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RESEARCH

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Bounds for blow-up time in a semilinear pseudo-parabolic equation with nonlocal source

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Abstract

This paper considers the following semilinear pseudo-parabolic equation with a nonlocal source:

$$u_t - \Delta u_t - \Delta u = u^p(x,t) \int_\Omega k(x,y) u^{p+1}(y,t)\,dy,$$

and it explores the characters of blow-up time for solutions, obtaining a lower bound as well as an upper bound for the blow-up time under different conditions, respectively. Also, we investigate a nonblow-up criterion and compute an exact exponential decay.

MSC: 60H15; 35L05; 35L70

Keywords: lower bound; blow up; upper bound

1 Introduction

In this paper, we deal with the blow-up problem for the following equation:

$$\begin{cases} u_t - \triangle u_t - \triangle u = u^p(x,t) \int_\Omega k(x,y) u^{p+1}(y,t)\,dy & \text{in } [0,T] \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } [0,T] \times \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$ is a bounded domain with smooth boundary, $u_0(x) \in H_0^1(\Omega)$, $T \in (0,\infty]$, and p satisfies

$$0 < p \leq \frac{2}{n-2}, \tag{1.2}$$

$k(x,y)$ is an integrable, real valued function satisfying

$$\begin{aligned} &k(x,y) = k(y,x), \qquad \int_\Omega \int_\Omega k^2(x,y)\,dx\,dy < \infty, \\ &\int_\Omega \int_\Omega k(x,y) u^{p+1}(x,t) u^{p+1}(y,t)\,dx\,dy > 0. \end{aligned} \tag{1.3}$$

The blow-up phenomena for problems similar to (1.1) have been extensively researched (see [1–7]). For instance,

$$\begin{cases} u_t - \triangle u = u^p & \text{in } [0,T] \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u(t,x) = 0 & \text{on } [0,T] \times \partial\Omega, \end{cases}$$

and bounds for the blow-up time have been explored [8, 9].

Recently, authors have begun to consider

$$u_t - \triangle u_t - \triangle u = f(u). \tag{1.4}$$

When $f(u) = u^p$ in (1.4), many results have been studied in [5, 6, 10–12] and the references therein, among which Xu [6] proved finite time blow up for solutions through the so-called potential well method, first introduced in [13]. The method has played an important role in dealing with parabolic and hyperbolic problems since it was discovered. Later on, confirmed by the same conditions as [6] that guarantee the occurrence of blow up, Luo [5] established a lower bound for the blow-up time. Furthermore, when $f(u) = u^p(x,t) \int_\Omega k(x,y) u^{p+1}(y,t)\,dy$ with $0 < p \leq \frac{2}{n-2}$ in (1.4), which is a new problem and has not been considered, by means of the potential well method, Yang [14] not only obtained the global existence and asymptotic behavior of solutions with deducing exponential decay, but also got the existence of solutions that blow up in finite time in $H_0^1(\Omega)$-norm with energy $J(u_0) \leq d$.

In the last several decades, an increasing number of researchers focused on exploring the upper bounds for the blow-up time. However, the authors have trouble getting lower bounds for the blow-up time, and therefore they received little attention. In this paper, we use the means of a differential inequality technique and present some results on the bounds for the blow-up time to problem (1.1) since little attention is paid to the bounds before we study. Our paper is organized as follows. In Section 2, we come up with the main results: First of all, a blow-up criterion and an upper bound for the blow-up time are determined. Second, we investigate the nonblow-up case. Finally, a lower bound for the blow-up time is obtained.

Before stating our principal theorem, we note that the Fréchet derivative f_u of the nonlinear function $f(u) = u^p(x,t) \int_\Omega k(x,y) u^{p+1}(y,t)\,dy$ is

$$\begin{aligned} f_u \cdot h(x,t) = {} & p u^{p-1}(x,t) h(x,t) \int_\Omega k(x,y) u^{p+1}(y,t)\,dy \\ & + (p+1) u^p(x,t) \int_\Omega k(x,y) u^p(y,t) h(y,t)\,dy, \quad \forall u \in H_0^1(\Omega). \end{aligned}$$

Clearly f_u is symmetric and bounded, so the potential F exists and is given by

$$\begin{aligned} F(u) &= \int_0^1 \bigl(f(\rho u), u\bigr)\,d\rho \\ &= \int_0^1 \int_\Omega \rho^p u^p(x,t) \biggl(\int_\Omega k(x,y) \rho^{p+1} u^{p+1}(y,t)\,dy \biggr) u(x,t)\,dx\,d\rho \\ &= \frac{1}{2p+2} \int_\Omega \int_\Omega k(x,y) u^{p+1}(x,t) u^{p+1}(y,t)\,dx\,dy. \end{aligned} \tag{1.5}$$

Differentiating (1.5) with respect to t, it follows that

$$\begin{aligned}\frac{d}{dt}F(u) &= \frac{1}{2p+2}\frac{d}{dt}\int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy \\ &= \frac{1}{2}\int_\Omega\int_\Omega k(x,y)u^{p}(x,t)u^{p+1}(y,t)u_t(x,t)\,dx\,dy \\ &\quad + \frac{1}{2}\int_\Omega\int_\Omega k(x,y)u^{p}(y,t)u^{p+1}(x,t)u_t(y,t)\,dx\,dy \\ &= \int_\Omega\int_\Omega k(x,y)u^{p}(x,t)u^{p+1}(y,t)u_t(x,t)\,dx\,dy = \bigl(f(u),u_t\bigr),\end{aligned} \tag{1.6}$$

where we have used the symmetry of $k(x,y)$.

To obtain the main results, we introduce the functionals

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2p+2}\int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy \tag{1.7}$$

and

$$I(u) = \|\nabla u\|_2^2 - \int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy. \tag{1.8}$$

We set out to establish local existence and uniqueness for (1.1).

Theorem 1.1 *Assume* (1.2) *holds. Then for any* $u_0 \in H_0^1(\Omega)$, *there exists a* $T > 0$ *for which* (1.1) *has a unique local solution* $u(t) \in L^\infty([0,T);H_0^1(\Omega))$ *with* $u_t(t) \in L^2([0,T);H_0^1(\Omega))$, *satisfying*

$$(u_t,v) + (\nabla u_t,\nabla v) + (\nabla u,\nabla v) = \left(u^p(x,t)\int_\Omega k(x,y)u^{p+1}(y,t)\,dy, v\right) \tag{1.9}$$

for all $v \in H_0^1(\Omega)$.

Proof Choose $\{\omega_j(x)\}$ as the basis functions of $H_0^1(\Omega)$. Construct the approximate solutions $u_m(x,t)$ of the problem (1.1)

$$u_m(x,t) = \sum_{j=1}^m g_j(t)\omega_j(x), \quad m = 1,2,\ldots$$

which satisfy

$$\begin{aligned}&(u_{mt},\omega_s) + (\nabla u_{mt},\nabla\omega_s) + (\nabla u_m,\nabla\omega_s) \\ &\quad = \left(u_m^p(x,t)\int_\Omega k(x,y)u_m^{p+1}(y,t)\,dy, \omega_s\right), \quad s = 1,2,\ldots,m,\end{aligned} \tag{1.10}$$

$$u_m(x,0) = \sum_{j=1}^m a_j\omega_j(x) \to u_0(x) \quad \text{in } H_0^1(\Omega). \tag{1.11}$$

Multiplying (1.10) by $g_s'(t)$, summing over s, and integrating with respect to t from 0 to t, we obtain

$$\int_0^t \left[\|u_{m\tau}\|_2^2 + \|\nabla u_{m\tau}\|_2^2\right] d\tau + J\bigl(u_m(t)\bigr) = J\bigl(u_m(0)\bigr) \le C. \tag{1.12}$$

In fact

$$\begin{aligned} J(u_0) &= \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{2p+2}\int_\Omega\int_\Omega k(x,y)u_0^{p+1}(x,t)u_0^{p+1}(y,t)\,dx\,dy \\ &\le \frac{1}{2}\|\nabla u_0\|_2^2 \le C, \end{aligned}$$

where we have used condition (1.3) and $u_0 \in H_0^1(\Omega)$. Thus for sufficiently large m, we get (1.12). Hence, by (1.12) and

$$J(u_m) = \frac{1}{2}\|\nabla u_m\|_2^2 - \frac{1}{2p+2}\int_\Omega\int_\Omega k(x,y)u_m^{p+1}(x,t)u_m^{p+1}(y,t)\,dx\,dy$$

we obtain

$$\begin{aligned} &\int_0^t \left[\|u_{m\tau}\|_2^2 + \|\nabla u_{m\tau}\|_2^2\right] d\tau + \frac{1}{2}\|u_m\|_{H_0^1}^2 \\ &\quad\le C + \frac{1}{2p+2}\int_\Omega\int_\Omega k(x,y)u_m^{p+1}(x,t)u_m^{p+1}(y,t)\,dx\,dy. \end{aligned} \tag{1.13}$$

We estimate the last term in the right-hand side using the Hölder and Sobolev inequalities (recall $u_m \in C^1([0,T],H_0^1(\Omega))$):

$$\begin{aligned} &\int_\Omega\int_\Omega k(x,y)u_m^{p+1}(x,t)u_m^{p+1}(y,t)\,dx\,dy \\ &\quad\le \int_\Omega u_m^{p+1}(x,t)\left(\int_\Omega k^2(x,y)\,dy\right)^{\frac{1}{2}}\left(\int_\Omega u_m^{2p+2}(y,t)\,dy\right)^{\frac{1}{2}}dx \\ &\quad= \|u_m\|_{2p+2}^{p+1}\int_\Omega u_m^{p+1}(x,t)\left(\int_\Omega k^2(x,y)\,dy\right)^{\frac{1}{2}}dx \\ &\quad\le \|u_m\|_{2p+2}^{p+1}\left(\int_\Omega u_m^{2p+2}(x,t)\,dx\right)^{\frac{1}{2}}\left(\int_\Omega\int_\Omega k^2(x,y)\,dy\,dx\right)^{\frac{1}{2}} \\ &\quad= \|u_m\|_{2p+2}^{2p+2}\left(\int_\Omega\int_\Omega k^2(x,y)\,dy\,dx\right)^{\frac{1}{2}} \\ &\quad= \kappa\|u_m\|_{2p+2}^{2p+2} \le \kappa C_*^{2p+2}\|u_m\|_{H_0^1}^{2p+2} \le C_T, \end{aligned} \tag{1.14}$$

here $\kappa = (\int_\Omega\int_\Omega k^2(x,y)\,dx\,dy)^{\frac{1}{2}} < \infty$, and C_* is the embedding constant for $H_0^1(\Omega) \hookrightarrow L^{2p+2}(\Omega)$. From (1.13) and (1.14) we obtain

$$\int_0^T \left[\|u_{m\tau}\|_2^2 + \|\nabla u_{m\tau}\|_2^2\right] d\tau + \frac{1}{2}\|u_m\|_{H_0^1}^2 \le C_T. \tag{1.15}$$

On the other hand, by using the Hölder and Sobolev inequalities, here $q = \frac{2p+2}{2p+1}$, we have

$$
\begin{aligned}
&\int_\Omega \left[u_m^p(x,t)\int_\Omega k(x,y)u_m^{p+1}(y,t)\,dy\right]^q dx \\
&\quad\le \int_\Omega \left|u_m(x,t)\right|^{pq}\left(\int_\Omega \left|k(x,y)\right|^2 dy\right)^{\frac{q}{2}}\left(\int_\Omega \left|u_m(y,t)\right|^{2p+2} dy\right)^{\frac{q}{2}} dx \\
&\quad\le \|u_m\|_{2p+2}^{q(p+1)}\left[\int_\Omega \left|u_m(x,t)\right|^{pq\cdot\frac{2p+1}{p}} dx\right]^{\frac{p}{2p+1}}\left[\int_\Omega \left(\int_\Omega \left|k(x,y)\right|^2 dy\right)^{\frac{q}{2}\cdot\frac{2p+1}{p+1}} dx\right]^{\frac{p+1}{2p+1}} \\
&\quad\le \|u_m\|_{2p+2}^{2p+2}\left[\int_\Omega\int_\Omega \left|k(x,y)\right|^2 dy\,dx\right]^{\frac{p+1}{2p+1}} \\
&\quad= \|u_m\|_{2p+2}^{2p+2}\kappa^{\frac{2p+2}{2p+1}} \le C_*^{2p+2}\|u_m\|_{H_0^1}^{2p+2}\kappa^{\frac{2p+2}{2p+1}}.
\end{aligned}
\tag{1.16}
$$

By (1.15) and (1.16), for sufficiently large m, we get

$$\|u_m\|_{H_0^1}^2 \le 2C_T, \tag{1.17}$$

$$\int_0^T \left[\|u_{m\tau}\|_2^2 + \|\nabla u_{m\tau}\|_2^2\right] d\tau < C_T, \tag{1.18}$$

$$\left\|u_m^p\int_\Omega k(x,y)u_m^{p+1}(y,t)\,dy\right\|_q^q \le C_*^{2p+2}(2C_T)^{p+1}\kappa^{\frac{2p+2}{2p+1}}. \tag{1.19}$$

Therefore, by these uniform estimates from (1.17)-(1.19), there exist u and a subsequence still denoted by $\{u_m\}$ such that, as $m \to \infty$,

$$
\begin{aligned}
&u_m \to u \quad \text{in } L^\infty\big([0,T],H_0^1(\Omega)\big) \text{ weakly star and a.e. in } \Omega\times[0,T], \\
&u_{mt} \to u_t \quad \text{in } L^2\big([0,T],H_0^1(\Omega)\big) \text{ weakly star}, \\
&u_m^p\int_\Omega k(x,y)u_m^{p+1}(y,t)\,dy \to u^p\int_\Omega k(x,y)u^{p+1}(y,t)\,dy \\
&\quad \text{in } L^\infty\big([0,T],L^q(\Omega)\big) \text{ weakly star.}
\end{aligned}
$$

Thus in (1.10), for s fixed, letting $m \to \infty$, one has

$$(u_t,\omega_s) + (\nabla u_t,\nabla\omega_s) + (\nabla u,\nabla\omega_s) = \left(u^p(x,t)\int_\Omega k(x,y)u^{p+1}(y,t)\,dy,\omega_s\right), \quad \forall s$$

and

$$
\begin{aligned}
&(u_t,v) + (\nabla u_t,\nabla v) + (\nabla u,\nabla v) = \left(u^p(x,t)\int_\Omega k(x,y)u^{p+1}(y,t)\,dy,v\right), \\
&\quad \forall v\in H_0^1(\Omega), t\in[0,T].
\end{aligned}
$$

Moreover, (1.11) gives $u(x,0) = u_0(x)$ in $H_0^1(\Omega)$. The existence of u solving (1.1) and satisfying (1.9) is so proved. □

2 Main results

I. Upper bound for blow-up time

Here, a condition to ensure blow-up at some finite time as well as an upper bound for the blow-up time is considered.

$J(u)$ is defined in (1.7) and we introduce

$$\alpha(t) = \|u(x,t)\|_{H_0^1}^2 = \int_\Omega u^2\,dx + \int_\Omega |\nabla u|^2\,dx. \tag{2.1}$$

A straightforward computation shows that

$$\int_\Omega uu_t\,dx + \int_\Omega \nabla u\cdot\nabla u_t\,dx + \int_\Omega |\nabla u|^2\,dx = \int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dy\,dx. \tag{2.2}$$

Combining (2.1) with (2.2), we obtain

$$\alpha'(t) = -2\int_\Omega |\nabla u|^2\,dx + 2\int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dy\,dx \ge \beta(t), \tag{2.3}$$

where

$$\beta(t) = -4(p+1)J(u) = -2(p+1)\|\nabla u\|_2^2 + 2\int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy, \tag{2.4}$$

for any $p>0$. We obtain the following by multiplying $u_t(x,t)$ on both sides of (1.1) and integrating by parts:

$$\begin{aligned}\int_\Omega |u_t|^2\,dx + \int_\Omega |\nabla u_t|^2\,dx = {}& -\int_\Omega \nabla u\cdot\nabla u_t\,dx \\ & + \int_\Omega\int_\Omega k(x,y)u^p(x,t)u^{p+1}(y,t)u_t(x,t)\,dy\,dx.\end{aligned} \tag{2.5}$$

From (2.5), we have

$$\begin{aligned}\beta'(t) &= 4(p+1)\biggl(-\int_\Omega \nabla u\cdot\nabla u_t\,dx \\ &\quad + \frac{1}{2p+2}\int_\Omega\int_\Omega k(x,y)(p+1)u^p(x,t)u^{p+1}(y,t)u_t(x,t)\,dy\,dx \\ &\quad + \frac{1}{2p+2}\int_\Omega\int_\Omega k(x,y)(p+1)u^p(y,t)u^{p+1}(x,t)u_t(y,t)\,dy\,dx\biggr) \\ &= 4(p+1)\biggl(-\int_\Omega \nabla u\cdot\nabla u_t\,dx + \int_\Omega\int_\Omega k(x,y)u^p(x,t)u^{p+1}(y,t)u_t(x,t)\,dy\,dx\biggr) \\ &= 4(p+1)\biggl(\int_\Omega |u_t|^2\,dx + \int_\Omega |\nabla u_t|^2\,dx\biggr),\end{aligned} \tag{2.6}$$

where we have used the symmetry property of $k(x,y)$ in the second step in (2.6).

By (2.1) and (2.6), one sees

$$\begin{aligned}\alpha(t)\beta'(t) &= 4(p+1)\biggl(\int_\Omega u^2\,dx + \int_\Omega |\nabla u|^2\,dx\biggr)\biggl(\int_\Omega |u_t|^2\,dx + \int_\Omega |\nabla u_t|^2\,dx\biggr) \\ &\ge 4(p+1)\biggl(\int_\Omega uu_t\,dx + \int_\Omega \nabla u\cdot\nabla u_t\,dx\biggr)^2 = (p+1)\bigl[\alpha'(t)\bigr]^2.\end{aligned} \tag{2.7}$$

In addition, it is indicated from (2.4) and (2.6) that $\beta(t)$ is a nondecreasing function of t. So if we assume $J(u_0) < 0$, then $\beta(t) > 0$ for all $t \geq 0$. And with (2.3), we have

$$\alpha(t)\beta'(t) \geq (p+1)\alpha'(t)\beta(t),$$

which becomes

$$\frac{\beta'(t)}{\beta(t)} \geq (p+1)\frac{\alpha'(t)}{\alpha(t)}. \tag{2.8}$$

By (2.3), we have

$$\frac{\alpha'(t)}{[\alpha(t)]^{p+1}} \geq \frac{\beta(0)}{[\alpha(0)]^{p+1}}. \tag{2.9}$$

Then

$$\frac{1}{[\alpha(t)]^p} \leq \frac{1}{[\alpha(0)]^p} - p\frac{\beta(0)}{[\alpha(0)]^{p+1}}t. \tag{2.10}$$

It is obvious that (2.10) implies u blows up at some finite time T. T is given by

$$T \leq \frac{1}{p}\frac{\alpha(0)}{\beta(0)} = \frac{\|u_0\|^2_{H_0^1}}{-4p(p+1)J(u_0)}. \tag{2.11}$$

The above result is presented in the following theorem.

Theorem 2.1 *For any $p > 0$, $u_0 \in H_0^1(\Omega) \cap L^{2p+2}(\Omega)$, $J(u_0) < 0$, then the solution $u(x,t)$ of* (1.1) *blows up in finite time, and T is bounded by* (2.11).

II. Nonblow-up case

In this section, we not only give a criterion which guarantees nonblow up, but also we deduce the exponential decay of $u(\cdot,t)$ in $H_0^1(\Omega)$-norm when $u_0(x)$ satisfies some conditions.

$\alpha(t)$ is defined in (2.1).

By a similar calculation to (1.14), we have

$$\int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dx\,dy \leq \kappa\|u\|^{2p+2}_{2p+2}. \tag{2.12}$$

By a straightforward computation, we have

$$\begin{aligned}\alpha'(t) &\leq -2\int_\Omega |\nabla u|^2\,dx + 2\kappa\|u\|^{2p+2}_{2p+2}\\ &\leq -2\int_\Omega |\nabla u|^2\,dx + 2\kappa C_*^{2p+2}\|\nabla u\|_2^{2p+2}\\ &\leq -2\int_\Omega |\nabla u|^2\,dx + 2\kappa C_*^{2p+2}\big[\varphi(t)\big]^{p+1}.\end{aligned} \tag{2.13}$$

The Poincaré inequality gives $\|\nabla u\|_2^2 \geq \lambda_1 \|u\|_2^2$ where λ_1 is the first eigenvalue of the problem

$$\begin{cases} \triangle\omega + \lambda\omega = 0 \quad \text{in } \Omega, \\ \omega = 0 \quad \text{on } \partial\Omega, \\ \omega > 0 \quad \text{in } \Omega. \end{cases} \tag{2.14}$$

Then

$$\|\nabla u\|_2^2 \geq \frac{\lambda_1}{1+\lambda_1} \|u\|_{H_0^1}^2. \tag{2.15}$$

From (2.13) and (2.15), we know that

$$\begin{aligned} \alpha'(t) &\leq -2\frac{\lambda_1}{1+\lambda_1}\|u\|_{H_0^1}^2 + 2\kappa C_*^{2p+2}\big[\alpha(t)\big]^{p+1} \\ &= -2\frac{\lambda_1}{1+\lambda_1}\alpha(t) + 2\kappa C_*^{2p+2}\big[\alpha(t)\big]^{p+1} \\ &\leq \frac{-2}{1+\lambda_1}\alpha(t)\big(\lambda_1 - \kappa(1+\lambda_1)C_*^{2p+2}\big[\alpha(t)\big]^p\big). \end{aligned} \tag{2.16}$$

Let

$$M = \left(\frac{\lambda_1}{\kappa(1+\lambda_1)C_*^{2p+2}}\right)^{\frac{1}{p}}.$$

If $\alpha(0) < M$, then we show that $\alpha(t) < M$.

In fact, if $\alpha(t) \geq M$, we know that there exists t_0 such that $\alpha(t_0) = M$ and $\alpha(t) < M$ for $0 \leq t < t_0$. From (2.16), we have $\alpha'(t) < 0$ for $0 \leq t < t_0$, from which we deduce $\alpha(0) \geq \alpha(t_0) = M$. It leads to a contradiction. This shows $\alpha'(t) < 0$, and blow up cannot occur in finite time.

Thus (2.16) becomes

$$\frac{\alpha'(t)}{\alpha(t)(\lambda_1 - \kappa(1+\lambda_1)C_*^{2p+2}[\alpha(t)]^p)} \leq \frac{-2}{1+\lambda_1}, \tag{2.17}$$

then

$$\int_{\alpha(0)}^{\alpha(t)} \frac{d\gamma}{\gamma(\lambda_1 - \kappa(1+\lambda_1)C_*^{2p+2}\gamma^p)} \leq \frac{-2}{1+\lambda_1}t, \tag{2.18}$$

through variable substitution $\eta = \gamma^p$, (2.18) turns to

$$\int_{[\alpha(0)]^p}^{[\alpha(t)]^p} \frac{d\eta}{p\eta(\lambda_1 - \kappa(1+\lambda_1)C_*^{2p+2}\eta)} \leq \frac{-2}{1+\lambda_1}t \tag{2.19}$$

and

$$\big\|u(x,t)\big\|_{H_0^1}^2 = \alpha(t) \leq \left(\frac{\lambda_1\|u_0\|_{H_0^1}^{2p}}{\lambda_1 - \kappa(1+\lambda_1)C_*^{2p+2}\|u_0\|_{H_0^1}^{2p}}\right)^{\frac{1}{p}} e^{-\frac{2\lambda_1}{1+\lambda_1}}.$$

We state this result in the following theorem.

Theorem 2.2 *If* $0 < p \leq \frac{2}{n-2}$, $\|u_0\|^2_{H_0^1} < M$, *then the solution of* (1.1) *cannot blow up in finite time in* $H_0^1(\Omega)$*-norm, and we have the exponential decay estimate*

$$\|u(x,t)\|^2_{H_0^1} \leq \left(\frac{\lambda_1\|u_0\|^{2p}_{H_0^1}}{\lambda_1 - \kappa(1+\lambda_1)C_*^{2p+2}\|u_0\|^{2p}_{H_0^1}}\right)^{\frac{1}{p}} e^{-\frac{2\lambda_1}{1+\lambda_1}}.$$

III. Lower bounds for blow-up time

This section is devoted to establishing a lower bound for T if the solution $u(x,t)$ blows up at $t = T$ under some conditions.

Let

$$d = \frac{p}{2(p+1)\kappa^{\frac{1}{p}}}\left(\frac{\lambda_1}{C_*^2(1+\lambda_1)}\right)^{1+\frac{1}{p}}. \tag{2.20}$$

For the problem (1.1) Yang [14] has proved Theorem 2.3.

Theorem 2.3 (Blow up for $J(u_0) < d$) ([14]) *Let* p *satisfy* (1.2), $u_0 \in H_0^1(\Omega)$. *Assume that* $J(u_0) < d$, $I(u_0) < 0$. *Then the weak solution* $u(t)$ *of problem* (1.1) *blows up in finite time*.

In view of Theorem 2.3, we see that when $J(u_0) < d$ and $I(u_0) < 0$, the solution $u(x,t)$ of problem (1.1) blows up in finite time T. To continue our study and estimate the lower bound for the blow-up time T, we assume $J(u_0) < d$, $I(u_0) < 0$ in this section.

We introduce $\alpha(t)$ in (2.1) and, calculating as (2.13), we have

$$\begin{aligned}\alpha'(t) &= -2\int_\Omega |\nabla u|^2\,dx + 2\int_\Omega\int_\Omega k(x,y)u^{p+1}(x,t)u^{p+1}(y,t)\,dy\,dx \\ &\leq 2\kappa C_*^{2p+2}\bigl[\alpha(t)\bigr]^{p+1}.\end{aligned} \tag{2.21}$$

Here, we claim $\alpha(t) > 0$. In fact, if there exists $t_0 \in [0,T)$ such that $\alpha(t_0) = 0$, then $\alpha(T) = 0$. It leads to a contradiction with the fact that $u(x,t)$ blows up at T in $H_0^1(\Omega)$-norm. Thus

$$\frac{\alpha'(t)}{(\alpha(t))^{p+1}} \leq 2\kappa C_*^{2p+2} \tag{2.22}$$

and

$$\bigl(\alpha(0)\bigr)^{-p} - \bigl(\alpha(t)\bigr)^{-p} \leq 2p\kappa C_*^{2p+2}t. \tag{2.23}$$

Letting $t \to T$ in (2.23), one obtains

$$T \geq \frac{\|u_0\|^{-2p}_{H_0^1}}{2p\kappa C_*^{2p+2}}. \tag{2.24}$$

We summarize this result in the following theorem.

Theorem 2.4 *If* $0 < p \le \frac{2}{n-2}$, $u_0 \in H_0^1(\Omega)$, $J(u_0) < d$, $I(u_0) < 0$, *then the solution* $u(x,t)$ *of* (1.1) *blows up in finite time* T *in* $H_0^1(\Omega)$*-norm, and* T *is bounded below by* (2.24).

Remark 2.1 We mention that the lower bound $\frac{\|u_0\|_{H_0^1}^{-2p}}{2p\kappa C_*^{2p+2}}$ is smaller than the upper bound $\frac{\|u_0\|_{H_0^1}^2}{-4p(p+1)J(u_0)}$ under the conditions in Theorem 2.4 and Theorem 2.1.

In fact,

$$J(u_0) \ge \int_\Omega |\nabla u_0|^2\,dx - \frac{1}{2p+2}\kappa\|u_0\|_{2p+2}^{2p+2} \ge -\frac{1}{2p+2}\kappa C_*^{2p+2}\|u_0\|_{H_0^1}^{2p+2},$$

thus

$$\frac{\|u_0\|_{H_0^1}^{-2p}}{2p\kappa C_*^{2p+2}} \le \frac{\|u_0\|_{H_0^1}^2}{-4p(p+1)J(u_0)}.$$

Remark 2.2 Suppose $p \in (0, \frac{2}{n-2}]$, $u_0 \in H_0^1(\Omega)$, $J(u_0) < d$, then it is well known through the results in [14] that problem (1.1) has a sharp condition: the case of $I(u_0) > 0$ admits a global weak solution and the case of $I(u_0) < 0$ does not admit any global weak solution. However, a more powerful condition $J(u_0) < 0$ is required in Theorem 2.1 and we can deduce $I(u_0) < 0$ from $J(u_0) < 0$. More importantly, we obtain the precise upper bound $\frac{\|u_0\|_{H_0^1}^2}{-4p(p+1)J(u_0)}$ for the blow-up time.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The article is a joint work of the two authors, who contributed equally to the final version of the paper. Both authors read and approved the final manuscript.

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