Aras-Gazić et al. *Journal of Inequalities and Applications* (2023) 2023:117
https://doi.org/10.1186/s13660-023-03028-7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

On generalization of Levinson's inequality involving averages of 3-convex functions

G. Aras-Gazić[1*], J. Pečarić[2] and A. Vukelić[3]

*Correspondence: garas@arhitekt.hr
[1]Faculty Architecture, University of Zagreb, Kačićeva 26, 10000 Zagreb, Croatia
Full list of author information is available at the end of the article

Abstract

By using an integral arithmetic mean, a generalization of Levinson's inequality given in (Pečarić et al. in Convex Functions, Partial Orderings, and Statistical Applications. Mathematics in Science and Engineering, vol. 187, 1992) and results from (Vukelić in Appl. Anal. Discrete Math. 14:670–684, 2020), we give extension of Wulbert's result from (Wulbert in Math. Comput. Model. 37:1383–1391, 2003). Also, we obtain inequalities with divided differences for the functions of higher order.

Mathematics Subject Classification: Primary 26D15; secondary 26D07; 26A51

Keywords: Levinson's inequality; n-convex function; (n,m)-convex function; Divided differences

1 Introduction

Suppose f is a continuous function defined on an interval I with a nonempty interior. Then, define

$$F(x,y) = \begin{cases} \frac{1}{y-x}\int_x^y f(t)\,dt, & x,y \in I,\ x \neq y, \\ f(x), & x = y \in I. \end{cases} \tag{1.1}$$

In a seminal work [14], Wulbert proved that the integral arithmetic mean F, defined in (1.1), exhibits convexity on the interval I^2 when the underlying function f is convex over the interval I. In a separate study [15], Zhang and Chu independently rediscovered this result without making any reference to Wulbert's findings. Their work revealed that the convexity of the integral arithmetic mean F hinges on the crucial condition that f must be convex on the interval I.

Since it will hold significant importance for our forthcoming analysis, let us take into consideration a real-valued function f defined on the interval $[a,b]$. The divided difference of order n for the function f at distinct points $x_0, x_1, \dots, x_n \in [a,b]$ is defined recursively (as elucidated in [1, 9]) in the following manner:

$$f[x_i] = f(x_i) \quad (i = 0, \dots, n)$$

and

$$f[x_0,\dots,x_n]=\frac{f[x_1,\dots,x_n]-f[x_0,\dots,x_{n-1}]}{x_n-x_0},\quad n\in\mathbb{N}_0.$$

The value $f[x_0,\dots,x_n]$ remains invariant regardless of the order in which the points $x_0,\dots,x_n$ are arranged.

The definition can be further extended to accommodate scenarios where some (or all) of the points coincide. Provided that $f^{(j-1)}(x)$ exists, we establish the following notation:

$$f[\underbrace{x,\dots,x}_{j\text{-times}}]=\frac{f^{(j-1)}(x)}{(j-1)!}. \tag{1.2}$$

In the context of divided differences, the following holds:

$$f[x_0,\dots,x_n]=\sum_{i=0}^{n}\frac{f(x_i)}{\omega'(x_i)},\quad \text{where } \omega(x)=\prod_{j=0}^{n}(x-x_j).$$

In conclusion, it is evident that the following property holds for divided differences:

$$f[x_0,\dots,x_n]=\sum_{i=0}^{n}\frac{f(x_i)}{\prod_{j=0,j\neq i}^{n}(x_i-x_j)}.$$

Under the condition that the function f has a continuous nth derivative on the interval $[a,b]$, we can represent the divided difference $f[x_0,\dots,x_n]$ using integral notation (refer to [9, p. 15]) as

$$f[x_0,\dots,x_n]=\int_{\Delta_n}f^{(n)}\left(\sum_{i=0}^{n}u_ix_i\right)du_0\cdots du_{n-1},$$

where

$$\Delta_n=\left\{(u_0,\dots,u_{n-1}):u_i\geq 0,\sum_{i=0}^{n-1}u_i\leq 1\right\}$$

and $u_n=1-\sum_{i=0}^{n-1}u_i$.

The notion of n-convexity is attributed to Popoviciu [10]. For the present study, we adhere to the definition as presented by Karlin [6].

Definition 1 A function $f:[a,b]\to\mathbb{R}$ is said to be n-convex on $[a,b]$, $n\geq 0$, if for all choices of $(n+1)$ distinct points in $[a,b]$, the nth order divided difference of f satisfies

$$f[x_0,\dots,x_n]\geq 0.$$

It is worth noting that Popoviciu's work demonstrated the fundamental result that any continuous n-convex function defined on the interval $[a,b]$ can be represented as the uniform limit of a sequence of n-convex polynomials. Moreover, [7] provides an extensive collection of related results and essential inequalities attributed to Favard, Berwald, and Steffensen.

The proof of the Jensen inequality for divided differences can be found in [4]:

Theorem 1 *Let f be an $(n+2)$-convex function on (a,b) and $\mathbf{x} \in (a,b)^{n+1}$. Then*

$$G(\mathbf{x}) = f[x_0, \dots, x_n]$$

is a convex function of the vector $\mathbf{x} = (x_0, \dots, x_n)$. Consequently,

$$f\left[\sum_{i=0}^{m} a_i x_0^i, \dots, \sum_{i=0}^{m} a_i x_n^i\right] \leq \sum_{i=0}^{m} a_i f\left[x_0^i, \dots, x_n^i\right] \quad (i \text{ is an upper index}) \tag{1.3}$$

holds for all $a_i \geq 0$ such that $\sum_{i=0}^{m} a_i = 1$.

In the context of future research, the notion of a generalized divided difference will hold relevance. Provided below is the definition for reference.

Consider a real-valued function $f(x,y)$ defined on $I \times J$ ($I = [a,b]$, $J = [c,d]$). The divided difference of order (n,m) for the function f at distinct points $x_0, \dots, x_n \in I$ and $y_0, \dots, y_m \in J$ is defined as follows (see [9, p. 18]):

$$\begin{aligned} f\begin{bmatrix} x_0, \dots, x_n \\ y_0, \dots, y_m \end{bmatrix} &= f[y_0, \dots, y_m][x_0, \dots, x_n] \\ &= f[x_0, \dots, x_n][y_0, \dots, y_m] \\ &= \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{f(x_i, y_j)}{\omega'(x_i) w'(y_j)}, \end{aligned} \tag{1.4}$$

where $\omega(x) = \prod_{i=0}^{n}(x - x_i)$, $w(y) = \prod_{j=0}^{m}(y - y_j)$.

Following the aforementioned definition, we can establish the concept of (n,m)-convexity, which is as follows (see [9, p. 18]):

Definition 2 A function $f: I \times J \to \mathbb{R}$ is said to be (n,m)-convex, or convex of order (n,m), if for all distinct points $x_0, \dots, x_n \in I, y_0, \dots, y_m \in J$,

$$f\begin{bmatrix} x_0, \dots, x_n \\ y_0, \dots, y_m \end{bmatrix} \geq 0. \tag{1.5}$$

If this inequality is strict, then f is said to be strictly (n,m)-convex.

In [11], Popoviciu presented and proved the following theorem:

Theorem 2 *If the partial derivative $f_{x^n y^m}^{(n+m)}$ $(\partial^{(n+m)} f / \partial x^n \partial y^m)$ of f exists, then f is (n,m)-convex iff*

$$f_{x^n y^m}^{(n+m)} \geq 0. \tag{1.6}$$

If the inequality in (1.6) is strict, then f is strictly (n,m)-convex.

In this research, we build upon the generalization of Levinson's inequality, and thus, we begin by stating the fundamental Levinson's inequality as follows (see [8] and [12]):

Theorem 3 *Let f be a real valued 3-convex function on $[0, 2a]$. Then for $0 \leq x_k \leq a$, $p_k > 0$ $(1 \leq k \leq n)$, and $P_k = \sum_{i=1}^{k} p_i$ $(2 \leq k \leq n)$ we have*

$$\begin{aligned} &\frac{1}{P_n}\sum_{k=1}^{n} p_k f(x_k) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right) \\ &\quad \leq \frac{1}{P_n}\sum_{k=1}^{n} p_k f(2a - x_k) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k (2a - x_k)\right). \end{aligned} \tag{1.7}$$

If $f''' > 0$, then the equality holds iff $x_1 = \cdots = x_n$.

In [2], Bullen provided a proof for the generalization of Theorem 3:

Theorem 4

a) *Let f be a real-valued 3-convex function on $[a, b]$ and x_k, y_k $(1 \leq k \leq n)$ be $2n$ points on $[a, b]$ such that*

$$\max\{x_1, \dots, x_n\} \leq \min\{y_1, \dots, y_n\}, \qquad x_1 + y_1 = \cdots = x_n + y_n. \tag{1.8}$$

If $p_k > 0$ $(1 \leq k \leq n)$, then

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k f(x_k) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right) \leq \frac{1}{P_n}\sum_{k=1}^{n} p_k f(y_k) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k y_k\right). \tag{1.9}$$

If f is strictly 3-convex there is equality in (1.9) if and only if $x_1 = \cdots = x_n$.

b) *If (1.9) holds for a continuous function f, (1.8) is satisfied by $2n$-distinct points and $p_k > 0$ for $k \in [1, n]$, then f is 3-convex.*

It is shown in [9] that the condition (1.8) can be weakened, i.e., the following result holds:

Theorem 5 *Let f be a 3-convex function on $[a, b]$, $p_i > 0$ $(1 \leq i \leq n)$, x_k, y_k $(1 \leq k \leq n)$ be points in $[a, b]$ such that*

$$x_1 + y_1 = \cdots = x_n + y_n = 2c \tag{1.10}$$

and

$$x_i + x_{n-i+1} \leq 2c, \tag{1.11}$$

$$(p_i x_i + p_{n-i+1} x_{n-i+1})/(p_i + p_{n-i+1}) \leq c, \quad \textit{for } 1 \leq i \leq n. \tag{1.12}$$

Then (1.9) is valid.

The primary objective of this paper is to provide an extension of Wulbert's result, as presented in [14], for 3-convex functions. We will also consider relevant findings from

[13]. Moreover, we aim to establish an inequality involving divided differences by utilizing the generalization of Levinson's inequality given in [9]. As a significant outcome, we will demonstrate the convexity of higher order for functions defined by divided differences.

2 Inequalities involving averages

Theorem 6 *Let f be a real-valued 3-convex function on $[a,b]$ and let F be defined in (1.1). Then for $p_i > 0$ $(1 \le i \le n)$, $a \le x_k, \tilde{x}_k, y_k, \tilde{y}_k \le b$ $(1 \le k \le n)$ such that*

$$x_1 + y_1 = \cdots = x_n + y_n = 2c, \qquad \tilde{x}_1 + \tilde{y}_1 = \cdots = \tilde{x}_n + \tilde{y}_n = 2c,$$
$$x_i + x_{n-i+1} \le 2c, \qquad \tilde{x}_i + \tilde{x}_{n-i+1} \le 2c,$$
$$\frac{p_i x_i + p_{n-i+1} x_{n-i+1}}{p_i + p_{n-i+1}} \le c, \qquad \frac{p_i \tilde{x}_i + p_{n-i+1} \tilde{x}_{n-i+1}}{p_i + p_{n-i+1}} \le c, \quad 1 \le i \le n,$$

and $P_k = \sum_{i=1}^{k} p_i$ $(2 \le k \le n)$ we have

$$\frac{1}{P_n}\sum_{k=1}^{n} p_k F(x_k, \tilde{x}_k) - F(\bar{x}, \bar{\tilde{x}}) \le \frac{1}{P_n}\sum_{k=1}^{n} p_k F(y_k, \tilde{y}_k) - F(\bar{y}, \bar{\tilde{y}}), \tag{2.1}$$

where $\bar{x} = \frac{1}{P_n}\sum_{k=1}^{n} p_k x_k$, $\bar{\tilde{x}} = \frac{1}{P_n}\sum_{k=1}^{n} p_k \tilde{x}_k$, $\bar{y} = \frac{1}{P_n}\sum_{k=1}^{n} p_k y_k$, and $\bar{\tilde{y}} = \frac{1}{P_n}\sum_{k=1}^{n} p_k \tilde{y}_k$.

Consequently, for $l + m = 3$ the integral arithmetic mean (1.1) is (l,m)-convex on $[a,b]^2$.

Proof Since the conditions

$$s\tilde{x}_1 + (1-s)x_1 + s\tilde{y}_1 + (1-s)y_1 = \cdots = s\tilde{x}_n + (1-s)x_n + s\tilde{y}_n + (1-s)y_n = 2c,$$
$$s\tilde{x}_i + (1-s)x_i + s\tilde{x}_{n-i+1} + (1-s)x_{n-i+1} \le 2c,$$
$$\frac{p_i(s\tilde{x}_i + (1-s)x_i) + p_{n-i+1}(s\tilde{x}_{n-i+1} + (1-s)x_{n-i+1})}{p_i + p_{n-i+1}} \le c, \quad 1 \le i \le n,$$

from Theorem 5 are satisfied, by using inequality (1.9), we get

$$\begin{aligned}
&\frac{1}{P_n}\sum_{k=1}^{n} p_k F(x_k, \tilde{x}_k) - F(\bar{x}, \bar{\tilde{x}}) \\
&= \frac{1}{P_n}\sum_{k=1}^{n} p_k \int_0^1 f\bigl(s\tilde{x}_k + (1-s)x_k\bigr)\,ds \\
&\quad - \int_0^1 f\left(s\frac{1}{P_n}\sum_{k=1}^{n} p_k \tilde{x}_k + (1-s)\frac{1}{P_n}\sum_{k=1}^{n} p_k x_k\right)ds \\
&= \int_0^1 \left[\frac{1}{P_n}\sum_{k=1}^{n} p_k f\bigl(s\tilde{x}_k + (1-s)x_k\bigr) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k\bigl(s\tilde{x}_k + (1-s)x_k\bigr)\right)\right]ds \\
&\le \int_0^1 \left[\frac{1}{P_n}\sum_{k=1}^{n} p_k f\bigl(s\tilde{y}_k + (1-s)y_k\bigr) - f\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k\bigl(s\tilde{y}_k + (1-s)y_k\bigr)\right)\right]ds \\
&= \frac{1}{P_n}\sum_{k=1}^{n} p_k \int_0^1 f\bigl(s\tilde{y}_k + (1-s)y_k\bigr)\,ds
\end{aligned}$$

$$- \int_0^1 f\left(s\frac{1}{P_n}\sum_{k=1}^n p_k\tilde{y}_k + (1-s)\frac{1}{P_n}\sum_{k=1}^n p_k y_k\right) ds$$
$$= \frac{1}{P_n}\sum_{k=1}^n p_k F(y_k,\tilde{y}_k) - F(\bar{y},\bar{\tilde{y}}).$$

Now, if we put $n = 2$, $x_1 = x$, $x_2 = y_2 = x + \frac{3h}{2}$, $y_1 = x + 3h$, $\tilde{x}_1 = \tilde{x}_2 = y$, $\tilde{y}_1 = \tilde{y}_2 = y$, $2x + 3h = 2y = 2c$, $p_1 = 1$, $p_2 = 2$, then inequality (2.1) reduces to

$$\frac{1}{3}F(x,y) - F(x+h,y) \le \frac{1}{3}F(x+3h,y) - F(x+2h,y).$$

Using the definition in (1.4), we get

$$2h^3\bigl(F[x,x+h,x+2h,x+3h]\bigr)[y] \ge 0.$$

It is a known fact that if this property holds for all possible $x, y, h > 0$, then F is $(3,0)$-convex, as stated in [11].

If we put $n = 2$, $x_1 = x$, $x_2 = x + 2h_1$, $y_1 = x + 2h_1$, $y_2 = x$, $\tilde{x}_1 = \tilde{x}_2 = y$, $\tilde{y}_1 = \tilde{y}_2 = y + h_2$, $p_1 = p_2 = 1$, $2x + 2h_1 = 2y + h_2 = 2c$ then inequality (2.1) reduces to

$$\frac{1}{2}\bigl(F(x,y) + F(x+2h_1,y)\bigr) - F(x+h_1,y)$$
$$\le \frac{1}{2}\bigl(F(x+2h_1,y+h_2) + F(x,y+h_2)\bigr) - F(x+h_1,y+h_2).$$

Using the definition in (1.4), we get

$$h_1^2h_2\bigl(F[x,x+h_1,x+2h_1]\bigr)[y,y+h_2] \ge 0.$$

Continuing the previous arguments, since this property holds for all possible $x, h_1, y, h_2 > 0$, we can deduce that F is $(2,1)$-convex.

The proofs for $(0,3)$- and $(1,2)$-convexity exhibit similarities, leading us to conclude that F is (l,m)-convex on $[a,b]^2$ when $l + m = 3$. □

Remark 1 Theorem 6 can be regarded as a generalization of Theorem 5 since inequality (2.1) for $x_k = \tilde{x}_k$ and $y_k = \tilde{y}_k$, $k = 1,\dots,n$ reproduces inequality (1.9).

For similar results regarding Jensen's inequality involving averages of convex functions, refer to [3] and [5].

3 Inequalities for divided differences

Theorem 7 *Let f be an $(n+3)$-convex function on $[a,b]$ and $\mathbf{x},\mathbf{y} \in [a,b]^{n+1}$. Then for x_k^i, y_k^i, $(0 \le k \le n)$ ("i" is an upper index), $a_i > 0$ $(0 \le i \le m)$, such that $\sum_{i=0}^m a_i = 1$,*

$$x_0^i + y_0^i = x_1^i + y_1^i = \cdots = x_n^i + y_n^i = 2c,$$
$$x_k^i + x_k^{m-i} \le 2c,$$
$$\bigl(a_i x_k^i + a_{m-i}x_k^{m-i}\bigr)/(a_i + a_{m-i}) \le c,$$

we have

$$\sum_{i=0}^{m} a_i f\left[x_0^i,\dots,x_n^i\right] - f\left[\sum_{i=0}^{m} a_i x_0^i,\dots,\sum_{i=0}^{m} a_i x_n^i\right]$$
$$\leq \sum_{i=0}^{m} a_i f\left[y_0^i,\dots,y_n^i\right] - f\left[\sum_{i=0}^{m} a_i y_0^i,\dots,\sum_{i=0}^{m} a_i y_n^i\right]. \tag{3.1}$$

Consequently,

$$G(\mathbf{x}) = f[x_0,x_1,x_2]$$

is an (l_1,l_2,l_3)*-convex function of the vector* $\mathbf{x} = (x_0,x_1,x_2)$*, when* $l_1 + l_2 + l_3 = 3$.

Proof Since the conditions

$$\sum_{j=0}^{n} u_j x_j^0 + \sum_{j=0}^{n} u_j y_j^0 = \cdots = \sum_{j=0}^{n} u_j x_j^m + \sum_{j=0}^{n} u_j y_j^m = 2c,$$
$$\sum_{j=0}^{n} u_j x_j^i + \sum_{j=0}^{n} u_j x_j^{m-i} \leq 2c,$$
$$\frac{a_i \sum_{j=0}^{n} u_j x_j^i + a_{m-i} \sum_{j=0}^{n} u_j x_j^{m-i}}{a_i + a_{m-i}} \leq c, \quad 0 \leq i \leq m,$$

from Theorem 5 are satisfied, by using inequality (1.9), for the 3-convex function $f^{(n)}$, we get

$$\begin{aligned}
&\sum_{i=0}^{m} a_i f\left[x_0^i,\dots,x_n^i\right] - f\left[\sum_{i=0}^{m} a_i x_0^i,\dots,\sum_{i=0}^{m} a_i x_n^i\right] \\
&= \sum_{i=0}^{m} a_i \int_{\Delta_n} f^{(n)}\left(\sum_{j=0}^{n} u_j x_j^i\right) du_0 \cdots du_{n-1} \\
&\quad - \int_{\Delta_n} f^{(n)}\left(\sum_{j=0}^{n} u_j \sum_{i=0}^{m} a_i x_j^i\right) du_0 \cdots du_{n-1} \\
&= \int_{\Delta_n}\left[\sum_{i=0}^{m} a_i f^{(n)}\left(\sum_{j=0}^{n} u_j x_j^i\right) - f^{(n)}\left(\sum_{i=0}^{m} a_i \sum_{j=0}^{n} u_j x_j^i\right)\right] du_0 \cdots du_{n-1} \\
&\leq \int_{\Delta_n}\left[\sum_{i=0}^{m} a_i f^{(n)}\left(\sum_{j=0}^{n} u_j y_j^i\right) - f^{(n)}\left(\sum_{i=0}^{m} a_i \sum_{j=0}^{n} u_j y_j^i\right)\right] du_0 \cdots du_{n-1} \\
&= \sum_{i=0}^{m} a_i \int_{\Delta_n} f^{(n)}\left(\sum_{j=0}^{n} u_j y_j^i\right) du_0 \cdots du_{n-1} \\
&\quad - \int_{\Delta_n} f^{(n)}\left(\sum_{j=0}^{n} u_j \sum_{i=0}^{m} a_i y_j^i\right) du_0 \cdots du_{n-1} \\
&= \sum_{i=0}^{m} a_i f\left[y_0^i,\dots,y_n^i\right] - f\left[\sum_{i=0}^{m} a_i y_0^i,\dots,\sum_{i=0}^{m} a_i y_n^i\right].
\end{aligned}$$

Now, if we put $n = 2$, $m = 1$,

$$\begin{aligned}
&x_0^0 = y_0, \qquad x_0^1 = y_0 + \frac{3h}{2},\\
&y_0^0 = y_0 + 3h, \qquad y_0^1 = y_0 + \frac{3h}{2},\\
&x_1^0 = x_1^1 = y_1^0 = y_1^1 = y_1,\\
&x_2^0 = x_2^1 = y_2^0 = y_2^1 = y_2,\\
&a_0 = \frac{1}{3}, \qquad a_1 = \frac{2}{3},
\end{aligned}$$

then inequality (3.1) reduces to

$$\begin{aligned}
&\frac{1}{3}G(y_0, y_1, y_2) - G(y_0 + h, y_1, y_2)\\
&\quad \le \frac{1}{3}G(y_0 + 3h, y_1, y_2) - G(y_0 + 2h, y_1, y_2).
\end{aligned}$$

Using the generalization of definition (1.4), we get

$$2h^3\big(\big(G[y_0, y_0 + h, y_0 + 2h, y_0 + 3h]\big)[y_1]\big)[y_2] \ge 0.$$

As observed in the proof of Theorem 6, since this property holds for all possible $y_0, y_1, y_2, h > 0$, we can conclude that G is $(3, 0, 0)$-convex.

If we put $n = 2$, $m = 1$,

$$\begin{aligned}
&x_0^0 = y_0^1 = y_0, \qquad x_0^1 = y_0^0 = y_0 + 2h_0,\\
&x_1^0 = x_1^1 = y_1,\\
&x_2^0 = x_2^1 = y_2^0 = y_2^1 = y_2,\\
&y_1^0 = y_1^1 = y_1 + h_1,\\
&a_0 = a_1 = \frac{1}{2}
\end{aligned}$$

then inequality (3.1) reduces to

$$\begin{aligned}
&\frac{1}{2}G(y_0, y_1, y_2) + \frac{1}{2}G(y_0 + 2h_0, y_1, y_2) - G(y_0 + h_0, y_1, y_2)\\
&\quad \le \frac{1}{2}G(y_0 + 2h_0, y_1 + h_1, y_2) + \frac{1}{2}G(y_0, y_1 + h_1, y_2) - G(y_0 + h_0, y_1 + h_1, y_2).
\end{aligned}$$

Using the generalization of definition (1.4), we get

$$h_0^2 h_1\big(\big(G[y_0, y_0 + h_0, y_0 + 2h_0]\big)[y_1, y_1 + h_1]\big)[y_2] \ge 0.$$

As before, since this holds for all possible $y_0, y_1, y_2, h_0, h_1 > 0$, G is $(2, 1, 0)$-convex.

If we put $n = 2$, $m = 3$,

$$\begin{aligned} &x_0^0 = x_0^2 = y_0^1 = y_0^3 = y_0, &&x_0^1 = x_0^3 = y_0^0 = y_0^2 = y_0 + h_0, \\ &x_1^0 = x_1^3 = y_1^1 = y_1^2 = y_1, &&x_1^1 = x_1^2 = y_1^0 = y_1^3 = y_1 + h_1 \\ &x_2^0 = x_2^1 = y_2^2 = y_2^3 = y_2, &&x_2^2 = x_2^3 = y_2^0 = y_2^1 = y_2 + h_2, \\ &a_0 = a_1 = a_2 = a_3 = \frac{1}{4}, \end{aligned}$$

then inequality (3.1) reduces to

$$\begin{aligned} &\frac{1}{4}G(y_0, y_1, y_2) + \frac{1}{4}G(y_0 + h_0, y_1 + h_1, y_2) \\ &\qquad + \frac{1}{4}G(y_0, y_1 + h_1, y_2 + h_2) + \frac{1}{4}G(y_0 + h_0, y_1, y_2 + h_2) \\ &\quad \le \frac{1}{4}G(y_0 + h_0, y_1 + h_1, y_2 + h_2) + \frac{1}{4}G(y_0, y_1, y_2 + h_2) \\ &\qquad + \frac{1}{4}G(y_0 + h_0, y_1, y_2) + \frac{1}{4}G(y_0, y_1 + h_1, y_2). \end{aligned}$$

Using the generalization of definition (1.4), we get

$$\frac{1}{4}h_0h_1h_2\big(\big(G[y_0, y_0 + h_0]\big)[y_1, y_1 + h_1]\big)[y_2, y_2 + h_2] \ge 0.$$

Continuing the previous arguments, since this property holds for all possible y_0, y_1, y_2, $h_0, h_1, h_2 > 0$, we can conclude that G is $(1,1,1)$-convex.

The proofs for $(0,3,0)$-, $(0,0,3)$-, $(1,2,0)$-, $(0,2,1)$-, $(0,1,2)$-, $(2,0,1)$-, and $(1,0,2)$-convexity share similarities. □

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
All authors reviewed and approved the final manuscript, providing valuable feedback and revisions to ensure its accuracy and quality. The decision to submit the manuscript for publication was made collectively by all authors in agreement with the content presented. The authors have contributed equally to this work.

Author details
[1]Faculty Architecture, University of Zagreb, Kačićeva 26, 10000 Zagreb, Croatia. [2]Croatian Academy of Sciences and Art, Trg Nikole Šubića Zrinskog 11, 10000 Zagreb, Croatia. [3]Faculty of Food Technology and Biotechnology, Mathematics Department, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia.

Received: 22 July 2023 Accepted: 1 September 2023 Published online: 18 September 2023

References

1. Atkinson, K.E.: An Introduction to Numerical Analysis, 2nd edn. Wiley, New York (1989)
2. Bullen, P.S.: An inequality of N. Levinson. Publ. Elektroteh. Fak. Univ. Beogr., Ser. Mat. Fiz. **412–460**, 109–112 (1973)

3. Čuljak, V., Franjić, I., Ghulam, R., Pečarić, J.: Schur-convexity of averages of convex functions. J. Inequal. Appl. **2011**, Article ID 581918 (2011)
4. Farwig, R., Zwick, D.: Some divided difference inequalities for *n*-convex functions. J. Math. Anal. Appl. **108**, 430–437 (1985)
5. Jakšetić, J., Pečarić, J., Roqia, G.: On Jensen's inequality involving averages of convex functions. Sarajevo J. Math. **8(20)**(1), 53–68 (2012)
6. Karlin, S.: Total Positivity. Stanford University Press, Stanford (1968)
7. Karlin, S., Studden, W.J.: Tchebycheff Systems: With Applications in Analysis and Statistics. Interscience, New York (1966)
8. Levinson, N.: Generalization of an inequality of Ky Fan. J. Math. Anal. Appl. **80**, 133–134 (1964)
9. Pečarić, J.E., Proschan, F., Tong, Y.L.: Convex Functions, Partial Orderings, and Statistical Applications. Mathematics in Science and Engineering, vol. 187. Academic Press, San Diego (1992)
10. Popoviciu, T.: Sur l'approximation des fonctions convexes d'ordre superieur. Mathematica **10**, 49–54 (1934)
11. Popoviciu, T.: Les Fonctions Convexes. Herman and Cie, Éditeurs, Paris (1944)
12. Popoviciu, T.: Sur certaines inégalités qui caracterisent les fonctions convexes. An. Sti. Univ. Al. I. Cuza Iasi Sect. I a Math. (N.S.) **11B**, 155–164 (1965)
13. Vukelić, A.: On Levinson's inequality involving averages of 3-convex functions. Appl. Anal. Discrete Math. **14**, 670–684 (2020)
14. Wulbert, D.E.: Favard's inequality on average values of convex functions. Math. Comput. Model. **37**, 1383–1391 (2003)
15. Zhang, X.M., Chu, Y.M.: Convexity of the integral arithmetic mean of a convex function. Rocky Mt. J. Math. **40**(3), 1061–1068 (2010)

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.