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A new reverse half-discrete Mulholland-type inequality with a nonhomogeneous kernel

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Abstract

In this paper, a new reverse half-discrete Mulholland-type inequality with the nonhomogeneous kernel of the form $h(v(x)\ln n)$ and the best possible constant factor is obtained by using the weight functions and the technique of real analysis. The equivalent reverses are considered. As corollaries, we deduce some new equivalent reverse inequalities with the homogeneous kernel of the form $k_\lambda(v(x),\ln n)$. A few particular cases are provided. Our new reverse half-discrete Mulholland-type inequality which has a nonhomogeneous kernel is more general than in the previous homogeneous kernel work. The harmonized integration will have more applications.

Mathematics Subject Classification: 26D15

Keywords: Mulholland-type inequality; Parameter; Weight function; Best possible constant factor; Reverse

1 Introduction

Hilbert-type inequalities are a class of mathematical inequalities that generalize the classical analytic inequality. They have applications in various areas of mathematics such as functional analysis, operator theory, and time scales [1–5].

Assuming that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and $a_m, b_n \geq 0$ are such that $0<\sum_{m=1}^{\infty} a_m^p<\infty$ and $0<\sum_{n=1}^{\infty} b_n^q<\infty$, we have the following Hardy–Hilbert inequality with the best possible constant factor $\pi/\sin(\frac{\pi}{p})$ (cf. [6, Theorem 315]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

We also have the following Mulholland's inequality with the same best possible constant factor $\pi/\sin(\frac{\pi}{p})$ (cf. [6, Theorem 343]):

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln mn}<\frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=2}^{\infty}m^{p-1}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}n^{q-1}b_n^q\right)^{\frac{1}{q}}. \tag{2}$$

If $f(x), g(y)\geq 0$, $0<\int_0^{\infty} f^p(x)\,dx<\infty$, and $0<\int_0^{\infty} g^q(y)\,dy<\infty$, then we have the following integral analogue of (1) with the best possible constant factor $\pi/\sin(\frac{\pi}{p})$, named

Hardy–Hilbert integral inequality (cf. [6, Theorem 316]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(y)\,dy\right)^{\frac{1}{2}}. \tag{3}$$

Inequalities (1)–(3) with their reverses play an important role in analysis and its applications. Some new extensions and applications were given in [7–14].

In 1934, a half-discrete Hilbert-type inequality with the nonhomogeneous kernel was given as follows (cf. [6, Theorem 351]): If $K(x)$ is decreasing, $0 < \phi(s) = \int_0^\infty K(x)x^{s-1}\,dx < \infty$, $f(x) \ge 0$, $0 < \int_0^\infty f^p(x) < \infty$, then

$$\sum_{m=1}^{\infty} n^{p-2}\left(\int_0^\infty K(nx)f(x)\,dx\right)^p < \phi^p\left(\frac{1}{q}\right)\int_0^\infty f^p(x)\,dx. \tag{4}$$

Recently, some new extensions of (4) were provided in [15–20].

In 2016, Hong [21] discussed an equivalent description of (1) with a general homogeneous kernel related to some parameters and the optimal constant factors. Similar works were considered in the papers [22]–[23]. Recently, Mulholland-type inequalities with homogeneous kernel were obtained in [24–26]. However, only a few reverse Mulholland-type inequalities with parameters were given in [27].

In this paper, by means of the weight functions and the techniques of real analysis, a new reverse half-discrete Mulholland-type inequality with a general nonhomogeneous kernel of the form $h(\nu(x)\ln n)$ is given. The best possible constant factor and some equivalent reverses are considered. As a corollary, we deduce some new equivalent reverse inequalities with a general homogeneous kernel of the form $k_\lambda(\nu(x), \ln n)$. A few particular cases are provided. Our new reverse Mulholland-type inequality with a nonhomogeneous kernel is more inclusive and encompasses previous studies that focused on homogeneous kernels. This advancement in harmonized integration is expected to have broader applications and implications.

2 Some lemmas

Definition 1 Suppose that $\sigma \in \mathbf{R} = (-\infty, \infty)$, $h(u)$ is a nonnegative measurable function of $u > 0$, $h(u)u^{\sigma-1}$ $(u > 0)$ is decreasing with

$$k(\sigma) := \int_0^\infty h(u)u^{\sigma-1}\,du \in \mathbf{R}_+ = (0, \infty),$$

$\nu(x) > 0$, $\nu'(x) > 0$ $(x > 0)$, with $\nu(0^+) = 0$ and $\nu(\infty) = \infty$. For $\mathbf{N} = \{1, 2, \ldots\}$, we define the following weight functions:

$$\omega(\sigma, n) := \ln^\sigma n \int_0^\infty h\big(\nu(x)\ln n\big)\frac{\nu'(x)}{(\nu(x))^{1-\sigma}}\,dx \quad \big(n \in \mathbf{N}\backslash\{1\}\big), \tag{5}$$

$$\varpi(\sigma, x) := \big(\nu(x)\big)^\sigma \sum_{n=2}^{\infty} h\big(\nu(x)\ln n\big)\frac{\ln^{\sigma-1} n}{n} \quad (x \in \mathbf{R}_+). \tag{6}$$

Setting $u = \nu(x)\ln n$, we find

$$\omega(\sigma, n) = k(\sigma) := \int_0^\infty h(u)u^{\sigma-1}\,du \quad \big(n \in \mathbf{N}\backslash\{1\}\big). \tag{7}$$

Lemma 1 *Under the assumptions of Definition 1, we have the following inequalities*:

$$k(\sigma)\big(1-\theta(x)\big)<\overline{\omega}(\sigma,x)<k(\sigma)\quad(x\in\mathbf{R}_+),\tag{8}$$

where $\theta(x):=\frac{1}{k(\sigma)}\int_0^{\nu(x)\ln 2}h(u)u^{\sigma-1}\,du\in(0,1)$ $(x>0)$.

Proof From the assumption that $h(u)u^{\sigma-1}$ $(u>0)$ is decreasing, we find that $\frac{1}{y}h(\nu(x)\ln y)\times\ln^{\sigma-1}y$ is strictly decreasing with respect to $y\in(1,\infty)$. In view of the decreasingness property of the series, for $x\in\mathbf{R}_+$, we have

$$\begin{aligned}
\overline{\omega}(\sigma,x)&<\big(\nu(x)\big)^{\sigma}\int_1^{\infty}h\big(\nu(x)\ln y\big)\frac{\ln^{\sigma-1}y}{y}\,dy\overset{u=\nu(x)\ln y}{=}\int_0^{\infty}h(u)u^{\sigma-1}\,du=k(\sigma),\\
\overline{\omega}(\sigma,x)&>\big(\nu(x)\big)^{\sigma}\int_2^{\infty}h\big(\nu(x)\ln y\big)\frac{\ln^{\sigma-1}y}{y}\,dy\overset{u=\nu(x)\ln y}{=}\int_{\nu(x)\ln 2}^{\infty}h(u)u^{\sigma-1}\,du\\
&=\int_0^{\infty}h(u)u^{\sigma-1}\,du-\int_0^{\nu(x)\ln 2}h(u)u^{\sigma-1}\,du\\
&=k(\sigma)\left(1-\frac{1}{k(\sigma)}\int_0^{\nu(x)\ln 2}h(u)u^{\sigma-1}\,du\right)\\
&=k(\sigma)\big(1-\theta(x)\big)>0,
\end{aligned}$$

where $\theta(x):=\frac{1}{k(\sigma)}\int_0^{\nu(x)\ln 2}h(u)u^{\sigma-1}\,du\in(0,1)$ $(x>0)$. Hence, we have (8).

The lemma is proved. □

Remark 1 (i) If $\nu(x)>0$, $\nu'(x)<0$, $\nu(0^+)=\infty$, $\nu(\infty)=0$, we can still obtain (7) and (8), only replacing $\nu'(x)$ by $|\nu'(x)|$ in (5). (ii) If $\sigma\le 1$, $h'(u)\le 0$, then the function $h(u)u^{\sigma-1}$ is still decreasing with respect to $u>0$.

Lemma 2 *Under the assumptions of Definition 1, and with* $p<1$ $(p\neq 0)$, $\frac{1}{p}+\frac{1}{q}=1$, *we suppose that* $f(x)$ *is a nonnegative measurable function in* $\mathbf{R}_+$, *and* $a_n\ge 0$ $(n\in\mathbf{N}\backslash\{1\})$ *is such that*

$$0<\int_0^{\infty}\frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}}f^p(x)\,dx<\infty\quad\textit{and}\quad 0<\sum_{n=2}^{\infty}\frac{\ln^{q(1-\sigma)-1}n}{n^{1-q}}a_n^q<\infty.$$

(i) For $p<0$ $(0<q<1)$, we have the following reverse inequalities:

$$\widetilde{J}_1:=\left[\sum_{n=2}^{\infty}\frac{\ln^{p\sigma-1}n}{n}\left(\int_0^{\infty}h\big(\nu(x)\ln n\big)f(x)\,dx\right)^p\right]^{\frac{1}{p}}>k(\sigma)\left[\int_0^{\infty}\frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}}f^p(x)\,dx\right]^{\frac{1}{p}},\tag{9}$$

$$\widetilde{J}_2:=\left[\int_0^{\infty}\big(\nu(x)\big)^{q\sigma-1}\nu'(x)\left(\sum_{n=2}^{\infty}h\big(\nu(x)\ln n\big)a_n\right)^q dx\right]^{\frac{1}{q}}>k(\sigma)\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\sigma)-1}n}{n^{1-q}}a_n^q\right]^{\frac{1}{q}};\tag{10}$$

(ii) For $0 < p < 1$ ($q < 0$), we have the following reverse inequalities:

$$\widehat{J}_1 := \left[\sum_{n=2}^{\infty} \frac{\ln^{p\sigma-1} n}{n} \left(\int_0^{\infty} h\big(\nu(x)\ln n\big) f(x)\,dx\right)^p\right]^{\frac{1}{p}}$$
$$> k(\sigma)\left[\int_0^{\infty} \big(1-\theta(x)\big) \frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{p}}, \tag{11}$$
$$\widehat{J}_2 := \left[\int_0^{\infty} \frac{(\nu(x))^{q\sigma-1}\nu'(x)}{(1-\theta(x))^{q-1}} \left(\sum_{n=2}^{\infty} h\big(\nu(x)\ln n\big) a_n\right)^q dx\right]^{\frac{1}{q}}$$
$$> k(\sigma)\left[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}}. \tag{12}$$

Proof (i) For $p < 0$ ($0 < q < 1$), by the reverse Hölder's inequality (cf. [28]), we have

$$\int_0^{\infty} h\big(\nu(x)\ln n\big) f(x)\,dx$$
$$= \int_0^{\infty} h\big(\nu(x)\ln n\big)\left[\frac{(\nu(x))^{\frac{1-\sigma}{q}} f(x)}{(\nu'(x))^{\frac{1}{q}} \ln^{\frac{1-\sigma}{p}} n}\right]\left[\frac{(\nu'(x))^{\frac{1}{q}} \ln^{\frac{1-\sigma}{p}} n}{(\nu(x))^{\frac{1-\sigma}{q}}}\right] dx$$
$$\geq \left[\int_0^{\infty} h\big(\nu(x)\ln n\big) \frac{(\nu(x))^{(1-\sigma)(p-1)}}{(\nu'(x))^{p-1} \ln^{1-\sigma} n} f^p(x)\,dx\right]^{\frac{1}{p}}$$
$$\times \left[\int_0^{\infty} h\big(\nu(x)\ln n\big) \frac{\nu'(x) \ln^{(1-\sigma)(q-1)} n}{(\nu(x))^{1-\sigma}}\,dx\right]^{\frac{1}{q}}$$
$$= \frac{(\omega(\sigma,n))^{\frac{1}{q}} n^{\frac{1}{p}}}{\ln^{\sigma-\frac{1}{p}} n}\left[\int_0^{\infty} h\big(\nu(x)\ln n\big) \frac{(\nu(x))^{(1-\sigma)(p-1)}}{(\nu'(x))^{p-1} n \ln^{1-\sigma} n} f^p(x)\,dx\right]^{\frac{1}{p}}.$$

For $p < 0$, by (8), we obtain $(\varpi(\sigma,x))^{\frac{1}{p}} > (k(\sigma))^{\frac{1}{p}}$, and

$$\widetilde{J}_1 \geq \big(k(\sigma)\big)^{\frac{1}{q}}\left[\sum_{n=2}^{\infty}\int_0^{\infty} h\big(\nu(x)\ln n\big) \frac{(\nu(x))^{(1-\sigma)(p-1)}}{(\nu'(x))^{p-1} n \ln^{1-\sigma} n} f^p(x)\,dx\right]^{\frac{1}{p}}$$
$$= \big(k(\sigma)\big)^{\frac{1}{q}}\left[\int_0^{\infty} \varpi(\sigma,x) \frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{p}}$$
$$> \big(k(\sigma)\big)\left[\int_0^{\infty} \frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{p}}.$$

Hence, (9) follows.

By the reverse Hölder's inequality and (6), we have

$$\sum_{n=2}^{\infty} h\big(\nu(x)\ln n\big) a_n$$
$$= \sum_{n=2}^{\infty} h\big(\nu(x)\ln n\big)\left[\frac{(\nu(x))^{\frac{1-\sigma}{q}}}{n^{\frac{1}{p}} \ln^{\frac{1-\sigma}{p}} n}\right]\left[\frac{n^{\frac{1}{p}} \ln^{\frac{1-\sigma}{p}} n}{(\nu(x))^{\frac{1-\sigma}{q}}} a_n\right]$$

$$\geq \left[\sum_{n=2}^{\infty} h\big(\nu(x)\ln n\big)\frac{(\nu(x))^{(1-\sigma)(p-1)}}{n\ln^{1-\sigma} n}\right]^{\frac{1}{p}}$$

$$\times \left[\sum_{n=2}^{\infty} h\big(\nu(x)\ln n\big)\frac{n^{1-q}\ln^{(1-\sigma)(q-1)} n}{(\nu(x))^{1-\sigma}}a_n^q\right]^{\frac{1}{q}}$$

$$= \big(\overline{\omega}(\sigma,x)\big)^{\frac{1}{p}}\frac{(\nu(x))^{\frac{1}{q}-\sigma}}{(\nu'(x))^{\frac{1}{q}}}\left[\sum_{n=2}^{\infty} h\big(\nu(x)\ln n\big)\frac{\nu'(x)n^{q-1}\ln^{(1-\sigma)(q-1)} n}{(\nu(x))^{1-\sigma}}a_n^q\right]^{\frac{1}{q}}.$$

Since $p < 0$, by (8), we obtain $(\overline{\omega}(\sigma,x))^{\frac{1}{p}} > (k(\sigma))^{\frac{1}{p}}$, and

$$\widetilde{J}_2 = \left[\int_0^{\infty}(\nu(x))^{q\sigma-1}\nu'(x)\left(\sum_{n=2}^{\infty}h\big(\nu(x)\ln n\big)a_n\right)^q dx\right]^{\frac{1}{q}}$$

$$> \big(k(\sigma)\big)^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\left(\int_0^{\infty}h\big(\nu(x)\ln n\big)\frac{\nu'(x)\ln^{(1-\sigma)(q-1)} n}{(\nu(x))^{1-\sigma}n^{1-q}}\,dx\right)a_n^q\right]^{\frac{1}{q}}$$

$$= \big(k(\sigma)\big)^{\frac{1}{p}}\left(\sum_{n=2}^{\infty}\omega(\sigma,n)\frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q\right)^{\frac{1}{q}}$$

$$= \big(k(\sigma)\big)^{\frac{1}{p}}\left[k(\sigma)\sum_{n=2}^{\infty}\frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q\right]^{\frac{1}{q}}$$

$$= k(\sigma)\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q\right]^{\frac{1}{q}}.$$

Then, (10) follows.

(ii) For $0 < p < 1$ ($q < 0$), by the reverse Hölder's inequality, we obtain

$$\int_0^{\infty}h\big(\nu(x)\ln n\big)f(x)\,dx$$

$$= \int_0^{\infty}h\big(\nu(x)\ln n\big)\left[\frac{(\nu(x))^{\frac{1-\sigma}{q}}f(x)}{(\nu'(x))^{\frac{1}{q}}\ln^{\frac{1-\sigma}{p}} n}\right]\left[\frac{(\nu'(x))^{\frac{1}{q}}\ln^{\frac{1-\sigma}{p}} n}{(\nu(x))^{\frac{1-\sigma}{q}}}\right]dx$$

$$\geq \left[\int_0^{\infty}h\big(\nu(x)\ln n\big)\frac{(\nu(x))^{(1-\sigma)(p-1)}}{(\nu'(x))^{p-1}\ln^{1-\sigma} n}f^p(x)\,dx\right]^{\frac{1}{p}}$$

$$\times \left[\int_0^{\infty}h\big(\nu(x)\ln n\big)\frac{\nu'(x)\ln^{(1-\sigma)(q-1)} n}{(\nu(x))^{1-\sigma}}\,dx\right]^{\frac{1}{q}}$$

$$= \big(\omega(\sigma,n)\big)^{\frac{1}{q}}\frac{n^{\frac{1}{p}}}{\ln^{\sigma-\frac{1}{p}} n}\left[\int_0^{\infty}h\big(\nu(x)\ln n\big)\frac{(\nu(x))^{(1-\sigma)(p-1)}f^p(x)}{(\nu'(x))^{p-1}n\ln^{1-\sigma} n}\,dx\right]^{\frac{1}{p}}.$$

In view of $0 < p < 1$ ($q < 0$) and (8), we obtain

$$\widehat{J}_1 \geq \big(k(\sigma)\big)^{\frac{1}{q}}\left[\sum_{n=2}^{\infty}\int_0^{\infty}h\big(\nu(x)\ln n\big)\frac{(\nu(x))^{(1-\sigma)(p-1)}f^p(x)}{(\nu'(x))^{p-1}n\ln^{1-\sigma} n}\,dx\right]^{\frac{1}{p}}$$

$$= \big(k(\sigma)\big)^{\frac{1}{q}}\left[\int_0^{\infty}\overline{\omega}(\sigma,x)\frac{(\nu(x))^{p(1-\sigma)-1}f^p(x)}{(\nu'(x))^{p-1}}\,dx\right]^{\frac{1}{p}}$$

$$> \big(k(\sigma)\big)^{\frac{1}{q}}\left[\int_0^\infty \big(k(\sigma)\big(1-\theta(x)\big)\frac{(\nu(x))^{p(1-\sigma)-1}f^p(x)}{(\nu'(x))^{p-1}}\,dx\right]^{\frac{1}{p}}$$
$$= \big(k(\sigma)\big)\left[\int_0^\infty \big(1-\theta(x)\big)\frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}}f^p(x)\,dx\right]^{\frac{1}{p}}.$$

Then, (11) follows.

By the reverse Hölder's inequality and (6), we have

$$\sum_{n=2}^\infty h\big(\nu(x)\ln n\big)a_n = \sum_{n=2}^\infty h\big(\nu(x)\ln n\big)\left[\frac{(\nu(x))^{\frac{1-\sigma}{q}}}{n^{\frac{1}{p}}\ln^{\frac{1-\sigma}{p}} n}\right]\left[\frac{n^{\frac{1}{p}}\ln^{\frac{1-\sigma}{p}} n}{(\nu(x))^{\frac{1-\sigma}{q}}}a_n\right]$$
$$\geq \left[\sum_{n=2}^\infty h\big(\nu(x)\ln n\big)\frac{(\nu(x))^{(1-\sigma)(p-1)}}{n\ln^{1-\sigma} n}\right]^{\frac{1}{p}}$$
$$\times \left[\sum_{n=2}^\infty h\big(\nu(x)\ln n\big)\frac{\ln^{(1-\sigma)(q-1)} n}{(\nu(x))^{1-\sigma}n^{1-q}}a_n^q\right]^{\frac{1}{q}}$$
$$= \big(\varpi(\sigma,x)\big)^{\frac{1}{p}}\frac{(\nu(x))^{\frac{1}{q}-\sigma}}{(\nu'(x))^{\frac{1}{q}}}\left[\sum_{n=2}^\infty h\big(\nu(x)\ln n\big)\frac{\nu'(x)\ln^{(1-\sigma)(q-1)} n}{(\nu(x))^{1-\sigma}n^{1-q}}a_n^q\right]^{\frac{1}{q}}.$$

Since $0 < p < 1$, by (8), we obtain $(\varpi(\sigma,x))^{\frac{1}{p}} > [k(\sigma)(1-\theta(x))]^{\frac{1}{p}}$, and

$$\widehat{J}_2 = \left[\int_0^\infty \frac{(\nu(x))^{q\sigma-1}\nu'(x)}{(1-\theta(x))^{q-1}}\left(\sum_{n=2}^\infty h\big(\nu(x)\ln n\big)a_n\right)^q dx\right]^{\frac{1}{q}}$$
$$> \big(k(\sigma)\big)^{\frac{1}{p}}\left[\sum_{n=2}^\infty\left(\int_0^\infty h\big(\nu(x)\ln n\big)\frac{\nu'(x)\ln^{(1-\sigma)(q-1)} n}{(\nu(x))^{1-\sigma}n^{1-q}}\,dx\right)a_n^q\right]^{\frac{1}{q}}$$
$$= \big(k(\sigma)\big)^{\frac{1}{p}}\left[\sum_{n=2}^\infty \omega(\sigma,n)\frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q\right]^{\frac{1}{q}}$$
$$= k(\sigma)\left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q\right]^{\frac{1}{q}}.$$

Hence, we have (12).

The lemma is proved. □

Lemma 3 *If $k(\sigma) = \int_0^\infty h(u)u^{\sigma-1}\,du$ is finite in an open interval $I \subset \mathbf{R}$, then $k(\sigma)$ is continuous at $\sigma \in I$.*

Proof For any $\sigma \in I$, $\varepsilon > \varepsilon_0 > 0$, such that $\sigma \pm \varepsilon_0 \in I$, we have

$$0 \leq k(\sigma \pm \varepsilon) = \int_0^1 h(u)u^{\sigma\pm\varepsilon-1}\,du + \int_1^\infty h(u)u^{\sigma\pm\varepsilon-1}\,du$$
$$\leq \int_0^1 h(u)u^{\sigma-\varepsilon-1}\,du + \int_1^\infty h(u)u^{\sigma+\varepsilon-1}\,du$$

$$\leq \int_0^1 h(u)u^{\sigma-\varepsilon_0-1}\,du + \int_1^\infty h(u)u^{\sigma+\varepsilon_0-1}\,du$$
$$\leq F(\sigma) := k(\sigma-\varepsilon_0) + k(\sigma+\varepsilon_0) < \infty,$$

where $F(\sigma)$ $(\sigma \in I)$ is called the dominating function. By Lebesgue dominated convergence theorem (cf. [29]), we have

$$k(\sigma \pm \varepsilon) \to k(\sigma) \quad (\varepsilon \to 0^+),$$

namely, $k(\sigma)$ is continuous with respect to $\sigma \in I$.

The lemma is proved. □

3 Main results

Theorem 1 *Suppose that* $p < 1$ $(p \neq 0)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sigma \in \mathbf{R}$, $v(x) > 0$, $v'(x) > 0$, $v(0^+) = 0$, $h(u) \geq 0$, $h(u)u^{\sigma-1}$ *is a decreasing function of* $u > 0$, $k(\sigma) = \int_0^\infty h(u)u^{\sigma-1}\,du \in \mathbf{R}_+$, *and* $f(x), a_n \geq 0$ *are such that*

$$0 < \int_0^\infty \frac{(v(x))^{p(1-\sigma)-1}}{(v'(x))^{p-1}} f^p(x)\,dx < \infty \quad \text{and} \quad 0 < \sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q < \infty.$$

(*i*) *If* $p < 0$ $(0 < q < 1)$, *then we have the following inequality equivalent to* (9) *and* (10):

$$I = \sum_{n=2}^\infty a_n \int_0^\infty h\big(v(x)\ln n\big) f(x)\,dx$$
$$> k(\sigma)\left[\int_0^\infty \frac{(v(x))^{p(1-\sigma)-1} f^p(x)}{(v'(x))^{p-1}}\,dx\right]^{\frac{1}{p}} \left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}}; \tag{13}$$

(*ii*) *If* $0 < p < 1$ $(q < 0)$, *then we have the following inequality equivalent to* (11) *and* (12):

$$I = \sum_{n=2}^\infty a_n \int_0^\infty h\big(v(x)\ln n\big) f(x)\,dx$$
$$> k(\sigma)\left[\int_0^\infty \big(1-\theta(x)\big)\frac{(v(x))^{p(1-\sigma)-1}}{(v'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{p}}$$
$$\times \left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}}, \tag{14}$$

where $\theta(x) := \frac{1}{k(\sigma)} \int_0^{v(x)\ln 2} h(u)u^{\sigma-1}\,du \in (0,1)$ $(x > 0)$.

Proof (i) "(9) $\Longrightarrow$ (13)" Since $p < 0$ $(0 < q < 1)$, by the reverse Hölder's inequality, we have

$$I = \sum_{n=2}^\infty \left[\frac{1}{n^{\frac{1}{p}} \ln^{\frac{1}{p}-\sigma} n} \int_0^\infty h\big(v(x)\ln n\big) f(x)\,dx\right]\left[n^{\frac{1}{p}} \ln^{\frac{1}{p}-\sigma} n a_n\right]$$
$$\geq \widetilde{J}_1 \left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}}. \tag{15}$$

In view of (9), we obtain (13).

"(13) $\Longrightarrow$ (9)" If (13) is valid, then we define the following function:

$$a_n := \frac{\ln^{p\sigma-1} n}{n}\left(\int_0^\infty h\big(\nu(x)\ln n\big)f(x)\,dx\right)^{p-1} \quad \big(n\in \mathbf{N}\backslash\{1\}\big).$$

If $\widetilde{J}_1 = \infty$, then (9) is naturally valid; if $\widetilde{J}_1 = 0$, then it is impossible to make (9) valid, namely, $\widetilde{J}_1 > 0$. Supposing that $0 < \widetilde{J}_1 < \infty$, by (13), we have

$$\begin{aligned}
\infty > \sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q = \widetilde{J}_1^p = I \\
> k(\sigma)\left[\int_0^\infty \frac{(\nu(x))^{p(1-\sigma)-1} f^p(x)}{(\nu'(x))^{p-1}}\,dx\right]^{\frac{1}{p}}\left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}} > 0, \\
\widetilde{J}_1 = \left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{p}} > k(\sigma)\left[\int_0^\infty \frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{p}},
\end{aligned}$$

namely, (9) follows. Hence, (13) and (9) are equivalent.

"(10) $\Longrightarrow$ (13)" By the reverse Hölder's inequality, we also have

$$\begin{aligned}
I &= \int_0^\infty \left[\frac{\nu(x)^{\frac{1}{q}-\sigma}}{(\nu'(x))^{\frac{1}{q}}} f(x)\,dx\right]\left[\frac{(\nu'(x))^{\frac{1}{q}}}{\nu(x)^{\frac{1}{q}-\sigma}}\sum_{n=2}^\infty h\big(\nu(x)\ln n\big)a_n\right] \\
&\geq \left[\int_0^\infty \frac{\nu(x)^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{p}} \widetilde{J}_2.
\end{aligned} \tag{16}$$

In view of (10), we obtain (13).

"(13) $\Longrightarrow$ (10)" If (13) is valid, then we consider the following function:

$$f(x) := \big(\nu(x)\big)^{q\sigma-1}\nu'(x)\left(\sum_{n=2}^\infty h\big(\nu(x)\ln n\big)a_n\right)^{q-1} \quad (x\in \mathbf{R}_+).$$

If $\widetilde{J}_2 = \infty$, (10) is naturally valid; if $\widetilde{J}_2 = 0$, it is impossible to make (10) valid, namely, $\widetilde{J}_2 > 0$. Suppose that $0 < \widetilde{J}_2 < \infty$. Then, by (13), we have

$$\begin{aligned}
\infty > \int_0^\infty \frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx = \widetilde{J}_2^q = I \\
> k(\sigma)\left[\int_0^\infty \frac{(\nu(x))^{p(1-\sigma)-1} f^p(x)}{(\nu'(x))^{p-1}}\,dx\right]^{\frac{1}{p}}\left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}} > 0, \\
\widetilde{J}_2 = \left[\int_0^\infty \frac{(\nu(x))^{p(1-\sigma)-1} f^p(x)}{(\nu'(x))^{p-1}}\,dx\right]^{\frac{1}{q}} > k(\sigma)\left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}},
\end{aligned}$$

namely, (10) follows. Then, (13) and (10) are equivalent.

Hence, (9), (10), and (13) are equivalent.

(ii) "(11) $\Longrightarrow$ (14)" If $0 < p < 1$ ($q < 0$), then by the reverse Hölder's inequality, we have

$$I = \sum_{n=2}^{\infty}\left[\frac{\ln^{\sigma-\frac{1}{p}} n}{n^{\frac{1}{p}}}\int_0^{\infty} h\bigl(\nu(x)\ln n\bigr)f(x)\,dx\right]\left[n^{\frac{1}{p}}\ln^{\frac{1}{p}-\sigma} n a_n\right]$$
$$\geq \widehat{J}_1\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q\right]^{\frac{1}{q}}. \tag{17}$$

In view of (11), we obtain (14).

"(14) $\Longrightarrow$ (11)". If (14) is valid, then we consider the following function:

$$a_n := \frac{\ln^{p\sigma-1} n}{n}\left(\int_0^{\infty} h\bigl(\nu(x)\ln n\bigr)f(x)\,dx\right)^{p-1} \quad \bigl(n \in \mathbf{N}\backslash\{1\}\bigr).$$

If $\widehat{J}_1 = \infty$, (11) is naturally valid; if $\widehat{J}_1 = 0$, then it is impossible to make (11) valid, namely, $\widehat{J}_1 > 0$. Supposing that $0 < \widehat{J}_1 < \infty$, by (14), we have

$$\infty > \sum_{n=2}^{\infty}\frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q = \widehat{J}_1^p = I$$
$$> k(\sigma)\left[\int_0^{\infty}\bigl(1-\theta(x)\bigr)\frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q\right]^{\frac{1}{q}}$$
$$> 0,$$
$$\widehat{J}_1 = \left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q\right]^{\frac{1}{p}}$$
$$> k(\sigma)\left[\int_0^{\infty}\bigl(1-\theta(x)\bigr)\frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}}f^p(x)\,dx\right]^{\frac{1}{p}},$$

namely, (11) follows. Hence, (14) and (11) are equivalent.

"(12) $\Longrightarrow$ (14)". By the reverse Hölder's inequality, we also have

$$I = \int_0^{\infty}\left[\bigl(1-\theta(x)\bigr)^{\frac{1}{p}}\frac{(\nu(x))^{\frac{1}{q}-\sigma}}{(\nu'(x))^{\frac{1}{q}}}f(x)\,dx\right]$$
$$\times\left[\frac{(\nu(x))^{\sigma-\frac{1}{q}}(\nu'(x))^{\frac{1}{q}}}{(1-\theta(x))^{\frac{1}{p}}}\sum_{n=2}^{\infty}h\bigl(\nu(x)\ln n\bigr)a_n\right]$$
$$\geq\left[\int_0^{\infty}\bigl(1-\theta(x)\bigr)\frac{\nu(x)^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}}f^p(x)\,dx\right]^{\frac{1}{p}}\widehat{J}_2. \tag{18}$$

In view of (12), we obtain (14).

"(14) $\Longrightarrow$ (12)". If (14) is valid, we set the following function:

$$f(x) := \frac{(\nu(x))^{q\sigma-1}\nu'(x)}{(1-\theta(x))^{q-1}}\left(\sum_{n=2}^{\infty}h\bigl(\nu(x)\ln n\bigr)a_n\right)^{q-1} \quad (x \in \mathbf{R}_+).$$

If $\widehat{J}_2 = \infty$, (12) is naturally valid; if $\widehat{J}_2 = 0$, it is impossible to make (12) valid, namely $\widehat{J}_2 > 0$. Supposing that $0 < \widehat{J}_2 < \infty$, by (14), we have

$$\begin{aligned}\infty &> \int_0^\infty \bigl(1-\theta(x)\bigr)\frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx = \widehat{J}_2^q = I \\ &> k(\sigma)\left[\int_0^\infty \bigl(1-\theta(x)\bigr)\frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{p}}\left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}} > 0, \\ \widehat{J}_2 &= \left[\int_0^\infty \bigl(1-\theta(x)\bigr)\frac{(\nu(x))^{p(1-\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{q}} \\ &> k(\sigma)\left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}},\end{aligned}$$

namely, (12) follows. Then, (12) and (14) are equivalent.

Hence, (11), (12), and (14) are equivalent.

The theorem is proved. □

Definition 2 If there exists a constant $M > 0$ such that

$$\left|\frac{\theta(x)}{\nu^a(x)}\right| \le M, \quad x \in (0,1),$$

then we write $\theta(x) = O(\nu^a(x))$, $x \in (0,1)$.

Theorem 2 *Under the assumptions of Theorem 1, consider $k(\sigma) \in \mathbf{R}_+$ ($\sigma \in I$) (I is an open interval).*

(i) If $p < 0$ ($0 < q < 1$), then the constant factor $k(\sigma)$ in (13), (9), and (10) is the best possible;

(ii) If $0 < p < 1$ ($q < 0$), there exists a constant $a > 0$ such that $\theta(x) = O(\nu^a(x))$ ($x \in (0,1)$), namely, the constant factor $k(\sigma)$ in (14), (11), and (12) is the best possible.

Proof For any $\varepsilon > 0$ such that $\sigma - \frac{\varepsilon}{q} \in I$, we set the following functions:

$$\widetilde{a}_n := \frac{\ln^{\sigma-\frac{\varepsilon}{q}-1} n}{n}, \quad n \in \mathbf{N}\backslash\{1\},$$

$$\widetilde{f}(x) = \begin{cases} (\nu(x))^{\sigma+\frac{\varepsilon}{p}-1}\nu'(x), & 0 < x \le 1, \\ 0, & x > 1. \end{cases}$$

(i) If $p < 0$ ($0 < q < 1$), then in view of the decreasingness property of the series, we have

$$\begin{aligned} L_1 &:= \left[\int_0^\infty \frac{(\nu(x))^{p(1-\sigma)-1}\widetilde{f}^p(x)}{(\nu'(x))^{p-1}}\,dx\right]^{\frac{1}{p}}\left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}\widetilde{a}_n^q\right]^{\frac{1}{q}} \\ &= \left[\int_0^1 \bigl(\nu(x)\bigr)^{\varepsilon-1}\nu'(x)\,dx\right]^{\frac{1}{p}}\left(\sum_{n=2}^\infty \frac{1}{n\ln^{\varepsilon+1} n}\right)^{\frac{1}{q}} \\ &\ge \left[\frac{(\nu(1))^\varepsilon}{\varepsilon}\right]^{\frac{1}{p}}\left(\int_2^\infty \frac{1}{y\ln^{\varepsilon+1} y}\,dy\right)^{\frac{1}{q}} = \frac{(\nu(1))^{\frac{\varepsilon}{p}}}{\varepsilon\ln^{\frac{\varepsilon}{q}} 2}, \end{aligned}$$

$$I_1 := \int_0^\infty \sum_{n=2}^\infty \widetilde{a}_n h\big(\nu(x)\ln n\big)\widetilde{f}(x)\,dx$$
$$= \int_0^1 \big(\nu(x)\big)^{\varepsilon-1}\nu'(x)\varpi\left(\sigma - \frac{\varepsilon}{q}, x\right)dx < \frac{1}{\varepsilon}\big(\nu(1)\big)^{\varepsilon} k\left(\sigma - \frac{\varepsilon}{q}\right).$$

If there exists a positive constant $k \geq k(\sigma)$ such that (13) is valid when we replace $k(\sigma)$ by k, then, in particular, we have $\varepsilon I_1 > \varepsilon k L_1$, namely

$$\big(\nu(1)\big)^{\varepsilon} k\left(\sigma - \frac{\varepsilon}{q}\right) > \frac{k \cdot (\nu(1))^{\frac{\varepsilon}{p}}}{\ln^{\frac{\varepsilon}{q}} 2}.$$

In view of Lemma 3, $k(\sigma)$ is continuous. When $\varepsilon \to 0^+$ in the above inequality, we find $k(\sigma) \geq k$. Hence, $k = k(\sigma)$ is the best possible constant factor in (13).

The constant factor $k(\sigma)$ in (9) (resp. (10)) is the best possible. Otherwise, by (15) (resp. (16)), we would reach a contradiction that the constant factor in (13) is not the best possible.

(ii) If $0 < p < 1$ $(q < 0)$, then in view of the assumption and the decreasingness property of the series, we obtain

$$L_2 := \left[\int_0^\infty \big(1 - \theta(x)\big)\frac{(\nu(x))^{p(1-\sigma)-1}\widetilde{f}^p(x)}{(\nu'(x))^{p-1}}\,dx\right]^{\frac{1}{p}}$$
$$\times \left[\sum_{n=2}^\infty n^{q-1}\ln^{q(1-\sigma)-1} n\widetilde{a}_n^q\right]^{\frac{1}{q}}$$
$$= \left[\int_0^1 \big(\nu(x)\big)^{\varepsilon-1}\nu'(x)\,dx - \int_0^1 O\big(\nu^{a+\varepsilon-1}(x)\big)\,dx\right]^{\frac{1}{p}}$$
$$\times \left(\frac{1}{2\ln^{\varepsilon+1} 2} + \sum_{n=3}^\infty \frac{1}{n\ln^{\varepsilon+1} n}\right)^{\frac{1}{q}}$$
$$\geq \left[\frac{1}{\varepsilon}\Big(\big(\nu(1)\big)^{\varepsilon} - \varepsilon O(1)\Big)\right]^{\frac{1}{p}}\left(\frac{1}{2\ln^{\varepsilon+1} 2} + \int_2^\infty \frac{dy}{y\ln^{\varepsilon+1} y}\right)^{\frac{1}{q}}$$
$$= \frac{1}{\varepsilon}\Big[\big(\nu(1)\big)^{\varepsilon} - \varepsilon O(1)\Big]^{\frac{1}{p}}\left(\frac{\varepsilon}{2\ln^{\varepsilon+1} 2} + \frac{1}{\ln^{\varepsilon} 2}\right)^{\frac{1}{q}},$$
$$I_1 = \int_0^1 \big(\nu(x)\big)^{\varepsilon-1}\nu'(x)\varpi\left(\sigma - \frac{\varepsilon}{q}, x\right)dx < \frac{1}{\varepsilon}\big(\nu(1)\big)^{\varepsilon} k\left(\sigma - \frac{\varepsilon}{q}\right).$$

If there exists a positive constant $k \geq k(\sigma)$ such that (14) is valid when we replace $k(\sigma)$ by k, then, in particular, we have $\varepsilon I_1 > \varepsilon k L_2$ and

$$\big(\nu(1)\big)^{\varepsilon} k\left(\sigma - \frac{\varepsilon}{q}\right) > k\Big[\Big(\big(\nu(1)\big)^{\varepsilon} - \varepsilon O\big(\nu(1)\big)\Big]^{\frac{1}{p}}\left(\frac{\varepsilon}{2\ln^{\varepsilon+1} 2} + \frac{1}{\ln^{\varepsilon} 2}\right)^{\frac{1}{q}}.$$

As $\varepsilon \to 0^+$, in view of the continuity of $k(\sigma)$, we find $k(\sigma) \geq k$. Hence, $k = k(\sigma)$ is the best possible constant factor of (14).

The constant factor $k(\sigma)$ in (11) (resp. (12)) is the best possible. Otherwise, by (17) (resp. (18)), we would reach a contradiction that the constant factor in (17) is not the best possible.

The theorem is proved. □

4 Some corollaries and particular cases

Replacing $\nu(x)$ by $\nu^{-1}(x)$ in Theorems 1 and 2, by Remark 1, setting

$$\theta_0(x) := \frac{1}{k(\sigma)} \int_0^{\nu^{-1}(x)\ln 2} h(u)u^{\sigma-1}\,du \in (0,1) \quad (x > 0),$$

we have the following corollary:

Corollary 1 *Under the assumptions of Theorem 1, consider $k(\sigma) \in \mathbf{R}_+$ ($\sigma \in I$).*

(i) If $p < 0$ ($0 < q < 1$), then we have the following equivalent inequalities with the best possible constant factor $k(\sigma)$:

$$\sum_{n=2}^{\infty} a_n \int_0^{\infty} h\big(\nu^{-1}(x)\ln n\big) f(x)\,dx > k(\sigma)\left[\int_0^{\infty} \frac{(\nu(x))^{p(1+\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}},$$

$$\left[\sum_{n=2}^{\infty} \frac{\ln^{p\sigma-1} n}{n} \left(\int_0^{\infty} h\big(\nu^{-1}(x)\ln n\big) f(x)\,dx\right)^p\right]^{\frac{1}{p}} > k(\sigma)\left[\int_0^{\infty} \frac{(\nu(x))^{p(1+\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{p}},$$

$$\left[\int_0^{\infty} \frac{\nu'(x)}{(\nu(x))^{q\sigma+1}} \left(\sum_{n=2}^{\infty} h\big(\nu^{-1}(x)\ln n\big) a_n\right)^q dx\right]^{\frac{1}{q}} > k(\sigma)\left[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}}.$$

(ii) If $0 < p < 1$ ($q < 0$), and there exists a constant $a > 0$ such that $\theta_0(x) = O(\frac{1}{\nu^a(x)})$ ($0 < x < 1$), then we have the following equivalent inequalities with the best possible constant factor $k(\sigma)$:

$$\sum_{n=2}^{\infty} a_n \int_0^{\infty} h\big(\nu^{-1}(x)\ln n\big) f(x)\,dx > k(\sigma)\left[\int_0^{\infty} \big(1-\theta_0(x)\big) \frac{(\nu(x))^{p(1+\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{p}} \times \left[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\right]^{\frac{1}{q}},$$

$$\left[\sum_{n=2}^{\infty} \frac{\ln^{p\sigma-1} n}{n} \left(\int_0^{\infty} h\big(\nu^{-1}(x)\ln n\big) f(x)\,dx\right)^p\right]^{\frac{1}{p}} > k(\sigma)\left[\int_0^{\infty} \big(1-\theta_0(x)\big) \frac{(\nu(x))^{p(1+\sigma)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\right]^{\frac{1}{p}},$$

$$\left[\int_0^\infty \frac{(1-\theta_0(x))^{1-q}\nu'(x)}{(\nu(x))^{q\sigma+1}}\left(\sum_{n=2}^\infty h\big(\nu^{-1}(x)\ln n\big)a_n\right)^q dx\right]^{\frac{1}{q}}$$
$$> k(\sigma)\left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q\right]^{\frac{1}{q}}.$$

Definition 3 If $\lambda \in \mathbf{R}$, the nonnegative measurable function $k_\lambda(x,y)$ ($(x,y) \in \mathbf{R}_+^2 = (0,\infty)\times(0,\infty)$) satisfies $k_\lambda(ux,uy) = u^{-\lambda}k_\lambda(x,y)$, for any $u,x,y \in \mathbf{R}_+$, then we call $k_\lambda(x,y)$ a homogeneous function of degree $-\lambda$.

Setting $h(u) = k_\lambda(1,u)$, $k_\lambda(\sigma) := \int_0^\infty k_\lambda(1,u)u^{\sigma-1}\,du$, $\mu = \lambda - \sigma$, defining

$$\theta_\lambda(x) := \frac{1}{k_\lambda(\sigma)}\int_0^{\nu^{-1}(x)\ln 2} k_\lambda(1,u)u^{\sigma-1}\,du \in (0,1) \quad (x>0),$$

and replacing $f(x)$ by $\nu^{-\lambda}(x)f(x)$ in Corollary 1, we have

Corollary 2 *Suppose that* $p<1$ ($p \neq 0$), $\frac{1}{p}+\frac{1}{q}=1$, $\mu,\sigma,\lambda \in \mathbf{R}$, $\mu+\sigma=\lambda$, $k_\lambda(1,u)>0$, $k_\lambda(1,u)u^{\sigma-1}$ *is decreasing for* $u>0$, $k_\lambda(\sigma) \in \mathbf{R}_+ (\sigma \in I)$, $\nu(x)>0$, $\nu'(x)>0$ $(x>0)$, *with* $\nu(0^+)=0$, $\nu(\infty)=\infty$, *and* $f(x), a_n \geq 0$, *are such that*

$$0<\int_0^\infty \frac{(\nu(x))^{p(1-\mu)-1}}{(\nu'(x))^{p-1}}f^p(x)\,dx<\infty \quad \textit{and} \quad 0<\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q<\infty.$$

(*i*) *If* $p<0$ $(0<q<1)$, *then we have the following equivalent reverse inequalities with the best possible constant factor* $k_\lambda(\sigma)$:

$$\sum_{n=2}^\infty a_n\int_0^\infty k_\lambda\big(\nu(x),\ln n\big)f(x)\,dx$$
$$> k_\lambda(\sigma)\left[\int_0^\infty \frac{(\nu(x))^{p(1-\mu)-1}}{(\nu'(x))^{p-1}}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q\right]^{\frac{1}{q}},$$

$$\left[\sum_{n=2}^\infty \frac{\ln^{p\sigma-1} n}{n}\left(\int_0^\infty k_\lambda\big(\nu(x),\ln n\big)f(x)\,dx\right)^p\right]^{\frac{1}{p}}$$
$$> k_\lambda(\sigma)\left[\int_0^\infty \frac{(\nu(x))^{p(1-\mu)-1}}{(\nu'(x))^{p-1}}f^p(x)\,dx\right]^{\frac{1}{p}},$$

$$\left[\int_0^\infty \frac{\nu'(x)}{(\nu(x))^{1-q\mu}}\left(\sum_{n=2}^\infty k_\lambda\big(\nu(x),\ln n\big)\right)a_n)^q\,dx\right]^{\frac{1}{q}}$$
$$> k_\lambda(\sigma)\left[\sum_{n=2}^\infty \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}}a_n^q\right]^{\frac{1}{q}}.$$

(*ii*) *If* $0<p<1$ $(q<0)$, *and there exists a constant* $a>0$ *such that* $\theta_\lambda(x) = O(\frac{1}{\nu^a(x)})$ $(0<x<1)$, *then we have the following equivalent reverse inequalities with the best possible constant* *factor* $k_\lambda(\sigma)$:

$$\begin{aligned}
&\sum_{n=2}^{\infty} a_n \int_0^\infty k_\lambda\big(\nu(x), \ln n\big) f(x)\,dx \\
&\quad > k_\lambda(\sigma)\bigg[\int_0^\infty \big(1-\theta_\lambda(x)\big)\frac{(\nu(x))^{p(1-\mu)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\bigg]^{\frac{1}{p}} \\
&\qquad \times \Bigg[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\Bigg]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
&\Bigg[\sum_{n=2}^{\infty} \frac{\ln^{p\sigma-1} n}{n}\bigg(\int_0^\infty k_\lambda\big(\nu(x), \ln n\big) f(x)\,dx\bigg)^p\Bigg]^{\frac{1}{p}} \\
&\quad > k_\lambda(\sigma)\bigg[\int_0^\infty \big(1-\theta_\lambda(x)\big)\frac{(\nu(x))^{p(1-\mu)-1}}{(\nu'(x))^{p-1}} f^p(x)\,dx\bigg]^{\frac{1}{p}},
\end{aligned}$$

$$\begin{aligned}
&\Bigg[\int_0^\infty \frac{(\nu(x))^{q\mu-1}\nu'(x)}{(1-\theta_\lambda(x))^{q-1}}\Bigg(\sum_{n=2}^{\infty} k_\lambda\big(\nu(x), \ln n\big)\Bigg)a_n)^q\,dx\Bigg]^{\frac{1}{q}} \\
&\quad > k_\lambda(\sigma)\Bigg[\sum_{n=2}^{\infty} \frac{\ln^{q(1-\sigma)-1} n}{n^{1-q}} a_n^q\Bigg]^{\frac{1}{q}}.
\end{aligned}$$

Example 1 (i) We set $h(u) = k_\lambda(1,u) = \frac{1}{(1+u)^\lambda}$ $(\lambda > 0)$, $\mu > 0$, $0 < \sigma \leq 1$, $\sigma + \mu = \lambda$. Then we find that $h(u) = \frac{1}{(1+u)^\lambda}$ is decreasing as a function of $u > 0$,

$$\begin{aligned}
&h(\big(\nu(x)\ln n\big) = \frac{1}{(1+\nu(x)\ln n)^\lambda}, \qquad k_\lambda\big(\nu(x), \ln n\big) = \frac{1}{(\nu(x)+\ln n)^\lambda}, \\
&k(\sigma) = k_\lambda(\sigma) = \int_0^\infty \frac{u^{\sigma-1}}{(1+u)^\lambda}\,du = B(\sigma, \lambda-\sigma) = B(\mu,\sigma) \in \mathbf{R}_+, \\
&0 < \theta(x) = \frac{1}{B(\mu,\sigma)}\int_0^{\nu(x)\ln 2} \frac{1}{(1+u)^\lambda} u^{\sigma-1}\,du \\
&\quad \leq \frac{1}{B(\mu,\sigma)}\int_0^{\nu(x)\ln 2} u^{\sigma-1}\,du = \frac{1}{\sigma B(\mu,\sigma)}\big(\nu(x)\ln 2\big)^\sigma, \\
&0 < \theta_\lambda(x) = \frac{1}{B(\mu,\sigma)}\int_0^{\nu^{-1}(x)\ln 2} \frac{1}{(1+u)^\lambda} u^{\sigma-1}\,du \\
&\quad \leq \frac{1}{B(\mu,\sigma)}\int_0^{\nu^{-1}(x)\ln 2} u^{\sigma-1}\,du = \frac{1}{\sigma B(\mu,\sigma)}\bigg(\frac{\ln 2}{\nu(x)}\bigg)^\sigma,
\end{aligned}$$

$a = \sigma > 0$, and $I = (0,\lambda)$. In view of Theorems 1 and 2, as well as Corollary 2, we have two classes of equivalent reverse inequalities with the particular kernels and the best possible constant factor $B(\mu,\sigma)$.

In particular, for $\nu(x) = \ln(1+x)$, the related inequalities are called half-discrete Mulholland-type inequalities.

(ii) We set $h(u) = k_\lambda(1,u) = \frac{\ln u}{u^\lambda - 1}$ $(\lambda > 0)$, $\mu > 0$, $0 < \sigma \le 1$, $\sigma + \mu = \lambda$. Then we find that $h(u) = \frac{\ln u}{u^\lambda - 1}$ is decreasing with respect to $u > 0$ (cf. [7]),

$$h\big(\nu(x)\ln n\big) = \frac{\ln(\nu(x)\ln n)}{(\nu(x)\ln n)^\lambda - 1}, k_\lambda\big(\nu(x),\ln n\big) = \frac{\ln(\nu(x)/\ln n)}{\nu^\lambda(x) - \ln^\lambda n},$$

$$\begin{aligned} k(\sigma) = k_\lambda(\sigma) &= \int_0^\infty \frac{u^{\sigma-1}\ln u}{u^\lambda - 1}\,du \\ &= \frac{1}{\lambda^2}\int_0^\infty \frac{v^{(\sigma/\lambda)-1}\ln v}{v-1}\,dv = \left[\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}\right]^2 \in \mathbf{R}_+. \end{aligned}$$

We obtain

$$\theta(x) = \left[\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}\right]^{-2}\int_0^{\nu(x)\ln 2}\frac{u^{\sigma/2}\ln u}{u^\lambda - 1}u^{\frac{\sigma}{2}-1}\,du.$$

Since $\frac{u^{\sigma/2}\ln u}{u^\lambda - 1} \to 0 (u \to 0^+)$, there exists a constant $M > 0$ such that $0 < \frac{u^{\sigma/2}\ln u}{u^\lambda - 1} \le M$ $(u \in (0, \nu(1)\ln 2])$. For $x \in (0,1)$, we have

$$\begin{aligned} 0 < \theta(x) &\le \left[\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}\right]^{-2} M\int_0^{\nu(x)\ln 2} u^{\frac{\sigma}{2}-1}\,du \\ &= \left[\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}\right]^{-2}\frac{2M}{\sigma}\big(\nu(x)\ln 2\big)^{\frac{\sigma}{2}}. \end{aligned}$$

In the same way, we can find that

$$0 < \theta_\lambda(x) \le \left[\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}\right]^{-2}\frac{2M}{\sigma}\left(\frac{\ln 2}{\nu(x)}\right)^{\frac{\sigma}{2}}.$$

So we set $a = \frac{\sigma}{2} > 0$ and $I = (0,\lambda)$. In view of Theorems 1 and 2, as well as Corollary 2, we have two classes of equivalent reverse inequalities with the particular kernels and the best possible constant factor $[\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}]^2$.

(iii) We set $h(u) = k_\lambda(1,u) = \frac{(\min\{1,u\})^\eta}{(\max\{1,u\})^{\lambda+\eta}}$ $(\mu,\sigma > -\eta, \sigma \le 1-\eta, \mu + \sigma = \lambda)$. Then we find that

$$\begin{aligned} h\big(\nu(x)\ln n\big) &= \frac{(\min\{1,\nu(x)\ln n\})^\eta}{(\max\{1,\nu(x)\ln n\})^{\lambda+\eta}}, \\ k_\lambda\big(\nu(x),\ln n\big) &= \frac{(\min\{\nu(x),\ln n\})^\eta}{(\max\{\nu(x),\ln n\})^{\lambda+\eta}}. \end{aligned}$$

In view of the assumptions, we still find that

$$h(u)u^{\sigma-1} = \frac{(\min\{1,u\})^\eta u^{\sigma-1}}{(\max\{1,u\})^{\lambda+\eta}} = \begin{cases} u^{\eta+\sigma-1}, & 0 < u < 1, \\ \frac{1}{u^{\mu+\eta+1}}, & u \ge 1, \end{cases}$$

is decreasing with respect to $u > 0$, and

$$k(\sigma) = k_\lambda(\sigma) = \int_0^\infty \frac{(\min\{1,u\})^\eta}{(\max\{1,u\})^{\lambda+\eta}}u^{\sigma-1}\,du$$

$$= \int_0^1 u^{\eta+\sigma-1}\,du + \int_1^{\infty} \frac{1}{u^{\lambda+\eta}} u^{\sigma-1}\,du$$
$$= \frac{1}{\eta+\sigma} + \frac{1}{\eta+\mu} = \frac{2\eta+\lambda}{(\eta+\sigma)(\eta+\mu)} \in \mathbf{R}_+.$$

We consider

$$\theta(x) = \frac{(\eta+\sigma)(\eta+\mu)}{2\eta+\lambda} \int_0^{\nu(x)\ln 2} \frac{(\min\{1,u\})^{\eta}}{(\max\{1,u\})^{\lambda+\eta}} u^{\sigma-1}\,du.$$

For $x \in (0,1)$, there exist constants $b, M > 0$ such that $b\nu(1)\ln 2 \leq 1$ and

$$0 < \theta(x) \leq \frac{(\eta+\sigma)(\eta+\mu)}{2\eta+\lambda} M \int_0^{b\nu(x)\ln 2} \frac{(\min\{1,u\})^{\eta} u^{\sigma-1}}{(\max\{1,u\})^{\lambda+\eta}}\,du$$
$$= \frac{(\eta+\sigma)(\eta+\mu)}{2\eta+\lambda} M \int_0^{b\nu(x)\ln 2} u^{\eta+\sigma-1}\,du = \frac{\eta+\mu}{2\eta+\lambda} M \bigl(b\nu(x)\ln 2\bigr)^{\eta+\sigma}.$$

In the same way, we can find that

$$0 < \theta_\lambda(x) \leq \frac{\eta+\mu}{2\eta+\lambda} M \left(\frac{b\ln 2}{\nu(x)}\right)^{\eta+\sigma}.$$

So we set $a = \eta+\sigma > 0$ and $I = (-\eta, \lambda+\eta)$. In view of Theorems 1 and 2, as well as Corollary 2, we have two classes of equivalent reverse inequalities with the particular kernels and the best possible constant factor $[\frac{\pi}{\lambda\sin(\pi\sigma/\lambda)}]^2$.

In particular, (a) for $\eta = 0$, we have $\mu, \sigma > 0$, $\sigma \leq 1$, $\mu+\sigma = \lambda$,

$$h(\bigl(\nu(x)\ln n\bigr) = \frac{1}{(\max\{1,\nu(x)\ln n\})^{\lambda}},$$
$$k_\lambda\bigl(\nu(x),\ln n\bigr) = \frac{1}{(\max\{\nu(x),\ln n\})^{\lambda}},$$

$k(\sigma) = k_\lambda(\sigma) = \frac{\lambda}{\sigma\mu} \in \mathbf{R}_+$, $a = \sigma > 0$, and $I = (0,\lambda)$.

(b) For $\eta = -\lambda$, we have $\mu, \sigma > \lambda$, $\sigma \leq 1$, $\mu+\sigma = \lambda$,

$$h(\bigl(\nu(x)\ln n\bigr) = \frac{1}{(\min\{1,\nu(x)\ln n\})^{\lambda}},$$
$$k_\lambda\bigl(\nu(x),\ln n\bigr) = \frac{1}{(\min\{\nu(x),\ln n\})^{\lambda}},$$

$k(\sigma) = k_\lambda(\sigma) = \frac{-\lambda}{\sigma\mu} \in \mathbf{R}_+$, $a = -\mu > 0$, and $I = (\lambda, 0)$ $(\lambda < 0)$.

(c) For $\lambda = 0$, $\eta > 0$, $-\eta < \sigma < \eta$, $\sigma \leq 1-\eta$, we have

$$h(u) = k_0(1,u) = \left(\frac{\min\{1,u\}}{\max\{1,u\}}\right)^{\eta},$$
$$k(\sigma) = k_0(\sigma) = \frac{2\eta}{\eta^2-\sigma^2} \in \mathbf{R}_+,$$
$$h\bigl(\nu(x)\ln n\bigr) = \left(\frac{\min\{1,\nu(x)\ln n\}}{\max\{1,\nu(x)\ln n\}}\right)^{\eta},$$

$$k_0\big(\nu(x), \ln n\big) = \left(\frac{\min\{\nu(x), \ln n\}}{\max\{\nu(x), \ln n\}}\right)^{\eta},$$

$a = \eta + \sigma > 0$, and $I = (-\eta, \eta)$.

5 Conclusions

In this paper, using the weight functions and the techniques of real analysis, a new reverse half-discrete Mulholland-type inequality with the nonhomogeneous kernel of the form $h(\nu(x)\ln n)$ is obtained in Theorem 1. The best possible constant factor and some equivalent reverses are considered in Theorem 2. In Corollary 2, we deduce some new equivalent reverse inequalities with the homogeneous kernel of the form $k_\lambda(\nu(x), \ln n)$. Some particular cases are provided in Example 1. These lemmas and theorems provide an extensive account of such inequalities.

In contrast to the extensive research conducted on the homogeneous kernel, the investigation of nonhomogeneous kernel is complex and applied to obtain a reverse half-discrete Hilbert-type inequality. The new reverse half-discrete Mulholland-type inequality with the nonhomogeneous kernel transforms the field of study into a multidimensional space. It becomes even more comprehensive in applications and consequences. The potential impact of our research is to inequalities involving higher-order derivative functions and multiple upper limit functions. It would be a remarkable achievement in the framework of a half-discrete Hilbert-type inequality with a nonhomogeneous kernel.

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Declarations

Competing interests
The authors declare no competing interests.

Author contributions
L.P. carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. R.A.R. and B.Y. participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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