|f\|_p\right)$$

in $L_p(\mathbb{R}_0^m)$. In [10–12, 14–16], we obtained approximation properties for positive and linear operators in Orlicz space. In particular, we acquired the direct theorem of multivariate Baskakov–Kantorovich operators in Orlicz space in [13].

In this paper, we will discover not only the direct theorem, but also the weak type inverse theorem for the multivariate Baskakov–Durrmeyer operators $V_{n,m}(f,\boldsymbol{x})$.

Our main results can be stated in the following two theorems.

Theorem 1 (Direct theorem) *Let $f \in L^*_\Phi(\mathbb{R}^m_0)$, $\Psi \in \Delta_2$, and $n > m$ for $n, m \in \mathbb{N}$. Then*

$$\left\| V_{n,m}(f) - f \right\|_\Phi \le C\left(\omega_{2,\varphi}\left(f, \frac{1}{\sqrt{n}}\right)_\Phi + \frac{\|f\|_\Phi}{n}\right).$$

Theorem 2 (Weak type inverse theorem) *Let $f \in L^*_\Phi(\mathbb{R}^m_0)$ and $n > m$ for $n, m \in \mathbb{N}$. Then*

$$\omega_{2,\varphi}\left(f, \frac{1}{n}\right)_\Phi \le \frac{C}{n}\sum_{k=1}^{n}\left\| V_{k,m}(f) - f \right\|_\Phi.$$

Remark 1 Theorem 1 is a generalization of [4, Theorem 2.2].

3 Proof of direct theorem

In order to prove the direct theorem, we need several lemmas.

Lemma 1 *Let $f \in L^*_\Phi(\mathbb{R}^m_0)$ and $n > m$ for $n, m \in \mathbb{N}$. Then*

$$\left\| V_{n,m}(f) \right\|_\Phi \le 2\|f\|_\Phi.$$

Proof Employing the decomposition formula

$$\begin{aligned} V_{n,m}(f, \boldsymbol{x}) = {} & \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)(n-1)\int_0^\infty p_{n,k_1}(u_1)\,\mathrm{d}u_1 \\ & \times \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)(n-2)\int_0^\infty p_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2 \cdots \\ & \times \sum_{k_m=0}^{\infty} p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{x_m}{1+\sum_{\ell=1}^{m-1}x_\ell}\right)(n-m) \\ & \times \int_0^\infty p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right) f(u_1, \ldots, u_m)\,\mathrm{d}u_m \end{aligned} \tag{4}$$

and Jensen's inequality, we obtain

$$\begin{aligned} \left\| V_{n,m}(f) \right\|_{(\Phi)} = \inf_{\lambda>0}\Bigg\{\lambda : {} & \int_{\mathbb{R}^m_0} \Phi\left(\frac{1}{\lambda}\left|\sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)(n-1)\int_0^\infty p_{n,k_1}(u_1)\,\mathrm{d}u_1\right.\right. \\ & \times \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)(n-2)\int_0^\infty p_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2 \cdots \\ & \times \sum_{k_m=0}^{\infty} p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{x_m}{1+\sum_{\ell=1}^{m-1}x_\ell}\right)(n-m) \\ & \times \int_0^\infty p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right) \\ & \left.\left. \times f(u_1, \ldots, u_m)\,\mathrm{d}u_m\right|\right)\mathrm{d}\boldsymbol{x} \le 1\Bigg\} \end{aligned}$$

$$
\begin{aligned}
&\leq \inf_{\lambda>0}\Bigg\{\lambda : \int_{\mathbb{R}_0^m} \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1) \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)\cdots \\
&\quad \times \sum_{k_m=0}^{\infty} p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{x_m}{1+\sum_{\ell=1}^{m-1}x_\ell}\right) \\
&\quad \times \Phi\Bigg(\prod_{k=1}^{m}(n-k)\int_0^\infty p_{n,k_1}(u_1)\,\mathrm{d}u_1 \int_0^\infty p_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2\cdots \\
&\quad \times \int_0^\infty p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right)\frac{|f(u_1,u_2,\ldots,u_m)|}{\lambda}\,\mathrm{d}u_m\Bigg)\,\mathrm{d}\boldsymbol{x} \leq 1\Bigg\} \\
&= \inf_{\lambda>0}\Bigg\{\lambda : \sum_{k_1=0}^{\infty}\int_0^\infty p_{n,k_1}(x_1)(1+x_1)\,\mathrm{d}x_1 \sum_{k_2=0}^{\infty}\int_0^\infty p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)(1+x_1 \\
&\quad + x_2)\,\mathrm{d}\left(\frac{x_2}{1+x_1}\right)\cdots \sum_{k_{m-1}=0}^{\infty}\int_0^\infty p_{n+\sum_{\ell=1}^{m-2}k_\ell,k_{m-1}}\left(\frac{x_{m-1}}{1+\sum_{k=1}^{m-2}x_k}\right) \\
&\quad \times \left(1+\sum_{k=1}^{m-1}x_k\right)\mathrm{d}\left(\frac{x_{m-1}}{1+\sum_{k=1}^{m-2}x_k}\right) \\
&\quad \times \sum_{k_m=0}^{\infty}\int_0^\infty p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{x_m}{1+\sum_{\ell=1}^{m-1}x_\ell}\right)\mathrm{d}\left(\frac{x_m}{1+\sum_{\ell=1}^{m-1}x_\ell}\right) \\
&\quad \times \Phi\Bigg(\prod_{k=1}^{m}(n-k)\int_0^\infty p_{n,k_1}(u_1)\,\mathrm{d}u_1 \int_0^\infty p_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2\cdots \\
&\quad \times \int_0^\infty p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right)\frac{|f(u_1,u_2,\ldots,u_m)|}{\lambda}\,\mathrm{d}u_m\Bigg) \leq 1\Bigg\} \\
&= \inf_{\lambda>0}\Bigg\{\lambda : \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{\prod_{\ell=1}^{m}(n-\ell)}\Phi\Bigg(\prod_{\ell=1}^{m}(n-\ell)\int_0^\infty p_{n,k_1}(u_1)\,\mathrm{d}u_1 \\
&\quad \times \int_0^\infty p_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2\cdots\int_0^\infty p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right) \\
&\quad \times \frac{|f(u_1,u_2,\ldots,u_m)|}{\lambda}\,\mathrm{d}u_m\Bigg) \leq 1\Bigg\} \\
&= \inf_{\lambda>0}\Bigg\{\lambda : \sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{\prod_{\ell=1}^{m}(n-\ell)}\Phi\Bigg(\prod_{\ell=1}^{m}(n-\ell)\int_0^\infty p_{n,k_1}(u_1)(1+u_1)\,\mathrm{d}u_1 \\
&\quad \times \int_0^\infty p_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right)(1+u_1+u_2)\,\mathrm{d}\left(\frac{u_2}{1+u_1}\right)\cdots\int_0^\infty\left(1+\sum_{k=1}^{m-1}u_k\right) \\
&\quad \times p_{n+\sum_{\ell=1}^{m-2}k_\ell,k_{m-1}}\left(\frac{u_{m-1}}{1+\sum_{\ell=1}^{m-2}u_\ell}\right)\mathrm{d}\left(\frac{u_{m-1}}{1+\sum_{\ell=1}^{m-2}u_\ell}\right) \\
&\quad \times \int_0^\infty p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right) \\
&\quad \times \frac{|f(u_1,u_2,\ldots,u_m)|}{\lambda}\,\mathrm{d}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right)\Bigg) \leq 1\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \inf_{\lambda>0}\Bigg\{\lambda : \sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{1}{(n-1)\cdots(n-m-1)}\int_0^{\infty} p_{n-m+1,k_1}(u_1)\,\mathrm{d}u_1 \\
&\quad\times (n+k_1-m+1)\int_0^{\infty} p_{n+k_1-m+2,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}\left(\frac{u_2}{1+u_1}\right)\cdots \\
&\quad\times \left(n-1+\sum_{\ell=1}^{m-1}k_\ell\right)\int_0^{\infty} p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right) \\
&\quad\times \Phi\left(\frac{|f(u_1,u_2,\ldots,u_m)|}{\lambda}\right)\mathrm{d}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right)\leq 1\Bigg\} \\
&= \inf_{\lambda>0}\Bigg\{\lambda : \sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\frac{n+k_1-m+1}{n-1}\,\frac{n+k_1+k_2-m+2}{n-2}\cdots \\
&\quad\times \frac{n+\sum_{\ell=1}^{m-1}k_\ell-1}{n-m+1}\int_0^{\infty} p_{n-m+1,k_1}(u_1)\,\mathrm{d}u_1\int_0^{\infty} p_{n+k_1-m+2,k_2}\left(\frac{u_2}{1+u_1}\right) \\
&\quad\times \mathrm{d}\left(\frac{u_2}{1+u_1}\right)\cdots\int_0^{\infty} p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right) \\
&\quad\times \Phi\left(\frac{|f(u_1,u_2,\ldots,u_m)|}{\lambda}\right)\mathrm{d}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right)\leq 1\Bigg\} \\
&= \inf_{\lambda>0}\left\{\lambda : \int_0^{\infty}\mathrm{d}u_1\int_0^{\infty}\mathrm{d}u_2\cdots\int_0^{\infty}\Phi\left(\frac{|f(u_1,u_2,\ldots,u_m)|}{\lambda}\right)\mathrm{d}u_m\leq 1\right\} \\
&= \|f\|_{(\Phi)}.
\end{aligned}
$$

By the double Inequality (1), we complete the proof of Lemma 1. □

Lemma 2 *Let $f \in L^*_\Phi(\mathbb{R}^2_0)$, $\Psi \in \Delta_2$, and $n > 2$. Then*

$$
\left\| V_{n,2}(f) - f \right\|_\Phi \leq \frac{C}{n}\left(\|f\|_\Phi + \sum_{i=1}^{2}\left\|\varphi_i^2 D_i^2 f\right\|_\Phi\right).
$$

Proof Let

$$
z = \frac{x_2}{1+x_1} \quad \text{and} \quad g_{u_1}(t) = f\big(u_1,(1+u_1)t\big)
$$

for $0 \leq t < \infty$. Utilizing the decomposition formula

$$
\begin{aligned}
V_{n,2}(f,\boldsymbol{x}) &= \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)(n-2)\int_0^{\infty} p_{n-1,k_1}(u_1)\,\mathrm{d}u_1 \\
&\quad\times \sum_{k_2=0}^{\infty} p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)(n+k_1-1) \\
&\quad\times \int_0^{\infty} p_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right) f\left(u_1,(1+u_1)\frac{u_2}{1+u_1}\right)\mathrm{d}\left(\frac{u_2}{1+u_1}\right)
\end{aligned}
$$

yields

$$V_{n,2}(f,\boldsymbol{x})-f(\boldsymbol{x})=\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)(n-2)\int_0^{\infty}p_{n-1,k_1}(u_1)\bigl(V_{n+k_1,1}(g_{u_1},z)\\ -g_{u_1}(z)\bigr)\,\mathrm{d}u_1+V_{n,1}^{*}\bigl(h(\cdot),x_1\bigr)-h(x_1), \tag{5}$$

where

$$h(u_1)=h(u_1,\boldsymbol{x})\triangleq f\biggl(u_1,\frac{(1+u_1)x_2}{1+x_1}\biggr),\quad 0\le u_1<\infty,$$

the notation $\triangleq$ means "define", and

$$V_{n,1}^{*}(g,x)=\sum_{i=0}^{\infty}p_{n,i}(x)(n-2)\int_0^{\infty}p_{n-1,i}(t)g(t)\,\mathrm{d}t.$$

Now we start out to estimate

$$J_1=\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)(n-2)\int_0^{\infty}p_{n-1,k_1}(u_1)\bigl(V_{n+k_1,1}(g_{u_1},z)-g_{u_1}(z)\bigr)\,\mathrm{d}u_1.$$

From [17], we obtain

$$\bigl|V_{n,1}(f,x)-f(x)\bigr|\le\frac{C}{n}\bigl[\bigl|f(x)\bigr|+\bigl|\varphi^2(x)f''(x)\bigr|\bigr]. \tag{6}$$

From the Inequality (6), Jensen's inequality, and the convexity of $\Phi(t)$, it follows

$$\begin{aligned}
&\int_0^{\infty}\int_0^{\infty}\Phi\biggl(\frac{1}{\lambda}|J_1|\biggr)\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&\quad=\int_0^{\infty}\int_0^{\infty}\Phi\biggl(\frac{1}{\lambda}\biggl|\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)(n-2)\\
&\qquad\times\int_0^{\infty}p_{n-1,k_1}(u_1)\bigl(V_{n+k_1,1}(g_{u_1},z)-g_{u_1}(z)\bigr)\,\mathrm{d}u_1\biggr|\biggr)\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&\quad\le\int_0^{\infty}\int_0^{\infty}\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)(n-2)\int_0^{\infty}p_{n-1,k_1}(u_1)\\
&\qquad\times\Phi\biggl(\frac{1}{\lambda}\bigl|V_{n+k_1,1}(g_{u_1},z)-g_{u_1}(z)\bigr|\biggr)\,\mathrm{d}u_1\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&\quad\le\int_0^{\infty}\int_0^{\infty}\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)(n-2)\int_0^{\infty}p_{n-1,k_1}(u_1)\\
&\qquad\times\Phi\biggl(\frac{C}{\lambda(n+k_1)}\bigl(\bigl|g_{u_1}(z)\bigr|+\bigl|\varphi^2(z)g_{u_1}''(z)\bigr|\bigr)\biggr)\,\mathrm{d}u_1\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&\quad=\sum_{k_1=0}^{\infty}\int_0^{\infty}p_{n,k_1}(x_1)(1+x_1)\,\mathrm{d}x_1(n-2)\int_0^{\infty}p_{n-1,k_1}(u_1)\,\mathrm{d}u_1
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^\infty \Phi\left(\frac{C}{\lambda(n+k_1)}\left(\left|g_{u_1}(z)\right| + \left|\varphi^2(z)g''_{u_1}(z)\right|\right)\right)\mathrm{d}z \\
&= \sum_{k_1=0}^\infty \frac{n+k_1-1}{n-1}\int_0^\infty p_{n-1,k_1}(u_1)\,\mathrm{d}u_1 \\
&\quad\times \int_0^\infty \Phi\left(\frac{C}{\lambda(n+k_1)}\left(\left|g_{u_1}(z)\right| + \left|\varphi^2(z)g''_{u_1}(z)\right|\right)\right)\mathrm{d}z \\
&\le \sum_{k_1=0}^\infty \int_0^\infty p_{n-1,k_1}(u_1)\,\mathrm{d}u_1 \\
&\quad\times \int_0^\infty \Phi\left(\frac{C(n+k_1-1)}{\lambda(n-1)(n+k_1)}\left(\left|g_{u_1}(z)\right| + \left|\varphi^2(z)g''_{u_1}(z)\right|\right)\right)\mathrm{d}z \\
&\le \sum_{k_1=0}^\infty \int_0^\infty p_{n-1,k_1}(u_1)\,\mathrm{d}u_1 \int_0^\infty \Phi\left(\frac{C}{n\lambda}\left(\left|g_{u_1}(z)\right| + \left|\varphi^2(z)g''_{u_1}(z)\right|\right)\right)\mathrm{d}z \\
&= \int_0^\infty \mathrm{d}u_1 \int_0^\infty \Phi\left(\frac{C}{n\lambda}\left(\left|g_{u_1}(z)\right| + \left|\varphi^2(z)g''_{u_1}(z)\right|\right)\right)\mathrm{d}z \\
&\le \frac{1}{2}\int_0^\infty \mathrm{d}u_1 \int_0^\infty \Phi\left(\frac{C}{n\lambda}\left|g_{u_1}(z)\right|\right)\mathrm{d}z \\
&\quad+ \frac{1}{2}\int_0^\infty \mathrm{d}u_1 \int_0^\infty \Phi\left(\frac{C}{n\lambda}\left|\varphi^2(z)g''_{u_1}(z)\right|\right)\mathrm{d}z.
\end{aligned}
$$

On the other hand, by definition, we can deduce

$$
\varphi^2(t)g''_{u_1}(t) = t(1+t)(1+u_1)^2 D_2^2 f\big(u_1,(1+u_1)t\big) = \big(\varphi_2^2 D_2^2 f\big)\big(u_1,(1+u_1)t\big)
$$

and

$$
\begin{aligned}
&\int_0^\infty \int_0^\infty \Phi\left(\frac{1}{\lambda}|J_1|\right)\mathrm{d}x_1\,\mathrm{d}x_2 \\
&\quad\le \frac{1}{2}\int_0^\infty \frac{\mathrm{d}u_1}{1+u_1}\int_0^\infty \Phi\left(\frac{C}{n\lambda}\left|f\left(u_1,\frac{(1+u_1)x_2}{1+x_1}\right)\right|\right)\mathrm{d}\left(\frac{(1+u_1)x_2}{1+x_1}\right) \\
&\qquad+ \frac{1}{2}\int_0^\infty \frac{\mathrm{d}u_1}{1+u_1}\int_0^\infty \Phi\left(\frac{C}{n\lambda}\left|\big(\varphi_2^2 D_2^2 f\big)\left(u_1,\frac{(1+u_1)x_2}{1+x_1}\right)\right|\right)\mathrm{d}\left(\frac{(1+u_1)x_2}{1+x_1}\right) \\
&\quad\le \frac{1}{2}\int_0^\infty \mathrm{d}u_1 \int_0^\infty \Phi\left(\frac{C}{n\lambda}\left|f(u_1,u_2)\right|\right)\mathrm{d}u_2 \\
&\qquad+ \frac{1}{2}\int_0^\infty \mathrm{d}u_1 \int_0^\infty \Phi\left(\frac{C}{n\lambda}\left|\big(\varphi_2^2 D_2^2 f\big)(u_1,u_2)\right|\right)\mathrm{d}u_2.
\end{aligned}
\tag{7}
$$

To estimate the second term $J_2 = V_{n,1}^*(h(\cdot),x_1) - h(x_1)$, we use a similar method as estimating (6) and acquire

$$
\left|V_{n,1}^*(f,x) - f(x)\right| \le \frac{C}{n}\left[\left|f(x)\right| + \left|\varphi^2(x)f''(x)\right|\right]. \tag{8}
$$

By the Inequality (8) and the convexity of $\Phi(t)$, we arrive at

$$\begin{aligned}\int_{\mathbb{R}_0^2}\Phi\left(\frac{1}{\lambda}|J_2|\right)\mathrm{d}\boldsymbol{x} &= \int_0^\infty\int_0^\infty \Phi\left(\frac{1}{\lambda}\left|V_{n,1}^*\left(h(\cdot),x_1\right)-h(x_1)\right|\right)\mathrm{d}x_1\,\mathrm{d}x_2\\ &\leq \int_0^\infty\int_0^\infty \Phi(\frac{C}{n\lambda}\left(\left|h(x_1)\right|+\left|\varphi^2(x_1)h''(x_1)\right|\right)\mathrm{d}x_1\,\mathrm{d}x_2\\ &\leq \frac{1}{2}\int_0^\infty\int_0^\infty \Phi\left(\frac{C}{n\lambda}\left|h(x_1)\right|\right)\mathrm{d}x_1\,\mathrm{d}x_2\\ &\quad+\frac{1}{2}\int_0^\infty\int_0^\infty \Phi\left(\frac{C}{n\lambda}\left|\varphi^2(x_1)h''(x_1)\right|\right)\mathrm{d}x_1\,\mathrm{d}x_2.\end{aligned}$$

When denoting $\varphi_{12}(\boldsymbol{x})=\varphi_{21}(\boldsymbol{x})\triangleq\sqrt{x_1x_2}$, $D_{12}^2=\frac{\partial^2}{\partial x_1\partial x_2}$, and $D_{21}^2=\frac{\partial^2}{\partial x_2\partial x_1}$, we can write

$$\begin{aligned}\left|\varphi^2(u)h''(u)\right| &= \left|u(1+u)\left[D_1^2f+\frac{x_2}{1+x_1}D_{12}^2f+\frac{x_2}{1+x_1}D_{21}^2f\right.\right.\\ &\quad\left.\left.+\frac{x_2^2}{(1+x_1)^2}D_{22}^2f\right]\left(u,(1+u)\frac{x_2}{1+x_1}\right)\right|\\ &= \left|\left(\frac{1+x_1}{1+x_1+x_2}\varphi_1^2D_1^2f+\varphi_{12}^2D_{12}^2f+\varphi_{21}^2D_{21}^2f\right.\right.\\ &\quad\left.\left.+\frac{u}{1+u}\frac{x_2}{1+x_1+x_2}\varphi_2^2D_2^2f\right)\left(u,(1+u)\frac{x_2}{1+x_1}\right)\right|.\end{aligned}$$

By virtue of the facts that $\varphi_{12}(\boldsymbol{x})$ is not bigger than $\varphi_1(\boldsymbol{x})$ or $\varphi_2(\boldsymbol{x})$ and that

$$\left|D_{12}^2f(\boldsymbol{x})\right|\leq\sup\left\{\left|D_1^2f(\boldsymbol{x})\right|,\left|D_2^2f(\boldsymbol{x})\right|\right\}$$

in [6, Lemma 2.1], we obtain

$$\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}\left(\varphi^2\left|h''\right|,x_1\right)\right)\mathrm{d}\boldsymbol{x}\leq\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}\sum_{i=1}^2\left|\left(\varphi_i^2D_i^2f\right)(\boldsymbol{x})\right|\right)\mathrm{d}\boldsymbol{x}$$

and

$$\begin{aligned}\int_{\mathbb{R}_0^2}\Phi\left(\frac{1}{\lambda}|J_2|\right)\mathrm{d}\boldsymbol{x} &\leq \frac{1}{2}\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}\left|f(\boldsymbol{x})\right|\right)\mathrm{d}\boldsymbol{x}\\ &\quad+\frac{1}{2}\int_{\mathbb{R}_0^2}\Phi\left(\frac{C}{\lambda n}\sum_{i=1}^2\left|\left(\varphi_i^2D_i^2f\right)(\boldsymbol{x})\right|\right)\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Combining the above inequality with (5) and (7) and paying attention to computation formulas of norm and the double Inequality (1) yield

$$\left\|V_{n,2}(f)-f\right\|_\Phi\leq\|J_1\|_\Phi+\|J_2\|_\Phi\leq\frac{C}{n}\left(\|f\|_\Phi+\sum_{i=1}^2\left\|\varphi_i^2D_i^2f\right\|_\Phi\right).$$

The proof of Lemma 2 is complete. □

Proof of Theorem 1 Our proof is based on induction on the dimension m and on a decomposition for the Baskakov–Durrmeyer operator.

For $m \geq 1$, the proof of Theorem 1 follows from combining Lemmas 1 and 2 with the estimates

$$\left\| V_{n,m}(f) - f \right\|_{\Phi} \leq C \begin{cases} \|f\|_{\Phi}, & f \in L_{\Phi}^{*}(\mathbb{R}_0^m), \\ \frac{1}{n} \sum_{i=1}^{m} \|\varphi_i^2 D_i^2 f\|_{\Phi} + \|f\|_{\Phi}, & f \in W_{\varphi}^{2,\Phi}(\mathbb{R}_0^m). \end{cases} \tag{9}$$

The first estimate in (9) can be derived from Lemma 1. By Lemma 2, the second estimate in (9) is valid for $m = 1, 2$. If the second estimate in (9) is valid for $m = r \geq 2$, that is

$$\left\| V_{n,r}(f) - f \right\|_{\Phi} \leq \frac{C}{n} \sum_{i=1}^{r} \left\| \varphi_i^2 D_i^2 f \right\|_{\Phi} + \|f\|_{\Phi}, \tag{10}$$

then we have to further verify its validity for $m = r + 1$.

Let

$$\begin{aligned}
&\boldsymbol{x}^* = (x_2, x_3, \dots, x_{r+1}), \qquad \boldsymbol{x} = (x_1, \boldsymbol{x}^*) \in \mathbb{R}_0^{r+1}, \\
&\boldsymbol{k}^* = (k_2, k_3, \dots k_{r+1}), \qquad \boldsymbol{k} = (k_1, \boldsymbol{x}^*) \in \mathbb{N}_0^{r+1}, \\
&\boldsymbol{z} = \frac{\boldsymbol{x}^*}{1 + x_1} = \left(\frac{x_2}{1 + x_1}, \dots, \frac{x_{r+1}}{1 + x_1} \right) = (z_1, \dots, z_r), \\
&\boldsymbol{t}^* = \frac{\boldsymbol{u}^*}{1 + u_1} = \left(\frac{u_2}{1 + u_1}, \dots, \frac{u_{r+1}}{1 + u_1} \right) = (t_1, \dots, t_r).
\end{aligned}$$

We claim that the decomposition formula

$$\begin{aligned}
V_{n,r+1}(f, \boldsymbol{x}) = {} & \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)(n-1) \int_0^{\infty} p_{n,k_1}(u_1) \, \mathrm{d}u_1 \\
& \times \sum_{\boldsymbol{k}^*=0}^{\infty} p_{n+k_1,\boldsymbol{k}^*} \left(\frac{\boldsymbol{x}^*}{1 + x_1} \right) (n-2) \cdots (n-r-1) \\
& \times \int_{\mathbb{R}_0^r} p_{n+k_1,\boldsymbol{k}^*} \left(\frac{\boldsymbol{u}^*}{1 + u_1} \right) f(u_1, \boldsymbol{u}^*) \, \mathrm{d}\boldsymbol{u}^* \\
= {} & \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)(n-2) \int_0^{\infty} p_{n-1,k_1}(u_1) \, \mathrm{d}u_1 \sum_{\boldsymbol{k}^*=0}^{\infty} p_{n+k_1,\boldsymbol{k}^*} \left(\frac{\boldsymbol{x}^*}{1 + x_1} \right) \\
& \times \frac{(n-3) \cdots (n-r-1)}{(n-2+k_1) \cdots (n+k_1-r)} (n+k_1-1)(n-2+k_1) \cdots (n+k_1-r) \\
& \times \int_{\mathbb{R}_0^r} p_{n+k_1,\boldsymbol{k}^*} \left(\frac{\boldsymbol{u}^*}{1 + u_1} \right) f\left(u_1, (1+u_1) \frac{\boldsymbol{u}^*}{1 + u_1} \right) \mathrm{d}\left(\frac{\boldsymbol{u}^*}{1 + u_1} \right) \\
\leq {} & \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)(n-2) \int_0^{\infty} p_{n-1,k_1}(u_1) \, \mathrm{d}u_1 \sum_{\boldsymbol{k}^*=0}^{\infty} p_{n+k_1,\boldsymbol{k}^*}(\boldsymbol{z})(n+k_1-1) \cdots
\end{aligned}$$

$$\times (n+k_1-r)\int_{\mathbb{R}_0^r} p_{n+k_1,\boldsymbol{k}^*}(\boldsymbol{t}^*) f\big(u_1,(1+u_1)\boldsymbol{t}^*\big)\,\mathrm{d}\boldsymbol{t}^*$$
$$=\sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)(n-2)\int_0^{\infty} p_{n-1,k_1}(u_1) V_{n+k_1,r}\big(g_{u_1}(\cdot),\boldsymbol{z}\big)\,\mathrm{d}u_1$$

is valid, where $g_{u_1}(t)=f(u_1,(1+u_1)t)$ for $0\le t<\infty$. From the above formula, it follows that

$$\begin{aligned} V_{n,r+1}(f,\boldsymbol{x})-f(\boldsymbol{x}) &\le \sum_{k_1=0}^{\infty} p_{n,k_1}(x_1)(n-2)\int_0^{\infty} p_{n-1,k_1}(u_1) \\ &\quad \times \big[V_{n+k_1,r}\big(g_{u_1}(\cdot),\boldsymbol{z}\big)-g_{u_1}(\boldsymbol{z})\big]\,\mathrm{d}u_1 \\ &\quad + V_{n,1}^*\big(h(\cdot),x_1\big)-h(x_1) \\ &\triangleq T_1+T_2, \end{aligned} \tag{11}$$

where

$$h(u_1)\triangleq h(x_1,\boldsymbol{x})\triangleq f\left(u_1,(1+u_1)\frac{\boldsymbol{x}^*}{1+x_1}\right),\quad 0\le u_1<\infty.$$

By the inequality

$$\begin{aligned} &\int_{\mathbb{R}_0^r}\Phi\left(\frac{1}{\lambda}\big|V_{n,r}(f,\boldsymbol{x})-f(\boldsymbol{x})\big|\right)\mathrm{d}\boldsymbol{x} \\ &\quad \le \int_{\mathbb{R}_0^r}\Phi\left(\frac{C}{n\lambda}\sum_{i=1}^{r}\big|(\varphi_i^2 D_i^2 f)(\boldsymbol{x})\big|\right)\mathrm{d}\boldsymbol{x}+\int_{\mathbb{R}_0^r}\Phi\left(\frac{C}{n\lambda}\big|f(\boldsymbol{x})\big|\right)\mathrm{d}\boldsymbol{x}, \end{aligned}$$

which can be obtained from (10) and Jensen's inequality, we arrive at

$$\begin{aligned} \int_{\mathbb{R}_0^{r+1}}\Phi\left(\frac{1}{\lambda}|T_1|\right)\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}_0^{r+1}}\Phi\left(\frac{1}{\lambda}\left|\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)(n-2)\int_0^{\infty}p_{n-1,k_1}(u_1)\right.\right. \\ &\quad \left.\left.\times\big[V_{n+k_1,r}\big(g_{u_1}(\cdot),\boldsymbol{z}\big)-g_{u_1}(\boldsymbol{z})\big]\,\mathrm{d}u_1\right|\right)\mathrm{d}\boldsymbol{x} \\ &\le \int_{\mathbb{R}_0^{r+1}}\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)(n-2)\int_0^{\infty}p_{n-1,k_1}(u_1) \\ &\quad \times\Phi\left(\frac{1}{\lambda}\big|V_{n+k_1,r}\big(g_{u_1}(\cdot),\boldsymbol{z}\big)-g_{u_1}(\boldsymbol{z})\big|\right)\mathrm{d}u_1\,\mathrm{d}\boldsymbol{x} \\ &= \sum_{k_1=0}^{\infty}\int_0^{\infty}p_{n,k_1}(x_1)(1+x_1)\,\mathrm{d}x_1(n-2)\int_0^{\infty}p_{n-1,k_1}(u_1)\,\mathrm{d}u_1 \\ &\quad \times\int_{\mathbb{R}_0^r}\Phi\left(\frac{1}{\lambda}\big|V_{n+k_1,r}\big(g_{u_1}(\cdot),\boldsymbol{z}\big)-g_{u_1}(\boldsymbol{z})\big|\right)\mathrm{d}\boldsymbol{z} \\ &= \sum_{k_1=0}^{\infty}\frac{n+k_1-1}{n-1}\int_0^{\infty}p_{n-1,k_1}(u_1)\,\mathrm{d}u_1 \end{aligned}$$

$$
\begin{aligned}
&\times \int_{\mathbb{R}_0^r} \Phi\left(\frac{1}{\lambda}\left|V_{n+k_1,r}\big(g_{u_1}(\cdot),\boldsymbol{z}\big)-g_{u_1}(\boldsymbol{z})\right|\right)\mathrm{d}\boldsymbol{z}\\
\le{}& \sum_{k_1=0}^{\infty}\int_0^{\infty} p_{n-1,k_1}(u_1)\,\mathrm{d}u_1\\
&\times \int_{\mathbb{R}_0^r} \Phi\left(\frac{n+k_1-1}{\lambda(n-1)}\left|V_{n+k_1,r}\big(g_{u_1}(\cdot),\boldsymbol{z}\big)-g_{u_1}(\boldsymbol{z})\right|\right)\mathrm{d}\boldsymbol{z}\\
\le{}& \sum_{k_1=0}^{\infty}\int_0^{\infty} p_{n-1,k_1}(u_1)\,\mathrm{d}u_1\\
&\times \left[\int_{\mathbb{R}_0^r} \Phi\left(\frac{C}{n\lambda}\sum_{i=1}^{r}\left|\big(\varphi_i^2 D_i^2 g_{u_1}\big)(\boldsymbol{z})\right|\right)\mathrm{d}\boldsymbol{z}\right.\\
&\left.+\int_{\mathbb{R}_0^r} \Phi\left(\frac{C}{n\lambda}\left|g_{u_1}(\boldsymbol{z})\right|\right)\mathrm{d}\boldsymbol{z}\right].
\end{aligned}
$$

On the other hand, by definition, we can deduce

$$
\begin{aligned}
\varphi_i^2(\boldsymbol{x})D_i^2 g_{u_1}(\boldsymbol{x}) &= x_i\big(1+|\boldsymbol{x}|\big)(1+u_1)^2 D_{i+1}^2 f\big(u_1,(1+u_1)\boldsymbol{x}\big)\\
&= \big(\varphi_{i+1}^2 D_{i+1}^2 f\big)\big(u_1,(1+u_1)\boldsymbol{x}\big).
\end{aligned}
$$

As a result, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}_0^{r+1}} \Phi\left(\frac{1}{\lambda}|T_1|\right)\mathrm{d}\boldsymbol{x}\\
&\quad\le \int_0^{\infty}\sum_{k_1=0}^{\infty} p_{n-1,k_1}(u_1)\,\mathrm{d}u_1\left[\int_{\mathbb{R}_0^r}\Phi\left(\frac{C}{n\lambda}\sum_{i=1}^{r+1}\right.\right.\\
&\qquad\times\left.\left|\big(\varphi_i^2 D_i^2 f\big)\big(u_1,(1+u_1)\boldsymbol{z}\big)\right|\right)\mathrm{d}\boldsymbol{z}+\left.\int_{\mathbb{R}_0^r}\Phi\left(\frac{C}{n\lambda}\left|f\big(u_1,(1+u_1)\boldsymbol{z}\big)\right|\right)\mathrm{d}\boldsymbol{z}\right]\\
&\quad\le \int_0^{\infty}\mathrm{d}u_1\int_{\mathbb{R}_0^r}\Phi\left(\frac{C}{n\lambda}\sum_{i=1}^{r+1}\left|\big(\varphi_i^2 D_i^2 f\big)\big(u_1,(1+u_1)\boldsymbol{z}\big)\right|\right)\mathrm{d}\big((1+u_1)\boldsymbol{z}\big)\\
&\qquad+\int_0^{\infty}\mathrm{d}u_1\int_{\mathbb{R}_0^r}\Phi\left(\frac{C}{n\lambda}\left|f\big(u_1,(1+u_1)\boldsymbol{z}\big)\right|\right)\mathrm{d}\big((1+u_1)\boldsymbol{z}\big)\\
&\quad= \int_{\mathbb{R}_0^{r+1}}\Phi\left(\frac{C}{n\lambda}\sum_{i=1}^{r+1}\left|\big(\varphi_i^2 D_i^2 f\big)(\boldsymbol{u})\right|\right)\mathrm{d}\boldsymbol{u}+\int_{\mathbb{R}_0^{r+1}}\Phi\left(\frac{C}{n\lambda}\left|f(\boldsymbol{u})\right|\right)\mathrm{d}\boldsymbol{u}.
\end{aligned}
\tag{12}
$$

By the Inequality (8) and the convexity of $\Phi(t)$, we acquire

$$
\begin{aligned}
\int_{\mathbb{R}_0^{r+1}} \Phi\left(\frac{1}{\lambda}|T_2|\right)\mathrm{d}\boldsymbol{x} &= \int_{\mathbb{R}_0^{r+1}} \Phi\left(\frac{1}{\lambda}\left|V_{n,1}^{*}\big(h(\cdot),x_1\big)-h(x_1)\right|\right)\mathrm{d}\boldsymbol{x}\\
&\le \int_{\mathbb{R}_0^{r+1}} \Phi\left(\frac{C}{n\lambda}\big(\left|h(x_1)\right|+\varphi^2(x_1)\left|h''(x_1)\right|\big)\right)\mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Denoting $\varphi_{ij}(\boldsymbol{x}) = \sqrt{x_i x_j}$ for $1 \le i < j \le r+1$ and $D_{ij}^2 = \frac{\partial^2}{\partial x_i \partial x_j}$, we have

$$\begin{aligned}\varphi^2(u)h''(u) &= u(1+u)\Bigg[D_1^2 f + \sum_{i=2}^{r+1} \frac{x_i}{1+x_1} D_{1i}^2 f + \sum_{i=2}^{r+1} \frac{x_i}{1+x_1} D_{i1}^2 f \\ &\quad + \sum_{i=2}^{r+1}\sum_{j=2}^{r+1} \frac{x_i x_j}{(1+x_1)^2} D_{ij}^2 f\Bigg]\left(u, \frac{(1+u)\boldsymbol{x}^*}{1+x_1}\right) \\ &= \Bigg(\frac{1+x_1}{1+|\boldsymbol{x}|}\varphi_1^2 D_1^2 f + \sum_{i=2}^{r+1} \varphi_{1i}^2 D_{1i}^2 f + \sum_{i=2}^{r+1} \varphi_{i1}^2 D_{i1}^2 f \\ &\quad + \sum_{i=2}^{r+1} \frac{u}{1+u}\frac{x_i}{1+|\boldsymbol{x}|}\varphi_i^2 D_i^2 f + \sum_{i,j=2, i\neq j}^{r+1} \frac{u}{1+u}\varphi_{ij}^2 D_{ij}^2 f\Bigg)\left(u, \frac{(1+u)\boldsymbol{x}^*}{1+x_1}\right).\end{aligned}$$

Recalling that $\varphi_{ij}(\boldsymbol{x})$ is not bigger than $\varphi_i(\boldsymbol{x})$ or $\varphi_j(\boldsymbol{x})$, and using the fact

$$\left|D_{ij}^2 f(\boldsymbol{x})\right| \le \sup_{1\le i\le r+1}\left|D_i^2 f(\boldsymbol{x})\right|$$

in [6, Lemma 2.1], we obtain

$$\begin{aligned}\int_{\mathbb{R}_0^{r+1}} \Phi\left(\frac{1}{\lambda}|T_2|\right)\mathrm{d}\boldsymbol{x} &\le \frac{1}{2}\int_{\mathbb{R}_0^{r+1}} \Phi\left(\frac{C}{n\lambda}\left|f(\boldsymbol{x})\right|\right)\mathrm{d}\boldsymbol{x} \\ &\quad + \int_{\mathbb{R}_0^{r+1}} \Phi\left(\frac{C}{n\lambda}\sum_{i=1}^{r+1}\left|\left(\varphi_i^2 D_i^2 f\right)(\boldsymbol{x})\right|\right)\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{13}$$

Combining the Inequalities (11), (12), and (13) and paying attention to computation of norm and the Inequality (1), we obtain the second estimate of (9) for any $m \ge 2$.

For $g \in W_\varphi^{2,\Phi}(\mathbb{R}_0^m)$, using (2), (9), and Lemma 1 gives

$$\begin{aligned}\left\|V_{n,m}(f) - f\right\|_\Phi &\le \left\|V_{n,m}(f) - V_{n,m}(g)\right\|_\Phi + \left\|V_{n,m}(g) - g\right\|_\Phi + \|f-g\|_\Phi \\ &\le C\|f-g\|_\Phi + \frac{C}{n}\left(\|g\|_\Phi + \sum_{i=1}^m \left\|\varphi_i^2 D_i^2 g\right\|_\Phi + \frac{1}{n}\sum_{i=1}^m\left\|D_i^2 g\right\|_\Phi\right) \\ &\le C\left(\|f-g\|_\Phi + \frac{1}{n}\sum_{i=1}^m\left\|\varphi_i^2 D_i^2 g\right\|_\Phi + \frac{1}{n^2}\sum_{i=1}^m\left\|D_i^2 g\right\|_\Phi\right) + \frac{C}{n}\|f\|_\Phi \\ &\le C\left[\omega_{2,\varphi}\left(f, \frac{1}{n^{1/2}}\right)_\Phi + \frac{1}{n}\|f\|_\Phi\right].\end{aligned}$$

The proof of Theorem 1 is complete. □

4 Proof of inverse theorem

In order to prove the inverse theorem, we need several lemmas.

Lemma 3 *Let $f \in W_\varphi^{2,\Phi}(\mathbb{R}_0^m)$ and $n > m$ for $n, m \in \mathbb{N}$. Then*

$$\left\|\varphi_i^2 D_i^2 V_{n,m}(f)\right\|_{(\Phi)} \le \left\|\varphi_i^2 D_i^2 f\right\|_{(\Phi)}, \quad i = 1, 2, \dots, m.$$

Proof By straight computation, we have

$$\begin{aligned}
\left|\varphi^2(x)V''_{n,1}(f,x)\right| &= \left|\varphi^2(x)\sum_{k=0}^{\infty}(n-1)p''_{n,k}(x)\int_0^{\infty}p_{n,k}(t)f(t)\,\mathrm{d}t\right| \\
&= \left|\varphi^2(x)\sum_{i=0}^{2}\sum_{k=i}^{\infty}(n-1)\frac{(n+k-1)!}{k!(n-1)!}\binom{2}{i}\left(D^ix^k\right)\right. \\
&\quad \left.\times\left(D^{2-i}(1+x)^{-n-k}\right)\int_0^{\infty}p_{n,k}(t)f(t)\,\mathrm{d}t\right| \\
&= \left|\sum_{k=0}^{\infty}(n-1)\frac{(n+k+1)(k+1)}{(k+2)(n+k)}p_{n,k+1}(x)\int_0^{\infty}p_{n,k+1}(t)\varphi^2(t)f''(t)\,\mathrm{d}t\right| \\
&\le \sum_{k=0}^{\infty}(n-1)p_{n,k+1}(x)\int_0^{\infty}p_{n,k+1}(t)\varphi^2(t)\left|f''(t)\right|\mathrm{d}t,
\end{aligned}$$

where $\frac{(n+k+1)(k+1)}{(k+2)(n+k)} \le 1$ for $k \ge 0$ and $n \in \mathbb{N}$. Using Jensen's inequality, we derive

$$\int_0^{\infty}\Phi\left(\frac{1}{\lambda}\left|\varphi^2(x)V''_{n,1}(f,x)\right|\right)\mathrm{d}x \le \int_0^{\infty}\Phi\left(\frac{1}{\lambda}\varphi^2(t)\left|f''(t)\right|\right)\mathrm{d}t. \tag{14}$$

Let

$$g_{u^*}(t) = f\left(u_1,u_2,\dots,u_{m-1},\left(1+\sum_{k=1}^{m-1}u_k\right)t\right), \quad 0 \le t < \infty$$

and $z = \frac{x_m}{1+\sum_{\ell=1}^{m-1}x_\ell}$. Then, by the Inequality (4) and for $m > 1$, we have

$$\begin{aligned}
V_{n,m}(f,\boldsymbol{x}) &= \sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)(n-1)\int_0^{\infty}p_{n,k_1}(u_1)\,\mathrm{d}u_1\sum_{k_2=0}^{\infty}p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right) \\
&\quad\times(n-2)\int_0^{\infty}p_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2\cdots \\
&\quad\times\sum_{k_m=0}^{\infty}p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{x_m}{1+\sum_{\ell=1}^{m-1}x_\ell}\right)(n-m) \\
&\quad\times\int_0^{\infty}p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right)f(u_1,\dots,u_m)\,\mathrm{d}u_m \\
&= \sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)(n-1)\int_0^{\infty}p_{n,k_1}(u_1)\,\mathrm{d}u_1\sum_{k_2=0}^{\infty}p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)(n-2) \\
&\quad\times\int_0^{\infty}p_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2\cdots\sum_{k_m=0}^{\infty}p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{x_m}{1+\sum_{\ell=1}^{m-1}x_\ell}\right) \\
&\quad\times(n-m)\int_0^{\infty}p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right)\left(1+\sum_{k=1}^{m-1}u_k\right)
\end{aligned}$$

$$\times f\left(u_1,\dots,u_{m-1},\left(1+\sum_{k=1}^{m-1}u_k\right)\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right)\mathrm{d}\left(\frac{u_m}{1+\sum_{\ell=1}^{m-1}u_\ell}\right)$$

$$=\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)(n-2)\int_0^\infty p_{n-1,k_1}(u_1)\,\mathrm{d}u_1\sum_{k_2=0}^{\infty}p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)(n-3)$$

$$\times\int_0^\infty p_{n+k_1-1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2\cdots$$

$$\times\sum_{k_{m-1}=0}^{\infty}p_{n+\sum_{\ell=1}^{m-2}k_\ell,k_{m-1}}\left(\frac{x_{m-1}}{1+\sum_{k=1}^{m-2}x_k}\right)(n-m)$$

$$\times\int_0^\infty p_{n+\sum_{\ell=1}^{m-2}k_\ell-1,k_{m-1}}\left(\frac{u_{m-1}}{1+\sum_{\ell=1}^{m-2}u_\ell}\right)\mathrm{d}u_{m-1}$$

$$\times\sum_{k_m=0}^{\infty}p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}(z)\left(n-1+\sum_{\ell=1}^{m-1}k_\ell\right)$$

$$\times\int_0^\infty p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}(t)f\left(u_1,\dots,u_{m-1},\left(1+\sum_{k=1}^{m-1}u_k\right)t\right)\mathrm{d}t$$

$$=\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)(n-2)\int_0^\infty p_{n-1,k_1}(u_1)\,\mathrm{d}u_1\sum_{k_2=0}^{\infty}p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)$$

$$\times(n-3)\int_0^\infty p_{n+k_1-1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2\cdots$$

$$\times\sum_{k_{m-1}=0}^{\infty}p_{n+\sum_{\ell=1}^{m-2}k_\ell,k_{m-1}}\left(\frac{x_{m-1}}{1+\sum_{k=1}^{m-2}x_k}\right)(n-m)$$

$$\times\int_0^\infty p_{n+\sum_{\ell=1}^{m-2}k_\ell-1,k_{m-1}}\left(\frac{u_{m-1}}{1+\sum_{\ell=1}^{m-2}u_\ell}\right)V_{n+\sum_{\ell=1}^{m-1}k_\ell,1}(g_{u^*},z)\,\mathrm{d}u_{m-1}. \tag{15}$$

Using the Inequalities (14), (15), and Jensen's inequality, we see that

$$\int_{\mathbb{R}_0^m}\Phi\left(\frac{1}{\lambda}\left|\varphi_m^2(\boldsymbol{x})D_m^2V_{n,m}(f,\boldsymbol{x})\right|\right)\mathrm{d}\boldsymbol{x}$$

$$\leq\int_0^\infty\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)\,\mathrm{d}x_1(n-2)\int_0^\infty p_{n-1,k_1}(u_1)\,\mathrm{d}u_1$$

$$\times\int_0^\infty\sum_{k_2=0}^{\infty}p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)\mathrm{d}x_2(n-2+k_1)\int_0^\infty p_{n+k_1-1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2\cdots$$

$$\times\int_0^\infty\sum_{k_{m-1}=0}^{\infty}p_{n+\sum_{\ell=1}^{m-2}k_\ell,k_{m-1}}\left(\frac{x_{m-1}}{1+\sum_{k=1}^{m-2}x_k}\right)\mathrm{d}x_{m-1}$$

$$\times\left(n-2+\sum_{\ell=1}^{m-2}k_\ell\right)\int_0^\infty p_{n+\sum_{\ell=1}^{m-2}k_\ell-1,k_{m-1}}\left(\frac{u_{m-1}}{1+\sum_{\ell=1}^{m-2}u_\ell}\right)\mathrm{d}u_{m-1}$$

$$\times\int_0^\infty\Phi\left(\frac{(n-3)(n-4)\cdots(n-m)(1+\sum_{k=1}^{m-1}x_k)^2}{\lambda(n-2+k_1)(n-2+k_1+k_2)\cdots(n-2+\sum_{\ell=1}^{m-2}k_\ell)}\right.$$

$$
\begin{aligned}
&\times \varphi_m^2(z)\left|V''_{n+\sum_{\ell=1}^{m-1}k_\ell,1}(g_{u^*},z)\right|\right)\left(1+\sum_{k=1}^{m-1}x_k\right)\mathrm{d}z \\
\leq{}& \int_0^\infty \sum_{k_1=0}^\infty p_{n-1,k_1}(x_1)\,\mathrm{d}x_1(n-2)\int_0^\infty p_{n-1,k_1}(u_1)\,\mathrm{d}u_1\int_0^\infty\sum_{k_2=0}^\infty(n-2 \\
&+k_1)p_{n+k_1-1,k_2}\left(\frac{x_2}{1+x_1}\right)\mathrm{d}\left(\frac{x_2}{1+x_1}\right)\int_0^\infty p_{n+k_1-1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2\cdots \\
&\times\int_0^\infty\sum_{k_{m-1}=0}^\infty p_{n+\sum_{\ell=1}^{m-2}k_\ell-1,k_{m-1}}\left(\frac{x_{m-1}}{1+\sum_{k=1}^{m-2}x_k}\right)\mathrm{d}\left(\frac{x_{m-1}}{1+\sum_{k=1}^{m-2}x_k}\right) \\
&\times\left(n-2+\sum_{\ell=1}^{m-2}k_\ell\right)\int_0^\infty p_{n+\sum_{\ell=1}^{m-2}k_\ell-1,k_{m-1}}\left(\frac{u_{m-1}}{1+\sum_{\ell=1}^{m-2}u_\ell}\right) \\
&\times\frac{n+\sum_{\ell=1}^{m-2}k_\ell-1}{n-1}\,\mathrm{d}u_{m-1}\int_0^\infty\sum_{k_m=0}^\infty p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m+1}(z)\,\mathrm{d}z \\
&\times\int_0^\infty\Phi\left(\frac{(n-3)(n-4)\cdots(n-m)}{\lambda(n-2+k_1)(n-2+k_1+k_2)\cdots(n-2+\sum_{\ell=1}^{m-2}k_\ell)}\right. \\
&\left.\times\varphi^2(t)\left|g''_{u^*}(t)\right|\right)\left(n-1+\sum_{\ell=1}^{m-1}k_\ell\right)p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m+1}(t)\,\mathrm{d}t \\
={}&\int_0^\infty\sum_{k_1=0}^\infty p_{n,k_1}(u_1)\,\mathrm{d}u_1\int_0^\infty\sum_{k_2=0}^\infty p_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2\cdots \\
&\times\int_0^\infty\sum_{k_{m-1}=0}^\infty\frac{n+\sum_{\ell=1}^{m-2}k_\ell-1}{n+\sum_{\ell=1}^{m-1}k_\ell}p_{n+\sum_{\ell=1}^{m-2}k_\ell,k_{m-1}}\left(\frac{u_{m-1}}{1+\sum_{\ell=1}^{m-2}u_\ell}\right)\mathrm{d}u_{m-1} \\
&\times\int_0^\infty\sum_{k_m=0}^\infty p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m+1}\left(\frac{u_m}{1+\sum_{k=1}^{m-1}u_k}\right) \\
&\times\Phi\left(\frac{(n-3)(n-4)\cdots(n-m)}{\lambda(n-2+k_1)(n-2+k_1+k_2)\cdots(n-2+\sum_{\ell=1}^{m-2}k_\ell)}\right. \\
&\left.\times\varphi_m^2(u_1,u_2,\ldots,u_m)\left|\frac{\partial^2}{\partial u_m^2}f(u_1,u_2,\ldots,u_m)\right|\right)\mathrm{d}u_m \\
\leq{}&\int_{\mathbb{R}_0^m}\Phi\left(\frac{1}{\lambda}\varphi_m^2(\boldsymbol{u})\left|D_m^2 f(\boldsymbol{u})\right|\right)\mathrm{d}\boldsymbol{u}.
\end{aligned}
$$

Hence, from the computation formula of the form, it follows that

$$\left\|\varphi_m^2 D_m^2 V_{n,m}(f)\right\|_{(\Phi)}\leq\left\|\varphi_m^2 D_m^2(f)\right\|_{(\Phi)}.$$

Similarly, we can prove the same results for $i=1,2,\ldots,m-1$. The proof of Lemma 3 is thus complete. □

Lemma 4 *Let $f\in L_\Phi^*(\mathbb{R}_0^m)$ and $n>m$ for $n,m\in\mathbb{N}$. Then*

$$\left\|\varphi_i^2 D_i^2 V_{n,m}(f)\right\|_{(\Phi)}\leq 4n\|f\|_{(\Phi)},\quad i=1,2,\ldots,m.$$

Proof By straight calculation, for $m = 1$, we have

$$\begin{aligned}
\left|\varphi^2(x)V''_{n,1}(f,x)\right| &= \left|\varphi^2(x)\sum_{k=0}^{\infty}(n-1)p''_{n,k}(x)\int_0^{\infty}p_{n,k}(t)f(t)\,\mathrm{d}t\right| \\
&= \left|\sum_{i=0}^{2}\sum_{k=i}^{\infty}\varphi^2(x)(n-1)\frac{(n+k-1)!}{k!(n-1)!}\binom{2}{i}\right. \\
&\quad \left.\times\left(D^i x^k\right)\left(D^{2-i}(1+x)^{-n-k}\right)\int_0^{\infty}p_{n,k}(t)f(t)\,\mathrm{d}t\right| \\
&= \left|\frac{n-1}{\varphi^2(x)}\sum_{k=0}^{\infty}p_{n,k}(x)\left[(nx-k)^2-k(2x+1)\right.\right. \\
&\quad \left.\left.+nx^2\right]\int_0^{\infty}p_{n,k}(t)f(t)\,\mathrm{d}t\right| \\
&\le \frac{n-1}{\varphi^2(x)}\sum_{k=0}^{\infty}p_{n,k}(x)\left[(nx-k)^2+k(2x+3)\right. \\
&\quad \left.+nx^2\right]\int_0^{\infty}p_{n,k}(t)\left|f(t)\right|\,\mathrm{d}t \\
&\triangleq 4n(n-1)\sum_{k=0}^{\infty}\beta_{n,k}(x)\int_0^{\infty}p_{n,k}(t)\left|f(t)\right|\,\mathrm{d}t, \qquad (16)
\end{aligned}$$

where

$$\beta_{n,k}(x) = \frac{p_{n,k}(x)[(nx-k)^2+k(2x+3)+nx^2]}{4n\varphi^2(x)}.$$

Moreover, we can verify that

$$\sum_{k=0}^{\infty}\beta_{n,k}(x) = 1 \quad \text{and} \quad \int_0^{\infty}\beta_{n,k}(x)\,\mathrm{d}x = \frac{1}{2}\left(\frac{1}{n}+\frac{1}{n-1}\right)$$

for $n \ge 2$. By the Inequalities (4) and (16), for $m > 1$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_0^m}\Phi\left(\frac{1}{\lambda}\left|\varphi_m^2(\boldsymbol{x})D_m^2V_{n,m}(f,\boldsymbol{x})\right|\right)\mathrm{d}\boldsymbol{x} \\
&= \int_0^{\infty}\sum_{k_1=0}^{\infty}p_{n,k_1}(x_1)\,\mathrm{d}x_1(n-2)\int_0^{\infty}p_{n-1,k_1}(u_1)\,\mathrm{d}u_1 \\
&\quad \times\int_0^{\infty}\sum_{k_2=0}^{\infty}p_{n+k_1,k_2}\left(\frac{x_2}{1+x_1}\right)\mathrm{d}x_2(n-2+k_1) \\
&\quad \times\int_0^{\infty}p_{n+k_1-1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2\cdots \\
&\quad \times\int_0^{\infty}\sum_{k_{m-1}=0}^{\infty}p_{n+\sum_{\ell=1}^{m-2}k_\ell,k_{m-1}}\left(\frac{x_{m-1}}{1+\sum_{k=1}^{m-2}x_k}\right)\mathrm{d}x_{m-1}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(n-2+\sum_{\ell=1}^{m-2} k_\ell\right) \int_0^\infty p_{n+\sum_{\ell=1}^{m-2} k_\ell -1, k_{m-1}}\left(\frac{u_{m-1}}{1+\sum_{\ell=1}^{m-2} u_\ell}\right) \mathrm{d}u_{m-1} \\
&\times \int_0^\infty \Phi\left(\frac{(n-3)(n-4)\cdots(n-m)(1+\sum_{k=1}^{m-1} x_k)^2}{\lambda(n-2+k_1)(n-2+k_1+k_2)\cdots(n-2+\sum_{\ell=1}^{m-2} k_\ell)}\right. \\
&\left.\times \varphi^2(z)\left|V''_{n+\sum_{\ell=1}^{m-1} k_\ell,1}(g_{u^*},z)\right|\right)\left(1+\sum_{k=1}^{m-1} x_k\right) \mathrm{d}z \\
\le{}& \int_0^\infty \sum_{k_1=0}^\infty p_{n-1,k_1}(x_1)\,\mathrm{d}x_1 (n-2)\int_0^\infty p_{n-1,k_1}(u_1)\,\mathrm{d}u_1 \\
&\times \int_0^\infty \sum_{k_2=0}^\infty p_{n+k_1-1,k_2}\left(\frac{x_2}{1+x_1}\right)(n-2+k_1)\,\mathrm{d}\left(\frac{x_2}{1+x_1}\right) \\
&\times \int_0^\infty p_{n+k_1-1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2 \cdots \int_0^\infty \left(n-2+\sum_{\ell=1}^{m-2} k_\ell\right) \\
&\times \sum_{k_{m-1}=0}^\infty p_{n+\sum_{\ell=1}^{m-2} k_\ell -1, k_{m-1}}\left(\frac{x_{m-1}}{1+\sum_{k=1}^{m-2} x_k}\right)\mathrm{d}\left(\frac{x_{m-1}}{1+\sum_{k=1}^{m-2} x_k}\right) \\
&\times \int_0^\infty p_{n+\sum_{\ell=1}^{m-2} k_\ell -1, k_{m-1}}\left(\frac{u_{m-1}}{1+\sum_{\ell=1}^{m-2} u_\ell}\right)\frac{n+\sum_{\ell=1}^{m-2} k_\ell -1}{n-1}\,\mathrm{d}u_{m-1} \\
&\times \int_0^\infty \Phi\left(\frac{4(n-3)(n-4)\cdots(n-m)(n+\sum_{\ell=1}^{m-1} k_\ell)(n-1+\sum_{\ell=1}^{m-1} k_\ell)}{\lambda(n-2+k_1)(n-2+k_1+k_2)\cdots(n-2+\sum_{\ell=1}^{m-2} k_\ell)}\right. \\
&\left.\times \sum_{k_m=0}^\infty \beta_{n+\sum_{\ell=1}^{m-1} k_\ell, k_m}(z)\int_0^\infty p_{n+\sum_{\ell=1}^{m-1} k_\ell, k_m}(t)\left|g_{u^*}(t)\right|\mathrm{d}t\right)\mathrm{d}z \\
\le{}& \int_0^\infty \sum_{k_1=0}^\infty p_{n-1,k_1}(x_1)\,\mathrm{d}x_1 (n-2)\int_0^\infty p_{n-1,k_1}(u_1)\,\mathrm{d}u_1 \int_0^\infty \sum_{k_2=0}^\infty (n-2 \\
&+k_1) p_{n+k_1-1,k_2}\left(\frac{x_2}{1+x_1}\right)\mathrm{d}\left(\frac{x_2}{1+x_1}\right)\int_0^\infty p_{n+k_1-1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2 \cdots \\
&\times \int_0^\infty \sum_{k_m=0}^\infty \frac{n+\sum_{\ell=1}^{m-2} k_\ell -1}{n-1}\beta_{n+\sum_{\ell=1}^{m-1} k_\ell, k_m}(z)\,\mathrm{d}z \\
&\times \left(n-1+\sum_{\ell=1}^{m-1} k_\ell\right)\int_0^\infty p_{n+\sum_{\ell=1}^{m-1} k_\ell, k_m}(t) \\
&\times \Phi\left(\frac{4(n-3)(n-4)\cdots(n-m)(n+\sum_{\ell=1}^{m-1} k_\ell)}{\lambda(n-2+k_1)(n-2+k_1+k_2)\cdots(n-2+\sum_{\ell=1}^{m-2} k_\ell)}\left|g_{u^*}(t)\right|\right)\mathrm{d}t \\
={}& \int_0^\infty \sum_{k_1=0}^\infty p_{n,k_1}(u_1)\,\mathrm{d}u_1 \int_0^\infty \sum_{k_2=0}^\infty p_{n+k_1,k_2}\left(\frac{u_2}{1+u_1}\right)\mathrm{d}u_2 \cdots \\
&\times \int_0^\infty \sum_{k_{m-1}=0}^\infty p_{n+\sum_{\ell=1}^{m-2} k_\ell, k_{m-1}}\left(\frac{u_{m-1}}{1+\sum_{\ell=1}^{m-2} u_\ell}\right)\frac{n+\sum_{\ell=1}^{m-2} k_\ell -1}{n+\sum_{\ell=1}^{m-1} k_\ell}\,\mathrm{d}u_{m-1} \\
&\times \int_0^\infty \sum_{k_m=0}^\infty \beta_{n+\sum_{\ell=1}^{m-1} k_\ell, k_m}\left(\frac{x_m}{1+\sum_{k=1}^{m-1} x_k}\right)\mathrm{d}\left(\frac{x_m}{1+\sum_{k=1}^{m-1} x_k}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(n-1+\sum_{\ell=1}^{m-1}k_\ell\right)\int_0^\infty p_{n+\sum_{\ell=1}^{m-1}k_\ell,k_m}\left(\frac{u_m}{1+\sum_{k=1}^{m-1}u_k}\right)\\
&\times\Phi\left(\frac{4(n-3)(n-4)\cdots(n-m)(n+\sum_{\ell=1}^{m-1}k_\ell)}{\lambda(n-2+k_1)\cdots(n-2+\sum_{\ell=1}^{m-2}k_\ell)}\left|f(u_1,\dots,u_m)\right|\right)\mathrm{d}u_m\\
\le{}&\int_0^\infty \mathrm{d}u_1\int_0^\infty \mathrm{d}u_2\cdots\int_0^\infty \Phi\left(\frac{4n}{\lambda}\left|f(u_1,\dots,u_m)\right|\right)\mathrm{d}u_m\\
={}&\int_{\mathbb{R}_0^m}\Phi\left(\frac{4n}{\lambda}\left|f(\boldsymbol{u})\right|\right)\mathrm{d}\boldsymbol{u}.
\end{aligned}
$$

Hence, from the computation formula of the form, it follows that

$$\left\|\varphi_m^2D_m^2V_{n,m}(f)\right\|_{(\Phi)}\le 4n\|f\|_{(\Phi)}.$$

Similarly, we can prove the same results for $i=1,2,\dots,m-1$. The proof of Lemma 4 is thus complete. □

Proof of Theorem 2 Let

$$\nu_n=\frac{1}{n}\left\|\varphi_i^2D_i^2V_{n,m}(f)\right\|_{(\Phi)},\quad i=1,2,\dots,m$$

and

$$\tau_k=4\left\|V_{k,m}(f)-f\right\|_{(\Phi)}.$$

It is obvious that $\nu_1=0$. From Lemmas 3 and 4, it follows that

$$
\begin{aligned}
\nu_n&\le\frac{1}{n}\left\|\varphi_i^2D_i^2V_{n,m}\big(V_{k,m}(f)\big)\right\|_{(\Phi)}+\frac{1}{n}\left\|\varphi_i^2D_i^2V_{n,m}\big(V_{k,m}(f)-f\big)\right\|_{(\Phi)}\\
&\le\frac{1}{n}\left\|\varphi_i^2D_i^2V_{k,m}(f)\right\|_{(\Phi)}+4\left\|V_{k,m}(f)-f\right\|_{(\Phi)}\\
&=\frac{k}{n}\nu_k+\tau_k.
\end{aligned}
$$

By [28, Lemma 2.1], we acquire $\nu_n\le\frac{C}{n}\sum_{k=1}^n\tau_k$. Therefore,

$$\left\|\varphi_i^2D_i^2V_{k,m}(f)\right\|_{(\Phi)}\le C\sum_{k=1}^n\left\|V_{k,m}(f)-f\right\|_{(\Phi)}.$$

Using the double Inequality (1), we obtain

$$\left\|\varphi_i^2D_i^2V_{k,m}(f)\right\|_{\Phi}\le C\sum_{k=1}^n\left\|V_{k,m}(f)-f\right\|_{\Phi}.$$

For $n\ge2$, there exists $s\in\mathbb{N}$ such that $\frac{n}{2}\le s\le n$ and

$$\left\|V_{s,m}(f)-f\right\|_{\Phi}\le\left\|V_{k,m}(f)-f\right\|_{\Phi},\quad\frac{n}{2}\le k\le n.$$

Accordingly, we have

$$\|V_{s,m}(f)-f\|_{\Phi} \le \frac{2}{n}\sum_{\frac{n}{2}\le k\le n}\|V_{k,m}(f)-f\|_{\Phi} \le \frac{2}{n}\sum_{k=1}^{n}\|V_{k,m}(f)-f\|_{\Phi}.$$

Hence, by the definition of the K–functional, we deduce

$$\begin{aligned}\tilde{K}_{2,\varphi}\left(f,\frac{1}{n}\right) &\le \|V_{s,m}(f)-f\|_{\Phi} + \frac{1}{n}\max_{1\le i\le m}\|\varphi_i^2 D_i^2 V_{s,m}(f)\|_{\Phi} \\ &\le \frac{C}{n}\sum_{k=1}^{n}\|V_{k,m}(f)-f\|_{\Phi}.\end{aligned}$$

Finally, using (3), we finish the proof of Theorem 2. □

5 Conclusions

In this paper, using the equivalent theorem between the modified K-functional and modulus of smoothness, employing a decomposition technique, and considering some properties of multivariate Baskakov–Durrmeyer operators in the form of Lemmas 1, 2, 3, and 4, we obtained a direct theorem and weak type inverse theorem in the Orlicz spaces $f \in L_{\Phi}^{*}(\mathbb{R}_0^m)$.

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References

1. Agrawal, P.N., Gupta, V., Sathish Kumar, A.: Generalized Baskakov–Durrmeyer type operators. Rend. Circ. Mat. Palermo (2) **63**(2), 193–209 (2014). https://doi.org/10.1007/s12215-014-0152-z
2. Baskakov, V.A.: An instance of a sequence of linear positive operators in the space of continuous functions. Dokl. Akad. Nauk SSSR (N.S.) **113**, 249–251 (1957) (Russian)
3. Berdysheva, E.E., Kurt, J.: Multivariate Bernstein–Durrmeyer operators with arbitrary weight functions. J. Approx. Theory **162**(3), 576–598 (2010). https://doi.org/10.1016/j.jat.2009.11.005
4. Cao, F., An, Y.: L^p approximation by multivariate Baskakov–Durrmeyer operator. J. Inequal. Appl. (2011). https://doi.org/10.1155/2011/158219
5. Cao, F., Ding, C., Xu, Z.: On multivariate Baskakov operators. J. Math. Anal. Appl. **307**(1), 274–291 (2005). https://doi.org/10.1016/j.jmaa.2004.10.061
6. Chen, W., Ditzian, Z.: Mixed and directional derivatives. Proc. Am. Math. Soc. **108**(1), 177–185 (1990). https://doi.org/10.2307/2047711

7. Cheng, W.-T., Mohiuddine, S.A.: Construction of a new modification of Baskakov operators on $(0,\infty)$. Filomat **37**(1), 139–154 (2023). https://doi.org/10.2298/FIL2301139C
8. Ditzian, Z., Totik, V.: Moduli of Smoothness. Springer Series in Computational Mathematics, vol. 9. Springer, New York (1987). https://doi.org/10.1007/978-1-4612-4778-4
9. Erençin, A.: Durrmeyer type modification of generalized Baskakov operators. Appl. Math. Comput. **218**(8), 4384–4390 (2011). https://doi.org/10.1016/j.amc.2011.10.014
10. Han, L., Wu, G.: Approximation by modified summation integral type operators in Orlicz spaces. Appl. Math. (Wuhan) **30**(3), 613–622 (2017)
11. Han, L.-X., Guo, B.-N.: Direct, inverse, and equivalence theorems for weighted Szász–Durrmeyer–Bézier operators in Orlicz spaces. Anal. Math. **47**(3), 569–592 (2021). https://doi.org/10.1007/s10476-021-0084-8
12. Han, L.-X., Guo, B.-N., Qi, F.: Equivalent theorem of approximation by linear combination of weighted Baskakov–Kantorovich operators in Orlicz spaces. J. Inequal. Appl. **2019**, 223 (2019). https://doi.org/10.1186/s13660-019-2174-8
13. Han, L.-X., Li, W.-H., Qi, F.: Approximation by multivariate Baskakov–Kantorovich operators in Orlicz spaces. Electron. Res. Arch. **28**(2), 721–738 (2020). https://doi.org/10.3934/era.2020037
14. Han, L.-X., Qi, F.: On approximation by linear combinations of modified summation operators of integral type in Orlicz spaces. Mathematics **7**(1), Article 6 (2019). https://doi.org/10.3390/math7010006
15. Han, L.X.: Equivalence theorem of approximation by Baskakov–Durrmeyer operators in Orlicz spaces $L^*_\Phi[0,\infty)$. J. Jilin Univ. Sci. Ed. **56**(2), 249–256 (2018) (Chinese)
16. Han, L.X., Wu, G.: Strong converse inequality of weighted simultaneous approximation for gamma operators in Orlicz spaces $L^*_\Phi(0,\infty)$. Appl. Math. J. Chin. Univ. Ser. A **31**(3), 366–378 (2016) (Chinese)
17. Heilmann, M.: Direct and converse results for operators of Baskakov–Durrmeyer type. Approx. Theory Appl. **5**(1), 105–127 (1989)
18. Kajla, A., Mohiuddine, S.A., Alotaibi, A.: Blending-type approximation by Lupaş–Durrmeyer-type operators involving Pólya distribution. Math. Methods Appl. Sci. **44**(11), 9407–9418 (2021). https://doi.org/10.1002/mma.7368
19. Kumar, A.S., Acar, T.: Approximation by generalized Baskakov–Durrmeyer–Stancu type operators. Rend. Circ. Mat. Palermo (2) **65**(3), 411–424 (2016). https://doi.org/10.1007/s12215-016-0242-1
20. Li, B.-Z.: Approximation by multivariate Bernstein–Durrmeyer operators and learning rates of least-squares regularized regression with multivariate polynomial kernels. J. Approx. Theory **173**, 33–55 (2013). https://doi.org/10.1016/j.jat.2013.04.007
21. Marshall, A.W., Olkin, I., Arnold, B.C.: Inequalities: Theory of Majorization and Its Applications, 2nd edn. Springer, New York (2011). https://doi.org/10.1007/978-0-387-68276-1
22. Mohiuddine, S.A., Singh, K.K., Alotaibi, A.: On the order of approximation by modified summation-integral-type operators based on two parameters. Demonstr. Math. **56**(1), 20220182 (2023). https://doi.org/10.1515/dema-2022-0182
23. Niculescu, C.P., Persson, L.-E.: Convex Functions and Their Applications: A Contemporary Approach, 2nd edn. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-78337-6
24. Ouimet, F., Qi, F.: Logarithmically complete monotonicity of a matrix-parametrized analogue of the multinomial distribution. Math. Inequal. Appl. **25**(3), 703–714 (2022). https://doi.org/10.7153/mia-2022-25-45
25. Qi, F.: Some properties of several functions involving polygamma functions and originating from the sectional curvature of the beta manifold. São Paulo J. Math. Sci. **14**(2), 614–630 (2020). https://doi.org/10.1007/s40863-020-00193-1
26. Qi, F., Guo, B.-N.: From inequalities involving exponential functions and sums to logarithmically complete monotonicity of ratios of gamma functions. J. Math. Anal. Appl. **493**(1), 124478 (2021). https://doi.org/10.1016/j.jmaa.2020.124478
27. Rao, M.M., Ren, Z.D.: Theory of Orlicz Space. Monographs and Textbooks in Pure and Applied Mathematics, vol. 146. Dekker, New York (1991)
28. van Wickeren, E.: Steckin–Marchaud-type inequalities in connection with Bernstein polynomials. Constr. Approx. **2**(4), 331–337 (1986). https://doi.org/10.1007/BF01893435
29. Verma, D.K., Gupta, V., Agrawal, P.N.: Some approximation properties of Baskakov–Durrmeyer–Stancu operators. Appl. Math. Comput. **218**(11), 6549–6556 (2012). https://doi.org/10.1016/j.amc.2011.12.031
30. Yadav, J., Mohiuddine, S.A., Kajla, A., Alotaibi, A.: Bivariate Lupaş–Durrmeyer type operators involving Pólya distribution. Filomat **37**(21), 7041–7056 (2023). https://doi.org/10.2298/FIL2321041Y
31. Zhang, P., Cao, F.-L., Xu, Z.-B.: Multivariate modulus of smoothness in Orlicz spaces and its application. Adv. Math. (China) **32**(6), 695–705 (2003)

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