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Bilateral series in terms of mixed mock modular forms

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Abstract

The number of strongly unimodal sequences of weight n is denoted by $u^*(n)$. The generating functions for $\{u^*(n)\}_{n=1}^{\infty}$ are $U^*(q) = \sum_{n=1}^{\infty} u^*(n)q^n$. Rhoades recently gave a precise asymptotic for $u^*(n)$ by expressing $U^*(q)$ as a mixed mock modular form. In this note, by revisiting the mixed mock modular form associated to $U^*(q)$, three new mixed mock modular forms are constructed by considering the bilateral series of $U^*(q)$ and the third order Ramanujan's mock theta function $f(q)$. The inner relationships among them are discussed although they are defined in different ways. These new mixed mock modular forms can be expressed in terms of Appell-Lerch sums. The related mock theta functions can be completed as harmonic weak Maass forms. As an application, we give a proof for the claim by Bajpai *et al.* that the bilateral series $B(f;q)$ of the third order mock theta function $f(q)$ is a mixed mock modular form.

MSC: 11F37; 11F03; 11F99

Keywords: bilateral series; mixed mock modular form; harmonic weak Maass form; Appell-Lerch sums

1 Introduction

Recall that the definition of the unimodal sequence of weight n is the following [1].

Definition 1.1 Assume there exists a sequence of integers $\{d_1, d_2, \dots, d_m\}$ such that:

(1) $\sum_{j=1}^{m} d_j = n$.
(2) For some j, we have $1 \le d_1 \le \cdots \le d_{j-1} \le d_j \ge d_{j+1} \ge \cdots \ge d_m \ge 1$.

Then the sequence of integers $\{d_1, d_2, \dots, d_m\}$ is called the unimodal sequence of weight n.

The number of the unimodal sequences of weight n is denoted by $u(n)$. For instance, $u(3) = 4$. The unimodal sequences of weight 3 are $\{1,1,1\}$, $\{1,2\}$, $\{2,1\}$, and $\{3\}$.

Definition 1.2 The strongly unimodal sequence of weight n is a sequence just with the condition (2) in Definition 1.1 replaced by:

(2)′ For some j, we have $0 < d_1 < \cdots < d_{j-1} < d_j > d_{j+1} > \cdots > d_m > 0$.

Denote the number of strongly unimodal sequences of weight n by $u^*(n)$. For example, $u^*(3) = 3$. The strongly unimodal sequences of weight 3 are $\{1,2\}$, $\{2,1\}$, and $\{3\}$.

If the unimodal sequences of weight n are not strongly unimodal sequences of weight n, then they are usually referred as weakly unimodal sequences. For example, the sequences of binomial coefficients $\{\binom{n}{j}\}_{j=0}^{n}$ are the unimodal sequences of weight 2^n. If n is even in the 2^n, then the associated sequences of binomial coefficients $\{\binom{n}{j}\}_{j=0}^{n}$ are strongly unimodal sequences of weight 2^n. If n is odd in the 2^n, then the associated sequences of binomial coefficients $\{\binom{n}{j}\}_{j=0}^{n}$ are weakly unimodal sequences of weight 2^n.

In order to introduce the result of Rhoades for $u^*(n)$, let us give some notations first.

Suppose $N := [n^{\frac{1}{2}}]$. We define the periodic function $((x))$ as follows [2]:

$$\big((x)\big) := \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \in \mathbb{R}\backslash\mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases} \tag{1}$$

Then the Dedekind sum is defined by [2]

$$s(h,k) := \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right), \tag{2}$$

it also can be represented as

$$s(h,k) := \sum_{\mu (\operatorname{mod} k)} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right). \tag{3}$$

Finally, by following Rhoades, we define $\omega_{h,k} := \exp(\pi i s(h,k))$.

Then Rhoades (Theorem 1.1 in [1]) gave a precise asymptotic for the strongly unimodal sequences as follows:

$$u^*(n) = \frac{1}{(24n-1)^{1/2}} \sum_{0<k<N, 2\nmid k} \frac{1}{k} \sum_{\ell=0}^{k-1} \mathcal{A}_k(\ell,n)\tilde{\mathcal{I}}_{\ell,k}(24n-1) + O\big(n^{1+\epsilon}\big), \tag{4}$$

where

$$\tilde{\mathcal{I}}_{\ell,k}(m) := \int_0^1 \frac{\cosh(\frac{\pi x}{\sqrt{3k}}(\ell - \frac{k-1}{2}))}{\cosh(\frac{\pi x}{2\sqrt{3}})} \sinh\left(\frac{\pi}{6k}\sqrt{m\big(1-x^2\big)}\right) dx, \quad \text{for } m \geq 1 \tag{5}$$

and

$$\mathcal{A}_k(\ell,n) := i^{\frac{1-k}{2}} \sum_{h (\operatorname{mod} k)} (-1)^{\frac{k+1}{2}(1-h)} \left(\frac{h}{k}\right) \omega_{h,k}^{-1} e^{-\pi i \frac{h}{k}(\ell - \frac{k-1}{2})^2 - \frac{2\pi i h n}{k}}, \tag{6}$$

with k odd, $0 \leq \ell < k$, $0 < h < k$, $(h,k) = 1$.

As $n \to \infty$, he also described the leading order asymptotic behavior of $u^*(n)$,

$$u^*(n) = \frac{\sqrt{3}}{2(24n-1)^{3/4}} \exp\left(\frac{\pi}{6}\sqrt{24n-1}\right)\left(1 - \frac{2\pi^2 + 9}{2^6\pi} \frac{1}{(24n-1)^{\frac{1}{2}}} + O\left(\frac{1}{n}\right)\right). \tag{7}$$

The generating functions for $\{u(n)\}_{n=1}^{\infty}$ and $\{u^*(n)\}_{n=1}^{\infty}$ are denoted by $U(q) = \sum_{n=1}^{\infty} u(n)q^n$ and $U^*(q) = \sum_{n=1}^{\infty} u^*(n)q^n$, respectively.

The key theory in the proof of the asymptotic for $u^*(n)$ relies on an identity expressing $U^*(q)$ as a mixed mock modular form. The function $f(q)$ is called a mixed mock modular form if it is the product of a modular form and a mock theta function. Indeed, both $U^*(q)$ and the related mixed mock modular form are associated with the Ramanujan mock theta functions. By using the pseudo-modularity of mixed mock modular forms, and a version of the circle method developed by Bringmann and Mahlburg [3], the asymptotic formulas for $u^*(n)$ are proved by Rhoades.

Auluck in 1951 [4] and Wright in 1971 [5] proved the following identity, respectively:

$$U(q) = \sum_{n=0}^{\infty} \frac{q^n}{(q)_{n-1}(q)_n}, \tag{8}$$

where $(x)_n = (x;q)_n := \prod_{j=0}^{n-1}(1 - xq^j)$.

On the other hand, Stanley [6] obtained a different form for $U(q)$,

$$U(q) = \frac{1}{(q)_\infty^2} \sum_{n\geq 1} (-1)^{n-1} q^{\frac{n(n+1)}{2}}, \tag{9}$$

where $(x)_\infty = (x;q)_\infty := \prod_{j=0}^{\infty}(1 - xq^j)$.

Rhoades explored the relationship between $u^*(n)$ and the weakly modular object $(-q)_\infty^2$ by using a pair of Ramanujan's mock theta functions. It is clear that

$$U^*(q) = \sum_{n=0}^{\infty} q^{n+1}(-q)_n^2. \tag{10}$$

Then he proved the following identity.

Proposition 1 (Theorem 1.3 in [1]) *Let*

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2} \tag{11}$$

be the third order mock theta function. Then

$$4U^*(q) = -f(q) + (-q)_\infty^2 \tilde{F}(q), \tag{12}$$

where $\tilde{F}(q) = \frac{2}{(q)_\infty(-q)_\infty^2} \sum_{n\in\mathbb{Z}} \frac{q^{\frac{n(n+1)}{2}}}{1+q^n}$ *is a mock theta function.*

Denote

$$F(q) := f(q) + 4U^*(q). \tag{13}$$

It is easy to see that

$$\tilde{F}(q) = (-q)_\infty^{-2} F(q) \tag{14}$$

and

$$F(q) = \frac{2}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{q^{\frac{n(n+1)}{2}}}{1+q^n}. \tag{15}$$

Remark 1

(a) It is easy to see that

$$U^*(q) = \frac{1}{2}(-q)^2_\infty \times \frac{1}{2}\bigl(\tilde{F}(q) - (-q)^{-2}_\infty f(q)\bigr), \tag{16}$$

where $\tilde{F}(q) - (-q)^{-2}_\infty f(q)$ is a mixed mock modular form.

(b) Andrews [7], Theorem 1, recently proved a similar identity for $U^*(q)$ by using a different pair of mock theta functions. Namely, he established

$$U^*(q) = \omega(-q) + 2(-q)^2_\infty \alpha(-q), \tag{17}$$

where $\omega(q) = \sum_{n=1}^\infty \frac{q^{n^2}}{(q;q^2)_n}$ is one of the third order Ramanujan's mock theta function [8–10], and $\alpha(q) = \sum_{n=0}^\infty \frac{q^{n+1}(-q^2;q^2)_n}{(q;q^2)_{n+1}}$ appears in Ramanujan's lost notebook and has been termed a second-order mock theta function by McIntosh [11].

The authors [12] also used this identity (17) to establish the inner relationships for the two different forms of the implied constant series $O(1)$ in the Ramanujan claim for the third order mock theta function $f(q)$.

(c) The function $F(q) = (-q)^2_\infty \tilde{F}(q)$ is referred to as a mixed mock modular form as it is the product of a modular form and a mock theta function.

2 Statement of results

For the fixed mock theta function $M(q) := \sum_{n\geq 0} c(n;q)$, we define its associated bilateral series by [13]

$$B(M;q) := \sum_{n=-\infty}^{\infty} c(n;q). \tag{18}$$

Then the tail of the bilateral series $B(M;q)$ is the following:

$$\sum_{n=-\infty}^{-1} c(n;q). \tag{19}$$

In this note, by considering the bilateral series $B(U^*;q)$ and $B(f;q)$, we define the following new functions:

$$H_1(q) = B\bigl(U^*;q\bigr) = \sum_{n\in\mathbb{Z}} u^*(n)q^n = \sum_{n\in\mathbb{Z}} q^{n+1}(-q)^2_n, \tag{20}$$

$$H_2(q) = \sum_{n=-\infty}^{-1} \frac{q^{n^2}}{(-q)^2_n} + 4\sum_{n=-\infty}^{-1} q^{n+1}(-q)^2_n, \tag{21}$$

$$H_3(q) = \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{(-q)^2_n} + 4\sum_{n\in\mathbb{Z}} q^{n+1}(-q)^2_n. \tag{22}$$

We mainly want to study the modular properties of these new series and the inner connections among them.

First, recall the Appell-Lerch sums defined by [9, 14]

$$m(x,q,z) := \frac{1}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{r(r-1)/2} z^r}{1-q^{r-1}xz}, \tag{23}$$

where

$$j(x;q) := (x)_\infty (q/x)_\infty (q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} x^n. \tag{24}$$

Then we give the following results.

Theorem 2.1 *Suppose we have all of the notation and hypotheses above. Then*

$$H_1(q) = \frac{1}{2(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{q^{\frac{n(n+1)}{2}}}{1+q^n} = (-q)_\infty^2 m(1,q,-1) \tag{25}$$

is a mixed mock modular form of weight 1/2.

Theorem 2.2 *Let* $B(f;q)$ *is the bilateral series of the third order Ramanujan's mock theta function* $f(q)$. *Then*

$$H_2(q) = 4B\bigl(U^*;q\bigr) = 4\sum_{n\in\mathbb{Z}} q^{n+1}(-q)_n^2 = 4H_1(q) = B(f;q) = \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{(-q)_n^2} \tag{26}$$

is a mixed mock modular form of weight 1/2. *Moreover, let* $\tilde{H}_2(q) = (-q)_\infty^{-2} H_2(q)$, *then* $\tilde{H}_2(q)$ *is a mock theta function.*

Theorem 2.3 *Suppose we have all of the notation and hypotheses above. Then*

$$H_3(q) = 2H_2(q) = 8H_1(q) = 8(-q)_\infty^2 m(1,q,-1) \tag{27}$$

is a mixed mock modular form of weight 1/2.

Theorem 2.4 *Let* $\tilde{H}_1(q) = (-q)_\infty^{-2} H_1(q)$. *Then*:

(1) $\tilde{H}_1(q) = m(1,q,-1)$ *is a mock theta function.*

(2) $q^{-1}\tilde{H}_1(q^8) + \frac{2}{q} g^*(q^8)$ *is a harmonic weak Maass form of weight* 1/2 *on* $\Gamma_1(1{,}024)$, *where* $g^*(q) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{(n+1)^2} (q;q^2)_n}{(-q;q^2)_{n+1}^2}$.

According to the above theorems and the results in the previous introduction, we can deduce the following corollaries directly.

Corollary 2.5 *Assume all of the notation and hypotheses above. Then*

$$\sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{(-q)_n^2} = 4\sum_{n\in\mathbb{Z}} q^{n+1}(-q)_n^2 = \frac{2}{(q)_\infty} \sum_{n\in\mathbb{Z}} \frac{q^{\frac{n(n+1)}{2}}}{1+q^n} \tag{28}$$

and

$$\sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{(-q)_n^2} + \sum_{n\in\mathbb{Z}} q^{n+1}(-q)_n^2 = 5iq^{\frac{1}{24}}\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^2 \mu\left(\frac{1}{2},\frac{1}{2};\tau\right), \tag{29}$$

where

$$\eta(\tau) := q^{1/24}\Pi_{n=1}^{\infty}(1-q^n), \tag{30}$$

and according to Zwegers [9], *then*

$$\mu(u,v;\tau) := \frac{e^{\pi iu}}{\vartheta(v;\tau)} \sum_{n\in\mathbb{Z}} \frac{(-1)^n e^{2\pi inv} q^{\frac{n(n+1)}{2}}}{1-e^{2\pi iu}q^n}, \tag{31}$$

$$\vartheta(v;\tau) := \sum_{v\in\frac{1}{2}+\mathbb{Z}} e^{\pi iv^2\tau+2\pi iv(v+\frac{1}{2})}. \tag{32}$$

Corollary 2.6 *In the notation above, we see that*

$$F(q) = H_2(q) = B(f;q) = \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{(-q)_n^2} = 4(-q)_\infty^2 m(1,q,-1) \tag{33}$$

is a mixed mock modular form of weight 1/2.

Remark 2 Bajpai *et al.* in [13] pointed out that $B(f;q)$ is a mixed mock modular form but they did not give the proof of their claim. In this note, we give a reasonable proof in agreement with them. We can express the $B(f;q)$ in terms of Appell-Lerch sums also.

3 Proofs of the theorems

In this section, we give the proofs of the four theorems. First, recall that the substitution $n \to -n$ in $(x;q)_n$ is the following:

$$(x;q)_{-n} := \frac{(-1)^n x^{-n} q^{n(n+1)/2}}{(q/x;q)_n}. \tag{34}$$

In the proof of Theorem 2.1, we consider

$$\begin{aligned} H_1(q) &= \sum_{n=-\infty}^{-1} q^{n+1}(-q)_n^2 + \sum_{n=0}^{\infty} q^{n+1}(-q)_n^2 \\ &= \sum_{n=1}^{\infty} q^{-n+1}(-q)_{-n}^2 + U^*(q) \\ &= \sum_{n=1}^{\infty} \frac{q^{n^2-2n+1}}{(-1;q)_n^2} + U^*(q) \\ &= \frac{1}{4}\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2} + U^*(q) \\ &= \frac{1}{4}f(q) + U^*(q) \\ &= \frac{1}{4}F(q). \end{aligned} \tag{35}$$

Rhoades used the following identity of Choi [15, 16] to prove the relation which is associated with Lerch sums,

$$\sum_{n=0}^{\infty}\frac{(ab)^n q^{n^2}}{(aq)_n(bq)_n}+\sum_{n=1}^{\infty}q^n\left(a^{-1}\right)_n\left(b^{-1}\right)_n=iq^{\frac{1}{8}}(1-a)a^{-\frac{1}{2}}b^{\frac{1}{2}}\left(a^{-1}q\right)_\infty\left(b^{-1}\right)_\infty\mu(u,v;\tau),\quad(36)$$

with $a=e^{2\pi iu}$ and $b=e^{2\pi iv}$.

Let $a=b=-1$, namely take $u=v=\frac{1}{2}$ in the above identity to obtain

$$\begin{aligned}F(q)&=f(q)+4U^*(q)\\&=4iq^{\frac{1}{24}}\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^2\mu\left(\frac{1}{2},\frac{1}{2};\tau\right)\\&=\frac{2}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{q^{\frac{n(n+1)}{2}}}{1+q^n}.\end{aligned}\quad(37)$$

But here we consider the following identity of Ramanujan [17] and Mortenson [18]. For $a,b\neq0$, we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{a^{-n-1}b^{-n}q^{n^2}}{(-a^{-1};q)_{n+1}(-qb^{-1};q)_n}+\sum_{n=1}^{\infty}(-aq;q)_{n-1}(-b;q)_nq^n\\&\quad=\frac{(-aq)_\infty}{b(q)_\infty(-qb^{-1})_\infty}j(-b;q)m(a/b,q,-b).\end{aligned}\quad(38)$$

Let $a=b=1$ in the above identity, it follows that

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-1;q)_{n+1}(-q;q)_n}+\sum_{n=1}^{\infty}(-q;q)_{n-1}(-1;q)_nq^n\\&\quad=\frac{(-q)_\infty}{(q)_\infty(-q)_\infty}j(-1;q)m(1,q,-1).\end{aligned}\quad(39)$$

Namely we have

$$\begin{aligned}\frac{1}{4}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n^2}+\sum_{n=0}^{\infty}q^{n+1}(-q;q)_n^2&=\frac{1}{4}f(q)+U^*(q)\\&=H_1(q)\\&=\frac{1}{2(q)_\infty}j(-1;q)m(1,q,-1)\\&=\frac{1}{2(q)_\infty}j(-1;q)\frac{1}{j(-1;q)}\sum_{n=-\infty}^{\infty}\frac{(-1)^{2n}q^{n(n-1)/2}}{1+q^{n-1}}\\&=\frac{1}{2(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{q^{\frac{n(n+1)}{2}}}{1+q^n}\\&=\frac{1}{4}F(q).\end{aligned}\quad(40)$$

Finally, the modularity of $H_1(q)$ can be proved by the mixed mock modular forms $F(q) = (-q)_\infty^2 \tilde{F}(q)$ of weight 1/2 since it is the product of a modular forms and a mock theta function.

According to the proof above, we have

$$\begin{aligned} H_1(q) &= iq^{\frac{1}{24}} \left(\frac{\eta(2\tau)}{\eta(\tau)} \right)^2 \mu\left(\frac{1}{2}, \frac{1}{2}; \tau \right) \\ &= iq^{\frac{1}{24}} q^{-\frac{1}{12}} (-q)_\infty^2 \mu\left(\frac{1}{2}, \frac{1}{2}; \tau \right) \\ &= iq^{-\frac{1}{24}} (-q)_\infty^2 \mu\left(\frac{1}{2}, \frac{1}{2}; \tau \right) \\ &= \frac{1}{2(q)_\infty} j(-1;q) m(1,q,-1) \\ &= \frac{1}{2(q)_\infty} (-1;q)_\infty (-q)_\infty (q)_\infty m(1,q,-1) \\ &= (-q)_\infty^2 m(1,q,-1). \end{aligned} \tag{41}$$

Remark 3 Therefore, we point out that the inner relationship between Lerch sums and Appell-Lerch sums is the following:

$$m(1,q,-1) = iq^{-\frac{1}{24}} \mu\left(\frac{1}{2}, \frac{1}{2}; \tau \right), \tag{42}$$

where $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$.

In the proof of Theorem 2.2, we see that

$$\begin{aligned} H_2(q) &= \sum_{n=-\infty}^{-1} \frac{q^{n^2}}{(-q)_n^2} + 4 \sum_{n=-\infty}^{-1} q^{n+1} (-q)_n^2 \\ &= \sum_{n=-\infty}^{-1} \frac{q^{n^2}}{(-q)_n^2} + \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2} \\ &= \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{(-q)_n^2} \\ &= B(f;q). \end{aligned} \tag{43}$$

On the other hand, we have

$$\begin{aligned} H_2(q) &= \sum_{n=-\infty}^{-1} \frac{q^{n^2}}{(-q)_n^2} + 4 \sum_{n=-\infty}^{-1} q^{n+1} (-q)_n^2 \\ &= 4 \sum_{n=0}^{\infty} q^{n+1} (-q)_n^2 + 4 \sum_{n=-\infty}^{-1} q^{n+1} (-q)_n^2 \\ &= 4 \sum_{n\in\mathbb{Z}} q^{n+1} (-q)_n^2 \\ &= 4B\bigl(U^*;q\bigr) \end{aligned}$$

$$
\begin{aligned}
&= 4H_1(q) \\
&= F(q). \qquad (44)
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
H_2(q) &= \sum_{n=-\infty}^{-1} \frac{q^{n^2}}{(-q)_n^2} + 4 \sum_{n=-\infty}^{-1} q^{n+1}(-q)_n^2 \\
&= B(f;q) \\
&= 4B\bigl(U^*;q\bigr) \\
&= 4H_1(q) \\
&= F(q). \qquad (45)
\end{aligned}
$$

Therefore, $H_2(q)$ is a mixed mock modular form of weight 1/2 from the modularity of $F(q)$. Using the results of Proposition 1, we know that $\tilde{H}_2(q) = (-q)_\infty^{-2} H_2(q) = (-q)_\infty^{-2} F(q)$ is a mock theta function.

In the proof of Theorem 2.3, we use the following identity of Mortenson in [14]:

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{a^{-n-1} b^{-n} q^{n^2}}{(-a^{-1};q)_{n+1}(-qb^{-1};q)_n} &= \sum_{n=-\infty}^{\infty} (-aq;q)_{n-1}(-b;q)_n q^n \\
&= \frac{(-aq)_\infty}{b(q)_\infty(-qb^{-1})_\infty} j(-b;q) m(a/b, q, -b). \qquad (46)
\end{aligned}
$$

If we let $a = b = 1$ in the above identity, then we have

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(-1;q)_{n+1}(-q;q)_n} &= \sum_{n=-\infty}^{\infty} (-q;q)_{n-1}(-1;q)_n q^n \\
&= \frac{1}{(q)_\infty} j(-1;q) m(1, q, -1), \qquad (47)
\end{aligned}
$$

namely we have

$$
\sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{2(-q;q)_n^2} = \frac{1}{(q)_\infty} j(-1;q) m(1, q, -1), \qquad (48)
$$

and

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} (-q;q)_{n-1}(-1;q)_n q^n &= \sum_{n=-\infty}^{\infty} (-q;q)_n(-1;q)_{n+1} q^{n+1} \\
&= 2 \sum_{n=-\infty}^{\infty} (-q;q)_n^2 q^{n+1} \\
&= \frac{1}{(q)_\infty} j(-1;q) m(1, q, -1). \qquad (49)
\end{aligned}
$$

Considering the definition of $H_3(q)$ and using the above two identities, we can see that

$$\begin{aligned} H_3(q) &= \sum_{n\in\mathbb{Z}} \frac{q^{n^2}}{(-q)_n^2} + 4\sum_{n\in\mathbb{Z}} q^{n+1}(-q)_n^2 \\ &= 2\frac{1}{(q)_\infty} j(-1;q)m(1,q,-1) + \frac{4}{2(q)_\infty} j(-1;q)m(1,q,-1) \\ &= 4\frac{1}{(q)_\infty} j(-1;q)m(1,q,-1). \end{aligned} \tag{50}$$

Combining with the definitions and the relationships of $H_1(q)$, $H_2(q)$, and $F(q)$, we deduce that

$$\begin{aligned} H_3(q) &= B(f;q) + 4B(U^*;q) = 2H_2(q) \\ &= 4H_1(q) + 4H_1(q) = 8H_1(q) \\ &= 2F(q). \end{aligned} \tag{51}$$

In the proof of Theorem 2.4, for the first assertion, by using the result of Remark 3, we have

$$\begin{aligned} \tilde{H}_1(q) &= (-q)_\infty^{-2} H_1(q) \\ &= (-q)_\infty^{-2} i q^{-\frac{1}{24}} (-q)_\infty^2 \mu\left(\frac{1}{2}, \frac{1}{2}; \tau\right) \\ &= i q^{-\frac{1}{24}} \mu\left(\frac{1}{2}, \frac{1}{2}; \tau\right) \\ &= m(1,q,-1) \\ &= i q^{-\frac{1}{24}} \mu\left(\frac{1}{2}, \frac{1}{2}; \tau\right). \end{aligned} \tag{52}$$

According to the theory of Zwegers, we know that $iq^{-\frac{1}{24}}\mu(\frac{1}{2}, \frac{1}{2}; \tau)$ is a mock theta function. So is $\tilde{H}_1(q)$.

On other hand, in view of Proposition 1, we know that $\tilde{F}(q)$ is a mock theta function, combining with the fact that

$$\tilde{F}(q) = (-q)_\infty^{-2} F(q) \quad \text{and} \quad F(q) = 4H_1(q), \tag{53}$$

we have

$$\tilde{H}_1(q) = (-q)_\infty^{-2} H_1(q) = \frac{1}{4}\tilde{F}(q). \tag{54}$$

Hence $\tilde{H}_1(q)$ is also a mock theta function.

For the second assertion, Mortenson in [14] proved the following identities:

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-x;q^2)_{n+1}(-q^2/x;q^2)_n} = m(x,q,-1) + \frac{J_{1,2}^2}{2j(-x;q)} \tag{55}$$

and

$$\left(1+x^{-1}\right)\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-qx,-q/x;q^2)_{n+1}} = m(x,q,-1) - \frac{J_{1,2}^2}{2j(-x,q)}, \tag{56}$$

where $J_{a,m} := j(q^a;q^m)$ for a and m being integers with m positive.

From the two identities above, we have

$$m(1,q,-1) = \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2} + \sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q^2)_{n+1}^2}. \tag{57}$$

Ken Ono [19, 20] defined the following series K' and K'':

$$K'(\omega;z) := \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(\omega q^2;q^2)_n(\omega^{-1}q^2;q^2)_n}, \tag{58}$$

and

$$K''(\omega;z) := \sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_{n-1}}{(\omega q;q^2)_n(\omega^{-1}q;q^2)_n}, \tag{59}$$

where $0 < a < c$.

Let $\zeta_c := e^{\frac{2\pi i}{c}}$ and $f_c := \frac{2c}{\gcd(c,4)}$. For $0 < a < c$, $q = e^{2\pi iz}$, denote

$$\tilde{K}(a,c;z) := \frac{1}{4}\csc\left(\pi\frac{a}{c}\right)q^{-\frac{1}{8}}K'\left(\zeta_c^a;z\right) + \sin\left(\pi\frac{a}{c}\right)q^{-\frac{1}{8}}K''\left(\zeta_c^a;z\right). \tag{60}$$

Then Ono proved that $\tilde{K}(a,c;2f_c^2z)$ is a weakly holomorphic modular form (harmonic weak Maass form) of weight 1/2 on $\Gamma_1(64f_c^4)$.

Take $a = 1$, $c = 2$ in the definition of $\tilde{K}(a,c;z)$, we get

$$\tilde{K}(1,2;z) = q^{-\frac{1}{8}}\left(\frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2} - \sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)^2}(q;q^2)_n}{(-q;q^2)_{n+1}^2}\right). \tag{61}$$

Combining the identities (57) with (61), we have

$$\tilde{K}(1,2;z) = q^{-\frac{1}{8}}m(1,q,-1) + 2q^{-\frac{1}{8}}K''(-1;z). \tag{62}$$

We can see that $f_c = 2$ if $c = 2$. According to the result of $\tilde{K}(a,c;2f_c^2z)$, we see that

$$\tilde{K}(1,2;8z) = q^{-1}m\left(1,q^8,-1\right) + 2q^{-1}K''(-1;8z) \tag{63}$$

is a weakly holomorphic modular form (harmonic weak Maass form) of weight 1/2 on $\Gamma_1(1{,}024)$. This is the end of the proofs.

Finally, the first assertion of Corollary 2.5 follows from the identities (25) and (26). The second assertion of Corollary 2.5 follows from the identities (41) and (45). The assertion of Corollary 2.6 follows from the identities (25) and (45).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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