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Generalized UH-stability of a nonlinear fractional coupling $(\wp_1, \wp_2)$-Laplacian system concerned with nonsingular Atangana–Baleanu fractional calculus

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Abstract

The classical $\wp$-Laplace equation is one of the special and significant second-order ODEs. The fractional-order $\wp$-Laplace ODE is an important generalization. In this paper, we mainly treat with a nonlinear coupling $(\wp_1, \wp_2)$-Laplacian systems involving the nonsingular Atangana–Baleanu (AB) fractional derivative. In accordance with the value range of parameters $\wp_1$ and $\wp_2$, we obtain sufficient criteria for the existence and uniqueness of solution in four cases. By using some inequality techniques we further establish the generalized UH-stability for this system. Finally, we test the validity and practicality of the main results by an example.

Mathematics Subject Classification: 34A08; 34D20; 37C25

Keywords: Coupling Laplacian system; AB-fractional calculus; Existence and uniqueness; Generalized UH-stability

1 Introduction

In this paper, we focus on the following nonlinear fractional coupling $(\wp_1, \wp_2)$-Laplacian systems involving a nonsingular Mittag-Leffler kernel:

$$\begin{cases} {}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\nu_1}[\Phi_{\wp_1}({}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(t))] = G_1(t, \mathcal{W}_1(t), \mathcal{W}_2(t)), \quad t \in (0,a], \\ {}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\nu_2}[\Phi_{\wp_2}({}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_2}\mathcal{W}_2(t))] = G_2(t, \mathcal{W}_1(t), \mathcal{W}_2(t)), \quad t \in (0,a], \\ \mathcal{W}_1(0) = u_1, \qquad \mathcal{W}_2(0) = u_2, \qquad {}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(0) = v_1, \qquad {}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_2}\mathcal{W}_2(0) = v_2, \end{cases} \tag{1.1}$$

where $u_1, u_2, v_1, v_2 \in \mathbb{R}$, $a > 0$, $0 < \mu_1, \mu_2, \nu_1, \nu_2 \le 1$, and $\wp_1, \wp_2 > 1$ are some constants, ${}^{\mathrm{AB}}\mathcal{D}_{0^+}^{*}$ is the $*$-order Atangana–Baleanu (AB) fractional derivative with nonsingular Mittag-Leffler kernel, $\Phi_{\wp_k}(z) = |z|^{\wp_k - 2}z$, $k = 1,2$, are $\wp_k$-Laplacian operators with inverses $\Phi_{\wp_k}^{-1} = \Phi_{q_k}$, provided that $\frac{1}{\wp_k} + \frac{1}{q_k} = 1$ and $G_k \in C([0,a] \times \mathbb{R}^2, \mathbb{R})$ are nonlinear.

In 2016, Atangana and Baleanu [1] raised a distinctive fractional calculus, later named Atangana–Baleanu (AB) fractional calculus, under common skeleton frame. The most prominent feature of AB-fractional calculus is the application of a special Mittag-Leffler function in the definition. The superiority of AB-fractional derivative to Riemann–Liouville (RL) and Riemann–Caputo (RC) fractional derivatives lies in nonsingularity. In fact, for all $0 < \gamma < 1$, $(t-\tau)^{-\gamma}$ and $\mathcal{E}_\gamma[-\frac{\gamma}{1-\gamma}(t-\tau)] = \sum_{n=0}^{\infty} \frac{[-\frac{\gamma}{1-\gamma}(t-\tau)]^n}{\Gamma(\gamma n+1)}$ are the kernels of RC- and AB-fractional derivatives of order γ, respectively. Decidedly, $(t-\tau)^{-\gamma}\to\infty$ (singular) and $\mathcal{E}_\gamma[-\frac{\gamma}{1-\gamma}(t-\tau)]\to 1$ (nonsingular) as $\tau\to t$. The nonsingularity of the AB-fractional derivative is very useful for solving some practical problems. In fact, Atangana and Baleanu [1] successfully solved a singular thermodynamic problem by applying the AB-fractional order model by proposing the AB-fractional derivative. Many scientists applied AB-fractional differential equation models to study practical problems such as controllability [2, 3], virus and bacterial transmission [4–7], neuroscience [8], nanofluid [9], ion flux [10] and thermo-diffusion [11]. Due to a wide application of AB-fractional differential equations, many scholars have attached great importance to the theory of AB-fractional differential system (see [12–27]). In addition, the $\wp$-Laplacian equation can describe turbulent flow phenomenon in fundamental fluid mechanics, and hence many papers have been published dealing with its theory and applications (see [28–35]).

In 1940s, Hyers and Ulam [36, 37] raised a new concept of stabilitym the Ulam and Hyers (UH) stability. Since then, the generalized UH-stability, Ulam–Hyers–Rassias stability, and generalized Ulam–Hyers–Rassias stability have also been proposed on the basis of UH-stability. Until now, the UH-type stability of various systems is still favored by scientists. As one of the important differential dynamic systems, the UH-type stability of fractional differential equations has also been focused on and achieved rich results ([14, 20, 21, 23, 27–30, 35, 38–43], among others). However, the UH-type stability of AB-fractional differential equations is rarely studied because the structure of the equations is more complex than that of a single differential equation. To the best my knowledge, there are no papers combining the AB-fractional derivative with coupling Laplacian system. Consequently, it is novel and interesting to probe the dynamic behavior of system (1.1). The importance of this paper is embodied in two aspects as follows: (i) Since nobody has studied the AB-fractional differential coupling Laplacian system yet, we first consider system (1.1) to fill this gap. (ii) We investigate the existence, uniqueness, and GUH-stability of system (1.1) and obtain some concise sufficient conditions.

The next framework of the paper is as follows. Section 2 reviews some necessary contents about AB-fractional calculus. In Sect. 3 the existence and uniqueness of a solution is obtained by the contraction mapping principle. In Sect. 4 the generalized UH-stability of (1.1) is further established. Section 5 gives an example illustrating the validity and availability of our main findings. A concise conclusion is made in Sect. 6.

2 Preliminaries

Definition 2.1 ([44]) For $0 < \gamma \le 1$, $a > 0$, and $\mathcal{W} : [0,a]\to\mathbb{R}$, the left-sided γth-order AB-fractional integral of $\mathcal{W}$ is defined by

$$ {}^{\mathrm{AB}}\mathcal{I}_0^\gamma \mathcal{W}(t) = \frac{1-\gamma}{\mathfrak{N}(\gamma)}\mathcal{W}(t) + \frac{\gamma}{\mathfrak{N}(\gamma)\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathcal{W}(s)\,ds, $$

where $\mathfrak{N}(\alpha)$ is a normalization constant with $\mathfrak{N}(0) = \mathfrak{N}(1) = 1$.

Definition 2.2 ([1]) For $0 < \gamma \le 1$, $a > 0$, and $\mathcal{W} \in C^1(0,a)$, the left-sided γth-order AB-fractional derivative of $\mathcal{W}$ is defined by

$$^{\mathrm{AB}}\mathcal{D}_{0^+}^{\gamma}\mathcal{W}(t) = \frac{\mathfrak{N}(\gamma)}{1-\gamma}\int_0^t \mathcal{E}\left[-\frac{\alpha}{1-\alpha}(t-s)\right]\mathcal{W}'(s)\,ds,$$

where $\mathcal{E}_\gamma(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\gamma n+1)}$ is the Mittag-Leffer special function with parameter γ.

Lemma 2.1 ([45]) *If $\mathcal{H} \in C[0,a]$, then the unique solution of the IVP*

$$\begin{cases} ^{\mathrm{AB}}\mathcal{D}_{0^+}^{\gamma}\mathcal{W}(t) = \mathcal{H}(t), & t \ge 0, 0 < \gamma \le 1, \\ \mathcal{W}(0) = \mathcal{W}_0, \end{cases}$$

is given by

$$\mathcal{W}(t) = \mathcal{W}_0 + \frac{1-\gamma}{\mathfrak{N}(\gamma)}\big[\mathcal{H}(t) - \mathcal{H}(0)\big] + \frac{\gamma}{\mathfrak{N}(\gamma)\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}\mathcal{H}(s)\,ds.$$

Lemma 2.2 ([30, 46]) *Let $p > 1$. The $\wp$-Laplacian operator $\Phi_{\wp}(z) = |z|^{\wp-2}z$ has the following properties*:

(i) *If $z \ge 0$, then $\Phi_{\wp}(z) = z^{\wp-1}$, and $\Phi_{\wp}(z)$ is increasing with respect to z;*
(ii) *$\Phi_{\wp}(zw) = \Phi_{\wp}(z)\Phi_{\wp}(w)$ for all $z, w \in \mathbb{R}$;*
(iii) *If $\frac{1}{\wp} + \frac{1}{q} = 1$, then $\Phi_{q}[\Phi_{\wp}(z)] = \Phi_{\wp}[\Phi_{q}(z)] = z$ for all $z \in \mathbb{R}$;*
(iv) *For all $z, w \ge 0$, $z \le w \Leftrightarrow \Phi_{q}(z) \le \Phi_{q}(w)$;*
(v) *$0 \le z \le \Phi_{q}^{-1}(w) \Leftrightarrow 0 \le \Phi_{q}(z) \le w$;*
(vi)

$$\big|\Phi_{q}(z) - \Phi_{q}(w)\big| \le \begin{cases} (q-1)\overline{M}^{q-2}|z-w|, & q \ge 2, 0 \le z, w \le \overline{M}, \\ (q-1)\underline{M}^{q-2}|z-w|, & 1 < q < 2, z, w \ge \underline{M} \ge 0. \end{cases}$$

Lemma 2.3 *Let $u_1, u_2, v_1, v_2 \in \mathbb{R}$, $a > 0$, $0 < \mu_1, \mu_2, \nu_1, \nu_2 \le 1$, and $\wp_1, \wp_2 > 1$ be some constants, and let $G_k \in C([0,a] \times \mathbb{R}^2, \mathbb{R})$, $k = 1, 2$. Then the nonlinear AB-fractional coupling Laplacian system* (1.1) *is equivalent to the integral system*

$$\begin{cases} \mathcal{W}_1(t) = u_1 + \dfrac{1-\mu_1}{\mathfrak{N}(\mu_1)}[\Phi_{q_1}(H_1(t, \mathcal{W}_1(t), \mathcal{W}_2(t))) - v_1] \\ \qquad + \dfrac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}\displaystyle\int_0^t (t-s)^{\mu_1-1}\Phi_{q_1}(H_1(s, \mathcal{W}_1(s), \mathcal{W}_2(s)))\,ds, \quad t \in [0,a], \\ \mathcal{W}_2(t) = u_2 + \dfrac{1-\mu_2}{\mathfrak{N}(\mu_2)}[\Phi_{q_2}(H_2(t, \mathcal{W}_1(t), \mathcal{W}_2(t))) - v_2] \\ \qquad + \dfrac{\mu_2}{\mathfrak{N}(\mu_2)\Gamma(\mu_2)}\displaystyle\int_0^t (t-s)^{\mu_1-1}\Phi_{q_2}(H_2(s, \mathcal{W}_1(s), \mathcal{W}_2(s)))\,ds, \quad t \in [0,a], \end{cases} \tag{2.1}$$

where $\frac{1}{\wp_k} + \frac{1}{q_k} = 1$ $(k = 1, 2)$, and

$$H_1\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big) = \Phi_{\wp_1}(v_1) + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\big[G_1\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big) - G_1(0, u_1, u_2)\big]$$
$$+ \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-\tau)^{\nu_1-1}G_1\big(\tau, \mathcal{W}_1(\tau), \mathcal{W}_2(\tau)\big)\,d\tau,$$

$$H_2\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) = \Phi_{p_2}(\upsilon_2) + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}\big[G_2\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) - F_2(0,u_1,u_2)\big] + \frac{\nu_2}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)}\int_0^t (t-\tau)^{\nu_2-1} G_2\big(\tau,\mathcal{W}_1(\tau),\mathcal{W}_2(\tau)\big)\,d\tau.$$

Proof Let $(\mathcal{W}_1(t),\mathcal{W}_2(t)) \in C([0,a],\mathbb{R}) \times C([0,a],\mathbb{R})$ be a solution of (1.1). Then from Lemma 2.1 and the first equation of (1.1) we have

$$\Phi_{p_1}\big({}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(t)\big) = \Phi_{p_1}\big({}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(0)\big) + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\big[G_1\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) - G_1\big(0,\mathcal{W}_1(0),\mathcal{W}_2(0)\big)\big] + \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-\tau)^{\nu_1-1} G_1\big(\tau,\mathcal{W}_1(\tau),\mathcal{W}_2(\tau)\big)\,d\tau. \tag{2.2}$$

In view of (2.2) and (iii) in Lemma 2.2, we have

$${}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(t) = \Phi_{q_1}\bigg(\Phi_{p_1}\big({}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(0)\big) + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\big[G_1\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) - G_1\big(0,\mathcal{W}_1(0),\mathcal{W}_2(0)\big)\big] + \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-\tau)^{\nu_1-1} G_1\big(\tau,\mathcal{W}_1(\tau),\mathcal{W}_2(\tau)\big)\,d\tau\bigg), \tag{2.3}$$

where $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $p_1 > 1$. Denote

$$H_1\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) = \Phi_{p_1}\big({}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(0)\big) + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\big[G_1\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big) - G_1\big(0,\mathcal{W}_1(0),\mathcal{W}_2(0)\big)\big] + \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-\tau)^{\nu_1-1} G_1\big(\tau,\mathcal{W}_1(\tau),\mathcal{W}_2(\tau)\big)\,d\tau. \tag{2.4}$$

By (2.3), (2.4), and Lemma 2.1 we obtain that

$$\mathcal{W}_1(t) = \mathcal{W}_1(0) + \frac{1-\mu_1}{\mathfrak{N}(\mu_1)}\big[\Phi_{q_1}\big(H_1\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big)\big) - \Phi_{q_1}\big(H_1\big(0,\mathcal{W}_1(0),\mathcal{W}_2(0)\big)\big)\big] + \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}\int_0^t (t-s)^{\mu_1-1}\Phi_{q_1}\big(H_1\big(s,\mathcal{W}_1(s),\mathcal{W}_2(s)\big)\big)\,ds. \tag{2.5}$$

By the second equation of (1.1), (2.2)–(2.5) are similar to

$$\mathcal{W}_2(t) = \mathcal{W}_2(0) + \frac{1-\mu_2}{\mathfrak{N}(\mu_2)}\big[\Phi_{q_2}\big(H_2\big(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\big)\big) - \Phi_{q_2}\big(H_2\big(0,\mathcal{W}_1(0),\mathcal{W}_2(0)\big)\big)\big] + \frac{\mu_2}{\mathfrak{N}(\mu_2)\Gamma(\mu_2)}\int_0^t (t-s)^{\mu_2-1}\Phi_{q_2}\big(H_2\big(s,\mathcal{W}_1(s),\mathcal{W}_2(s)\big)\big)\,ds, \tag{2.6}$$

where $\frac{1}{p_2} + \frac{1}{q_2} = 1$, $p_2 > 1$, and

$$H_2\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big) = \Phi_{p_2}\big({}^{AB}\mathcal{D}_{0^+}^{\mu_2}\mathcal{W}_2(0)\big) + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}\big[G_2\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big) - G_2\big(0, \mathcal{W}_1(0), \mathcal{W}_2(0)\big)\big] + \frac{\nu_2}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)}\int_0^t (t-\tau)^{\nu_2-1}G_2\big(\tau, \mathcal{W}_1(\tau), \mathcal{W}_2(\tau)\big)\,d\tau. \tag{2.7}$$

We substitute the initial values $\mathcal{W}_1(0) = u_1$, $\mathcal{W}_2(0) = u_2$, ${}^{AB}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(0) = v_1$, and ${}^{AB}\mathcal{D}_{0^+}^{\mu_2}\mathcal{W}_2(0) = v_2$ into (2.4)–(2.7) to get (2.1), which means that $(\mathcal{W}_1(t), \mathcal{W}_2(t)) \in C([0,a],\mathbb{R}) \times C([0,a],\mathbb{R})$ is also a solution of (2.1). Since $z \to \Phi_p(z)$ is reversible, the above derivation is completely reversible. Conversely, if $(\mathcal{W}_1(t), \mathcal{W}_2(t)) \in C([0,a],\mathbb{R}) \times C([0,a],\mathbb{R})$ is a solution of (2.1), then it is also a solution of (1.1). The proof is completed. □

3 Existence and uniqueness

This section concentrates on the solvability of system (1.1) thanks to the following contraction fixed point theorem.

Lemma 3.1 ([47]) *Let $\mathbb{X}$ be a Banach space, and let $\phi \neq \mathbb{X}_1 \subset \mathbb{X}$ be closed. If $\mathcal{F} : \mathbb{X}_1 \to \mathbb{X}_1$ is a contraction, then $\mathcal{F}$ admits a unique fixed point $u^* \in \mathbb{E}$.*

According to Lemma 2.3, we take $\mathbb{X} = C([0,a],\mathbb{R}) \times C([0,a],\mathbb{R})$. Then $(\mathbb{X}, \|\cdot\|)$ is a Banach space equipped with the norm $\|w\| = \|(w_1,w_2)\| = \max\{\sup_{0\le t\le l}|w_1(t)|, \sup_{0\le t\le l}|w_2(t)|\}$, $w = (w_1,w_2) \in \mathbb{X}$. Accordingly, we will study the solvability and stability of (1.1) on $(\mathbb{X}, \|\cdot\|)$. For convenience, we introduce the following conditions and symbols.

(A_1) The real constants satisfy $u_1 \neq 0$ or $u_2 \neq 0$, $a, v_1, v_2 > 0$, $0 < \mu_1, \mu_2, \nu_1, \nu_2 \le 1$, and $p_1, p_2 > 1$; $G_k \in C([0,a] \times \mathbb{R}^2, \mathbb{R})$, $k = 1,2$.

(A_2) For all $t \in [0,a]$ and $w_1, w_2 \in \mathbb{R}$, there areconstants $m_k, M_k > 0$ such that

$$m_k \le G_k(t, w_1, w_2) \le M_k, \quad k = 1,2.$$

(A_3) For all $t \in [0,a]$ and $w_1, \overline{w}_1, w_2, \overline{w}_2 \in \mathbb{R}$, there are functions $0 \le \mathcal{L}_{k1}(t), \mathcal{L}_{k2}(t) \in C[0,a]$ such that

$$\big|G_k(t, w_1, w_2) - G_k(t, \overline{w}_1, \overline{w}_2)\big| \le \mathcal{L}_{k1}(t)|w_1 - \overline{w}_1| + \mathcal{L}_{k2}(t)|w_2 - \overline{w}_2|.$$

Denote

$$\underline{\mathcal{M}_k} = v_k^{p_k-1} - \frac{1-\nu_k}{\mathfrak{N}(\nu_k)}(M_k - m_k), \qquad \overline{\mathcal{M}_k} = v_k^{p_k-1} + \frac{1-\nu_k}{\mathfrak{N}(\nu_k)}(M_k - m_k) + \frac{M_k a^{\nu_k}}{\mathfrak{N}(\nu_k)\Gamma(\nu_k)},$$

$$\Theta_k = \frac{1}{\mathfrak{N}(\mu_k)\mathfrak{N}(\nu_k)}\left[(1-\mu_k)(1-\nu_k) + \frac{(1-\mu_k)a^{\nu_k}}{\Gamma(\nu_k)} + \frac{(1-\nu_k)a^{\mu_k}}{\Gamma(\mu_k)} + \frac{\mu_k\nu_k a^{\mu_k+\nu_k}}{\Gamma(\mu_k+\nu_k)}\right],$$

$$\overline{\xi_k} = \Theta_k(q_k - 1)\overline{\mathcal{M}_k}^{q_k-2}\big(\|\mathcal{L}_{k1}\|_a + \|\mathcal{L}_{k2}\|_a\big),$$

$$\underline{\xi_k} = \Theta_k(q_k - 1)\underline{\mathcal{M}_k}^{q_k-2}\big(\|\mathcal{L}_{k1}\|_a + \|\mathcal{L}_{k2}\|_a\big),$$

$$\|\mathcal{L}_{k1}\|_a = \max_{t\in[0,a]}\mathcal{L}_{k1}(t), \qquad \|\mathcal{L}_{k2}\|_a = \max_{t\in[0,a]}\mathcal{L}_{k2}(t), \quad k = 1,2.$$

(A$_4$) One of the following conditions is satisfied: $\varphi_1, \varphi_2 \geq 2$, $\overline{\xi_1}, \overline{\xi_2} < 1$; or $\varphi_1 \geq 2$, $1 < \varphi_2 < 2$, $\overline{\xi_1}, \underline{\xi_2} < 1$; or $1 < \varphi_1 < 2$, $\varphi_2 \geq 2$, $\underline{\xi_1}, \overline{\xi_2} < 1$; or $1 < \varphi_1, \varphi_2 < 2$, $\underline{\xi_1}, \underline{\xi_2} < 1$.

Theorem 3.1 *Assume that* (A$_1$)–(A$_4$) *hold and* $\underline{\mathcal{M}_k} > 0$ $(k = 1, 2)$. *Then system* (1.1) *has a unique nonzero solution* $(\mathcal{W}_1^*(t), \mathcal{W}_2^*(t)) \in \mathbb{X}$.

Proof Obviously, $(\mathcal{W}_1(0), \mathcal{W}_2(0)) = (u_1, u_2) \neq (0, 0)$, that is, $(\mathcal{W}_1(t), \mathcal{W}_2(t)) \not\equiv (0, 0)$ for all $t \in [0, a]$. For all $(\mathcal{W}_1, \mathcal{W}_2) \in \mathbb{X}$, in light of Lemma 2.3, define the vector operator $\mathcal{F} : \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{F}(\mathcal{W}_1, \mathcal{W}_2)(t) = \bigl(\mathcal{F}_1(\mathcal{W}_1, \mathcal{W}_2)(t), \mathcal{F}_2(\mathcal{W}_1, \mathcal{W}_2)(t)\bigr), \quad t \in [0, a], \tag{3.1}$$

where

$$\begin{aligned}\mathcal{F}_1(\mathcal{W}_1, \mathcal{W}_2)(t) = {}& u_1 + \frac{1-\mu_1}{\mathfrak{N}(\mu_1)}\bigl[\Phi_{\varphi_1}\bigl(H_1\bigl(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\bigr)\bigr) - v_1\bigr] \\ & + \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} \int_0^t (t-s)^{\mu_1 - 1} \Phi_{\varphi_1}\bigl(H_1\bigl(s, \mathcal{W}_1(s), \mathcal{W}_2(s)\bigr)\bigr)\, ds, \end{aligned} \tag{3.2}$$

$$\begin{aligned}\mathcal{F}_2(\mathcal{W}_1, \mathcal{W}_2)(t) = {}& u_2 + \frac{1-\mu_2}{\mathfrak{N}(\mu_2)}\bigl[\Phi_{\varphi_2}\bigl(H_2\bigl(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\bigr)\bigr) - v_2\bigr] \\ & + \frac{\mu_2}{\mathfrak{N}(\mu_2)\Gamma(\mu_2)} \int_0^t (t-s)^{\mu_2 - 1} \Phi_{\varphi_2}\bigl(H_2\bigl(s, \mathcal{W}_1(s), \mathcal{W}_2(s)\bigr)\bigr)\, ds \end{aligned} \tag{3.3}$$

with $H_1(t, \mathcal{W}_1(t), \mathcal{W}_2(t))$ and $H_2(t, \mathcal{W}_1(t), \mathcal{W}_2(t))$ the same as in (2.1).

For all $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)$ and $t \in [0, a]$, from (2.1), (A$_1$), and (A$_2$) we have

$$\begin{aligned} H_1\bigl(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\bigr) &\leq v_1^{p_1 - 1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}(M_1 - m_1) + \frac{\nu_1 M_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \int_0^t (t-\tau)^{\nu_1 - 1}\, d\tau \\ &= v_1^{p_1 - 1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}(M_1 - m_1) + \frac{\nu_1 M_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)\nu_1} t^{\nu_1} \\ &\leq v_1^{p_1 - 1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}(M_1 - m_1) + \frac{M_1 a^{\nu_1}}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} = \overline{\mathcal{M}_1}. \end{aligned} \tag{3.4}$$

In the same way, we obtain

$$H_1\bigl(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\bigr) \geq v_1^{p_1 - 1} - \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}(M_1 - m_1) = \underline{\mathcal{M}_1}, \tag{3.5}$$

$$H_2\bigl(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\bigr) \leq v_2^{p_2 - 1} + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}(M_2 - m_2) + \frac{M_2 a^{\nu_2}}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)} = \overline{\mathcal{M}_2}, \tag{3.6}$$

and

$$H_2\bigl(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\bigr) \geq v_2^{p_2 - 1} - \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}(M_2 - m_2) = \underline{\mathcal{M}_2}. \tag{3.7}$$

Obviously, $\underline{\mathcal{M}_1} \leq \overline{\mathcal{M}_1}$, $\underline{\mathcal{M}_2} \leq \overline{\mathcal{M}_2}$. In line with (3.2), (3.4), (3.5), (A$_3$), and (vi) of Lemma 2.2, for all $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2)$, $\overline{\mathcal{W}} = (\overline{\mathcal{W}}_1, \overline{\mathcal{W}}_2) \in \mathbb{X}$, $t \in [0, a]$, we get

$$\begin{aligned} &\bigl|\mathcal{F}_1(\mathcal{W}_1, \mathcal{W}_2)(t) - \mathcal{F}_1(\overline{\mathcal{W}}_1, \overline{\mathcal{W}}_2)(t)\bigr| \\ &\quad = \biggl|\frac{1-\mu_1}{\mathfrak{N}(\mu_1)}\bigl[\Phi_{\varphi_1}\bigl(H_1\bigl(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\bigr)\bigr) - \Phi_{\varphi_1}\bigl(H_1\bigl(t, \overline{\mathcal{W}}_1(t), \overline{\mathcal{W}}_2(t)\bigr)\bigr)\bigr] \end{aligned}$$

$$+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} \int_0^t (t-s)^{\mu_1-1} \big[\Phi_{\varphi_1}\big(H_1\big(s, \mathcal{W}_1(s), \mathcal{W}_2(s)\big)\big)$$

$$- \Phi_{\varphi_1}\big(H_1\big(s, \overline{\mathcal{W}}_1(s), \overline{\mathcal{W}}_2(s)\big)\big)\big] \, ds \bigg|$$

$$\leq \frac{1-\mu_1}{\mathfrak{N}(\mu_1)} \big| \Phi_{\varphi_1}\big(H_1\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big)\big) - \Phi_{\varphi_1}\big(H_1\big(t, \overline{\mathcal{W}}_1(t), \overline{\mathcal{W}}_2(t)\big)\big)\big|$$

$$+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} \int_0^t (t-s)^{\mu_1-1} \big| \Phi_{\varphi_1}\big(H_1\big(s, \mathcal{W}_1(s), \mathcal{W}_2(s)\big)\big)$$

$$- \Phi_{\varphi_1}\big(H_1\big(s, \overline{\mathcal{W}}_1(s), \overline{\mathcal{W}}_2(s)\big)\big)\big| \, ds. \tag{3.8}$$

When $\varphi_1 \geq 2$, (3.8) leads to

$$\big|\mathcal{F}_1(\mathcal{W}_1, \mathcal{W}_2)(t) - \mathcal{F}_1(\overline{\mathcal{W}}_1, \overline{\mathcal{W}}_2)(t)\big|$$

$$\leq \frac{1-\mu_1}{\mathfrak{N}(\mu_1)} (\varphi_1 - 1) \overline{\mathcal{M}_1}^{\varphi_1 - 2} \big| H_1\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big) - H_1\big(t, \overline{\mathcal{W}}_1(t), \overline{\mathcal{W}}_2(t)\big)\big|$$

$$+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} (\varphi_1 - 1) \overline{\mathcal{M}_1}^{\varphi_1 - 2} \int_0^t (t-s)^{\mu_1-1} \big| H_1\big(s, \mathcal{W}_1(s), \mathcal{W}_2(s)\big)$$

$$- H_1\big(s, \overline{\mathcal{W}}_1(s), \overline{\mathcal{W}}_2(s)\big)\big| \, ds$$

$$\leq \frac{1-\mu_1}{\mathfrak{N}(\mu_1)} (\varphi_1 - 1) \overline{\mathcal{M}_1}^{\varphi_1 - 2} \bigg[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)} \big| G_1\big(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\big) - G_1\big(t, \overline{\mathcal{W}}_1(t), \overline{\mathcal{W}}_2(t)\big)\big|$$

$$+ \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \int_0^t (t-\tau)^{\nu_1-1} \big| G_1\big(\tau, \mathcal{W}_1(\tau), \mathcal{W}_2(\tau)\big) - G_1\big(\tau, \overline{\mathcal{W}}_1(\tau), \overline{\mathcal{W}}_2(\tau)\big)\big| \, d\tau \bigg]$$

$$+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} (\varphi_1 - 1) \overline{\mathcal{M}_1}^{\varphi_1 - 2} \int_0^t (t-s)^{\mu_1-1} \bigg[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)} \big| G_1\big(s, \mathcal{W}_1(s), \mathcal{W}_2(s)\big)$$

$$- G_1\big(s, \overline{\mathcal{W}}_1(s), \overline{\mathcal{W}}_2(s)\big)\big| + \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \int_0^s (s-\tau)^{\nu_1-1} \big| G_1\big(\tau, \mathcal{W}_1(\tau), \mathcal{W}_2(\tau)\big)$$

$$- G_1\big(\tau, \overline{\mathcal{W}}_1(\tau), \overline{\mathcal{W}}_2(\tau)\big)\big| \, d\tau \bigg] \, ds$$

$$\leq \frac{1-\mu_1}{\mathfrak{N}(\mu_1)} (\varphi_1 - 1) \overline{\mathcal{M}_1}^{\varphi_1 - 2} \bigg[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)} \big[\mathcal{L}_{11}(t) \big| \mathcal{W}_1(t) - \overline{\mathcal{W}}_1(t)\big| + \mathcal{L}_{12}(t) \big| \mathcal{W}_2(t) - \overline{\mathcal{W}}_2(t)\big| \big]$$

$$+ \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \int_0^t \big[\mathcal{L}_{11}(\tau) \big| \mathcal{W}_1(\tau) - \overline{\mathcal{W}}_1(\tau)\big| + \mathcal{L}_{12}(\tau) \big| \mathcal{W}_2(\tau) - \overline{\mathcal{W}}_2(\tau)\big| \big] \, d\tau \bigg]$$

$$+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} (\varphi_1 - 1) \overline{\mathcal{M}_1}^{\varphi_1 - 2} \int_0^t (t-s)^{\mu_1-1} \bigg[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)} \big[\mathcal{L}_{11}(s) \big| \mathcal{W}_1(s) - \overline{\mathcal{W}}_1(s)\big|$$

$$+ \mathcal{L}_{12}(s) \big| \mathcal{W}_2(s) - \overline{\mathcal{W}}_2(s)\big| \big] + \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \int_0^s (s-\tau)^{\nu_1-1} \big[\mathcal{L}_{11}(\tau) \big| \mathcal{W}_1(\tau) - \overline{\mathcal{W}}_1(\tau)\big|$$

$$+ \mathcal{L}_{12}(\tau) \big| \mathcal{W}_2(\tau) - \overline{\mathcal{W}}_2(\tau)\big| \big] \, d\tau \bigg] \, ds$$

$$\leq \frac{1-\mu_1}{\mathfrak{N}(\mu_1)} (\varphi_1 - 1) \overline{\mathcal{M}_1}^{\varphi_1 - 2} \bigg[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)} \big[\|\mathcal{L}_{11}\|_a \cdot \|\mathcal{W} - \overline{\mathcal{W}}\| + \|\mathcal{L}_{12}\|_a \cdot \|\mathcal{W} - \overline{\mathcal{W}}\| \big]$$

$$+ \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \int_0^t (t-\tau)^{\nu_1-1} \big[\|\mathcal{L}_{11}\|_a \cdot \|\mathcal{W} - \overline{\mathcal{W}}\| + \|\mathcal{L}_{12}\|_a \cdot \|\mathcal{W} - \overline{\mathcal{W}}\| \big] \, d\tau \bigg]$$

$$+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} (\varphi_1 - 1) \overline{\mathcal{M}_1}^{\varphi_1 - 2} \int_0^t (t-s)^{\mu_1-1} \bigg[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)} \big[\|\mathcal{L}_{11}\|_a \cdot \|\mathcal{W} - \overline{\mathcal{W}}\|$$

$$
\begin{aligned}
&+\|\mathcal{L}_{12}\|_a\cdot\|\mathcal{W}-\overline{\mathcal{W}}\|\Big]+\frac{\nu_1}{\mathfrak{N}(\nu_1)(\Gamma(\nu_1))}\int_0^s(s-\tau)^{\nu_1-1}\big[\|\mathcal{L}_{11}\|_a\cdot\|\mathcal{W}-\overline{\mathcal{W}}\| \\
&+\|\mathcal{L}_{12}\|_a\cdot\|\mathcal{W}-\overline{\mathcal{W}}\|\big]\,d\tau\Bigg]\,ds \\
=&\Bigg[\frac{(1-\mu_1)(1-\nu_1)}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)}+\frac{(1-\mu_1)\nu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^t(t-\tau)^{\nu_1-1}\,d\tau \\
&+\frac{(1-\nu_1)\mu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\mu_1)}\int_0^t(t-s)^{\mu_1-1}\,ds \\
&+\frac{\mu_1\nu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\mu_1)\Gamma(\nu_1)}\int_0^t(t-s)^{\mu_1-1}\bigg(\int_0^s(s-\tau)^{\nu_1-1}\,d\tau\bigg)\,ds\Bigg] \\
&\times(\wp_1-1)\overline{\mathcal{M}_1}^{\wp_1-2}\big(\|\mathcal{L}_{11}\|_a+\|\mathcal{L}_{12}\|_a\big)\|\mathcal{W}-\overline{\mathcal{W}}\| \\
=&\Bigg[\frac{(1-\mu_1)(1-\nu_1)}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)}+\frac{1-\mu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\nu_1)}t^{\nu_1}+\frac{1-\nu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\mu_1)}t^{\mu_1} \\
&+\frac{\mu_1\nu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\mu_1+\nu_1)}t^{\mu_1+\nu_1}\Bigg](\wp_1-1)\overline{\mathcal{M}_1}^{\wp_1-2}\big(\|\mathcal{L}_{11}\|_a+\|\mathcal{L}_{12}\|_a\big)\|\mathcal{W}-\overline{\mathcal{W}}\| \\
\le&\frac{1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)}\bigg[(1-\mu_1)(1-\nu_1)+\frac{(1-\mu_1)a^{\nu_1}}{\Gamma(\nu_1)}+\frac{(1-\nu_1)a^{\mu_1}}{\Gamma(\mu_1)}+\frac{\mu_1\nu_1a^{\mu_1+\nu_1}}{\Gamma(\mu_1+\nu_1)}\bigg] \\
&\times(\wp_1-1)\overline{\mathcal{M}_1}^{\wp_1-2}\big(\|\mathcal{L}_{11}\|_a+\|\mathcal{L}_{12}\|_a\big)\|\mathcal{W}-\overline{\mathcal{W}}\| \\
=&\overline{\xi_1}\|\mathcal{W}-\overline{\mathcal{W}}\|.
\end{aligned}
\tag{3.9}
$$

When $1<\wp_1<2$, as in (3.9), (3.8) gives

$$
\begin{aligned}
&\big|\mathcal{F}_1(\mathcal{W}_1,\mathcal{W}_2)(t)-\mathcal{F}_1(\overline{\mathcal{W}}_1,\overline{\mathcal{W}}_2)(t)\big| \\
&\le\frac{1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)}\bigg[(1-\mu_1)(1-\nu_1)+\frac{(1-\mu_1)a^{\nu_1}}{\Gamma(\nu_1)}+\frac{(1-\nu_1)a^{\mu_1}}{\Gamma(\mu_1)}+\frac{\mu_1\nu_1a^{\mu_1+\nu_1}}{\Gamma(\mu_1+\nu_1)}\bigg] \\
&\quad\times(\wp_1-1)\underline{\mathcal{M}_1}^{\wp_1-2}\big(\|\mathcal{L}_{11}\|_a+\|\mathcal{L}_{12}\|_a\big)\|\mathcal{W}-\overline{\mathcal{W}}\| \\
&=\underline{\xi_1}\|\mathcal{W}-\overline{\mathcal{W}}\|.
\end{aligned}
\tag{3.10}
$$

Similarly to (3.8)–(3.10),

$$
\begin{aligned}
&\big|\mathcal{F}_2(\mathcal{W}_1,\mathcal{W}_2)(t)-\mathcal{F}_2(\overline{\mathcal{W}}_1,\overline{\mathcal{W}}_2)(t)\big| \\
&\le\frac{1}{\mathfrak{N}(\mu_2)\mathfrak{N}(\nu_2)}\bigg[(1-\mu_2)(1-\nu_2)+\frac{(1-\mu_2)a^{\nu_2}}{\Gamma(\nu_2)}+\frac{(1-\nu_2)a^{\mu_2}}{\Gamma(\mu_2)}+\frac{\mu_2\nu_2a^{\mu_2+\nu_2}}{\Gamma(\mu_2+\nu_2)}\bigg] \\
&\quad\times(\wp_2-1)\overline{\mathcal{M}_2}^{\wp_2-2}\big(\|\mathcal{L}_{21}\|_a+\|\mathcal{L}_{22}\|_a\big)\|\mathcal{W}-\overline{\mathcal{W}}\| \\
&=\overline{\xi_2}\|\mathcal{W}-\overline{\mathcal{W}}\|,\quad \wp_2\ge 2,
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
&\big|\mathcal{F}_2(\mathcal{W}_1,\mathcal{W}_2)(t)-\mathcal{F}_2(\overline{\mathcal{W}}_1,\overline{\mathcal{W}}_2)(t)\big| \\
&\le\frac{1}{\mathfrak{N}(\mu_2)\mathfrak{N}(\nu_2)}\bigg[(1-\mu_2)(1-\nu_2)+\frac{(1-\mu_2)a^{\nu_2}}{\Gamma(\nu_2)}+\frac{(1-\nu_2)a^{\mu_2}}{\Gamma(\mu_2)}+\frac{\mu_2\nu_2a^{\mu_2+\nu_2}}{\Gamma(\mu_2+\nu_2)}\bigg]
\end{aligned}
$$

$$\begin{aligned}&\times (\mathscr{q}_2 - 1)\underline{\mathcal{M}_2}^{\mathscr{q}_2-2}\bigl(\|\mathcal{L}_{21}\|_a + \|\mathcal{L}_{22}\|_a\bigr)\|\mathcal{W} - \overline{\mathcal{W}}\| \\ &= \underline{\xi_2}\|\mathcal{W} - \overline{\mathcal{W}}\|, \quad 1 < \mathscr{q}_2 < 2. \end{aligned} \tag{3.12}$$

It follows from (3.9)–(3.12) that

$$\bigl\|\mathcal{F}(\mathcal{W}_1, \mathcal{W}_2)(t) - \mathcal{F}(\overline{\mathcal{W}}_1, \overline{\mathcal{W}}_2)(t)\bigr\| \le \begin{cases} \max\{\overline{\xi_1}, \overline{\xi_2}\} \cdot \|\mathcal{W} - \overline{\mathcal{W}}\|, & \mathscr{q}_1, \mathscr{q}_2 \ge 2, \\ \max\{\overline{\xi_1}, \underline{\xi_2}\} \cdot \|\mathcal{W} - \overline{\mathcal{W}}\|, & \mathscr{q}_1 \ge 2, 1 < \mathscr{q}_2 < 2, \\ \max\{\underline{\xi_1}, \overline{\xi_2}\} \cdot \|\mathcal{U} - \overline{\mathcal{W}}\|, & 1 < \mathscr{q}_1 < 2, \mathscr{q}_2 \ge 2, \\ \max\{\underline{\xi_1}, \underline{\xi_2}\} \cdot \|\mathcal{W} - \overline{\mathcal{W}}\|, & 1 < \mathscr{q}_1, \mathscr{q}_2 < 2. \end{cases} \tag{3.13}$$

Let $\xi_k \in \{\overline{\xi_k}, \underline{\xi_k}\}$, $k = 1, 2$. Then (A_4) implies that $0 < \max\{\xi_1, \xi_2\} < 1$. Thus (3.13) indicates that $\mathcal{F} : \mathbb{X} \to \mathbb{X}$ is a contraction. From Lemmas 3.1 and 2.2 we know that $\mathcal{F}$ has a unique fixed point $\mathcal{W}^*(t) = (\mathcal{W}_1^*(t), \mathcal{W}_2^*(t)) \in \mathbb{X}$, which is the unique solution of (1.1). The proof is completed. □

4 Generalized UH-stability

For $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2) \in \mathbb{X}$ and $\epsilon > 0$, the latter definition of stability requires the inequalities

$$\begin{cases} {}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\nu_1}[\Phi_{\mathscr{p}_1}({}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(t))] - G_1(t, \mathcal{W}_1(t), \mathcal{W}_2(t)) \le \epsilon, & t \in (0, a], \\ {}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\nu_2}[\Phi_{\mathscr{p}_2}({}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_2}\mathcal{W}_2(t))] - G_2(t, \mathcal{W}_1(t), \mathcal{W}_2(t)) \le \epsilon, & t \in (0, a], \\ \mathcal{W}_1(0) = \mathscr{u}_1, \quad \mathcal{W}_2(0) = \mathscr{u}_2, \quad {}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(0) = \mathscr{v}_1, \quad {}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_2}\mathcal{W}_2(0) = \mathscr{v}_2. \end{cases} \tag{4.1}$$

Definition 4.1 Suppose that for all $\epsilon > 0$ and $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2) \in \mathbb{X}$ satisfying (4.1), there are a unique $\mathcal{W}^* = (\mathcal{W}_1^*, \mathcal{W}_2^*) \in \mathbb{X}$ satisfying (1.1) and a constant $\omega_1 > 0$ such that

$$\bigl\|\mathcal{W}(t) - \mathcal{W}^*(t)\bigr\| \le \omega_1\epsilon.$$

Then problem (1.1) is said to be Ulam–Hyers (UH) stable.

Definition 4.2 Suppose that for all $\epsilon > 0$ and $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2) \in \mathbb{X}$ satisfying (4.1), there are a unique $\mathcal{W}^* = (\mathcal{W}_1^*, \mathcal{W}_2^*) \in \mathbb{X}$ satisfying (1.1) and $\varpi \in C(\mathbb{R}, \mathbb{R}^+)$ with $\varpi(0) = 0$ such that

$$\bigl\|\mathcal{W}(t) - \mathcal{W}^*(t)\bigr\| \le \varpi(\epsilon).$$

Then problem (1.1) is said to be generalized Ulam–Hyers (GUH) stable.

Remark 4.1 $\mathcal{W} = (\mathcal{W}_1, \mathcal{W}_2) \in \mathbb{X}$ is a solution of inequality (4.1) iff there exists $\phi = (\phi_1, \phi_2) \in \mathbb{X}$ such that

(a) $|\phi_1(t)| \le \epsilon$, $|\phi_2(t)| \le \epsilon$, $0 < t \le a$;
(b) ${}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\nu_1}[\Phi_{\mathscr{p}_1}({}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(t))] = G_1(t, \mathcal{W}_1(t), \mathcal{W}_2(t)) + \phi_1(t)$, $0 < t \le a$;
(c) ${}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\nu_2}[\Phi_{\mathscr{p}_2}({}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_2}\mathcal{W}_2(t))] = G_2(t, \mathcal{W}_1(t), \mathcal{W}_2(t)) + \phi_2(t)$, $0 < t \le a$;
(d) $\mathcal{W}_1(0) = \mathscr{u}_1$, $\mathcal{W}_2(0) = \mathscr{u}_2$, ${}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(0) = \mathscr{v}_1$, ${}^{\mathrm{AB}}\mathcal{D}_{0^+}^{\mu_2}\mathcal{W}_2(0) = \mathscr{v}_2$.

Theorem 4.1 *Under* (A_1)–(A_4), *problem* (1.1) *is generalized UH-stable*.

Proof In view of Lemma 2.3 and Remark 4.1, to solve inequality (4.1), we have

$$\begin{cases} \mathcal{W}_1(t) = u_1 + \frac{1-\mu_1}{\mathfrak{N}(\mu_1)}[\Phi_{q_1}(H_1^\phi(t,\mathcal{W}_1(t),\mathcal{W}_2(t))) - v_1] \\ \qquad + \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}\int_0^t (t-s)^{\mu_1-1}\Phi_{q_1}(H_1^\phi(s,\mathcal{W}_1(s),\mathcal{W}_2(s)))\,ds, \\ \mathcal{W}_2(t) = u_2 + \frac{1-\mu_2}{\mathfrak{N}(\mu_2)}[\Phi_{q_2}(H_2^\phi(t,\mathcal{W}_1(t),\mathcal{W}_2(t))) - v_2] \\ \qquad + \frac{\mu_2}{\mathfrak{N}(\mu_2)\Gamma(\mu_2)}\int_0^t (t-s)^{\mu_2-1}\Phi_{q_2}(H_2^\phi(s,\mathcal{W}_1(s),\mathcal{W}_2(s)))\,ds, \end{cases} \tag{4.2}$$

$$\begin{aligned} &H_1^\phi\bigl(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\bigr) \\ &\quad = \Phi_{p_1}(v_1) + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\bigl[G_1\bigl(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\bigr) + \phi_1(t) - G_1(0,u_1,u_2) - \phi_1(0)\bigr] \\ &\quad + \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^t (t-\tau)^{\nu_1-1}\bigl[G_1\bigl(\tau,\mathcal{W}_1(\tau),\mathcal{W}_2(\tau)\bigr) + \phi_1(\tau)\bigr]\,d\tau, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &H_2^\phi\bigl(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\bigr) \\ &\quad = \Phi_{p_2}(v_2) + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}\bigl[G_2\bigl(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\bigr) + \phi_2(t) - G_2(0,u_1,u_2) - \phi_2(0)\bigr] \\ &\quad + \frac{\nu_2}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)}\int_0^t (t-\tau)^{\nu_2-1}\bigl[G_2\bigl(\tau,\mathcal{W}_1(\tau),\mathcal{W}_2(\tau)\bigr) + \phi(\tau)\bigr]\,d\tau. \end{aligned} \tag{4.4}$$

By Theorem 3.1 and Lemma 2.3 the unique solution $\mathcal{W}^*(t) = (\mathcal{W}_1^*(t),\mathcal{W}_2^*(t)) \in \mathbb{X}$ of (1.1) also meets (2.1). For all $\epsilon > 0$ small enough, from (A_1), (A_2), and (a) in Remark 4.1 it follows that (3.4)–(3.7) are similar to

$$\begin{aligned} H_1^\phi\bigl(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\bigr) &\le v_1^{p_1-1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}(M_1 - m_1 + 2\epsilon) + \frac{M_1 a^{\mu_1}}{\mathfrak{N}(\nu_1)\Gamma(\mu_1)}(M_1+\epsilon) \\ &= \overline{\mathcal{M}_1}(\epsilon), \end{aligned} \tag{4.5}$$

$$H_1^\phi\bigl(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\bigr) \ge v_1^{p_1-1} - \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}(M_1 - m_1 + 2\epsilon) = \underline{\mathcal{M}_1}(\epsilon) > 0, \tag{4.6}$$

$$\begin{aligned} H_2^\phi\bigl(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\bigr) &\le v_2^{p_2-1} + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}(M_2 - m_2 - 2\epsilon) + \frac{M_2 a^{\nu_2}}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)}(M_2+\epsilon) \\ &= \overline{\mathcal{M}_2}(\epsilon), \end{aligned} \tag{4.7}$$

and

$$H_2^\phi\bigl(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\bigr) \ge v_2^{p_2-1} - \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}(M_2 - m_2 + 2\epsilon) = \underline{\mathcal{M}_2}(\epsilon) > 0. \tag{4.8}$$

Clearly, $0 < \underline{\mathcal{M}_1}(\epsilon) < \underline{\mathcal{M}_1} < \overline{\mathcal{M}_1} < \overline{\mathcal{M}_1}(\epsilon)$ and $0 < \underline{\mathcal{M}_2}(\epsilon) < \underline{\mathcal{M}_2} < \overline{\mathcal{M}_2} < \overline{\mathcal{M}_2}(\epsilon)$.

Similarly to (3.8) and (3.9), when $q_1 \ge 2$, we draw from (2.1), (4.2), (4.3), and (4.5) that

$$\begin{aligned} &\bigl|\mathcal{W}_1(t) - \mathcal{W}_1^*(t)\bigr| \\ &\quad = \left|\frac{1-\mu_1}{\mathfrak{N}(\mu_1)}\bigl[\Phi_{q_1}\bigl(H_1^\phi\bigl(t,\mathcal{W}_1(t),\mathcal{W}_2(t)\bigr)\bigr) - \Phi_{q_1}\bigl(H_1\bigl(t,\mathcal{W}_1^*(t),\mathcal{W}_2^*(t)\bigr)\bigr)\bigr]\right. \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} \int_0^t (t-s)^{\mu_1-1} \bigl[\Phi_{\varphi_1}\bigl(H_1^{\phi}\bigl(s, \mathcal{W}_1(s), \mathcal{W}_2(s)\bigr)\bigr) \\
&- H_1\bigl(s, \mathcal{W}_1^*(s), \mathcal{W}_2^*(s)\bigr)\bigr] \, ds \Bigg| \\
\leq{}& \frac{1-\mu_1}{\mathfrak{N}(\mu_1)} \bigl| \Phi_{\varphi_1}\bigl(H_1^{\phi}\bigl(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\bigr)\bigr) - \Phi_{\varphi_1}\bigl(H_1\bigl(t, \mathcal{W}_1^*(t), \mathcal{W}_2^*(t)\bigr)\bigr)\bigr| \\
&+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} \int_0^t (t-s)^{\mu_1-1} \bigl| \Phi_{\varphi_1}\bigl(H_1^{\phi}\bigl(s, \mathcal{W}_1(s), \mathcal{W}_2(s)\bigr)\bigr) \\
&- H_1\bigl(s, \mathcal{W}_1^*(s), \mathcal{W}_2^*(s)\bigr)\bigr| \, ds \\
\leq{}& \frac{1-\mu_1}{\mathfrak{N}(\mu_1)} (\varphi_1 - 1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2} \bigl| H_1^{\phi}\bigl(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\bigr) - H_1\bigl(t, \mathcal{W}_1^*(t), \mathcal{W}_2^*(t)\bigr)\bigr| \\
&+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} (\varphi_1 - 1)\overline{\mathcal{M}_1}(\epsilon)^{q_1-2} \\
&\times \int_0^t (t-s)^{\mu_1-1} \bigl| H_1^{\phi}\bigl(s, \mathcal{W}_1(s), \mathcal{W}_2(s)\bigr) - H_1\bigl(s, \mathcal{W}_1^*(s), \mathcal{W}_2^*(s)\bigr)\bigr| \, ds \\
\leq{}& \frac{1-\mu_1}{\mathfrak{N}(\mu_1)} (\varphi_1 - 1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2} \\
&\times \biggl[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)} \bigl[\bigl| G_1\bigl(t, \mathcal{W}_1(t), \mathcal{W}_2(t)\bigr) - G_1\bigl(t, \mathcal{W}_1^*(t), \mathcal{W}_2^*(t)\bigr)\bigr| + 2\epsilon \bigr] + \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \\
&\times \int_0^t (t-\tau)^{\nu_1-1} \bigl[\bigl| G_1\bigl(\tau, \mathcal{W}_1(\tau), \mathcal{W}_2(\tau)\bigr) - G_1\bigl(\tau, \mathcal{W}_1^*(\tau), \mathcal{W}_2^*(\tau)\bigr)\bigr| + 2\epsilon \bigr] \, d\tau \biggr] \\
&+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} (\varphi_1 - 1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2} \int_0^t (t-s)^{\mu_1-1} \biggl[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)} \bigl[\bigl| G_1\bigl(s, \mathcal{W}_1(s), \mathcal{W}_2(s)\bigr) \\
&- G_1\bigl(s, \mathcal{W}_1^*(s), \mathcal{W}_2^*(s)\bigr)\bigr| + 2\epsilon \bigr] + \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \int_0^s (s-\tau)^{\nu_1-1} \bigl[\bigl| G_1\bigl(\tau, \mathcal{W}_1(\tau), \mathcal{W}_2(\tau)\bigr) \\
&- G_1\bigl(\tau, \mathcal{W}_1^*(\tau), \mathcal{W}_2^*(\tau)\bigr)\bigr| + 2\epsilon \bigr] \, d\tau \biggr] \, ds \\
\leq{}& \frac{1-\mu_1}{\mathfrak{N}(\mu_1)} (\varphi_1 - 1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2} \\
&\times \biggl[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)} \bigl[\mathcal{L}_{11}(t) \bigl| \mathcal{W}_1(t) - \mathcal{W}_1^*(t) \bigr| + \mathcal{L}_{12}(t) \bigl| \mathcal{W}_2(t) - \mathcal{W}_2^*(t) \bigr| + 2\epsilon \bigr] \\
&+ \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \int_0^t (t-\tau)^{\nu_1-1} \bigl[\mathcal{L}_{11}(\tau) \bigl| \mathcal{W}_1(\tau) - \mathcal{W}_1^*(\tau) \bigr| \\
&+ \mathcal{L}_{12}(\tau) \bigl| \mathcal{W}_2(\tau) - \mathcal{W}_2^*(\tau) \bigr| + 2\epsilon \bigr] \, d\tau \biggr] \\
&+ \frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)} (\varphi_1 - 1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2} \int_0^t (t-s)^{\mu_1-1} \biggl[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)} \bigl[\mathcal{L}_{11}(s) \bigl| \mathcal{W}_1(s) - \mathcal{W}_1^*(s) \bigr| \\
&+ \mathcal{L}_{12}(s) \bigl| \mathcal{W}_2(s) - \mathcal{W}_2^*(s) \bigr| + 2\epsilon \bigr] \\
&+ \frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \int_0^s (s-\tau)^{\nu_1-1} \bigl[\mathcal{L}_{11}(\tau) \bigl| \mathcal{W}_1(\tau) - \mathcal{W}_1^*(\tau) \bigr| \\
&+ \mathcal{L}_{12}(\tau) \bigl| \mathcal{W}_2(\tau) - \mathcal{W}_2^*(\tau) \bigr| + 2\epsilon \bigr] \, d\tau \biggr] \, ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1-\mu_1}{\mathfrak{N}(\mu_1)}(\varphi_1-1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2}\bigg[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\big[\|\mathcal{L}_{11}\|_a\cdot\|\mathcal{W}-\mathcal{W}^*\| \\
&\quad+\|\mathcal{L}_{12}\|_a\cdot\|\mathcal{W}-\mathcal{W}^*\|+2\epsilon\big] \\
&\quad+\frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^t(t-\tau)^{\nu_1-1}\big[\|\mathcal{L}_{11}\|_a\cdot\|\mathcal{W}-\mathcal{W}^*\| \\
&\quad+\|\mathcal{L}_{12}\|_l\cdot\|\mathcal{W}-\mathcal{W}^*\|+2\epsilon\big]\,d\tau\bigg] \\
&\quad+\frac{\mu_1}{\mathfrak{N}(\mu_1)\Gamma(\mu_1)}(\varphi_1-1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2}\int_0^t(t-s)^{\mu_1-1}\bigg[\frac{1-\nu_1}{\mathfrak{N}(\nu_1)}\big[\|\mathcal{L}_{11}\|_a\cdot\|\mathcal{W}-\mathcal{W}^*\| \\
&\quad+\|\mathcal{L}_{12}\|_a\cdot\|\mathcal{W}-\mathcal{W}^*\|+2\epsilon\big]+\frac{\nu_1}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^s(s-\tau)^{\nu_1-1}\big[\|\mathcal{L}_{11}\|_a\cdot\|\mathcal{W}-\mathcal{W}^*\| \\
&\quad+\|\mathcal{L}_{12}\|_a\cdot\|\mathcal{W}-\mathcal{W}^*\|+2\epsilon\big]\,d\tau\bigg]\,ds \\
&=\bigg[\frac{(1-\mu_1)(1-\nu_1)}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)}+\frac{(1-\mu_1)\nu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\nu_1)}\int_0^t(t-\tau)^{\nu_1-1}\,d\tau \\
&\quad+\frac{(1-\nu_1)\mu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\mu_1)}\int_0^t(t-s)^{\mu_1-1}\,ds \\
&\quad+\frac{\mu_1\nu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\mu_1)\Gamma(\nu_1)}\int_0^t(t-s)^{\mu_1-1}\bigg(\int_0^s(s-\tau)^{\nu_1-1}\,d\tau\bigg)\,ds\bigg] \\
&\quad\times(\varphi_1-1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2}\big[\big(\|\mathcal{L}_{11}\|_a+\|\mathcal{L}_{12}\|_a\big)\|\mathcal{W}-\overline{\mathcal{W}}\|+2\epsilon\big] \\
&=\bigg[\frac{(1-\mu_1)(1-\nu_1)}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)}+\frac{1-\mu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\nu_1)}t^{\nu_1}+\frac{1-\nu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\mu_1)}t^{\mu_1} \\
&\quad+\frac{\mu_1\nu_1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)\Gamma(\mu_1+\nu_1)}t^{\mu_1+\nu_1}\bigg](\varphi_1-1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2} \\
&\quad\times\big[\big(\|\mathcal{L}_{11}\|_a+\|\mathcal{L}_{12}\|_a\big)\|\mathcal{W}-\overline{\mathcal{W}}\|+2\epsilon\big] \\
&\le\frac{1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)}\bigg[(1-\mu_1)(1-\nu_1)+\frac{(1-\mu_1)a^{\nu_1}}{\Gamma(\nu_1)}+\frac{(1-\nu_1)a^{\mu_1}}{\Gamma(\mu_1)}+\frac{\mu_1\nu_1a^{\mu_1+\nu_1}}{\Gamma(\mu_1+\nu_1)}\bigg] \\
&\quad\times(\varphi_1-1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2}\big[\big(\|\mathcal{L}_{11}\|_a+\|\mathcal{L}_{12}\|_a\big)\|\mathcal{W}-\overline{\mathcal{W}}\|+2\epsilon\big] \\
&=\overline{\Upsilon_1}(\epsilon)\|\mathcal{W}-\overline{\mathcal{W}}\|+2\epsilon\overline{\Lambda_1}(\epsilon),
\end{aligned}
\tag{4.9}
$$

where $\overline{\Upsilon_1}(\epsilon)=\Theta_1(\varphi_1-1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2}(\|\mathcal{L}_{11}\|_a+\|\mathcal{L}_{12}\|_a)$ and $\overline{\Lambda_1}(\epsilon)=\Theta_1(\varphi_1-1)\overline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2}$.

Similarly to (4.9), we apply (4.6)–(4.8) to obtain

$$
\big|\mathcal{W}_2(t)-\mathcal{W}_2^*(t)\big|\le\overline{\Upsilon_2}(\epsilon)\|\mathcal{W}-\mathcal{W}^*\|+2\epsilon\overline{\Lambda_2}(\epsilon),\quad q_2\ge 2, \tag{4.10}
$$

$$
\big|\mathcal{W}_1(t)-\mathcal{W}_1^*(t)\big|\le\underline{\Upsilon_1}(\epsilon)\|\mathcal{W}-\mathcal{W}^*\|+2\epsilon\underline{\Lambda_1}(\epsilon),\quad 1<q_1<2, \tag{4.11}
$$

and

$$
\big|\mathcal{W}_2(t)-\mathcal{W}_2^*(t)\big|\le\underline{\Upsilon_2}(\epsilon)\|\mathcal{W}-\mathcal{W}^*\|+2\epsilon\underline{\Lambda_2}(\epsilon),\quad 1<q_2<2, \tag{4.12}
$$

where $\overline{\Upsilon_2}(\epsilon)=\Theta_2(\varphi_2-1)\overline{\mathcal{M}_2}(\epsilon)^{\varphi_2-2}(\|\mathcal{L}_{21}\|_a+\|\mathcal{L}_{22}\|_a)$, $\overline{\Lambda_2}(\epsilon)=\Theta_2(\varphi_2-1)\overline{\mathcal{M}_2}(\epsilon)^{\varphi_2-2}$, $\underline{\Upsilon_1}(\epsilon)=\Theta_1(\varphi_1-1)\underline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2}(\|\mathcal{L}_{11}\|_a+\|\mathcal{L}_{12}\|_a)$, $\underline{\Lambda_1}(\epsilon)=\Theta_1(\varphi_1-1)\underline{\mathcal{M}_1}(\epsilon)^{\varphi_1-2}$, $\underline{\Upsilon_2}(\epsilon)=\Theta_2(\varphi_2-1)\underline{\mathcal{M}_2}(\epsilon)^{\varphi_2-2}(\|\mathcal{L}_{21}\|_a+\|\mathcal{L}_{22}\|_a)$, and $\underline{\Lambda_2}(\epsilon)=\Theta_2(\varphi_2-1)\underline{\mathcal{M}_2}(\epsilon)^{\varphi_2-2}$.

For all $\epsilon > 0$ small enough, we have $0 < \overline{\Upsilon_1}(\epsilon), \underline{\Upsilon_1}(\epsilon), \overline{\Upsilon_2}(\epsilon), \underline{\Upsilon_2}(\epsilon) < 1$. Take $\Upsilon_k(\epsilon) \in \{\overline{\Upsilon_k}(\epsilon), \underline{\Upsilon_k}(\epsilon)\}$ and $\Lambda_k(\epsilon) \in \{\overline{\Lambda_k}(\epsilon), \underline{\Lambda_k}(\epsilon)\}$, $k = 1, 2$. Then it follows from (4.9)–(4.12) that

$$\|\mathcal{W} - \mathcal{W}^*\| \le \frac{2\max\{\Lambda_1(\epsilon), \Lambda_2(\epsilon)\}}{1 - \max\{\Upsilon_1(\epsilon), \Upsilon_2(\epsilon)\}}\epsilon. \tag{4.13}$$

Consequently, we claim that problem (1.1) is generalized UH-stable in accordance with (4.13) and Definition 4.2. The proof is completed. □

5 A verification example

In this section, we inspect the correctness and applicability of our findings by using the following example:

$$\begin{cases} {}^{AB}\mathcal{D}_{0^+}^{0.6}[\Phi_{\wp_1}({}^{AB}\mathcal{D}_{0^+}^{0.7}\mathcal{W}_1(t))] = \frac{2+\cos(\mathcal{W}_1(t))}{100} + \frac{1}{50}|\sin(t)|\frac{\mathcal{W}_2(t)}{1+\mathcal{W}_2(t)^2}, \quad t \in (0, \sqrt{2}], \\ {}^{AB}\mathcal{D}_{0^+}^{0.4}[\Phi_{\wp_2}({}^{AB}\mathcal{D}_{0^+}^{0.2}\mathcal{W}_2(t))] = \frac{2+\sin(3t)}{100}[\frac{3\pi}{4} + \arctan(\mathcal{W}_1(t) + \mathcal{W}_2(t))], \quad t \in (0, \sqrt{2}], \\ \mathcal{W}_1(0) = -1, \qquad \mathcal{W}_2(0) = 1, \qquad {}^{AB}\mathcal{D}_{0^+}^{0.7}\mathcal{W}_1(0) = 2, \qquad {}^{AB}\mathcal{D}_{0^+}^{0.2}\mathcal{W}_2(0) = 3. \end{cases} \tag{5.1}$$

Obviously, $a = \sqrt{2}$, $\mu_1 = 0.7$, $\nu_1 = 0.6$, $\mu_2 = 0.2$, $\nu_2 = 0.4$, $u_1 = -1$, $u_2 = 1$, $v_1 = 2$, $v_2 = 3$, $G_1(t, w_1, w_2) = \frac{2+\cos(w_1)}{100} + \frac{1}{50}|\sin(t)|\frac{w_2}{1+w_2^2}$, and $G_2(t, w_1, w_2) = \frac{2+\sin(3t)}{200}[\frac{3\pi}{4} + \arctan(w_1 + w_2)]$. Choose $\mathfrak{N}(x) = 1 - x + \frac{x}{\Gamma(x)}$, $0 < x \le 1$. Then $\mathfrak{N}(0) = \mathfrak{N}(1) = 1$. By a simple calculation we have

$$\begin{aligned} &\frac{1}{100} \le G_1(t, w_1, w_2) \le \frac{4}{100}, \qquad \frac{\pi}{800} \le G_2(t, w_1, w_2) \le \frac{15\pi}{800}, \\ &\left|G_1(t, w_1, w_2) - G_1(t, \overline{w}_1, \overline{w}_2)\right| \le \frac{1}{100}|w_1 - \overline{w}_1| + \frac{|\sin(t)|}{100}|w_2 - \overline{w}_2|, \\ &\left|G_2(t, w_1, w_2) - G_2(t, \overline{w}_1, \overline{w}_2)\right| \le \frac{2 + \sin(3t)}{200}\big[|w_1 - \overline{w}_1| + |w_2 - \overline{w}_2|\big]. \end{aligned}$$

Therefore conditions (A_1)–(A_3) are fulfilled. Furthermore, $m_1 = \frac{1}{100}$, $M_1 = \frac{4}{100}$, $m_2 = \frac{\pi}{800}$, $M_2 = \frac{15\pi}{800}$, $\mathcal{L}_{11}(t) = \frac{1}{100}$, $\mathcal{L}_{12}(t) = \frac{|\sin(t)|}{100}$, $\mathcal{L}_{21}(t) = \mathcal{L}_{22}(t) = \frac{2+\sin(3t)}{200}$, $\|\mathcal{L}_{11}\|_a = \frac{1}{100}$, $\|\mathcal{L}_{12}\|_a = \frac{\sin(\sqrt{2})}{100}$, $\|\mathcal{L}_{21}\|_a = \|\mathcal{L}_{22}\|_a = \frac{3}{200}$, and

$$\begin{aligned} \Theta_1 &= \frac{1}{\mathfrak{N}(\mu_1)\mathfrak{N}(\nu_1)}\left[(1-\mu_1)(1-\nu_1) + \frac{(1-\mu_1)a^{\nu_1}}{\Gamma(\nu_1)} + \frac{(1-\nu_1)a^{\mu_1}}{\Gamma(\mu_1)} + \frac{\mu_1\nu_1 a^{\mu_1+\nu_1}}{\Gamma(\mu_1+\nu_1)}\right] \\ &\approx 2.2188, \\ \Theta_2 &= \frac{1}{\mathfrak{N}(\mu_2)\mathfrak{N}(\nu_2)}\left[(1-\mu_2)(1-\nu_2) + \frac{(1-\mu_2)a^{\nu_2}}{\Gamma(\nu_2)} + \frac{(1-\nu_2)a^{\mu_2}}{\Gamma(\mu_2)} + \frac{\mu_2\nu_2 a^{\mu_2+\nu_2}}{\Gamma(\mu_2+\nu_2)}\right] \\ &\approx 1.6718, \end{aligned}$$

Case 1: When $\wp_1 = \frac{3}{2}$ and $\wp_2 = \frac{5}{4}$, we have $q_1 = 3 > 2$, $q_2 = 5 > 2$, and

$$\begin{aligned} \underline{\mathcal{M}_1} &= v_1^{\wp_1 - 1} - \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}(M_1 - m_1) \approx 1.3993 > 0, \\ \underline{\mathcal{M}_2} &= v_2^{\wp_2 - 1} - \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}(M_2 - m_2) \approx 1.2738 > 0, \end{aligned}$$

$$\overline{\mathcal{M}_1} = \upsilon_1^{\wp_1-1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}(M_1 - m_1) + \frac{M_1 a^{\nu_1}}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \approx 1.4703,$$

$$\overline{\mathcal{M}_2} = \upsilon_2^{\wp_2-1} + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}(M_2 - m_2) + \frac{M_2 a^{\nu_2}}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)} \approx 1.3974,$$

$$\overline{\xi_1} = \Theta_1(\varphi_1 - 1)\overline{\mathcal{M}_1}^{\varphi_1-2}\left(\|\mathcal{L}_{11}\|_a + \|\mathcal{L}_{12}\|_a\right) \approx 0.1297 < 1,$$

$$\overline{\xi_2} = \Theta_2(\varphi_2 - 1)\overline{\mathcal{M}_2}^{\varphi_2-2}\left(\|\mathcal{L}_{21}\|_a + \|\mathcal{L}_{22}\|_a\right) \approx 0.5474 < 1.$$

So (A_4) also holds. By Theorems 3.1 and 4.1 we obtain that system (5.1) has a unique generalized UH-stable solution.

Case 2: When $p_1 = \frac{3}{2}$ and $p_2 = 5$, we have $q_1 = 3 > 2$ and $1 < q_2 = \frac{5}{4} < 2$ with the same values of $\underline{\mathcal{M}_1}$, $\overline{\mathcal{M}_1}$, and $\overline{\xi_1}$ as in Case 1. In addition,

$$\underline{\mathcal{M}_2} = \upsilon_2^{\wp_2-1} - \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}(M_2 - m_2) \approx 80.9577 > 0,$$

$$\overline{\mathcal{M}_2} = \upsilon_2^{\wp_2-1} + \frac{1-\nu_2}{\mathfrak{N}(\nu_2)}(M_2 - m_2) + \frac{M_2 a^{\nu_2}}{\mathfrak{N}(\nu_2)\Gamma(\nu_2)} \approx 81.0814,$$

$$\underline{\xi_2} = \Theta_2(\varphi_2 - 1)\underline{\mathcal{M}_2}^{\varphi_2-2}\left(\|\mathcal{L}_{21}\|_a + \|\mathcal{L}_{22}\|_a\right) \approx 4.6457 \times 10^{-4} < 1.$$

So (A_4) also holds. By Theorems 3.1 and 4.1 we obtain that system (5.1) has a unique generalized UH-stable solution.

Case 3: When $p_1 = 3$ and $p_2 = \frac{5}{4}$, we have $1 < q_1 = \frac{3}{2} < 2$ and $q_2 = 5 > 2$ with the same values of $\underline{\mathcal{M}_2}$, $\overline{\mathcal{M}_2}$, and $\overline{\xi_2}$ as in Case 1. In addition,

$$\underline{\mathcal{M}_1} = \upsilon_1^{\wp_1-1} - \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}(M_1 - m_1) \approx 3.9851 > 0,$$

$$\overline{\mathcal{M}_1} = \upsilon_1^{\wp_1-1} + \frac{1-\nu_1}{\mathfrak{N}(\nu_1)}(M_1 - m_1) + \frac{M_1 a^{\nu_1}}{\mathfrak{N}(\nu_1)\Gamma(\nu_1)} \approx 4.0561,$$

$$\underline{\xi_1} = \Theta_1(\varphi_1 - 1)\underline{\mathcal{M}_1}^{\varphi_1-2}\left(\|\mathcal{L}_{11}\|_a + \|\mathcal{L}_{12}\|_a\right) \approx 0.0110 < 1,$$

So (A_4) also holds. By Theorems 3.1 and 4.1 we obtain that system (5.1) has a unique generalized UH-stable solution.

Case 4: When $p_1 = 3$ and $p_2 = 5$, we have $1 < q_1 = \frac{3}{2} < 2$ and $1 < q_2 = \frac{5}{4} < 2$. From Cases 2 and 3 we know that $\underline{\mathcal{M}_1} \approx 3.9851 > 0$, $\overline{\mathcal{M}_1} \approx 4.0561$, $\underline{\mathcal{M}_2} \approx 80.9577 > 0$, $\overline{\mathcal{M}_2} \approx 81.0814$, $\underline{\xi_1} \approx 0.0110 < 1$, and $\underline{\xi_2} \approx 4.6457 \times 10^{-4} < 1$. So (A_4) also holds. By Theorems 3.1 and 4.1 we declare that system (5.1) has a unique generalized UH-stable solution.

6 Conclusions

AB-fractional differential equations are good mathematical models in many scientific and engineering fields. As far as we know, there have no works dealing with the nonlinear AB-fractal differential coupled equations with Laplacian. So we investigated system (1.1) to fill this gap. The existence, uniqueness, and generalized UH-stability of solution are obtained. Our outcomes snow that the Laplacian parameters $(\wp_1, \wp_2)$, the fractional orders μ_k, ν_k $(k = 1, 2)$, the initial conditions ${}^{AB}\mathcal{D}_{0^+}^{\mu_1}\mathcal{W}_1(0) = \upsilon_1$ and ${}^{AB}\mathcal{D}_{0^+}^{\mu_2}\mathcal{W}_2(0) = \upsilon_2$, and the performances of $G_k(t, \cdot, \cdot)$ $(k = 1, 2)$ have an effect on the existence and stability of (1.1). The techniques and methods in the paper can be applied to other types of fractional differential systems. In addition, inspired by recent published papers [48–55], we will investigate the Lyapunov stability of fractional differential equations, the coincidence theory of fractional differential equations, and diffusion fractional partial differential equations in the future.

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