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Existence results for the families of multi-mappings with applications to integral and functional equations

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Abstract

The aim of this manuscript is to prove some new fixed-point results for the two families of multivalued dominated-contractive maps defined on a closed ball in a complete multiplicative-metric space. The fixed-point theorems for multigraph-dominated maps are established. We also investigate the existence of fixed points of the two families of graphic contractions in a multiplicative-metric space. For illustration, an example is provided to clarify that the hypothesis of the obtained result is valid. Moreover, the applications of the obtained results to show the existence of solutions to the coupled systems of nonlinear Volterra-type integral equations and mappingal equations are presented as a confirmation of the novelty of our results.

Keywords: Generalized F-contraction; Closed ball; α_*-dominated multivalued maps; Graphical contraction; Integral and mapping equations

1 Introduction and preliminaries

In modern mathematical analysis, the inequalities involving mappings or operators are considered important for a solution of fixed-point problems. In this regard, Banach [4] (1922) proved an important idea known as the contraction principle. Its significance can be assessed by the number of its generalizations in the literature (see [1–39]). The concept of multiplicative calculus [5] paved the path towards multiplicative metrics. Moreover, Bashirov et al. [6] claimed that multiplicative ODEs are more general than ODEs. Florack and Assen [12] provided a rigorous analysis of multiplicative calculus and explained its prospective usage in biomedical-image analysis. Abbas et al. [22] presented a fixed-point problem satisfying locally contraction and an application to resolve the unique solution to the multiplicative BVP.

Wardowski [39] investigated a generalization of the Banach contraction called F-contraction. The F-contraction proved an elegant and applicable contraction principle that led many researchers to produce different multiplications of F-contraction and hence established different significant fixed-point results (see [10, 17, 20–22, 31, 32, 35]). Recently, Rasham et al. [33] introduced the families of multivalued mappings satisfying a general inequality in double controlled dislocated quasimetric spaces. Mehmood et al. [17]

proved a result for set-valued F-contractive family of maps on a closed ball and derived an application to show the existence of the unique solution to nonlinear integral equation. In this article, we prove some new fixed-point results for the two families of multivalued dominated-contractive maps defined on a closed ball in a complete multiplicative-metric space. The fixed-point theorems for multigraph-dominated maps are established. We also investigate the existence of fixed points of the two families of graphic contractions in a multiplicative-metric space. For illustration, an example is provided to clarify that the hypothesis of the obtained result is valid. Moreover, the applications of the obtained results to show the existence of solutions to the coupled systems of nonlinear Volterra-type integral equations and mapping equations are presented as a confirmation of the novelty of our results. These results generalize the results that appeared in [23] and can be applied to obtain the results presented in [18].

Let us focus on the following preliminary notions.

Definition 1.1 ([24]) Let $\check{Z}$ be a nonempty set. The mapping $\eta : \check{Z}^2 \longrightarrow \mathbb{R}$ is said to be a multiplicative metric on $\check{Z}$ if for all $\varrho, \upsilon, q \in \check{Z}$, the axioms (i)–(iii) hold:

(i) $\eta(\varrho, \upsilon) > 1$ with $\varrho \neq \upsilon$ and $\eta(\varrho, \upsilon) = 1 \iff \varrho = \upsilon$;

(ii) $\eta(\varrho, \upsilon) = \eta(\upsilon, \varrho)$;

(iii) $\eta(\varrho, q) \leq [\eta(\varrho, \upsilon) \cdot \eta(\upsilon, q)]$.

The pair $(\check{Z}, \eta)$ is known as a multiplicative metric space or briefly a M^* space.

Suppose that e_0 is any point that belongs to $\check{Z}$ and $r > 1$, an open ball $\overline{B_{\eta_m}(\kappa_0, r)}$ in M^* space with center e_0 and radius r is defined by $\{y \in \check{Z} : \eta(y, e_0) < r\}$.

Definition 1.2 ([24]) Let $(\check{Z}, \eta)$ be a M^* space.

(i) A sequence $\{q_n\}$ in $\check{Z}$ is called a multiplicative Cauchy sequence if for $\epsilon > 1$, there exists $p \in \mathbb{N}$ such that $\eta(q_l, q_k) \leq \epsilon$ for every $k, l > p$ or equivalently, $\eta(q_n, q_m) \to 1$ as $m, n \to \infty$.

(ii) If the multiplicative metric space $(\check{Z}, \eta)$ has the property that every Cauchy sequence $\{q_n\}$ in $\check{Z}$ converges to a point in $\check{Z}$, then it is known as a complete space.

Example 1.3 Let $\check{Z} = [0, \infty)$ and define the mapping $\eta : \check{Z}^2 \to [1, \infty)$ by

$$\eta(t, y) = \nu^{|t-y|}; \quad \nu > 1.$$

Then, $(\check{Z}, \eta)$ is a M^* space.

Remark 1.4 Every M^* space $(\check{Z}, \eta)$ can be generated from a metric space $(\check{Z}, d)$ by the following relation:

$$\eta(l, f) = e^{d(l,f)}.$$

Definition 1.5 Let $(\check{Z}, \eta)$ be a M^* space and $K \subseteq \check{Z}$ be nonempty and $s \in \check{Z}$. A point $p_0 \in K$ is called a best approximation of s in K if

$$\eta(s, K) = \eta(s, p_0); \quad \eta(s, K) = \inf\{\eta(s, p_0) : p_0 \in K\}.$$

The set K is called a compact set if for every $s \in \check{Z}$, there exists a point of best approximation in K.

Definition 1.6 Let $(\check{Z}, \eta)$ be a M^* space. The mapping $H_\eta : P(\check{Z})^2 \to [0, \infty)$, defined by

$$H_\eta(C, D) = \max\Big\{\sup_{l \in C} \eta(l, D), \ \sup_{u \in D} \eta(C, u)\Big\},$$

satisfies all the axioms of multiplicative metric. This special metric H_η is called a Hausdorff multiplicative metric on $P(\check{Z})$.

Definition 1.7 ([34]) Let $L, R : \check{Z} \to P(\check{Z})$ be set-valued mappings and $\beta : \check{Z} \times \check{Z} \to [0, +\infty)$ be a mapping of positive real numbers. Then, L and R are called $\beta_\star$-admissible if for each $u, v \in \check{Z}$

$$\beta(u, v) \geq 1 \quad \Rightarrow \quad \beta_\star(Lu, Rv) \geq 1 \quad \text{and} \quad \beta_\star(Rv, Lu) \geq 1,$$

where $\beta_\star(Lu, Re) = \inf\{\beta(b, f) : b \in Lu, f \in Re\}$. If L intersects R, we obtain the definition of an α_*-admissible map [3].

Definition 1.8 ([26]) Let $\check{Z}$ be a nonempty set and $\alpha : \check{Z}^2 \to [0, +\infty)$ be a mapping. The multivalued map $W : \check{Z} \to P(\check{Z})$ is known as α_*-dominated on $V \subseteq \check{Z}$ if for every $p \in V$, $\alpha_*(p, Wp) = \inf\{\alpha(p, t) : t \in Wp\} > 1$.

Definition 1.9 ([39]) A self-map S defined on $(\check{Z}, d)$ is called a $G^\star$-contraction if there exists $\tau > 0$ such that for each $r, s \in \check{Z}$, $d(Sr, Ss) > 0$ implies

$$\tau + G^\star\big(d(Sr, Ss)\big) \leq G^\star\big(d(r, s)\big),$$

where $G^\star : (0, \infty) \to \mathbb{R}$ satisfies the following axioms:

($F1$) For each $\ell, \nu > 0$, $\ell < \nu \iff G^\star(\ell) < G^\star(\nu)$;
($F2$) $\lim_{n \to +\infty} g_n = 0 \iff \lim_{n \to +\infty} G^\star(g_n) = -\infty$, for every positive sequence $\{g_n\}_{n=1}^\infty$;
($F3$) for any $q \in (0, 1)$, $\lim_{g \to 0^+} g^q G^\star(g) = 0$.

Let $S^\star$ denotes the set of maps satisfying ($F1$)–($F3$).

Lemma 1.10 ([28]) *Let $(P(\check{Z}), H_\eta)$ be a multiplicative Hausdorff-metric space. If, for each $u \in L$ and for all $L, C \in P(\check{Z})$ there is $f_l \in C$ such that $\eta(u, C) = \eta(u, f_l)$. Then, the inequality $H_\eta(L, C) \geq \eta(u, f_l)$ holds.*

Example 1.11 ([26]) Assume that $\check{Z} = \mathbb{R}$ and define a mapping $\alpha : \check{Z}^2 \to [0, +\infty)$ by

$$\alpha(\ell, \nu) = \begin{cases} 1 & \text{if } \ell > \nu, \\ \frac{1}{2} & \text{if } \ell \leq \nu. \end{cases}$$

Define $R, K : \mathbb{R} \to P(\mathbb{R})$ by

$$Rt = [u - 2, u - 1] \quad \text{and} \quad Kr = [r - 5, r - 4].$$

Then, R and K are not α_*-admissible but are α_*-dominated.

2 Main results

Let $(\check{Z},\eta)$ be a multiplicative metric space, $f_0 \in \check{Z}$, and $\{L_\varsigma : \varsigma \in \Theta\}$, $\{M_\xi : \xi \in \Delta\}$ be two families of multivalued maps from $\check{Z}$ to $\mathcal{F}(\check{Z})$, where the $\mathcal{F}(\check{Z})$ contains all the closed and bounded subsets of $\check{Z}$. Let $f_1 \in L_a f_0$ be an element satisfying $\eta(f_0, L_a f_0) = \eta(f_0,f_1)$, and $f_2 \in M_b f_1$ satisfies the equation $\eta(f_1, M_b f_1) = \eta(f_1,f_2)$. We can find $f_3 \in L_a f_2$ satisfying the equation $\eta(f_2, L_a f_2) = \eta(f_2,f_3)$. In this way, we can construct a sequence $\{M_\xi L_\varsigma(f_n)\}$ in $\check{Z}$, where $f_{2n+1} \in L_i f_{2n}, f_{2n+2} \in M_j f_{2n+1}$, $n \in \mathbb{N}$, $i \in \Theta$ and $j \in \Delta$. Also, $\eta(f_{2n}, L_i f_{2n}) = \eta(f_{2n},f_{2n+1})$, $\eta(f_{2n+1}, M_j f_{2n+1}) = \eta(f_{2n+1},f_{2n+2})$. Then, $\{M_\xi L_\varsigma(f_n)\}$ is called a sequence in $\check{Z}$ generated by f_0. If $\{L_\varsigma : \varsigma \in \Theta\} = \{M_\xi : \xi \in \Delta\}$, then we say $\{\check{Z}L_\varsigma(f_n)\}$ instead of $\{M_\xi L_\varsigma(f_n)\}$. For $e, \nu \in \check{Z}$, $\lambda \in (0,\frac{1}{2})$, we define $M^*_{(\varsigma,\xi)}(e,y)$ as

$$M^*_{(\varsigma,\xi)}(e,y) = \left(\max\left\{\begin{array}{c}\eta(e,y), \eta(e,L_\varsigma e), \eta(y,M_\xi y),\\ \frac{\eta^2(e,L_\varsigma e).\eta(y,M_\xi y)}{1+\eta^2(e,y)}\end{array}\right\}\right)^\lambda.$$

Theorem 2.1 *Let $\check{Z}$ be a non-empty set and there exists a mapping $\alpha : \check{Z} \times \check{Z} \to [0,\infty)$, and $r > 0$, $\kappa_0 \in \overline{B_{\eta_m}(\kappa_0,r)} \subseteq \check{Z}$. Let $\{L_\varsigma : \varsigma \in \Theta\}$, $\{M_\xi : \xi \in \Delta\}$ be two families of α_*-dominated multimappings on $\overline{B_{\eta_m}(\kappa_0,r)}$. If there exist $\tau > 0$, $\lambda \in (0,\frac{1}{2})$ with $h = \frac{\lambda}{1-\lambda}$ and a strictly increasing mapping $G^\star$ such that*:

$$\tau + G^\star\big(H_\eta(L_\varsigma e, M_\xi y)\big) \le G^\star\big(M^*_{(\varsigma,\xi)}(e,y)\big)^\lambda, \tag{2.1}$$

for all $e, y \in \overline{B_{\eta_m}(\kappa_0,r)} \cap \{M_\xi L_\varsigma(\kappa_n)\}$ with $\alpha(e,y) \ge 1$, $\varsigma \in \Theta$, $\xi \in \Delta$, and $H_\eta(L_\varsigma e, M_\xi y) > 0$. If

$$\eta(\kappa_0, L_\varsigma \kappa_0) \le r^{1-h}. \tag{2.2}$$

Then, $\{M_\xi L_\varsigma(\kappa_n)\}$ is a sequence in $\overline{B_{\eta_m}(\kappa_0,r)}$, and if $(\check{Z},\eta)$ is complete multiplicative metric space, then $\{M_\xi L_\varsigma(\kappa_n)\} \to u \in \overline{B_{\eta_m}(\kappa_0,r)}$. Moreover, if $\alpha(\kappa_n,u) \ge 1$ and $\alpha(u,\kappa_n) \ge 1$ for all integers $n \ge 0$, then L_ς and M_ξ admit a common fixed point u in $\overline{B_{\eta_m}(\kappa_0,r)}$ for all $\varsigma \in \Theta$ and $\xi \in \Delta$.

Proof Let $\{M_\xi L_\varsigma(\kappa_n)\}$ be a sequence as constructed above. By (2.2), we have

$$\eta(\kappa_0,\kappa_1) = \eta(\kappa_0, L_\varsigma \kappa_0) \le r^{1-h} < r.$$

It follows that,

$$\kappa_1 \in \overline{B_{\eta_m}(\kappa_0,r)}.$$

Let $\kappa_2,\dots,\kappa_j \in \overline{B_{\eta_m}(\kappa_0,r)}$, for if $j = 2\grave{\imath} + 1$ for some $\grave{\imath} \in \mathbb{N}$ and since $\{L_\varsigma : \varsigma \in \Theta\}$ and $\{M_\xi : \xi \in \Delta\}$ be two families of α_*-dominated multimappings on $\overline{B_{\eta_m}(\kappa_0,r)}$, so $\alpha_*(\kappa_{2\grave{\imath}}, L_\varsigma \kappa_{2\grave{\imath}}) \ge 1$ and $\alpha_*(\kappa_{2\grave{\imath}+1}, M_\xi \kappa_{2\grave{\imath}+1}) \ge 1$ for all $\varsigma \in \Theta$ and $\xi \in \Delta$. As $\alpha_*(\kappa_{2\grave{\imath}}, L_\varsigma \kappa_{2\grave{\imath}}) \ge 1$, this tends to $\inf\{\alpha(\kappa_{2\grave{\imath}}, b) : b \in L_\varsigma \kappa_{2\grave{\imath}}\} \ge 1$. Also, $\kappa_{2\grave{\imath}+1} \in L_f \kappa_{2\grave{\imath}}$, $\kappa_{2\grave{\imath}+2} \in M_g \kappa_{2\grave{\imath}+1}$ for some $f \in \Theta$, and $g \in \Delta$ so $\alpha(\kappa_{2\grave{\imath}}, \kappa_{2\grave{\imath}+1}) \ge 1$. By Lemma 1.10, we have

$$\tau + G^\star\big(\eta(\kappa_{2\grave{\imath}+1},\kappa_{2\grave{\imath}+2})\big) \le \tau + G^\star\big(H_\eta(L_f \kappa_{2\grave{\imath}}, M_g \kappa_{2\grave{\imath}+1})\big) \le G^\star\big(M^*_{(f,g)}(\kappa_{2\grave{\imath}}, \kappa_{2\grave{\imath}+1})\big)^\lambda$$

$$\leq G^{\star}\left(\max\left\{\begin{array}{c}\eta(\kappa_{2\grave{\imath}},\kappa_{2\grave{\imath}+1}),\eta(\kappa_{2\grave{\imath}},\kappa_{2\grave{\imath}+1}),\\ \eta(\kappa_{2\grave{\imath}+1},\kappa_{2\grave{\imath}+2}),\frac{\eta(\kappa_{2\grave{\imath}},\kappa_{2\grave{\imath}+1}).\eta(\kappa_{2\grave{\imath}+1},\kappa_{2\grave{\imath}+2})}{1+\eta(\kappa_{2i},\kappa_{2i+1})}\end{array}\right\}^{\lambda}\right)$$
$$\leq G^{\star}\left(\max\left\{\eta(\kappa_{2\grave{\imath}},\kappa_{2\grave{\imath}+1}),\eta(\kappa_{2\grave{\imath}+1},\kappa_{2\grave{\imath}+2})\right\}^{\lambda}\right).$$

Thus,

$$\tau + G^{\star}\left(\eta(\kappa_{2\grave{\imath}+1},\kappa_{2\grave{\imath}+2})\right) \leq G^{\star}\left(\eta(\kappa_{2\grave{\imath}},\kappa_{2i+1})\right)^{h}. \tag{2.3}$$

Since $G^{\star} \in S^{\star}$ and for all $i \in \mathbb{N}$, where $h = \frac{\lambda}{1-\lambda}$. By (2.3), we have

$$\eta(\kappa_{2\grave{\imath}+1},\kappa_{2\grave{\imath}+2}) < \eta(\kappa_{2\grave{\imath}},\kappa_{2i+1})^{h}. \tag{2.4}$$

Similarly, if j is even, we have

$$\eta(\kappa_{2\grave{\imath}+2},\kappa_{2\grave{\imath}+3}) < \eta(\kappa_{2\grave{\imath}+1},\kappa_{2i+2})^{h}. \tag{2.5}$$

Now, we have

$$\eta(\kappa_{j},\kappa_{j+1}) < \eta(\kappa_{j-1},\kappa_{j})^{h} \quad \text{for all } j \in \mathbb{N}. \tag{2.6}$$

Therefore,

$$\begin{aligned}\eta(\kappa_{j},\kappa_{j+1}) &< \eta(\kappa_{j-1},\kappa_{j})^{h} < \eta(\kappa_{j-2},\kappa_{j-1})^{h^2} < \eta(\kappa_{j-3},\kappa_{j-2})^{h^3}\\ &< \eta(\kappa_{j-4},\kappa_{j-3})^{h^4} < \cdot \ldots \cdot < \eta(\kappa_{0},\kappa_{1})^{j}.\end{aligned} \tag{2.7}$$

Now,

$$\begin{aligned}\eta(\kappa_{0},\kappa_{j+1}) &\leq \eta(\kappa_{0},\kappa_{1})\cdot\eta(\kappa_{1},\kappa_{2})\cdot\eta(\kappa_{2},\kappa_{3})\cdot\ldots\cdot\eta(\kappa_{j},\kappa_{j+1})\\ &\leq \eta(\kappa_{0},\kappa_{1})\cdot\eta(\kappa_{0},\kappa_{1})^{h}\cdot\eta(\kappa_{0},\kappa_{1})^{h^2}\cdot\eta(\kappa_{0},\kappa_{1})^{h^3}\\ &\quad\times\eta(\kappa_{0},\kappa_{1})^{h^4}\cdot\ldots\cdot\eta(\kappa_{0},\kappa_{1})^{h^j}\\ &\leq \eta(\kappa_{0},\kappa_{1})^{(h^0+h^1+h^2+h^3+\cdots+h^j)}\\ &\leq \eta(\kappa_{0},\kappa_{1})^{\frac{(1-h^{j+1})}{1-h}}.\end{aligned}$$

Then, we have

$$\eta(\kappa_{0},\kappa_{j+1}) \leq r^{1-h\times\frac{(1-h^{j+1})}{1-h}} \leq r^{(1-h^{j+1})} < r,$$

which leads to $\kappa_{j+1} \in \overline{B_{\eta_m}(\kappa_0,r)}$. Thus, by induction $\kappa_n \in \overline{B_{\eta_m}(\kappa_0,r)}$ for all $n \in \mathbb{N}$. Also, $\alpha(\kappa_n,\kappa_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$. Now,

$$\eta(\kappa_{n},\kappa_{n+1}) < \eta(\kappa_{0},\kappa_{1})^{h^n} \quad \text{for all } n \in \mathbb{N}. \tag{2.8}$$

For any $n, m \in \mathbb{N}$, consider

$$\begin{aligned}
\eta(\kappa_m,\kappa_n) &\leq \eta(\kappa_m,\kappa_{m+1})\cdot\eta(\kappa_{m+1},\kappa_{m+2})\cdot\eta(\kappa_{m+2},\kappa_{m+3}) \\
&\quad \cdot \ldots \cdot \eta(\kappa_{n-1},\kappa_n) \\
&< \eta(\kappa_0,\kappa_1)^{h^m}\cdot\eta(\kappa_0,\kappa_1)^{h^{m+1}}\cdot\ldots\cdot \\
&\quad \eta(\kappa_0,\kappa_1)^{h^{n-1}}, (\text{by } (2.8)) \\
&< \eta(\kappa_0,\kappa_1)^{(h^m+h^{m+1}+h^{m+2}+\cdots+h^{n-1})} \\
\eta(\kappa_m,\kappa_n) &< \eta(\kappa_0,\kappa_1)^{(\frac{h^n}{1-h})}.
\end{aligned}$$

Clearly, $\eta(\kappa_m,\kappa_n) \to 1$ as $m, n \to +\infty$. Hence, $\{M_\xi L_\varsigma(\kappa_n)\}$ is a Cauchy sequence in a complete M^* space $(\overline{B_{\eta_m}(\kappa_0,r)},\eta)$ so there is $u \in \overline{B_{\eta_m}(\kappa_0,r)}$ and $\{M_\xi L_\varsigma(\kappa_n)\} \to u$ as $n \to +\infty$, then

$$\lim_{n\to+\infty}\eta(\kappa_n,u) = 1. \tag{2.9}$$

Now,

$$\eta(u,M_\xi u) \leq \eta(u,\kappa_{2n+1})\cdot\eta(\kappa_{2n+1},M_\xi u).$$

Hence, there exists some $e \in \zeta$ such that $\kappa_{2n+1} \in S_e\kappa_{2n}$ and $\eta(\kappa_{2n},S_e\kappa_{2n}) = \eta(\kappa_{2n},\kappa_{2n+1})$. By using (2.1) and Lemma 1.10, we obtain

$$\eta(u,M_\xi u) \leq \eta(u,\kappa_{2n+1})\cdot H_\eta(S_e\kappa_{2n},M_\xi u). \tag{2.10}$$

Given $\alpha(\kappa_n,u) \geq 1$, on the contrary, assume that $\eta(u,M_\xi u) > 0$, then $\eta(\kappa_n,M_\xi u) > 0$ for each $n \geq k$ and for $n \geq k$, we have

$$\begin{aligned}
&< \eta(u,\kappa_{2n+1}).\left(\max\begin{array}{c}\eta(\kappa_{2n},u),\eta(\kappa_{2n},M_\xi u), \\ \eta(\kappa_{2n+1},M_\xi u), \\ \dfrac{\eta(\kappa_{2n},\kappa_{2n+1}).\eta(\kappa_{2n+1},M_\xi u)}{1+\eta(\kappa_{2n},\kappa_{2n+1})}\end{array}\right)^{\lambda}, \\
&< \eta(u,\kappa_{2n+1})\cdot\left(\max\left\{\eta(\kappa_{2n},u),\eta(\kappa_{2n+1},M_\xi u)\right\}\right)^{\lambda}.
\end{aligned} \tag{2.11}$$

By (2.9) and the limit $n \to +\infty$ on both sides of (2.10), we obtain $\eta(u,M_\xi u) < \eta(u,M_\xi u)^\lambda$, a contradiction. Hence, $\eta(u,M_\xi u) = 1$ or $u \in M_\xi u$. Similarly, by using Lemma 1.10 and inequality (2.9) we obtain $\eta(u,L_\varsigma u) = 1$ or $u \in L_\varsigma u$. Hence, L_ς and M_ξ have a common fixed point u in $\overline{B_{\eta_m}(\kappa_0,r)}$. Now,

$$\eta(u,u) \leq \left[\eta(u,L_\varsigma u).\eta(L_\varsigma u,u)\right].$$

This tends to $\eta(u,u) = 1$. □

Example 2.2 Let $\check{Z} = [0, \infty)$ and define the mapping $\eta : [0, \infty)^2 \to [0, \infty)$ by

$$\eta(r, s) = \exp\bigl(|r - s|\bigr) \quad \text{for all } r, s \in \check{Z}.$$

Then, $(\check{Z}, \eta)$ is a complete multiplicative metric space. Define $L_\varsigma, M_\xi : [0, \infty) \to P([0, \infty))$ by

$$L_m\kappa = \begin{cases} [\frac{\kappa}{5m}, \frac{2}{5m}\kappa] & \text{if } \kappa \in [0, 15] \cap \check{Z}, \\ [2\kappa m, 3mc] & \text{if } \kappa \in (15, \infty) \cap \check{Z}, \end{cases} \quad \text{where } m = 1, 2, 3, \ldots,$$

$$M_n\kappa = \begin{cases} [\frac{\kappa}{7n}, \frac{3}{7n}\kappa] & \text{if } \kappa \in [0, 15] \cap \check{Z}, \\ [4n\kappa, 5n\kappa] & \text{if } \kappa \in (15, \infty) \cap \check{Z}, \end{cases} \quad \text{where } n = 1, 2, 3, \ldots.$$

Assume that, $\kappa_0 = 1$, $r = 15$, then $\overline{B_\eta(\kappa_0, r)} = [0, 15] \cap \check{Z}$. Now, $\eta(\kappa_0, L_1\kappa_0) = \eta(1, L_1 1) = \eta(1, \frac{1}{5})$. Hence, $\kappa_1 = \frac{1}{5}$. Now, $\eta(\kappa_1, M_1\kappa_1) = \eta(\frac{1}{5}, M_1\frac{1}{5}) = \eta(\frac{1}{5}, \frac{1}{35})$. Hence, $\kappa_2 = \frac{1}{35}$. Now, $\eta(\kappa_2, L_2\kappa_2) = \eta(\frac{1}{35}, L_2\frac{1}{35}) = \eta(\frac{1}{35}, \frac{1}{175})$. Hence, $\kappa_3 = \frac{1}{175}$. Continuing in this way, we have $\{M_n L_m(\kappa_n)\} = \{1, \frac{1}{5}, \frac{1}{35}, \frac{1}{175} \ldots.\}$. Moreover, taking $\lambda = \frac{7}{23} \in (0, \frac{1}{2})$ and $h = \frac{7}{17} \in (0, 1)$. Also, from (2.2), we obtain $\eta(\kappa_0, L_1\kappa_0) \leq 14^{1 - \frac{7}{17}} = 14^{\frac{10}{17}}$, which holds

$$\eta(\kappa_0, L_1\kappa_0) = e^{|1 - \frac{1}{3}|} < 14^{\frac{10}{17}}.$$

Consider the map $\alpha : [0, \infty)^2 \to [0, \infty)$ defined by

$$\alpha(a, b) = \begin{cases} 1 & \text{if } a > b, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Now, if $\kappa, \nu \in \overline{B_{\eta_m}(\kappa_0, r)} \cap \{M_\xi L_\varsigma(\kappa_n)\}$ with $\alpha(\kappa, \nu) \geq 1$, we have

$$\begin{aligned} H_\eta(L_m\kappa, M_n\nu) &= \max\Bigl\{\sup_{a \in L_m\kappa} \eta(a, M_n\nu), \sup_{b \in M_n\nu} \eta(L_m\kappa, b)\Bigr\} \\ &= \max\left\{\sup_{a \in S_m\kappa} \eta\left(a, \left[\frac{\nu}{4n}, \frac{3\nu}{4n}\right]\right), \sup_{b \in T_n\nu} \eta\left(\left[\frac{\kappa}{3m}, \frac{2\kappa}{3m}\right], b\right)\right\} \\ &= \max\left\{\eta\left(\frac{2\kappa}{3m}, \left[\frac{\nu}{4n}, \frac{3\nu}{4n}\right]\right), \eta\left(\left[\frac{\kappa}{3m}, \frac{2\kappa}{3m}\right], \frac{3\nu}{4n}\right)\right\} \\ &= \max\left\{\eta\left(\frac{2\kappa}{3m}, \frac{\nu}{4n}\right), \eta\left(\frac{\kappa}{3m}, \frac{3\nu}{4n}\right)\right\} \\ &= \max\bigl\{e^{|\frac{2\kappa}{3m} - \frac{\nu}{4n}|}, e^{|\frac{\kappa}{3m} - \frac{3\nu}{4n}|}\bigr\} \\ &< \max\begin{pmatrix} e^{|\kappa - \nu|}, e^{|\kappa - \frac{\kappa}{3m}|}, e^{|\nu - \frac{\nu}{4n}|}, \\ \frac{e^{|\kappa - \frac{\kappa}{3m}|^2} \cdot e^{|\nu - \frac{\nu}{4n}|^2}}{1 + e^{|\kappa - \frac{\kappa}{3m}|^2}} \end{pmatrix}^\lambda \\ &< \max\begin{pmatrix} \eta(\kappa, \nu), \frac{\eta(\kappa, [\frac{\kappa}{3m}, \frac{2}{3m}\kappa]) . \eta(\nu, [\frac{\nu}{4n}, \frac{3}{4n}\nu])}{1 + \eta(\kappa, \nu)}, \\ \eta(\kappa, [\frac{\kappa}{3m}, \frac{2}{3m}\kappa]), \eta(\kappa, [\frac{\nu}{4n}, \frac{3}{4n}\nu]) \end{pmatrix}^\lambda. \end{aligned}$$

Thus,

$$H_\eta(L_m\kappa, M_n\nu)) < \big(M^*_{(\varsigma,\xi)}(\kappa,\nu)\big),$$

we can find $0 < \tau \leq \frac{12}{95}$ and a mapping $G^\star$ defined by $G^\star(s) = \ln s$, we have

$$\tau + G^\star\big(H_\eta(L_m\kappa, M_n\nu)\big) \leq G^\star\big(M^*_{(\varsigma,\xi)}(\kappa,\nu)\big).$$

Note that, for 15, $16 \in \check{Z}$, then $\alpha(16,15) \geq 1$. However, we have

$$\tau + G^\star\big(H_\eta(L_2 16, M_1 15)\big) > G^\star\big(M^*_{(\varsigma,\xi)}(16,15)\big).$$

Hence, all the assumptions in the statement of Theorem 2.1 have been verified. Hence, L_ς and M_ξ admit a common fixed point for each $\varsigma \in \Theta$ and $\xi \in \Delta$ that is 0.

Corollary 2.3 *Assume that there exists a mapping $\alpha : \check{Z} \times \check{Z} \to [0,\infty)$, and $r > 0$, $\kappa_0 \in \overline{B_{\eta_m}(\kappa_0, r)} \subseteq \check{Z}$. Let $\{L_\varsigma : \varsigma \in \Theta\}$, $\{M_\xi : \xi \in \Delta\}$ be two families of α_*-dominated multimappings on $\overline{B_{\eta_m}(\kappa_0, r)}$. If there exist $\tau > 0$, $\lambda \in (0, \frac{1}{2})$ with $h = \frac{\lambda}{1-\lambda}$ and strictly increasing mapping $G^\star$ so that*:

$$\tau + G^\star\big(\eta(L_\varsigma e, M_\xi y)\big) \leq G^\star\big(M^*_{(\varsigma,\xi)}(e,y)\big)^\lambda. \tag{2.12}$$

For each $e, y \in \overline{B_{\eta_m}(\kappa_0, r)} \cap \{\kappa_n\}$, $\alpha(e,y) \geq 1$, $\varsigma \in \Theta$, $\xi \in \Delta$, and $\eta(L_\varsigma e, M_\xi y) > 0$ such that,

$$\eta(\kappa_0, L_\varsigma \kappa_0) \leq r^{1-h}.$$

Then, $\{M_\xi L_\varsigma(\kappa_n)\}$ is a sequence in $\overline{B_{\eta_m}(\kappa_0, r)}$, and if $(\check{Z}, \eta)$ is complete multiplicative metric space, then $\{M_\xi L_\varsigma(\kappa_n)\} \to u \in \overline{B_{\eta_m}(\kappa_0, r)}$. Moreover, if $\alpha(\kappa_n, u) \geq 1$ and $\alpha(u, \kappa_n) \geq 1$ for all integers $n \geq 0$, then L_ς and M_ξ admit a common fixed point u in $\overline{B_{\eta_m}(\kappa_0, r)}$ for all $\varsigma \in \Theta$ and $\xi \in \Delta$.

Corollary 2.4 *Assume that there exists a mapping $\alpha : \check{Z} \times \check{Z} \to [0,\infty)$, and $r > 0$, $\kappa_0 \in \overline{B_{\eta_m}(\kappa_0, r)} \subseteq \check{Z}$. Let $\{L_\varsigma : \varsigma \in \Theta\}$, $\{M_\xi : \xi \in \Delta\}$ be two families of α_*-dominated multimappings on $\overline{B_{\eta_m}(\kappa_0, r)}$. If there exist $\tau > 0$, $\lambda \in (0, \frac{1}{2})$ with $h = \frac{\lambda}{1-\lambda}$ and a strictly increasing mapping $G^\star$ so that*:

$$\tau + G^\star\big(H_\eta(L_\varsigma e, S_\beta y)\big) \leq G^\star\big(M^*_{(\varsigma,\xi)}(e,y)\big)^\lambda, \tag{2.13}$$

for all $e, y \in \overline{B_{\eta_m}(\kappa_0, r)} \cap \{\check{Z} L_\varsigma(\kappa_n)\}$, $\alpha(e,y) \geq 1$, $\varsigma \in \Theta$, $\xi \in \Delta$, and $H_\eta(L_\varsigma e, S_\beta y) > 0$ such that:

$$\eta(\kappa_0, L_\varsigma \kappa_0) \leq r^{1-h}.$$

Then, $\{M_\xi L_\varsigma(\kappa_n)\}$ is a sequence in $\overline{B_{\eta_m}(\kappa_0, r)}$, $\alpha(\kappa_n, \kappa_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\{M_\xi L_\varsigma(\kappa_n)\} \to u \in \overline{B_{\eta_m}(\kappa_0, r)}$. Also, if $\alpha(\kappa_n, u) \geq 1$ and $\alpha(u, \kappa_n) \geq 1$ for all integers $n \geq 0$, then L_ς and M_ξ admit a fixed point u in $\overline{B_{\eta_m}(\kappa_0, r)}$ for all $\varsigma \in \Theta$ and $\xi \in \Delta$.

3 Fixed-point theorems involving families of multigraph-dominated mappings

In this section, we give an application of Theorem 2.1 in graph theory. The details about the use of graphs in fixed-point theory can be seen in [9, 16].

Definition 3.1 Let $Y \neq \Phi$ and $\Omega = (V(\Omega), \Xi(\Omega))$; $V(\Omega) = X$, $B \subseteq Y$. We say the mapping $F : Y \to P(Y)$ is multigraph dominated on B if $(r, y) \in \Xi(\Omega)$, for each $y \in Fr$ and $r \in B$.

Theorem 3.2 *Let $(\check{Z}, \eta)$ be a complete multiplicative metric space associated to Ω. Let $r > 0$, $\kappa_0 \in \overline{B_{\eta_m}(\kappa_0, r)}$ and $\{L_\varsigma : \varsigma \in \Theta\}$, $\{M_\xi : \xi \in \Delta\}$ be two families of multimappings from $\check{A}$ to $P(\check{A})$. Assume the following*:

(i) *$\{L_\varsigma : \varsigma \in \Theta\}$, $\{M_\xi : \xi \in \Delta\}$ be defined on $\overline{B_{\eta_m}(\kappa_0, r)} \cap \{M_\xi L_\varsigma(\kappa_n)\}$.*

(ii) *There exist $\tau > 0$ and $G^\star$ is a monotonic mapping satisfying*;

$$\tau + G^\star\bigl(H_\eta(L_\varsigma a, M_\xi b)\bigr) \leq G^\star\bigl(M^*_{(\varsigma,\xi)}(a, b)\bigr), \tag{3.1}$$

for all $a, b \in \overline{B_{\eta_m}(\kappa_0, r)} \cap \{M_\xi L_\varsigma(\kappa_n)\}$, $(a, b) \in \Xi(\Omega)$, $\varsigma \in \Theta$, $\xi \in \Delta$ and $H_\eta(L_\varsigma a, M_\xi b) > 0$.

(iii) *$\eta(\kappa_0, L_\varsigma \kappa_0) \leq r^{1-h}$. Then, $\{M_\xi L_\varsigma(\kappa_n)\}$ is a sequence in $\overline{B_{\eta_m}(\kappa_0, r)}$, $(\kappa_n, \kappa_{n+1}) \in \Xi(\Omega)$ and $\{M_\xi L_\varsigma(\kappa_n)\} \to f$. Also, if f satisfies (3.1) and $(\kappa_n, f) \in \Xi(\Omega)$ or $(f, \kappa_n) \in \Xi(\Omega)$ for all naturals where $n = 1, 2, 3, \ldots$, L_ς and M_ξ have a common fixed point f in $\overline{B_{\eta_m}(\kappa_0, r)}$.*

Proof Define the mapping $\alpha : \check{A} \times \check{A} \to [0, \infty)$ by

$$\alpha(a, b) = \begin{cases} 1 & \text{if } a \in \overline{B_{\eta_m}(\kappa_0, r)},\ (a, b) \in \Xi(\Omega), \\ 0 & \text{otherwise.} \end{cases}$$

Since L_ς and M_ξ are graph dominated on $\overline{B_{\eta_m}(\kappa_0, r)}$, for $e \in \overline{B_{\eta_m}(\kappa_0, r)}$, $(a, b) \in \Xi(\Omega)$ for each $b \in L_\varsigma a$ and $(a, b) \in \Xi(\Omega)$ for all $b \in M_\xi a$. Hence, $\alpha(a, b) = 1$ for all $b \in L_\varsigma a$ and $\alpha(a, b) = 1$ for each $b \in M_\xi a$. This leads us to $\inf\{\alpha(a, b) : b \in L_\varsigma a\} = 1$ and $\inf\{\alpha(a, b) : b \in M_\xi a\} = 1$. Hence, $\alpha_*(a, L_\varsigma a) = 1$, $\alpha_*(a, M_\xi a) = 1$ for each $a \in \overline{B_{\eta_m}(\kappa_0, r)}$. Hence, $L_\varsigma, M_\xi : \check{Z} \to P(\check{Z})$ are the families of α_*-dominated multivalued mappings on $\overline{B_{\eta_m}(\kappa_0, r)}$. Furthermore, inequality (3.1) can be rewritten as

$$\tau + G^\star\bigl(H_\eta(L_\varsigma a, M_\xi b)\bigr) \leq G^\star\bigl(M^*_{(\varsigma,\xi)}(a, b)\bigr),$$

for all $a, b \in \overline{B_{\eta_m}(\kappa_0, r)} \cap \{M_\xi L_\varsigma(\kappa_n)\}$, $\alpha(a, b) \geq 1$ and $H_\eta(L_\varsigma a, M_\xi b) > 0$. Also, (iii) holds. Then, from Theorem 2.1, we have $\{M_\xi L_\varsigma(\kappa_n)\}$ is a sequence in $\overline{B_{\eta_m}(\kappa_0, r)}$ and $\{M_\xi L_\varsigma(\kappa_n)\} \to f \in \overline{B_{\eta_m}(\kappa_0, r)}$. Now, $\kappa_n, f \in \overline{B_{\eta_m}(\kappa_0, r)}$ and either $(\kappa_n, f) \in \Xi(\Omega)$ or $(f, \kappa_n) \in \Xi(\Omega)$ leads to either $\alpha(\kappa_n, f) \geq 1$ or $\alpha(f, \kappa_n) \geq 1$. Hence, all the conditions of Theorem 2.1 are satisfied. Hence, by Theorem 2.1, L_ς and M_ξ have a common fixed point f in $\overline{B_{\eta_m}(\kappa_0, r)}$ and $\eta(f, f) = 0$. □

For single-valued mappings, we have the following theorem:

Theorem 3.3 *Let $\kappa_0 \in \check{Z}$ and $\{L_\varsigma : \varsigma \in \Theta\}$, $\{M_\xi : \xi \in \Delta\}$ be two families of maps defined on the complete multiplicative metric space $(\check{Z}, \eta)$. Suppose there exist $\tau > 0$ and a monotonically increasing mapping $G^\star : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following condition*:

$$\tau + G^\star\bigl(\eta(L_\varsigma a, M_\xi b)\bigr) \le G^\star\bigl(M^*_{(\varsigma,\xi)}(a,b)\bigr),$$

whenever $a, b \in \{\kappa_n\}$, $\varsigma \in \Theta$, $\xi \in \Delta$ and $\eta(L_\varsigma a, M_\xi b) > 0$. Then, L_ς and M_ξ admit a unique common fixed point in $\check{Z}$ for all $\varsigma \in \Theta$ and $\xi \in \Delta$.

Proof The proof of Theorem 3.3 can be derived from the proof of Theorem 2.1 by taking $\mathcal{F}(\check{Z}) = \check{Z}$ and the mapping $\alpha : \check{Z} \times \check{Z} \to [0,\infty)$ is defined by $\alpha(z_1, z_2) = 1$ for all $z_1, z_2 \in \check{Z}$. Under this setting we have $H_\eta(L_\varsigma a, M_\xi b) = \eta(L_\varsigma a, M_\xi b)$ and the proof follows. □

4 Application to integral equations

The theory of integral equations may be traced back at least to Fourier's discovery of the theorem concerning integrals that bears his name; indeed, while not Fourier's point of view, this theorem can be seen as a statement of the solution of a certain first-order integral equation. However, Abel and Liouville, as well as others after them, began to study exceptional integral equations in a fully conscious manner, and many of them recognized the critical role the theory was destined to play. We intend to apply Theorem 3.3 to show the existence of the solution to the following family of nonlinear Volterra-type integral equations:

$$f(k) = \int_0^k H_\varsigma(k, h, f)\, dh, \tag{4.1}$$

$$\varkappa(k) = \int_0^k G_\xi(k, h, \varkappa))\, dh, \tag{4.2}$$

for all $k \in [0,1]$, $\varsigma \in \Theta$, $\xi \in \Delta$ and H_ς, G_ξ are mappings defined on $[0,1]^2 \times C([0,1], \mathbb{R}_+)$ to $\mathbb{R}$. We show the existence of the solution to (4.1) and (4.2). For $f \in C([0,1], \mathbb{R}_+)$, define norm as: $\|f\|_\tau = \sup_{k\in[0,1]}\{e^{|f(k)|}e^{-\tau k}\}$, $\tau > 0$. Then, define

$$\eta_\tau(f, \varkappa) = \Bigl[\sup_{k\in[0,1]}\bigl\{e^{|f(k)-\varkappa(k)|}e^{-\tau k}\bigr\}\Bigr] = e^{\|f-\varkappa\|_\tau},$$

for all $f, \varkappa \in C([0,1], \mathbb{R}_+)$, with these settings, $(C([0,1], \mathbb{R}_+), \eta_\tau)$ becomes a complete multiplicative metric space.

The following theorem describes the criteria for the existence of the solution to integral equations.

Theorem 4.1 *Suppose that the following are satisfied*:

(i) $\{H_\varsigma, \varsigma \in \Theta\}$, $\{G_\xi, \xi \in \Delta\}$ *are two families of maps from* $[0,1] \times [0,1] \times C([0,1], \mathbb{R}_+)$ *to* $\mathbb{R}$;

(ii) *Define*

$$(L_\varsigma f)(k) = \int_0^k H_\varsigma(k, h, f)\, dh,$$

$$(M_{\xi}\varkappa)(k) = \int_0^k G_{\xi}(k,h,\varkappa)\,dh.$$

If there exist $\tau > 0$, *such that*

$$e^{|H_{\varsigma}(k,h,f)-G_{\xi}(k,h,c)|} \leq \frac{\tau M^{*}_{(\varsigma,\xi)}(f,c)}{\tau M^{*}_{(\varsigma,\xi)}(f,c)+1}$$

for every $k,h \in [0,1]$ *and* $f,\varkappa \in C([0,1],\mathbb{R})$, *where*

$$M^{*}_{(\varsigma,\xi)}(f,\varkappa) = \max\left\{\begin{array}{c} e^{\|f-\varkappa\|_{\tau}}, e^{\|f-L_{\varsigma}f)\|_{\tau}}, e^{\|\varkappa-M_{\xi}\varkappa)\|_{\tau}}, \\ \frac{e^{\|f-L_{\varsigma}f)\|^{2}_{\tau}}.e^{\|\varkappa-M_{\xi}\varkappa)\|_{\tau}}}{1+e^{\|f(h)-\varkappa(h)\|^{2}_{\tau}}}, \end{array}\right\}.$$

Then, integral equations (4.1) *and* (4.2) *admit a unique solution in* $C([0,1],\mathbb{R}_{+})$.

Proof By (ii)

$$\begin{aligned} e^{|L_{\varsigma}f-M_{\xi}\varkappa|} &= \int_0^k e^{|H_{\varsigma}(k,h,f)-G_{\xi}(k,h,\varkappa)|}\,dh, \\ &\leq \int_0^k \frac{\tau M^{*}_{(\varsigma,\xi)}(f,\varkappa)}{\tau M^{*}_{(\varsigma,\xi)}(f,\varkappa)+1} e^{\tau h}\,dh \\ &\leq \frac{\tau M^{*}_{(\varsigma,\xi)}(f,\varkappa)}{\tau M^{*}_{(\varsigma,\xi)}(f,\varkappa)+1}\int_0^k e^{\tau h}\,dh \\ &\leq \frac{M^{*}_{(\varsigma,\xi)}(f,\varkappa)}{\tau M^{*}_{(\varsigma,\xi)}(f,\varkappa)+1} e^{\tau k}. \end{aligned}$$

This implies

$$\begin{aligned} &e^{|L_{\varsigma}f-M_{\xi}\varkappa|}e^{-\tau k} \leq \frac{M^{*}_{(\varsigma,\xi)}(f,\varkappa)}{\tau M^{*}_{(\varsigma,\xi)}(f,\varkappa)+1}, \\ &e^{\|L_{\varsigma}f-M_{\xi}\varkappa\|_{\tau}} \leq \frac{M^{*}_{(\varsigma,\xi)}(f,\varkappa)}{\tau M^{*}_{(\varsigma,\xi)}(f,\varkappa)+1}, \\ &\frac{\tau M^{*}_{(\varsigma,\xi)}(f,\varkappa)+1}{M^{*}_{(\varsigma,\xi)}(f,\varkappa)} \leq \frac{1}{e^{\|L_{\varsigma}f-M_{\xi}\varkappa\|_{\tau}}}, \\ &\tau + \frac{1}{M^{*}_{(\varsigma,\xi)}(f,\varkappa)} \leq \frac{1}{e^{\|L_{\varsigma}f-M_{\xi}\varkappa\|_{\tau}}}, \end{aligned}$$

which further implies

$$\tau - \frac{1}{e^{\|L_{\varsigma}f-M_{\xi}\varkappa\|_{\tau}}} \leq \frac{-1}{M^{*}_{(\varsigma,\xi)}(f,\varkappa)}.$$

Hence, all the conditions of Theorem 3.3 are satisfied for $G^{\star}(\varkappa) = \frac{-1}{\varkappa}$; $\varkappa > 0$ and $d_{\tau}(f,\varkappa) = e^{\|f-\varkappa\|_{\tau}}$. Hence, the given two families of integral equations in (4.1) and (4.2) have a unique common solution. □

5 Application to functional equations

Equations in which the unknowns are mappings rather than traditional variables are known as functional equations. The methods for solving functional equations, on the other hand, can differ significantly from those for isolating a classical variable. Here, we present an application of Theorem 3.3 to show the existence of the solution to a functional equation in dynamical programming.

Let $\bar{O}$ and Γ be two Banach spaces, $\$ \subseteq \bar{O}$, $\mho \subseteq \Gamma$ and

$$\begin{aligned} \sigma: &\quad \$ \times \mho \to \$,\\ \hbar, \ddot{y}: &\quad \$ \times \mho \to \mathbb{R},\\ C, K: &\quad \$ \times \mho \times \mathbb{R} \to \mathbb{R}. \end{aligned}$$

Further useful results relevant to dynamic programming are shown in ([7, 8, 25]). We assume that $\$$ and $\mho$ show only for the decisions spaces. The problem related to dynamical programming is to find the solution of the given equations:

$$p(\alpha) = \sup_{\alpha \in \mho} \left\{ \hbar(\alpha, \theta) + C\left(\alpha, \theta, p\left(\sigma(\alpha, \theta)\right)\right) \right\}, \tag{5.1}$$

$$q(\alpha) = \sup_{\theta \in \mho} \left\{ \ddot{y}(\alpha, \theta) + K\left(\alpha, \theta, q\left(\sigma(\alpha, \theta)\right)\right) \right\}, \tag{5.2}$$

for $\alpha \in \$$. We want to show the equations (5.1) and (5.2) have a unique solution. Suppose $R(\$)$ represents the class of all positive-valued mappings on $\$$. Consider,

$$\eta(\upsilon, w) = \left\| e^{\upsilon - w} \right\|_{\infty} = \sup_{\alpha \in N} e^{|\upsilon(\alpha) - w(\alpha)|}, \tag{5.3}$$

for all $\upsilon, w \in R(\$)$, and $(R(\$), \eta)$ becomes a complete multiplicative metric space. Assume that

($\hat{C}1$): $C, K, \hbar$, and $\ddot{y}$ are bounded.

($\hat{C}2$): For $\alpha \in \$$, $\upsilon \in R(\$)$, and two families of mappings $\Upsilon_{\varsigma}, \bar{O}_{\xi} : R(\$) \to R(\$)$, take

$$C\upsilon(\alpha) = \sup_{\theta \in \mho} \left\{ \hbar(\alpha, \theta) + C\left(\alpha, \theta, \upsilon\left(\sigma(\alpha, \theta)\right)\right) \right\}, \tag{5.4}$$

$$\bar{O}_{\xi}\upsilon(\alpha) = \sup_{\theta \in \mho} \left\{ \ddot{y}(\alpha, \theta) + K\left(\alpha, \theta, \upsilon\left(\sigma(\alpha, \theta)\right)\right) \right\}. \tag{5.5}$$

Furthermore, for each $(\alpha, \theta) \in \$ \times \mho$, $\upsilon, w \in R(\$)$, $t \in \$$ and $\tau > 0$,

$$\left| C\left(\alpha, \theta, \upsilon(t)\right) - K(\alpha, \theta, w(t) \right| \le M^{*}_{(\zeta, \xi)}(\upsilon, w) e^{-\tau}, \tag{5.6}$$

where

$$M^{*}_{(\zeta, \xi)}(\upsilon, w) = \sup \left\{ \begin{pmatrix} e^{|\upsilon(t) - w(t)|}, \\ e^{|\upsilon(t) - \Upsilon_{\varsigma}\upsilon(t)|}, e^{|\upsilon(t) - \bar{O}_{\xi} w(t)|} \\ \frac{e^{|\upsilon(t) - \Upsilon_{\varsigma}\upsilon(t)|^2} . e^{|\upsilon(t) - \bar{O}_{\xi} w(t)|}}{1 + e^{|\upsilon(t) - w(t)|^2}}, \end{pmatrix}^{\lambda} \right\}.$$

Theorem 5.1 *Suppose that* ($\hat{C}1$), ($\hat{C}2$), *and* (5.6) *hold. Then, the equations* (5.1) *and* (5.2) *have a unique common and bounded solution in* $R(\$)$.

Proof Take any $c > 0$. By (5.4) and (5.5), there are $\upsilon_1, \upsilon_2 \in R(\$)$, and $\theta_1, \theta_2 \in \mho$ such that

$$(\Upsilon_\varsigma \upsilon_1) < \hbar(\alpha,\theta_1) + C\bigl(\alpha,\theta_1,\upsilon_1\bigl(\sigma(\alpha,\theta_1)\bigr)\bigr) + c, \tag{5.7}$$

$$(\bar{O}_\xi \upsilon_2) < \hbar(\alpha,\theta_2) + K\bigl(\alpha,\theta_2,\upsilon_2\bigl(\sigma(\alpha,\theta_2)\bigr)\bigr) + c. \tag{5.8}$$

Using the definition of supremum, we obtain

$$(\Upsilon_\varsigma \upsilon_1) \geq \hbar(\alpha,\theta_2) + C\bigl(\alpha,\theta_2,\upsilon_1\bigl(\sigma(\alpha,\theta_2)\bigr)\bigr), \tag{5.9}$$

$$(\bar{O}_\xi \upsilon_2) \geq \hbar(\alpha,\theta_1) + K\bigl(\alpha,\theta_1,\upsilon_2\bigl(\sigma(\alpha,\theta_1)\bigr)\bigr). \tag{5.10}$$

Then, from (5.6), (5.7), and (5.10), we have

$$\begin{aligned}
&(\Upsilon_\varsigma \upsilon_1)(\alpha) - (\bar{O}_\xi \upsilon_2)(\alpha)\\
&\quad \leq C\bigl(\alpha,\theta_1,\upsilon_1\bigl(\sigma(\alpha,\theta_1)\bigr)\bigr) - K\bigl(\alpha,\theta_1,\upsilon_2\bigl(\sigma(\alpha,\theta_1)\bigr)\bigr) + c\\
&\quad \leq \bigl|C\bigl(\alpha,\theta_1,\upsilon_1\bigl(\sigma(\alpha,\theta_1)\bigr)\bigr) - K\bigl(\alpha,\theta_1,\upsilon_2\bigl(\sigma(\alpha,\theta_1)\bigr)\bigr)\bigr| + c\\
&\quad \leq M^*_{(\varsigma,\xi)}(\upsilon,w)e^{-\tau} + c.
\end{aligned}$$

Since, $c > 0$ is arbitrary, we obtain

$$\begin{aligned}
&\bigl|\Upsilon_\varsigma \upsilon_1(\alpha) - \bar{O}_\xi \upsilon_2(\alpha)\bigr| \leq M^*_{(\varsigma,\xi)}(\upsilon,w)e^{-\tau},\\
&e^\tau\bigl|\Upsilon_\varsigma \upsilon_1(\alpha) - \bar{O}_\xi \upsilon_2(\alpha)\bigr| \leq M^*_{(\varsigma,\xi)}(\upsilon,w).
\end{aligned}$$

This implies that

$$\tau + \ln\bigl|\Upsilon_\varsigma \upsilon_1(\alpha) - \bar{O}_\xi \upsilon_2(\alpha)\bigr| \leq \ln\bigl(M^*_{(\varsigma,\xi)}(\upsilon,w)\bigr).$$

Thus, all the requirements of Theorem 5.1 hold for $T(\ddot{y}) = \ln\ddot{y}; \ddot{y} > 0$ and $\eta_\tau(\upsilon,w) = e^{\|\upsilon-w\|_\tau}$. Hence, C and K both have a common fixed point $\upsilon^* \in R(\$)$ and $\upsilon^*(\alpha)$ is the unique solution of both (5.1) and (5.2). □

6 Conclusion

In this manuscript, we achieved some new fixed-point results for two families of set-valued maps satisfying a generalized contractive conditions only on a closed ball with an intersection of an iterative sequence in a complete multiplicative-metric space. A strictly increasing map $G^\star$ has been used instead of the class of maps that were used by Wardowski [39]. We apply dominated maps to obtain some new results for fixed points. The notion of two families of multigraph-dominated mappings is introduced. Furthermore, some new fixed-point result on a closed ball are obtained for graphic contraction in a multiplicative-metric space. Some applications are given to approximate the unique common bounded solution for coupled system of nonlinear integral equations and functional equations in dynamical programming. Our results extend and generalize many results appearing in the literature such as Rasham et al. [17, 29–31, 33], Wordowski's result [39], Acar et al. [21] and many classical results. This work can be reproduced in the directions given in [11, 18, 37].

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