Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fuzzy prime ideals redefined

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Abstract

In order to generalize the notions of a $(\in, \in \vee q)$-fuzzy subring and various $(\in, \in \vee q)$-fuzzy ideals of a ring, a (λ, μ)-fuzzy subring and a (λ, μ)-fuzzy ideal of a ring are defined. The concepts of (λ, μ)-fuzzy semiprime, prime, semiprimary and primary ideals are introduced, and the characterizations of such fuzzy ideals are obtained based on a (λ, μ)-cut set.

MSC: 16L99; 03E72

Keywords: (λ, μ)-fuzzy subring; (λ, μ)-fuzzy ideal; (λ, μ)-fuzzy prime ideal; (λ, μ)-fuzzy primary ideal

1 Introduction

The concept of a fuzzy set introduced by Zadeh [1] was applied to the group theory by Rosenfeld [2] and the ring theory by Liu [3]. Since then, many scholars have studied the theories of fuzzy subrings and various fuzzy prime ideals [4–6]. It is worth pointing out that Bhakat and Das introduced the concept of an (α, β)-fuzzy subgroup by using the 'belongs to' relation and 'quasi-coincident with' relation between a fuzzy point and a fuzzy subset, and gave the concepts of an $(\in, \in \vee q)$-fuzzy subgroup and an $(\in, \in \vee q)$-fuzzy subring [7, 8]. It is well known that a fuzzy subset A of a group G is a Rosenfeld's fuzzy subgroup if and only if $A_t = \{x \in G \mid A(x) \geq t\}$ is a subgroup of G for all $t \in (0,1]$ (for our convenience, here $\emptyset$ is regarded as a subgroup of G). Similarly, A is an $(\in, \in \vee q)$-fuzzy subgroup if and only if A_t is a subgroup of G for all $t \in (0, 0.5]$. A corresponding result should be considered when A_t is a subgroup of G for all $t \in (a, b]$, where $(a, b]$ is an arbitrary subinterval of [0,1]. Motivated by the above problem, Yuan *et al.* [9] introduced a fuzzy subgroup with the thresholds of a group. In order to generalize the concepts of an $(\in, \in \vee q)$-fuzzy subring and an $(\in, \in \vee q)$-fuzzy ideal of a ring, Yao [10] introduced the notions of a (λ, μ)-fuzzy subring and a (λ, μ)-fuzzy ideal and discussed their fundamental properties. In this paper, we will introduce the concepts of (λ, μ)-fuzzy prime, fuzzy semiprime, fuzzy primary and fuzzy semiprimary ideals of a ring.

2 (λ, μ)-fuzzy ideal

Let X be a nonempty set. By a fuzzy subset A of X, we mean a map from X to the interval $[0,1]$, $A : X \to [0,1]$. If A is a fuzzy subset of X and $t \in [0,1]$, then the cut set A_t and the open cut set $A_{\langle t\rangle}$ of A are defined as follows:

$$A_t = \{x \in X \mid A(x) \geq t\}, \qquad A_{\langle t\rangle} = \{x \in X \mid A(x) > t\}.$$

First, we recall some definitions and results for the sake of completeness.

Definition 1 Let $x \in X$, $t \in (0,1]$. A fuzzy subset A of X of the form

$$A(y) = \begin{cases} t, & \text{if } y = x, \\ 0, & \text{if } y \neq x \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

Definition 2 [11] A fuzzy point x_t is said to belong to (resp. be quasi-coincident with) a fuzzy subset A, written as $x_t \in A$ (resp. $x_t qA$), if

$$A(x) \geq t \quad \bigl(\text{resp. } A(x) + t > 1\bigr),$$

$x_t \in A$ or $x_t qA$ will be denoted by $x_t \in \vee qA$.

In the following discussions, R always stands for an associate ring, λ and μ are constant numbers such that $0 \leq \lambda < \mu \leq 1$, and N denotes the set of all positive integers.

Definition 3 [8] A fuzzy subset A of R is said to be an $(\in, \in \vee q)$-fuzzy subring of R if for all $x, y \in R$ and $t, r \in (0,1]$,
(1) $x_t, y_r \in A \Longrightarrow (x+y)_{t\wedge r} \in \vee qA$,
(2) $x_t \in A \Longrightarrow (-x)_t \in \vee qA$,
(3) $x_t, y_r \in A \Longrightarrow (xy)_{t\wedge r} \in \vee qA$.

Definition 4 [8] A fuzzy subset A of R is said to be an $(\in, \in \vee q)$-fuzzy ideal of R if
(1) A is an $(\in, \in \vee q)$-fuzzy subring of R,
(2) $x_t \in A, y \in R \Longrightarrow (xy)_t, (yx)_t \in \vee qA, \forall t \in (0,1]$.

According to Definition 3 and Definition 4, we have that a fuzzy subset A of R is a (λ, μ)-fuzzy subring (ideal) of R if and only if for all $x, y \in R$,
(1) $A(x-y) \geq A(x) \wedge A(y) \wedge 0.5$,
(2) $A(xy) \geq A(x) \wedge A(y) \wedge 0.5$ ((2)$'$ $A(xy) \geq (A(x) \vee A(y)) \wedge 0.5$).
In order to give more general concepts of a fuzzy subring and a fuzzy ideal of R, we introduce the following definitions.

Definition 5 A fuzzy subset A of R is said to be a (λ, μ)-fuzzy addition subgroup of R if for all $x, y \in R$,

$$A(x+y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu, \qquad A(-x) \vee \lambda \geq A(x) \wedge \mu.$$

Clearly, a fuzzy subset A of R is a (λ, μ)-fuzzy addition subgroup of R if and only if for all $x, y \in R$, $A(x-y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$.

Definition 6 [10] A fuzzy subset A of R is said to be a (λ, μ)-fuzzy subring of R if for all $x, y \in R$,
(1) $A(x-y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$,
(2) $A(xy) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$.

Definition 7 [10] A fuzzy subset A of R is said to be a (λ,μ)-fuzzy left ideal (resp. fuzzy right ideal) of R if for all $x, y \in R$,

(1) $A(x-y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$,

(2) $A(xy) \vee \lambda \geq A(y) \wedge \mu$ (resp. $A(xy) \vee \lambda \geq A(x) \wedge \mu$).

A is said to be a (λ,μ)-fuzzy ideal of R if it is both a (λ,μ)-fuzzy left ideal and a (λ,μ)-fuzzy right ideal of R.

According to the above definitions, a (λ,μ)-fuzzy left ideal or a (λ,μ)-fuzzy right ideal of R must be a (λ,μ)-fuzzy subring. A fuzzy subset A of R is a (λ,μ)-fuzzy ideal of R if and only if for all $x, y \in R$,

(1) $A(x-y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$,

(2) $A(xy) \vee \lambda \geq (A(x) \vee A(y)) \wedge \mu$.

Obviously, an $(\in, \in \vee q)$-fuzzy subring (fuzzy ideal) of R is a (λ,μ)-fuzzy subring (fuzzy ideal) of R with $\lambda = 0$ and $\mu = 0.5$.

The following theorem is obvious.

Theorem 1 *Let A, B be (λ,μ)-fuzzy left ideals (fuzzy right ideals, fuzzy ideals, fuzzy subrings) of R. Then $A \cap B$ is also a fuzzy left ideal (fuzzy right ideal, fuzzy ideal, fuzzy subring) of R.*

Theorem 2 *Let A, B be (λ,μ)-fuzzy left ideals (fuzzy right ideals, fuzzy ideals) of R. Then $A+B$ is also a (λ,μ)-fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of R, where*

$$(A+B)(x) = \sup\{A(x_1) \wedge B(x_2) \mid x = x_1 + x_2\}, \quad \forall x \in R.$$

Proof We only prove the case of a (λ,μ)-fuzzy left ideal.

For all $x, y \in R$, we have

$$\begin{aligned}
&(A+B)(x-y) \vee \lambda \\
&\quad \geq \sup\{A(x_1 - x_2) \wedge B(y_1 - y_2) \mid x = x_1 + y_1, y = x_2 + y_2\} \vee \lambda \\
&\quad \geq \sup\{A(x_1) \wedge A(x_2) \wedge B(y_1) \wedge B(y_2) \wedge \mu \mid x = x_1 + y_1, y = x_2 + y_2\} \\
&\quad = \sup\{A(x_1) \wedge B(y_1) \mid x = x_1 + y_1\} \wedge \sup\{A(x_2) \wedge B(y_2) \mid y = x_2 + y_2\} \wedge \mu \\
&\quad = (A+B)(x) \wedge (A+B)(y) \wedge \mu, \\
&(A+B)(xy) \vee \lambda \\
&\quad \geq \sup\{A(xy_1) \wedge B(xy_2) \mid y = y_1 + y_2\} \vee \lambda \\
&\quad = \sup\{\big(A(xy_1) \vee \lambda\big) \wedge \big(B(xy_2) \vee \lambda\big) \mid y = y_1 + y_2\} \\
&\quad \geq \sup\{\big(A(y_1) \wedge \mu\big) \wedge \big(B(y_2) \wedge \mu\big) \mid y = y_1 + y_2\} \\
&\quad = \sup\{A(y_1) \wedge B(y_2) \mid y = y_1 + y_2\} \wedge \mu \\
&\quad = (A+B)(y) \wedge \mu.
\end{aligned}$$

So, $A+B$ is a (λ,μ)-fuzzy left ideal of R.

Let A, B be fuzzy subsets of R. Then the fuzzy subset $A \odot B$ is defined as follows: $\forall x \in R$,

$$(A \odot B)(x) = \begin{cases} \sup\{\inf_{1\le i\le n}(A(x_i) \wedge B(y_i)) \mid x = \sum_{i=1}^{n} x_i y_i, x_i, y_i \in R, n \in N\}, \\ \qquad \text{if } x \text{ can be expressed as } x = \sum x_i y_i, x_i, y_i \in R, \\ 0, \quad \text{otherwise.} \end{cases}$$

□

Theorem 3 *Let A be a (λ, μ)-fuzzy left ideal, and let B be a fuzzy subset of R. Then $A \odot B$ is a (λ, μ)-fuzzy left ideal of R.*

Proof For all $z_1, z_2 \in R$, we have

$$\begin{aligned}
&(A \odot B)(z_1 - z_2) \vee \lambda \\
&\quad \ge \sup\left\{ \inf_{1\le i\le n, 1\le j\le m} A(x_i) \wedge B(y_i) \wedge A(-x_j') \wedge B(y_j') \mid, \right. \\
&\qquad \left. z_1 = \sum_{i=1}^{n} x_i y_i, z_2 = \sum_{j=1}^{m} x_j' y_j', m, n \in N \right\} \vee \lambda \\
&\quad \ge \sup\left\{ \inf_{1\le i\le n} A(x_i) \wedge B(y_i) \,\middle|\, z_1 = \sum_{i=1}^{n} x_i y_i, n \in N \right\} \\
&\qquad \wedge \sup\left\{ \inf_{1\le j\le m} A(x_j') \wedge B(y_j') \,\middle|\, z_2 = \sum_{j=1}^{m} x_j' y_j', m \in N \right\} \wedge \mu \\
&\quad = (A \odot B)(z_1) \wedge (A \odot B)(z_2) \wedge \mu, \\
&(A \odot B)(z_1 z_2) \vee \lambda \ge \sup\left\{ \inf_{1\le i\le n} A(z_1 x_i) \wedge B(y_i) \,\middle|\, z_2 = \sum_{i=1}^{n} x_i y_i, n \in N \right\} \vee \lambda \\
&\quad \ge \sup\left\{ \inf_{1\le i\le n} A(x_i) \wedge B(y_i) \,\middle|\, z_2 = \sum_{i=1}^{n} x_i y_i, n \in N \right\} \wedge \mu \\
&\quad = (A \odot B)(z_2) \wedge \mu.
\end{aligned}$$

So, $A \odot B$ is a $(\in, \in \vee q)$-fuzzy left ideal of R. □

Similarly, we have the following theorem.

Theorem 4 *Let A be a fuzzy subset, and let B be a (λ, μ)-fuzzy right ideal of R. Then $A \odot B$ is a (λ, μ)-fuzzy right ideal of R.*

The following theorem is an immediate consequence of Theorem 3 and Theorem 4.

Theorem 5 *Let A be a (λ, μ)-fuzzy left ideal, and let B be a (λ, μ)-fuzzy right ideal of R. Then $A \odot B$ is a (λ, μ)-fuzzy ideal of R.*

One of the most common methods of studying a fuzzy subring and a fuzzy ideal is by using their cut sets. Now we give the relation between a (λ, μ)-fuzzy subring (fuzzy ideal) with its cut set or open cut set.

Theorem 6 [10] *A fuzzy subset A of R is a (λ,μ)-fuzzy subring (fuzzy ideal) of R if and only if for all $t \in (\lambda,\mu]$, A_t is a subring (ideal) of R or $A_t = \emptyset$.*

Theorem 7 *A fuzzy subset A of R is a (λ,μ)-fuzzy subring (fuzzy ideal) of R if and only if for all $t \in [\lambda,\mu)$, $A_{\langle t\rangle}$ is a subring (ideal) of R or $A_{\langle t\rangle} = \emptyset$.*

Proof We only prove the case of a (λ,μ)-fuzzy subring.

Let A be a (λ,μ)-fuzzy subring of R, and let $t \in [\lambda,\mu)$. If $x,y \in A_{\langle t\rangle}$, then $A(x) \wedge A(y) > t$, $A(x-y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu > t$. Considering $\lambda \leq t$, we have $A(x-y) > t$ and $x-y \in A_{\langle t\rangle}$. Similarly, $xy \in A_{\langle t\rangle}$. It follows that $A_{\langle t\rangle}$ is a subring of R.

Conversely, assume that $A_{\langle t\rangle}$ is a subring of R for all $t \in [\lambda,\mu)$. If possible, let $A(x_0 - y_0) \vee \lambda < A(x_0) \wedge A(y_0) \wedge \mu$ for some $x_0, y_0 \in R$. Put $t = A(x_0 - y_0) \vee \lambda$, then $t \in [\lambda,\mu)$, and $A(x_0 - y_0) \leq t$, $A(x_0) \wedge A(y_0) > t$. So, $x_0, y_0 \in A_{\langle t\rangle}$, and $x_0 - y_0 \notin A_{\langle t\rangle}$. This is a contradiction to the fact that $A_{\langle t\rangle}$ is a subring of R. This shows that $A(x-y) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$ holds for all $x,y \in R$.

Similarly, $A(xy) \vee \lambda \geq A(x) \wedge A(y) \wedge \mu$, $\forall x,y \in R$. That is, $A_{\langle t\rangle}$ is a subring of R. □

In the following theorems, it is shown that the homomorphism image (preimage) of a (λ,μ)-fuzzy subring is also a (λ,μ)-fuzzy subring. Similar result can be obtained for a (λ,μ)-fuzzy ideal under some conditions.

Theorem 8 [10] *Let $f : R \to R'$ be a homomorphism of rings. If A is a (λ,μ)-fuzzy subring of R, then $f(A)$ is a (λ,μ)-fuzzy subring of R'. Particularly, if A is a (λ,μ)-fuzzy ideal of R and f is onto, then $f(A)$ is a (λ,μ)-fuzzy ideal of R', where*

$$f(A)(y) = \begin{cases} \sup\{A(x) \mid f(x) = y\}, & \text{if } f^{-1}(y) \neq \emptyset, \\ 0, & \text{otherwise} \end{cases} \quad \forall y \in R'.$$

Theorem 9 [10] *Let $f : R \to R'$ be a homomorphism of rings. If B is a (λ,μ)-fuzzy subring (fuzzy ideal) of R', then $f^{-1}(B)$ is a (λ,μ)-fuzzy subring (fuzzy ideal) of R, where*

$$f^{-1}(B)(x) = B\bigl(f(x)\bigr), \quad \forall x \in R.$$

3 (λ,μ)-cut set

Based on the notion of an $(\in, \in \vee q)$-level subset defined in [12], we introduce the concept of a (λ,μ)-cut set of a fuzzy subset. Let A be a fuzzy subset of a set X and $t \in [0,1]$. Then the subset $A_t^{(\lambda,\mu)}$ of X defined by

$$A_t^{(\lambda,\mu)} = \bigl\{x \in X \mid A(x) \vee \lambda \geq t \wedge \mu \text{ or } A(x) > (2\mu - t) \vee \lambda\bigr\}$$

is said to be a (λ,μ)-cut set of A. We denote $x_t \in_{(\lambda,\mu)} A$ if $x \in A_t^{(\lambda,\mu)}$.

Obviously, if A is a fuzzy subset of X and $t \in [0,1]$, then

$$A_t^{(\lambda,\mu)} = \begin{cases} X, & t \leq \lambda, \\ A_t, & \lambda < t \leq \mu, \\ A_{\langle (2\mu - t)\vee\lambda\rangle}, & t > \mu. \end{cases}$$

Moreover, $x_t \in \vee qA$ coincides with $x_t \in_{(0,0.5)} A$, and $x_t \in \vee q_k A$ [12] coincides with $x_t \in_{(0,\frac{k}{2})} A$.

Lemma 1 *Let A, B be fuzzy subsets of X. Then for all* $t \in [0,1]$,

$$A \subseteq B \quad \Longrightarrow \quad A_t^{(\lambda,\mu)} \subseteq B_t^{(\lambda,\mu)}.$$

Proof The proof is straightforward. □

Theorem 10 *Let A, B be fuzzy subsets of X and* $t \in [0,1]$. *Then*

(1) $(A \cap B)_t^{(\lambda,\mu)} = A_t^{(\lambda,\mu)} \cap B_t^{(\lambda,\mu)}$,

(2) $(A \cup B)_t^{(\lambda,\mu)} = A_t^{(\lambda,\mu)} \cup B_t^{(\lambda,\mu)}$.

Proof

(1) Obviously, we have $(A \cap B)_t^{(\lambda,\mu)} \subseteq A_t^{(\lambda,\mu)} \cap B_t^{(\lambda,\mu)}$ from Lemma 1. If $x \in A_t^{(\lambda,\mu)} \cap B_t^{(\lambda,\mu)}$, then $x \in A_t^{(\lambda,\mu)}$ and $x \in B_t^{(\lambda,\mu)}$. So, we have the following four cases.

Case 1, if $A(x) \vee \lambda \geq t \wedge \mu$ and $B(x) \vee \lambda \geq t \wedge \mu$, then $(A \cap B)(x) \vee \lambda \geq t \wedge \mu$. So, $x \in (A \cap B)_t^{(\lambda,\mu)}$.

Case 2, if $A(x) > (2\mu - t) \vee \lambda$ and $B(x) > (2\mu - t) \vee \lambda$, then $(A \cap B)(x) > (2\mu - t) \vee \lambda$. So, $x \in (A \cap B)_t^{(\lambda,\mu)}$.

Case 3, if $A(x) \vee \lambda \geq t \wedge \mu$ and $B(x) > (2\mu - t) \vee \lambda$, then when $t \leq \mu$, we have $B(x) \vee \lambda > (2\mu - t) \vee \lambda \geq \mu \geq t \wedge \mu$ and hence $(A \cap B)(x) \vee \lambda \geq t \wedge \mu$. When $t > \mu$, we have $A(x) \geq \mu > (2\mu - t) \vee \lambda$ and hence $(A \cap B)(x) > (2\mu - t) \vee \lambda$. It means that $x \in (A \cap B)_t^{(\lambda,\mu)}$.

Case 4, if $A(x) > (2\mu - t) \vee \lambda$ and $B(x) \vee \lambda \geq t \wedge \mu$, then we can also obtain that $x \in (A \cap B)_t^{(\lambda,\mu)}$ just as in Case 3.

Therefore, $(A \cap B)_t^{(\lambda,\mu)} = A_t^{(\lambda,\mu)} \cap B_t^{(\lambda,\mu)}$.

(2) Obviously, we have $(A \cup B)_t^{(\lambda,\mu)} \supseteq A_t^{(\lambda,\mu)} \cup B_t^{(\lambda,\mu)}$ from Lemma 1.

Let $x \in (A \cup B)_t^{(\lambda,\mu)}$, then either $A(x) \vee B(x) \vee \lambda \geq t \wedge \mu$ or $A(x) \vee B(x) > (2\mu - t) \vee \lambda$. It means that $A(x) \vee \lambda \geq t \wedge \mu$, or $B(x) \vee \lambda \geq t \wedge \mu$, or $A(x) > (2\mu - t) \vee \lambda$, or $B(x) > (2\mu - t) \vee \lambda$. So $x \in A_t^{(\lambda,\mu)}$ or $x \in B_t^{(\lambda,\mu)}$. That is, $x \in A_t^{(\lambda,\mu)} \cup B_t^{(\lambda,\mu)}$. Hence, $(A \cup B)_t^{(\lambda,\mu)} = A_t^{(\lambda,\mu)} \cup B_t^{(\lambda,\mu)}$. □

Theorem 11 *Let A, B, C be fuzzy subsets of X and* $t \in [0,1]$. *Then*

(1) $[A \cap (B \cup C)]_t^{(\lambda,\mu)} = (A \cap B)_t^{(\lambda,\mu)} \cup (A \cap C)_t^{(\lambda,\mu)}$,

(2) $[A \cup (B \cap C)]_t^{(\lambda,\mu)} = (A \cup B)_t^{(\lambda,\mu)} \cap (A \cup C)_t^{(\lambda,\mu)}$.

Proof The proof can be obtained immediately from Theorem 10. □

Theorem 12 *Let A be a* (λ,μ)*-fuzzy subring (fuzzy ideal) of R. Then for all* $t \in [0,1]$, $A_t^{(\lambda,\mu)}$ *is a subring (ideal) of R or* $A_t^{(\lambda,\mu)} = \emptyset$.

Proof We only prove the case of a (λ,μ)-fuzzy subring.

If $t \leq \lambda$, then $A_t^{(\lambda,\mu)} = R$. If $\lambda < t \leq \mu$, then $A_t^{(\lambda,\mu)} = A_t$ and A_t is a subring of R from Theorem 6. If $t > \mu$, then $A_t^{(\lambda,\mu)} = A_{\langle(2\mu-t)\vee\lambda\rangle}$ and $(2\mu - t) \vee \lambda \in [\lambda,\mu)$. So, $A_t^{(\lambda,\mu)}$ is a subring of R from Theorem 7. □

Theorem 13 *Let A be a fuzzy subset of R. If for all* $t \in (\lambda,\mu]$, $A_t^{(\lambda,\mu)}$ *is a subring (ideal) of R or* $A_t^{(\lambda,\mu)} = \emptyset$, *then A is a* (λ,μ)*-fuzzy subring (fuzzy ideal) of R.*

Proof The proof can be obtained from Theorem 6. □

4 (λ, μ)-fuzzy prime and fuzzy primary ideal

There are several deferent definitions of a fuzzy prime ideal and a fuzzy primary ideal of R. In this section, by a prime ideal S of R, we mean an ideal of R such that $ab \in S \Longrightarrow a \in S$ or $b \in S$.

Bhakat and Das [8] defined fuzzy prime, fuzzy semiprime, fuzzy primary and fuzzy semiprimary ideals in a ring which must be $(\in, \in \vee q)$-fuzzy ideals first.

Definition 8 [8] An $(\in, \in \vee q)$-fuzzy ideal A of R is said to be

(1) fuzzy semiprime if for all $x \in R$ and $t \in (0,1]$, $(x^2)_t \in A \Longrightarrow x_t \in \vee qA$,
(2) fuzzy prime if for all $x, y \in R$ and $t \in (0,1]$, $(xy)_t \in A \Longrightarrow x_t \in \vee qA$ or $y_t \in \vee qA$,
(3) fuzzy semiprimary if for all $x, y \in R$ and $t \in (0,1]$, $(xy)_t \in A \Longrightarrow x_t^m \in \vee qA$ or $y_t^n \in \vee qA$ for some $m, n \in N$,
(4) fuzzy primary if for all $x, y \in R$ and $t \in (0,1]$, $(xy)_t \in A \Longrightarrow x_t \in \vee qA$ or $y_t^m \in \vee qA$ for some $m \in N$.

In order to generalize these notions, we introduce (λ, μ)-fuzzy prime, (λ, μ)-fuzzy semiprime, (λ, μ)-fuzzy primary and (λ, μ)-fuzzy semiprimary ideals.

Definition 9 A (λ, μ)-fuzzy ideal A of R is said to be

(1) (λ, μ)-fuzzy semiprime if for all $x \in R$ and $t \in (0,1]$, $(x^2)_t \in A \Longrightarrow x_t \in_{(\lambda,\mu)} A$,
(2) (λ, μ)-fuzzy prime if for all $x, y \in R$ and $t \in (0,1]$, $(xy)_t \in A \Longrightarrow x_t \in_{(\lambda,\mu)} A$ or $y_t \in_{(\lambda,\mu)} A$,
(3) (λ, μ)-fuzzy semiprimary if for all $x, y \in R$ and $t \in (0,1]$, $(xy)_t \in A \Longrightarrow x_t^m \in_{(\lambda,\mu)} A$ or $y_t^n \in_{(\lambda,\mu)} A$ for some $m, n \in N$,
(4) (λ, μ)-fuzzy primary if for all $x, y \in R$ and $t \in (0,1]$, $(xy)_t \in A \Longrightarrow x_t \in_{(\lambda,\mu)} A$ or $y_t^m \in_{(\lambda,\mu)} A$ for some $m \in N$.

In the following four theorems, we give the equivalence condition of a (λ, μ)-fuzzy prime (semiprime, primary, semiprimary) ideal.

Theorem 14 *A (λ, μ)-fuzzy ideal A of R is (λ, μ)-fuzzy prime if and only if for all $x, y \in R$,*

$$\lambda \vee A(x) \vee A(y) \geq A(xy) \wedge \mu.$$

Proof Let A be (λ, μ)-fuzzy prime. If possible, let $\lambda \vee A(x_0) \vee A(y_0) < A(x_0 y_0) \wedge \mu$ for some $x_0, y_0 \in R$. Put $t = A(x_0 y_0) \wedge \mu$, then $A(x_0) \vee A(y_0) < t$ and $(x_0 y_0)_t \in A$. So, $\lambda \vee A(x_0) < t = t \wedge \mu$ and $\lambda \vee A(y_0) < t = t \wedge \mu$. From $\lambda \vee A(x_0) < t \wedge \mu$, we have $A(x_0) \leq \lambda \vee A(x_0) < t \leq \mu \leq 2\mu - t$ and hence $x_0 \notin A_t^{(\lambda,\mu)}$. Similarly, $y_0 \notin A_t^{(\lambda,\mu)}$. This is a contradiction. Therefore, for all $x, y \in R$, we have $\lambda \vee A(x) \vee A(y) \geq A(xy) \wedge \mu$.

Conversely, if for all $x, y \in R$, $\lambda \vee A(x) \vee A(y) \geq A(xy) \wedge \mu$ and $(xy)_t \in A$, then $\lambda \vee A(x) \vee A(y) \geq t \wedge \mu$. So, $\lambda \vee A(x) \geq t \wedge \mu$ or $\lambda \vee A(y) \geq t \wedge \mu$. That is, $x \in_{(\lambda,\mu)} A$ or $y \in_{(\lambda,\mu)} A$. Hence, A is (λ, μ)-fuzzy prime. □

Theorem 15 *A (λ, μ)-fuzzy ideal A of R is (λ, μ)-fuzzy semiprime if and only if for all $x \in R$,*

$$\lambda \vee A(x) \geq A\bigl(x^2\bigr) \wedge \mu.$$

Proof The proof is similar to that of Theorem 14. □

Theorem 16 *A* (λ,μ)*-fuzzy ideal A of R is* (λ,μ)*-fuzzy primary if and only if for all* $x,y\in R$, $\exists m_0\in N$ *such that*

$$\lambda\vee A(x)\vee A(y^{m_0})\geq A(xy)\wedge\mu.$$

Proof Let A be (λ,μ)-fuzzy primary. If possible, there exist $x_0,y_0\in R$ such that $\lambda\vee A(x_0)\vee A(y_0^m)<A(x_0y_0)\wedge\mu$ for all $m\in N$, then $\lambda\vee A(x_0)\vee A(y_0^m)<t\leq\mu$ and $(x_0y_0)_t\in A$, where $t=A(x_0y_0)\wedge\mu$. So, $\lambda\vee A(x_0)<t=t\wedge\mu$ and $\lambda\vee A(y_0^m)<t=t\wedge\mu$. From $\lambda\vee A(x_0)<t$, we have $A(x_0)\leq\lambda\vee A(x_0)<t\leq\mu\leq 2\mu-t$ and hence $x_0\notin A_t^{(\lambda,\mu)}$. Similarly, $y_0^m\notin A_t^{(\lambda,\mu)}$. This is a contradiction. Therefore, for all $x,y\in R$, $\exists m_0\in N$ such that $\lambda\vee A(x)\vee A(y^{m_0})\geq A(xy)\wedge\mu$.

Conversely, if for all $x,y\in R$, $\exists m_0\in N$ such that $\lambda\vee A(x)\vee A(y^{m_0})\geq A(xy)\wedge\mu$, then from $(xy)_t\in A$, we have $\lambda\vee A(x)\vee A(y^{m_0})\geq t\wedge\mu$. So, $\lambda\vee A(x)\geq t\wedge\mu$ or $\lambda\vee A(y^{m_0})\geq t\wedge\mu$. That is, $x_t\in_{(\lambda,\mu)}A$ or $y_t^{m_0}\in_{(\lambda,\mu)}A$. Hence, A is (λ,μ)-fuzzy primary. □

Theorem 17 *A* (λ,μ)*-fuzzy ideal A of R is* (λ,μ)*-fuzzy semiprimary if and only if for all* $x,y\in R$, $\exists m_0,n_0\in N$ *such that*

$$\lambda\vee A(x^{m_0})\vee A(y^{n_0})\geq A(xy)\wedge\mu.$$

Proof The proof is similar to that of Theorem 16. □

Now, we characterize the (λ,μ)-fuzzy prime (semiprimary) ideal by using its cut set.

Theorem 18 *A fuzzy subset A of R is a* (λ,μ)*-fuzzy prime* (*fuzzy semiprime*) *ideal if and only if for all* $t\in(\lambda,\mu]$, A_t *is a prime* (*semiprime*) *ideal of R or* $A_t=\emptyset$.

Proof We only prove the case of a (λ,μ)-fuzzy prime ideal.

Let A be a (λ,μ)-fuzzy prime ideal of R. Then A is a (λ,μ)-fuzzy ideal of R. So, A_t is an ideal of R or $A_t=\emptyset$ from Theorem 6. For all $t\in(\lambda,\mu]$, if $xy\in A_t$, then $\lambda\vee A(x)\vee A(y)\geq A(xy)\wedge\mu\geq t\wedge\mu=t$. Considering $\lambda<t$, we have $A(x)\vee A(y)\geq t$. It follows that $x\in A_t$ or $y\in A_t$. Hence, A_t is a prime ideal of R.

Conversely, assume A_t is a prime ideal of R for all $t\in(\lambda,\mu]$ whenever $A_t\neq\emptyset$, then A_t is an ideal of R, and hence A is a (λ,μ)-fuzzy ideal from Theorem 6. Let $t\in(0,1]$ and $(xy)_t\in A$. If $t\leq\lambda$, then it is clear that $x_t\in_{(\lambda,\mu)}A$. If $t\in(\lambda,\mu]$, then $x\in A_t$ or $y\in A_t$ since A_t is a prime ideal of R. So, $x_t\in_{(\lambda,\mu)}A$ or $y_t\in_{(\lambda,\mu)}A$. If $t>\mu$, then $xy\in A_\mu$. It implies that $x\in A_\mu$ or $y\in A_\mu$, since A_μ is a prime ideal of R. Furthermore, we have

$$x\in A_\mu\Longrightarrow A(x)\geq\mu=t\wedge\mu\Longrightarrow x\in A_t^{(\lambda,\mu)}\Longrightarrow x_t\in_{(\lambda,\mu)}A.$$

Similarly,

$$y\in A_\mu\Longrightarrow y_t\in_{(\lambda,\mu)}A.$$

It follows that A is a (λ,μ)-fuzzy prime ideal of R. □

Theorem 19 *Let A be a* (λ,μ)*-fuzzy ideal of R such that* $A_\mu \neq \emptyset$*, and let B be a* (λ,μ)*-fuzzy prime ideal of* A_μ*. Then* $A \cap B$ *is a* (λ,μ)*-fuzzy prime ideal of* A_μ*.*

Proof From Theorem 1 and Theorem 6, A_μ is a subring of R and $A \cap B$ is a (λ,μ)-fuzzy ideal of A_μ. For all $x, y \in A_\mu$, $t \in (0,1]$, we have $A(x) \geq \mu$ and $A(y) \geq \mu$. Hence, $x, y \in A_t^{(\lambda,\mu)}$. If $(xy)_t \in A \cap B$, then $x \in B_t^{(\lambda,\mu)}$ or $y \in B_t^{(\lambda,\mu)}$, since B is a (λ,μ)-fuzzy prime ideal of A_μ. So $x \in A_t^{(\lambda,\mu)} \cap B_t^{(\lambda,\mu)} = (A \cap B)_t^{(\lambda,\mu)}$, or $y \in A_t^{(\lambda,\mu)} \cap B_t^{(\lambda,\mu)} = (A \cap B)_t^{(\lambda,\mu)}$. It follows that $A \cap B$ is a (λ,μ)-fuzzy prime ideal of A_μ. □

Similarly, we have the following theorem.

Theorem 20 *Let A be a* (λ,μ)*-fuzzy ideal of R such that* $A_\mu \neq \emptyset$*, and let B be a* (λ,μ)*-fuzzy semiprime (fuzzy primary, fuzzy semiprimary) ideal of* A_μ*. Then* $A \cap B$ *is a* (λ,μ)*-fuzzy semiprime (fuzzy primary, fuzzy semiprimary) ideal of* A_μ*.*

The following theorem gives the relation between a (λ,μ)-fuzzy prime ideal with its preimage under a ring homomorphism.

Lemma 2 *Let* $f : R \longrightarrow R'$ *be a homomorphism of rings, and let B be a* (λ,μ)*-fuzzy subring of* R'*. Then for all* $t \in (0,1]$, $f^{-1}(B)_t^{(\lambda,\mu)} = f^{-1}(B_t^{(\lambda,\mu)})$*.*

Theorem 21 *Let* $f : R \longrightarrow R'$ *be a homomorphism of rings, and let B be a* (λ,μ)*-fuzzy prime ideal of* R'*. Then* $f^{-1}(B)$ *is a* (λ,μ)*-fuzzy prime ideal of R.*

Proof From Theorem 9, $f^{-1}(B)$ is a (λ,μ)-fuzzy ideal of R. Let $x, y \in R$ and $t \in (0,1]$. If $(xy)_t \in f^{-1}(B)$, then $(f(x)f(y))_t \in B$. Considering B is a (λ,μ)-fuzzy prime ideal of R', we have $f(x) \in B_t^{(\lambda,\mu)}$ or $f(y) \in B_t^{(\lambda,\mu)}$. Hence $x \in f^{-1}(B)_t^{(\lambda,\mu)}$ or $y \in f^{-1}(B)_t^{(\lambda,\mu)}$ from Lemma 2. It follows that $f^{-1}(B)$ is a (λ,μ)-fuzzy prime ideal of R. □

Similarly, we can obtain corresponding conclusions about a (λ,μ)-fuzzy semiprime ideal, a (λ,μ)-fuzzy primary ideal and a (λ,μ)-fuzzy semiprimary ideal. But in general, the homomorphism image $f(A)$ of a (λ,μ)-fuzzy prime ideal A of R may not be (λ,μ)-fuzzy prime even if f is a surjective homomorphism.

5 Conclusion

In this paper, we proposed the concept of a (λ,μ)-fuzzy ideal which can be regarded as the generalization of a common fuzzy ideal introduced by Liu [11]. In the meantime, we also proposed several concepts of various (λ,μ)-fuzzy ideals such as a (λ,μ)-fuzzy prime ideal and a (λ,μ)-fuzzy primary ideal, and then we characterized their properties and obtained several equivalence conditions of a (λ,μ)-fuzzy prime ideal and a (λ,μ)-fuzzy primary ideal.

Competing interests
The author declares that they have no competing interests.

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Acknowledgements
This work was supported by the National Natural Science Foundation of China (No. 71140008).

Received: 24 April 2012 Accepted: 3 September 2012 Published: 19 September 2012

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doi:10.1186/1029-242X-2012-203
Cite this article as: Yao: **Fuzzy prime ideals redefined.** *Journal of Inequalities and Applications* 2012 **2012**:203.