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Mean ergodic theorem for semigroups of linear operators in multi-Banach spaces

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Abstract
In this paper, by using Rodé's method, we extend Yosida's theorem to semigroups of linear operators in multi-Banach spaces.
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1 Introduction

In 1938, Yosida [1] proved the following mean ergodic theorem for linear operators: Let E be a real Banach space and T_j $(j = 1, 2, \dots)$ be linear operators of E into itself such that there exists a constant C with $\|(T_1^n, \dots, T_j^n)\| \leq C$ for $n = 1, 2, 3, \dots$, and T_j is weakly completely continuous, *i.e.*, T_j maps the closed unite ball of E into a weakly compact subset of E. Then the Cesaro means

$$S_{n,j}x = \frac{1}{n}\sum_{k=1}^{n} T_j^k x$$

converges strongly as $n \to +\infty$ to a fixed point of T_j for each $x \in E$.

On the other hand, in 1975, Baillon [2] proved the following nonlinear ergodic theorem. Let X be a Banach space and C be a closed convex subset of X. The mappings $T_j : C \to C$ $(j = 1, 2, \dots)$ are called nonexpansive on C if

$$\|T_j x - T_j y\| \leq \|x - y\| \quad \forall x, y \in C.$$

Let $F(T_j)$ be the set of fixed points of T_j. If X is strictly convex, $F(T_j)$ is closed and convex. In [2], Baillon proved the first nonlinear ergodic theorem such that if X is a real Hilbert space and $F(T_j) \neq \emptyset$, then for each $x \in C$, the sequence $\{S_{n,j}x\}$ defined by

$$S_{n,j}x = \left(\frac{1}{n}\right)\left(x + T_j x + \cdots + T_j^{n-1}x\right)$$

converges weakly to a fixed point of T_j. It was also shown by Pazy [3] that if X is a real Hilbert space and $S_{n,j}x$ converges weakly to $y \in C$, then $y \in F(T_j)$.

Recently, Rodé [4] and Takahashi [5] tried to extend this nonlinear ergodic theorem to semigroup, generalizing the Cesaro means on $\mathbf{N} = \{1, 2, \dots\}$, such that the corresponding sequence of mappings converges to a projection onto the set of common fixed points. In this paper, by using Rodé's method, we extend Yosida's theorem to semigroups of linear operators in multi-Banach spaces. The proofs employ the methods of Yosida [1], Rodé [4], Greenleaf [6] and Takahashi [7, 8]. Our paper is motivated from ideas in [9].

2 Multi-Banach spaces

The notion of multi-normed space was introduced by Dales and Polyakov in [10]. This concept is somewhat similar to operator sequence space and has some connections with operator spaces and Banach lattices. Motivations for the study of multi-normed spaces and many examples are given in [10–16].

Let $(E, \|\cdot\|)$ be a complex normed space, and let $k \in \mathbf{N}$. We denote by E^k the linear space $E \oplus \cdots \oplus E$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$. The linear operations on E^k are defined coordinate-wise. The zero element of either E or E^k is denoted by 0. We denote by $\mathbf{N}_k$ the set $\{1, 2, \ldots, k\}$ and by Σ_k the group of permutations on k symbols.

Definition 2.1 Let E be a linear space, and take $k \in \mathbf{N}$. For $\sigma \in \Sigma_k$, define

$$A_\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)}), \quad x = (x_1, \ldots, x_k) \in E^k.$$

For $\alpha = (\alpha_i) \in \mathbf{C}^k$, define

$$M_\alpha(x) = (\alpha_i x_i), \quad x = (x_1, \ldots, x_k) \in E^k.$$

Definition 2.2 Let $(E, \|\cdot\|)$ be complex (respectively, real) normed space, and take $n \in \mathbf{N}$. A multi-norm of level n on $\{E^k : k \in \mathbf{N}_n\}$ is a sequence $(\|\cdot\|_k : k \in \mathbf{N}_n)$ such that $\|\cdot\|$ is a norm on E^k for each $k \in \mathbf{N}_n$, such that $\|x\|_1 = \|x\|$ for each $x \in E$ (so that $\|\cdot\|_1$ is the initial norm), and such that the following axioms (A1)-(A4) are satisfied for each $k \in \mathbf{N}_n$ with $k \geq 2$:

(A1) for each $\sigma \in \Sigma_k$ and $x \in E^k$, we have

$$\|A_\sigma(x)\|_k = \|x\|_k;$$

(A2) for each $\alpha_1, \ldots, \alpha_k \in \mathbf{C}$ (respectively, each $\alpha_1, \ldots, \alpha_k \in \mathbf{R}$) and $x \in E^k$, we have

$$\|M_\alpha(x)\|_k \leq \Bigl(\max_{i \in \mathbf{N}_k} |\alpha_i|\Bigr) \|x\|_k;$$

(A3) for each $x_1, \ldots, x_{k-1}$, we have

$$\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1};$$

(A4) for each $x_1, \ldots, x_{k-1} \in E$

$$\|(x_1, \ldots, x_{k-2}, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1}, x_{k-1})\|_{k-1}.$$

In this case, $((E^k, \|\cdot\|_k) : k \in \mathbf{N}_n)$ is a multi-normed space of level n.

A multi-norm on $\{E^k : k \in \mathbf{N}\}$ is a sequence

$$\left(\|\cdot\|_k\right) = \left(\|\cdot\|_k : k \in \mathbf{N}\right)$$

such that $(\|\cdot\|_k : k \in \mathbf{N}_n)$ is a multi-norm of level n for each $n \in \mathbf{N}$. In this case, $((E^n, \|\cdot\|_n) : n \in \mathbf{N})$ is a multi-normed space.

Lemma 2.3 [12] *Suppose that* $((E^k, \|\cdot\|_k) : k \in \mathbf{N})$ *is a multi-normed space, and take* $k \in \mathbf{N}_n$. *Then*

(a) $\|(x, \dots, x)\|_k = \|x\|$ $(x \in E)$;
(b) $\max_{i \in \mathbf{N}_k} \|x_i\| \leq \|(x_1, \dots, x_k)\|_k \leq \sum_{i=1}^k \|x_i\| \leq k \max_{i \in \mathbf{N}_k} \|x_i\|$ $(x_1, \dots, x_k \in E)$.

It follows from (b) that, if $(E, \|\cdot\|)$ is a Banach space, then $(E^k, \|\cdot\|_k)$ is a Banach space for each $k \in \mathbf{N}$; in this case $((E^k, \|\cdot\|_k) : k \in \mathbf{N})$ is a multi-Banach space.

Now we state two important examples of multi-norms for an arbitrary normed space E; *cf.* [10].

Example 2.4 The sequence $(\|\cdot\|_k : k \in \mathbf{N})$ on $\{E^k : k \in \mathbf{N}\}$ defined by

$$\left\|(x_1, \dots, x_k)\right\|_k := \max_{i \in \mathbf{N}_k} \|x_i\| \quad (x_1, \dots, x_k \in E)$$

is a multi-norm called the minimum multi-norm. The terminology 'minimum' is justified by property (b).

Example 2.5 Let $\{(\|\cdot\|_k^\alpha : k \in \mathbf{N}) : \alpha \in A\}$ be the (non-empty) family of all multi-norms on $\{E^k : k \in \mathbf{N}\}$. For $k \in \mathbf{N}$, set

$$\left\|(x_1, \dots, x_k)\right\|_k := \sup_{\alpha \in A} \left\|(x_1, \dots, x_k)\right\|_k^\alpha \quad (x_1, \dots, x_k \in E).$$

Then $(\|\cdot\|_k : k \in \mathbf{N})$ is a multi-norm on $\{E^k : k \in \mathbf{N}\}$, called the maximum multi-norm.

We need the following observation, which can easily be deduced from the triangle inequality for the norm $\|\cdot\|_k$ and the property (b) of multi-norms.

Lemma 2.6 *Suppose that* $k \in \mathbf{N}$ *and* $(x_1, \dots, x_k) \in E^k$. *For each* $j \in \{1, \dots, k\}$, *let* $(x_n^j)_{n=1,2,\dots}$ *be a sequence in E such that* $\lim_{n \to \infty} x_n^j = x_j$. *Then for each* $(y_1, \dots, y_k) \in E^k$ *we have*

$$\lim_{n \to \infty} \left(x_n^1 - y_1, \dots, x_n^k - y_k\right) = (x_1 - y_1, \dots, x_k - y_k).$$

Definition 2.7 Let $((E^k, \|\cdot\|_k) : k \in \mathbf{N})$ be a multi-normed space. A sequence (x_n) in E is a *multi-null* sequence if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$\sup_{k \in \mathbf{N}} \left\|(x_n, \dots, x_{n+k-1})\right\|_k < \varepsilon \quad (n \geq n_0).$$

Let $x \in \mathcal{E}$. We say that the sequence (x_n) is *multi-convergent* to $x \in E$ and write

$$\lim_{n \to \infty} x_n = x$$

if $(x_n - x)$ is a multi-null sequence.

3 Preliminaries and lemmas

Let E a real Banach space and let E^* be the conjugate space of E, that is, the space of all continuous linear functionals on E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. We denote by $\operatorname{co} D$ the convex hull of D, $\overline{\operatorname{co}}\, D$ the closure of $\operatorname{co} D$.

Let U be a linear continuous operator of E into itself. Then we denote by U^* the conjugate operator of U.

Assumption (A) Let $(E^j, \|\cdot\|_j)$ be a multi-Banach space and $\{T_{j,t} : t \in G\}$ $(j = 1, 2, \ldots)$ be a family of linear continuous operators of a real Banach space E_j into itself such that there exists a real number C with $\|(T_{1,t}, \ldots, T_{j,t})\|_j \leq C$ for all $t \in G$ and the weak closure of $\{T_{j,t}x : t \in G\}$ is weakly compact, for each $x \in E$. The index set G is a topological semigroup such that $T_{j,st} = T_{j,s} \cdot T_{j,t}$ for all $s, t \in G$ and T_j is continuous with respect to the weak operator topology: $\langle T_{j,s}x, x^* \rangle \to \langle T_{j,t}x, x^* \rangle$ for all $x \in E$ and $x^* \in E^*$ if $s \to t$ in G.

We denote by $m_j(G)$ the Banach space of all bounded continuous real valued functions on the topological semigroup G with the supremum norm. For each $s \in G$ and $f_j \in m_j(G)$, we define elements $l_s f_j$ and $r_s f_j$ in $m_j(G)$ given by $l_s f_j(t) = f_j(st)$ and $r_s f_j(t) = f_j(ts)$ for all $t \in G$. An element $\mu_j \in m_j(G)^*$ (the conjugate space of $m_j(G)$) is called a mean on G if $\|\mu_j\| = \mu_j(1) = 1$ moreover, we have $\|(\mu_1, \ldots, \mu_j)\|_j = \frac{\sum_{i=1}^{j} \mu_i(1)}{j} = 1$. A mean μ_j on G is called left (right) invariant if $\mu_j(l_s f_j) = \mu_j(f_j)$ $(\mu_j(r_s f_j) = \mu_j(f_j))$ for all $f_j \in m_j(G)$ and $s \in G$. An invariant mean is a left and right invariant mean. We know that $\mu_j \in m_j(G)^*$ is a mean on G if and only if

$$\inf\{f_j(t) : t \in G\} \leq \mu_j(f_j) \leq \sup\{f_j(t) : t \in G\}$$

for every $f_j \in m_j(G)$; see [6, 17–20].

Let $\{T_{j,t} : t \in G\}$ be a family of linear continuous operators of E into itself satisfying Assumption (A) and μ_j be a mean on G. Fix $x \in E$. Then, for $x^* \in E^*$, the real valued function $t \to \langle T_{j,t}x, x^* \rangle$ is in $m_j(G)$. Denote by $\mu_{j,t}\langle T_{j,t}x, x^* \rangle$ the value of μ_j at this function. By linearity of μ_j and of $\langle \cdot, \cdot \rangle$, this is linear in x^*; moreover, since

$$\begin{aligned}
&\left| \left(\mu_{1,t}\langle T_{1,t}x, x^* \rangle, \ldots, \mu_{j,t}\langle T_{j,t}x, x^* \rangle \right) \right| \\
&\quad \leq \left\| (\mu_1, \ldots, \mu_j) \right\|_j \cdot \sup_t \left| \left(\langle T_{1,t}x, x^* \rangle, \ldots, \mu_{j,t}\langle T_{j,t}x, x^* \rangle \right) \right| \\
&\quad \leq \sup_t \left\| (T_1 x, \ldots, T_j x) \right\|_j \cdot \left\| x^* \right\|_j \\
&\quad \leq C \cdot \|x\|_j \cdot \left\| x^* \right\|_j
\end{aligned}$$

it is continuous in x^*. Hence we find that $\mu_{j,t}\langle T_{j,t}x, \cdot \rangle$ is an element of E^{**}. So, from weak compactness of $\overline{\operatorname{co}}\{T_{j,t}x : t \in G\}$ such that $\mu_{j,t}\langle T_{j,t}x, x^* \rangle = \langle T_{j,\mu_j}x, x^* \rangle$ for every $x^* \in E^*$.

Put $K = \overline{\operatorname{co}}\{T_{j,t}x : t \in G\}$ and suppose that the element $\mu_{j,t}\langle T_{j,t}x, \cdot \rangle$ is not contained in the $n(K)$, where n is the natural embedding of the Banach space E into its second conjugate space E^{**}. Then, since the convex set $n(K)$ is compact in the *weak** topology of E^{**}, there exists an element $y^* \in E^*$ such that

$$\mu_{j,t}\langle T_{j,t}x, y^* \rangle < \inf\{\langle y^*, z^{**} \rangle : z^{**} \in n(k)\}.$$

Hence, we have

$$\begin{aligned}\mu_{j,t}\langle T_{j,t}x, y^*\rangle &< \inf\{\langle y^*, z^{**}\rangle : z^{**} \in n(k)\}\\ &\le \inf\{\langle T_{j,t}x, y^*\rangle : t \in G\}\\ &\le \mu_{j,t}\langle T_{j,t}x, y^*\rangle.\end{aligned}$$

This is a contradiction. Thus, for a mean μ_j on G, we can define a linear continuous operator T_{j,μ_j} of E into itself such that $\|(T_{1,\mu_1}, \ldots, T_{j,\mu_j})\|_j \le C$, $T_{j,\mu_j}x \in \overline{\mathrm{co}}\{T_{j,t}x : t \in G\}$ for all $x \in E$, and $\mu_{j,t}\langle T_{j,t}x, x^*\rangle = \langle T_{j,\mu_j}x, x^*\rangle$ for all $x \in E$ and $x^* \in E^*$. We denote by $F_j(G)$ the set all common fixed points of the mappings $T_{j,t}$, $t \in G$.

Lemma 3.1 *Assume that a left invariant mean μ_j exists on G, then $T_{j,\mu_j}(E) \subset F_j(G)$. Especially, $F_j(G)$ is then not empty.*

Proof Let $x \in E$ and μ be a left invariant mean on G. Then since, for $s \in G$ and x^*,

$$\begin{aligned}\langle T_{j,s}T_{j,\mu_J}x, x^*\rangle &= \langle T_{j,\mu_j}x, T^*_{j,s}x^*\rangle\\ &= \mu_{j,t}\langle T_{j,t}x, T^*_{j,s}x^*\rangle = \mu_{j,t}\langle T_{j,s}T_{j,t}x, x^*\rangle\\ &= \mu_{j,t}\langle T_{j,st}x, x^*\rangle = \mu_{j,t}\langle T_{j,t}x, x^*\rangle\\ &= \langle T_{j,\mu_j}x, x^*\rangle,\end{aligned}$$

we have $T_{j,s}T_{j,\mu_j}x = T_{j,\mu_j}x$. Hence, $T_{j,\mu_j}(E) \subset F_j(G)$. □

Lemma 3.2 *Let λ_j be an invariant mean on G. Then $T_{j,\lambda_j}T_{j,s} = T_{j,s}T_{j,\lambda_j} = T_{j,\lambda_j}$ for each $s \in G$ and $T_{j,\lambda_j}T_{j,\mu_j} = T_{j,\mu_j}T_{j,\lambda_j} = T_{j,\lambda_j}$ for each mean μ_j on G. Especially, T_{j,λ_j} is a projection of E onto $F(G)$.*

Proof Let $s \in G$. Then, since

$$\begin{aligned}\langle T_{j,\lambda_j}T_{j,s}x, x^*\rangle &= \lambda_{j,t}\langle T_{j,t}T_{j,s}x, x^*\rangle = \lambda_{j,t}\langle T_{j,ts}x, x^*\rangle\\ &= \lambda_{j,t}\langle T_{j,t}x, x^*\rangle = \langle T_{j,\lambda_j}x, x^*\rangle\end{aligned}$$

for $x \in E$ and $x^* \in E^*$, we have $T_{j,\lambda_j}T_{j,s} = T_{j,\lambda_j}$. It is obvious from Lemma 3.1 that $T_{j,s}T_{j,\lambda_j} = T_{j,\lambda_j}$ for each $s \in G$. Let μ_j be a mean on G. Then, since

$$\begin{aligned}\langle T_{j,\mu_j}T_{j,\lambda_j}x, x^*\rangle &= \mu_{j,t}\langle T_{j,t}T_{j,\lambda_j}x, x^*\rangle = \langle \mu_{j,t}T_{j,\lambda_j}x, x^*\rangle\\ &= \langle T_{j,\lambda_j}x, x^*\rangle\end{aligned}$$

and

$$\begin{aligned}\langle T_{j,\lambda_j}T_{j,\mu_j}x, x^*\rangle &= \langle T_{j,\mu_j}x, T^*_{j,\lambda_j}x^*\rangle = \mu_{j,t}\langle T_{j,t}x, T^*_{j,\lambda_j}x^*\rangle\\ &= \mu_{j,t}\langle T_{j,\lambda_j}T_{j,t}x, x^*\rangle = \mu_{j,t}\langle T_{j,\lambda_j}x, x^*\rangle\\ &= \langle T_{j,\lambda_j}x, x^*\rangle\end{aligned}$$

for $x \in E$ and $x^* \in E^*$, we have $T_{j,\mu_j} T_{j,\lambda_j} = T_{j,\lambda_j} T_{j,\mu_j} = T_{j,\lambda_j}$. Putting $\mu_j = \lambda_j$, we have $T^2_{\lambda_j} = T_{\lambda_j}$ and hence T_{λ_j} is a projection of E onto $F_j(G)$. □

As direct consequence of Lemma 3.2, we have the following.

Lemma 3.3 *Let μ_j and λ_j be invariant means on G. Then $T_{j,\mu_j} = T_{j,\lambda_j}$.*

Lemma 3.4 *Assume that an invariant mean exists on G. Then, for each $x \in E$, the set $\overline{\mathrm{co}}\{T_{j,t}x : t \in G\} \cap F_j(G)$ consists of a single point.*

Proof Let $x \in E$ and μ_j be an invariant mean on G. Then we know that $T_{j,\mu_j} x \in F_j(G)$ and $T_{j,\mu_j} x \in \overline{\mathrm{co}}\{T_{j,t}x : t \in G\}$. So, we show that $\overline{\mathrm{co}}\{T_{j,t}x : t \in G\} \cap F_j(G) = \{T_{j,\mu_j} x\}$. Let $x_0 \in \overline{\mathrm{co}}\{T_{j,t}x : t \in G\} \cap F_j(G)$ and $\epsilon > 0$. Then, for $x^* \in E^*$, there exists an element $\sum_{i=1}^n \alpha_i T_{j,t_i} x$ in the set $\mathrm{co}\{T_{j,t}x : t \in G\}$ such that $\epsilon > C \cdot \|x^*\|_j \cdot \| \sum_{i=1}^n \alpha_i T_{j,t_i} x - x_0 \|_j$. Hence, we have

$$\begin{aligned}
\epsilon &> C \cdot \|x^*\|_j \cdot \left\| \sum_{i=1}^n \alpha_i T_{j,t_i} x - x_0 \right\|_j \\
&\geq \sup_t \|T_{j,t}\|_j \cdot \left\| \sum_{i=1}^n \alpha_i T_{j,t_i} x - x_0 \right\|_j \cdot \|x^*\|_j \\
&\geq \sup_t \left\| \sum_{i=1}^n \alpha_i T_{j,t} T_{j,t_i} x - x_0 \right\|_j \cdot \|x^*\|_j \\
&\geq \left| \left\langle \sum_{i=1}^n \alpha_i T_{j,t} T_{j,t_i} x - x_0, x^* \right\rangle \right| \\
&= \left| \sum_{i=1}^n \alpha_i \mu_{j,t} \langle T_{j,tt_i} x - x_0, x^* \rangle \right| \\
&= \left| \mu_{j,t} \langle T_{j,t} x - x_0, x^* \rangle \right| \\
&= \left| \langle T_{j,\mu_j} x - x_0, x^* \rangle \right|.
\end{aligned}$$

Since ϵ is arbitrary, we have $\langle T_{j,\mu_j} x, x^* \rangle = \langle x_0, x^* \rangle$ for every $x^* \in E^*$ and hence $T_{j,\mu_j} x = x_0$. □

4 Ergodic theorems

Now, we can prove mean ergodic theorems for semigroups of linear continuous operators in multi-Banach space.

Theorem 4.1 *Let $\{T_{j,t} : t \in G\}$ be a family of linear continuous operators in a real Banach space E satisfying Assumption* (A). *If a net $\{\mu_j^\alpha : \alpha \in I\}$ of means on G is asymptotically invariant, i.e.,*

$$\mu_j^\alpha - r_s^* \mu_j^\alpha \quad \textit{and} \quad \mu_j^\alpha - l_s^* \mu_j^\alpha$$

converge to 0 *in the weak* topology of $m_j(G)^*$ for each $s \in G$, then there exists a projection Q_j of E on to $F_j(G)$ such that $\|(Q_1, \ldots, Q_j)\|_j \leq C$, $T_{j,\mu_j^\alpha} x$ converges weakly to $Q_j x$ for each $x \in E$,*

$Q_jT_{j,t} = T_{j,t}Q_j = Q_j$ for each $t \in G$, and $Q_jx \in \overline{\text{co}}\{T_{j,t}x : t \in G\}$ for each $x \in E$. Furthermore, the projection Q_j onto $F_j(G)$ is the same for all asymptotically invariant nets.

Proof Let μ_j be a cluster point of net $\{\mu_j^\alpha : \alpha \in I\}$ in the *weak** topology of $m_j(G)^*$. Then μ_j is an invariant mean on G. Hence, by Lemma 3.2, T_{j,μ_j} is a projection of E onto $F_j(G)$ such that $\|(T_{1,\mu_1},\dots,T_{j,\mu_j})\|_j \le C$, $T_{j,\mu_j}T_{j,t} = T_{j,t}T_{j,\mu_j} = T_{j,\mu_j}$ for each $t \in G$ and $T_{j,\mu_j}x \in \overline{\text{co}}\{T_{j,t}x : t \in G\}$ for each $x \in E$. Setting $Q_j = T_{j,\mu_j}$, we show that $T_{j,\mu_j^\alpha}x$ converges weakly to Q_jx for each $x \in E$. Since $T_{j,\mu_j^\alpha}x \in \overline{\text{co}}\{T_{j,t}x : t \in G\}$ for all $\alpha \in I$ and $\overline{\text{co}}\{T_{j,t}x : t \in G\}$ is weakly compact, there exists a subnet $\{T_{j,\mu_j^\beta}x : \beta \in J\}$ of $\{T_{j,\mu_j^\alpha}x : \alpha \in I\}$ such that $T_{j,\mu_j^\beta}x$ converges weakly to an element $x_0 \in \overline{\text{co}}\{T_{j,t}x : t \in G\}$. To show that $T_{j,\mu_j^\alpha}x$ converges weakly to Q_jx, it is sufficient to show $x_0 = Q_jx$. Let $x^* \in E^*$ and $s \in G$. Since $T_{j,\mu_j^\beta}x \to x_0$ weakly, we have $\mu_{j,t}^\beta\langle T_{j,t}x, x^*\rangle \to \langle x_0, x^*\rangle$ and $\mu_{j,t}^\beta\langle T_{j,t}x, T_{j,s}^*x^*\rangle \to \langle x_0, T_{j,s}^*x^*\rangle = \langle T_{j,s}x_0, x^*\rangle$. On the other hand, since $\mu_j^\beta - l_s^*\mu_j^\beta \to 0$ in the *weak** topology, we have

$$\begin{aligned}
&\mu_{j,t}^\beta\langle T_{j,t}x, x^*\rangle - l_s^*\mu_{j,t}^\beta\langle T_{j,t}x, x^*\rangle \\
&\quad = \mu_{j,t}^\beta\langle T_{j,t}x, x^*\rangle - \mu_{j,t}^\beta\langle T_{j,st}x, x^*\rangle \\
&\quad = \mu_{j,t}^\beta\langle T_{j,t}x, x^*\rangle - \mu_{j,t}^\beta\langle T_{j,t}x, T_{j,s}^*x^*\rangle \\
&\quad \to 0.
\end{aligned}$$

Hence, we have $\langle x_0, x^*\rangle = \langle T_{j,s}x_0, x^*\rangle$ and hence $x_0 \in F_j(G)$. So, we obtain $Q_jx = T_{j,\mu_j}x = x_0$ from Lemma 3.4. That the projection Q_j is the same for all asymptotically invariant nets is obvious from Lemma 3.3. □

As direct consequence of Theorem 4.1, we have the following.

Corollary 4.2 *Let $\{T_{j,t} : t \in G\}$ be as in Theorem* 4.1 *and assume that an invariant mean exists on G. Then there exists a projection Q_j of E onto F_j such that $\|(Q_1,\dots,Q_j)\|_j \le C$, $Q_jT_{j,t} = T_{j,t}Q_j = Q_j$ for each $t \in G$ and $Q_jx \in \overline{\text{co}}\{T_{j,t}x : t \in G\}$ for each $x \in E$.*

Theorem 4.3 *Let $\{T_{j,t} : t \in G\}$ be as in Theorem* 4.1. *If a net $\{\mu_j^\alpha : \alpha \in I\}$ of means on G is asymptotically invariant and further $\mu_j^\alpha - r_s^*\mu_j^\alpha$ converges to* 0 *in the strong topology of $m_j(G)^*$, then there exists a projection Q_j of E onto $F_j(G)$ such that $\|(Q_1,\dots,Q_j)\|_j \le C$, $T_{j,\mu_j^\alpha}x$ converges strongly to Q_jx for each $x \in E$, $Q_jT_{j,t} = T_{j,t}Q_j = Q_j$ for each $t \in G$, and $Q_jx \in \overline{\text{co}}\{T_{j,t}x : t \in G\}$ for each $x \in E$.*

Proof As in the proof of Theorem 4.1, let $Q_j = T_{j,\mu_j}$, where μ_j is a cluster point of the net $\{\mu_j^\alpha : \alpha \in I\}$ in the *weak** topology of $m_j(G)^*$. We show that $T_{j,\mu_j^\alpha}x$ converges strongly to Q_jx for each $x \in E$.

Let $E_0 = \overline{\text{co}}\{y - T_{j,t}y : y \in E, t \in G\}$. Then, for any $z \in E_0$, $T_{j,\mu_j^\alpha}z$ converges strongly to 0. In fact, if $z = y - T_{j,s}y$, then since, for any $y^* \in E^*$,

$$\begin{aligned}
\bigl|\langle T_{j,\mu_j^\alpha}z, y^*\rangle\bigr| &= \bigl|\mu_{j,t}^\alpha\langle T_{j,t}(y - T_{j,s}y), y^*\rangle\bigr| \\
&= \bigl|\mu_{j,t}^\alpha\langle T_{j,t}y, y^*\rangle - \mu_{j,t}^\alpha\langle T_{j,ts}y, y^*\rangle\bigr| \\
&= \bigl|\bigl(\mu_{j,t}^\alpha - r_s^*\mu_{j,t}^\alpha\bigr)\langle T_{j,t}y, y^*\rangle\bigr|
\end{aligned}$$

$$\begin{aligned}&\leq \left\|\left(\mu_1^\alpha - r_s^*\mu_1^\alpha,\dots,\mu_j^\alpha - r_s^*\mu_j^\alpha\right)\right\|_j \cdot \sup_t \left|\langle T_{j,t}y, y^*\rangle\right| \\ &\leq \left\|\left(\mu_1^\alpha - r_s^*\mu_1^\alpha,\dots,\mu_j^\alpha - r_s^*\mu_j^\alpha\right)\right\|_j \cdot C \cdot \|y\|_j \cdot \left\|y^*\right\|_j,\end{aligned}$$

we have $\|(T_{1,\mu_1^\alpha}z,\dots,T_{j,\mu_j^\alpha}z)\|_j \leq C \cdot \|(\mu_1^\alpha - r_s^*\mu_1^\alpha,\dots,\mu_j^\alpha - r_s^*\mu_j^\alpha)\|_j \cdot \|y\|_j$. Using this inequality, we show that $T_{j,\mu_j^\alpha}z$ converges strongly to 0 for any $z \in E_0$. Let z be any element of E_0 and ϵ be any positive number. By the definition of E_0, there exists an element $\sum_{i=1}^n a_i(y_i - T_{j,s_i}y_i)\epsilon$ in the set $\operatorname{co}\{y - T_{j,s}y : y \in E, s \in G\}$ such that $\epsilon > 2C \cdot \|(z - \sum_{i=1}^n a_i(y_i - T_{1,s_i}y_i),\dots,z - \sum_{i=1}^n a_i(y_i - T_{j,s_i}y_i))\|_j$. On the other hand, from $\|(\mu_1^\alpha - r_s^*\mu_1^\alpha,\dots,\mu_j^\alpha - r_s^*\mu_j^\alpha)\|_j \to 0$ for all $s \in G$, there exists $a_0 \in I$ such that, for all $\alpha \geq \alpha_0$ and $i = 1,2,\dots,n$,

$$\epsilon > \left\|\left(\mu_1^\alpha - r_{s_i}^*\mu_1^\alpha,\dots,\mu_j^\alpha - r_{s_i}^*\mu_j^\alpha\right)\right\|_j \cdot 2C\|y_i\|_j.$$

This yields

$$\begin{aligned}&\left\|(T_{1,\mu_1^\alpha}z,\dots,T_{j,\mu_j^\alpha}z)\right\|_j \\ &\quad\leq \left\|\left(T_{1,\mu_1^\alpha}z - T_{1,\mu_1^\alpha}\left(\sum_{i=1}^n a_i(y_i - T_{1,s_i}y_i)\right),\right.\right. \\ &\qquad \left.\left.\dots, T_{j,\mu_j^\alpha}z - T_{j,\mu_j^\alpha}\left(\sum_{i=1}^n a_i(y_i - T_{j,s_i}y_i)\right)\right)\right\|_j \\ &\qquad + \left\|\left(T_{1,\mu_1^\alpha}\left(\sum_{i=1}^n a_i(y_i - T_{1,s_i}y_i)\right),\dots,T_{j,\mu_j^\alpha}\left(\sum_{i=1}^n a_i(y_i - T_{j,s_i}y_i)\right)\right)\right\|_j \\ &\quad\leq \left\|(T_{1,\mu_1^\alpha},\dots,T_{j,\mu_j^\alpha})\right\|_j \\ &\qquad \cdot \left\|\left(z - \sum_{i=1}^n a_i(y_i - T_{1,s_i}y_i),\dots,z - \sum_{i=1}^n a_i(y_i - T_{j,s_i}y_i)\right)\right\|_j \\ &\qquad + \sum_{i=1}^n \left\|\left(T_{1,\mu_j^\alpha}(y_i - T_{1,s_i}y_i),\dots,T_{j,\mu_j^\alpha}(y_i - T_{j,s_i}y_i)\right)\right\|_j \\ &\quad\leq C \cdot \left\|\left(z - \sum_{i=1}^n a_i(y_i - T_{1,s_i}y_i),\dots,z - \sum_{i=1}^n a_i(y_i - T_{j,s_i}y_i)\right)\right\|_j \\ &\qquad + \sup_i \left\|\left(\mu_1^\alpha - r_{s_i}^*\mu_1^\alpha,\dots,\mu_j^\alpha - r_{s_i}^*\mu_j^\alpha\right)\right\|_j \cdot C \cdot \|y_i\|_j \\ &\quad< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Hence, $T_{j,\mu_j^\alpha}Z$ converges strongly to 0 for each $z \in E_0$.

Next, assume that $x - T_{j,\mu_j}x$ for some $x \in E$ is not contained in the set E_0. Then, by the Hahn-Banach theorem, there exists a linear continuous functional y^* such that $\langle x - T_{j,\mu_j}x, y^*\rangle = 1$ and $\langle z, y^*\rangle = 0$ for all $z \in E_0$. So since $x - T_{j,t}x \in E_0$ for all $t \in G$, we have

$$\left\langle x - T_{j,\mu_j}x, y^*\right\rangle = \mu_{j,t}\left\langle x - T_{j,t}x, y^*\right\rangle = 0.$$

This is a contradiction. Hence, $x - T_{j,\mu_j}$ for all $x \in E$ are contained in E_0. Therefore we find that $T_{j,\mu_j^\alpha}x - T_{j,\mu_j}x = T_{j,\mu_j^\alpha}(x - T_{j,\mu_j})$ converges strongly to 0 for all $x \in E$. This completes the proof. □

By using Theorem 4.3, we can obtain the following corollary.

Corollary 4.4 *Let E be a real Banach space and T_j be a linear operator of E into itself such that exists a constant C with $\|(T_1^n, \dots, T_j^n)\|_j \leq C$ for $n = 1, 2, \dots$, and T_j is weakly completely continuous, i.e., T_j maps the closed unit ball of E into a weakly compact subset of E. Then there exists a projection Q_j of E onto the set $F_j(T)$ of all fixed point of T_j such that $\|(Q_1, \dots, Q_j)\|_j \leq C$, the Cesaro means $S_{j,n} = \frac{1}{n}\sum_{k=1}^n T_j^k x$ converges strongly to $Q_j x$ for each $x \in E$, and $T_j Q_j = Q_j T_j = Q_j$.*

Proof Let $x \in E$. Then, since $\{T_j^n x : n = 1, 2, \dots\} = T_j(\{T^{n-1}x : n = 1, 2, \dots\}) \subset T_j(B(0, \|x\| \cdot (c+1)))$, where $B(x, r)$ means the closed ball with center x and radius r, the weak closure of $\{T_j^n x : n = 1, 2, \dots\}$ is weakly compact. On the other hand, let $G = \{1, 2, 3, \dots\}$ with the discrete topology and μ_j^n be a mean on G such that $\mu_j^n(f_j) = \sum_{i=1}^n (\frac{1}{n}) f_j(i)$ for each $f_j \in m_j(G)$. Then it is obvious that $\|(\mu_1^n - r_k^* \mu_1^n, \dots, \mu_j^n - r_k^* \mu_j^n)\|_j \leq \frac{2k}{n} \to 0$ for all $k \in G$. So, it follows from Theorem 4.3 that Corollary 4.4 is true. □

If $G = [0, \infty)$ with the natural topology, then we obtain the corresponding result.

Corollary 4.5 *Let E be a real Banach space and $\{T_{j,t} : t \in [0, \infty)\}$ be a family of linear operators of E into itself satisfying Assumption* (A). *Then there exists a projection Q_j of E onto $F_j(G)$ such that $\|(Q_1, \dots, Q_j)\|_j \leq C$, $\frac{1}{T}\int_0^T T \int_{j,t} x\,dt$ converges strongly to $Q_j x$ for each $x \in E$, and $T_{j,t} Q_j = Q_j T_{j,t} = Q_j$ for each $t \in [0, \infty)$.*

Remark 4.6 $\frac{1}{T}\int_0^T T \int_{j,t} x\,dt$ are weak vector valued integrals with respect to means on $G = [0, \infty)$. As in Section IV of Rodé [4], we can also obtain the strong convergence of the sequences

$$(1-r)\sum_{k=1}^{\infty} r^k T_j^k x, \quad r \to 1-$$

and

$$\lambda \int_0^{\infty} e^{-\lambda t} T_{j,t} x\,dt, \quad \lambda \to 0+.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

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