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Regularity theory on A-harmonic system and A-Dirac system

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Abstract

In this paper, we show the regularity theory on an A-harmonic system and an A-Dirac system. By the method of the removability theorem, we explain how an A-harmonic system arises from an A-Dirac system and establish that an A-harmonic system is in fact the real part of the corresponding A-Dirac system.

Keywords: A-harmonic system; A-Dirac system; Caccioppoli estimate; natural growth condition; removable theorem

1 Introduction

In this paper, we consider the regularity theory on an A-Dirac system,

$$-D\tilde{A}(x,u,Du) = f(x,u,Du), \quad \text{in } \Omega, \tag{1.1}$$

and an A-harmonic system,

$$-\operatorname{div} A(x,u,\nabla u) = f(x,u,\nabla u), \quad \text{in } \Omega. \tag{1.2}$$

Here Ω is a bounded domain in R^n $(n \geq 2)$, $A(x,u,\nabla u)$ and $f(x,u,\nabla u)$ are measurable functions defined on $\Omega \times R^n \times R^{nN}$, N is an integer with $N > 1$, $u : \Omega \to R^n$ is a vector valued function. Furthermore, $A(x,u,\nabla u)$ and $f(x,u,\nabla u)$ satisfy the following structural conditions with $m > 2$:

(H1) $A(x,u,p)$ are differentiable functions in p and there exists a constant $C > 0$ such that

$$\left|\frac{\partial A(x,u,p)}{\partial p}\right| \leq C\left(1+|p|^2\right)^{\frac{m-2}{2}} \quad \text{for all } (x,u,p) \in \Omega \times R^n \times R^{nN}.$$

(H2) $A(x,u,p)$ are uniformly strongly elliptic, that is, for some $\lambda > 0$ we have

$$\left(\frac{\partial A(x,u,p)}{\partial p}\nu_i^\alpha\right)\nu_j^\beta \geq \lambda\left(1+|p|^2\right)^{\frac{m-2}{2}}|\nu|^2.$$

(H3) There exist $\beta \in (0,1)$ and $K : [0,\infty) \mapsto [0,\infty)$ monotone nondecreasing such that

$$\left|A(x,u,p) - A(\tilde{x},\tilde{u},p)\right| \leq K\left(|u|\right)\left(|x-\tilde{x}|^m + |u-\tilde{u}|^m\right)^{\frac{\beta}{m}}\left(1+|p|\right)^{\frac{m}{2}}$$

for all $x,\tilde{x} \in \Omega$, $u,\tilde{u} \in R^n$, and $p \in R^{nN}$. Without loss of generality, we take $K \geq 1$.

(H4) There exist constants C_1 and C_2 such that

$$\left|f(x,u,p)\right| \leq C_1|p|^m + C_2.$$

(H1) and (H2) imply

$$\left|A(x,u,p) - A(x,u,\xi)\right| \leq C\left(1 + |p|^2 + |\xi|^2\right)^{\frac{m-2}{2}} |p-\xi|; \tag{1.3}$$

$$\left(A(x,u,p) - A(x,u,\xi)\right)(p-\xi) \geq \lambda\left(1 + |p|^2 + |\xi|^2\right)^{\frac{m-2}{2}} |p-\xi|^2 \tag{1.4}$$

for all $x \in \Omega$, $u \in R^n$ and $p, \xi \in R^{nN}$, where $\lambda > 0$ is a constant.

Definition 1.1 We say that a function $u \in W^{1,m}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ is a weak solution to (1.2), if the equality

$$\int_\Omega A(x,u,\nabla u)\nabla\phi\, dx = \int_\Omega f(x,u,\nabla u)\phi\, dx \tag{1.5}$$

holds for all $\phi \in W^{1,m}_0(\Omega)$ with compact support.

In this paper, we assume that the solutions of the A-harmonic system (1.1) and the A-Dirac system (1.2) exist [1] and establish the regularity result directly. In other words, the main purpose of this paper is to show the regularity theory on an A-harmonic system and the corresponding A-Dirac system. It means that we should know the properties of an A-harmonic operator and an A-Dirac operator. This main context will be stated in Section 2. Further discussion can be found in [2–10] and the references therein.

In order to prove the main result, we also need a suitable Caccioppoli estimation (see Theorem 3.1). Then by the technique of removable singularities, we can find that solutions to an A-harmonic system satisfying a Lipschitz condition or in the case of a bounded mean oscillation can be extended to Clifford valued solutions to the corresponding A-Dirac system.

The technique of removable singularities was used in [2] to remove singularities for monogenic functions with modulus of continuity $\omega(r)$, where the sets $r^n\omega(r)$ and Hausdorff measure are removable. Kaufman and Wu [11] used the method in the case of Hölder continuous analytic functions. In fact, under a certain geometric condition related to the Minkowski dimension, sets can be removable for A-harmonic functions in Hölder and bounded mean oscillation classes [12]. Even in the case of Hölder continuity, a precise removable sets condition was stated [13]. In [7], the author showed that under a certain oscillation condition, sets satisfying a generalized Minkowski-type inequality were removable for solutions to the A-Dirac system. The general result can be found in [14].

Motivated by these facts, one ask: Does a similar result hold for the more general case of the systems (1.1) and (1.2)? We will answer this question in this paper and obtain the following result.

Theorem 1.2 *Let E be a relatively closed subset of Ω. Suppose that $u \in L^m_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$ has distributional first derivatives in Ω, u is a solution to the scalar part of A-Dirac system* (1.1) *under the structure conditions* (H1)-(H4) *in* $\Omega \setminus E$*, and* u *is of the type of an* m,k*-oscillation in* $\Omega \setminus E$*. If for each compact subset* K *of* E

$$\int_{\Omega \setminus K} d(x,K)^{m(k-1)-k} < \infty, \tag{1.6}$$

then u *extends to a solution of the* A*-Dirac system in* Ω.

2 *A*-Dirac system

In this section, we would introduce the A-Dirac system. Thus the definition of the A-Dirac operator is necessary. We first present the definitions and notations as regards the Clifford algebra at first [7].

We write $\mathcal{U}_n$ for the real universal Clifford algebra over R^n. The Clifford algebra is generated over R by the basis of the reduced products

$$\{e_1, e_2, \dots, e_1e_2, \dots, e_1 \cdots e_n\}, \tag{2.1}$$

where $\{e_1, e_2, \dots, e_n\}$ is an orthonormal basis of R^n with the relation $e_ie_j + e_je_i = -2\delta_{ij}$. We write e_0 for the identity. The dimension of $\mathcal{U}_n$ is 2^n. We have an increasing tower $R \subset C \subset H \subset \mathcal{U}_3 \subset \cdots$. The Clifford algebra $\mathcal{U}_n$ is a graded algebra as $\mathcal{U}_n = \bigoplus_l \mathcal{U}_n^l$, where $\mathcal{U}_n^l$ are those elements whose reduced Clifford products have length l.

For $A \in \mathcal{U}_n$, $Sc(A)$ denotes the scalar part of A, that is, the coefficient of the element e_0.

Throughout this paper, $\Omega \subset R^n$ is a connected and open set with boundary $\partial\Omega$. A Clifford-valued function $u : \Omega \to \mathcal{U}_n$ can be written as $u = \sum_\alpha u_\alpha e_\alpha$, where each u_α is real-valued and e_α are reduced products. The norm used here is given by $|\sum_\alpha u_\alpha e_\alpha| = (\sum_\alpha u_\alpha^2)^{\frac{1}{2}}$, which is sub-multiplicative, $|AB| \le C|A||B|$.

The Dirac operator defined here is

$$D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}. \tag{2.2}$$

Also $D^2 = -\triangle$. Here $\triangle$ is Laplace operator.

Throughout, Q is a cube in Ω with volume $|Q|$. We write σQ for the cube with the same center as Q and with side length σ times that of Q. For $q > 0$, we write $L^q(\Omega,\mathcal{U}_n)$ for the space of Clifford-valued functions in Ω whose coefficients belong to the usual $L^q(\Omega)$ space. Also, $W^{1,m}(\Omega,\mathcal{U}_n)$ is the space of Clifford valued functions in Ω whose coefficients as well as their first distributional derivatives are in $L^q(\Omega)$. We also write $L^q_{\mathrm{loc}}(\Omega,\mathcal{U}_n)$ for $\bigcap L^q(\Omega',\mathcal{U}_n)$, where the intersection is over all Ω' compactly contained in Ω. We similarly write $W^{1,m}_{\mathrm{loc}}(\Omega,\mathcal{U}_n)$. Moreover, we write $\mathcal{M}_\Omega = \{u : \Omega \to \mathcal{U}_n | Du = 0\}$ for the space of monogenic functions in Ω.

Furthermore, we define the Dirac Sobolev space

$$W^{D,m}(\Omega) = \left\{ u \in \mathcal{U}_n \;\Big|\; \int_\Omega |u|^m + \int_\Omega |Du|^m < \infty \right\}. \tag{2.3}$$

The local space $W^{D,m}_{\mathrm{loc}}$ is similarly defined. Notice that if u is monogenic, then $u \in L^m(\Omega)$ if and only if $u \in W^{D,m}(\Omega)$. Also it is immediate that $W^{1,m}(\Omega) \subset W^{D,m}(\Omega)$.

With those definitions and notations and also of the A-Dirac operator, we define the linear isomorphism $\theta : R^n \to \mathcal{U}_n^1$ by

$$\theta(\omega_1, \dots, \omega_n) = \sum_{i=1}^{n} \omega_i e_i. \tag{2.4}$$

For $x, y \in R^n$, Du is defined by $\theta(\nabla\phi) = D\phi$ for a real-valued function ϕ, and we have

$$-Sc\big(\theta(x)\theta(y)\big) = \langle x, y\rangle, \tag{2.5}$$

$$\big|\theta(x)\big| = |x|. \tag{2.6}$$

Here $\tilde{A}(x, \xi, \eta) : \Omega \times \mathcal{U}_1 \times \mathcal{U}_n \to \mathcal{U}_n$ is defined by

$$\tilde{A}(x, u, \eta) = \theta A\big(x, u, \theta^{-1}\eta\big), \tag{2.7}$$

which means that (1.5) is equivalent to

$$\begin{aligned}\int_\Omega Sc\big(\theta A(x, u, \nabla u)\theta(\nabla\phi)\big)\,dx &= \int_\Omega Sc\big(\tilde{A}(x, u, Du)D\phi\big)\,dx \\ &= \int_\Omega Sc\big(f(x, u, Du)\phi\big)\,dx.\end{aligned} \tag{2.8}$$

For the Clifford conjugation $\overline{(e_{j1} \cdots e_{jl})} = (-1)^l e_{jl} \cdots e_{j1}$, we define a Clifford-valued inner product as $\bar{\alpha}\beta$. Moreover, the scalar part of this Clifford inner product $Sc(\bar{\alpha}\beta)$ is the usual inner product $\langle\alpha, \beta\rangle$ in R^{2^n}, when α and β are identified as vectors.

For convenience, we replace $\tilde{A}$ with A and recast the structure systems above and define the operator:

$$A(x, \xi, \eta) : \Omega \times \mathcal{U}_1 \times \mathcal{U}_n \to \mathcal{U}_n, \tag{2.9}$$

where A preserves the grading of the Clifford algebra, $x \to A(x, \xi, \eta)$ is measurable for all ξ, η, and $\xi \to A(x, \xi, \eta)$, $\eta \to A(x, \xi, \eta)$ are continuous for a.e. $x \in \Omega$.

Definition 2.1 A Clifford valued function $u \in W^{D,m}_{\mathrm{loc}}(\Omega, \mathcal{U}_n^k) \cap L^\infty(\Omega, \mathcal{U}_n^k)$, for $k = 0, 1, \dots, n$, is a weak solution to system (1.1) under conditions (H1)-(H4). If for all $\phi \in W_0^{1,m}(\Omega, \mathcal{U}_n^k)$, then we have

$$\int_\Omega \overline{A(x, u, Du)} D\phi\,dx = \int_\Omega \overline{f(x, u, Du)}\phi\,dx. \tag{2.10}$$

3 Proof of the main results

In this section, we will establish the main results. At first, a suitable Caccioppoli estimate [7, 15] for solutions to (2.10) is necessary.

Theorem 3.1 *Let u be weak solutions to the scalar part of system* (1.1) *with $\lambda > 2C_1M$ and where* (H1)-(H4) *are satisfied. Then for every $x_0 \in \Omega$, $u_0 \in \mathcal{U}_1^k$, $p_0 \in \mathcal{U}_n^k$, and arbitrary $\sigma > 1$*

we have

$$\begin{aligned}
&\int_Q \bigl[\bigl(1+|p_0|^2\bigr)^{\frac{m-2}{2}}|Du-p_0|^2+|Du-p_0|^m\bigr]\,dx \\
&\quad\le C\biggl\{\frac{1}{(\sigma|Q|)^{2/n}}\int_{\sigma Q}\bigl(1+|p_0|^2\bigr)^{\frac{m-2}{2}}|u-P|^2\,dx \\
&\qquad+\frac{1}{(\sigma|Q|)^{m/n}}\int_{\sigma Q}|u-P|^m\,dx+\int_{\sigma Q}G^2\,dx\biggr\},
\end{aligned} \tag{3.1}$$

where $P = u(x) - u_0 + p_0(x - x_0)$ *and*

$$\begin{aligned}
\int_{\sigma Q} G^2\,dx = {}&\sigma|Q|\bigl\{\bigl[K\bigl(|u_0|+|p_0|\bigr)\bigl(1+|p_0|\bigr)^{\frac{m}{2}}\bigr]^{\frac{2}{1-\beta}}\bigl(\sigma|Q|\bigr)^{\frac{2\beta}{n}} \\
&+\bigl(C_2^2+C_1^2|p_0|^{2m}\bigr)\bigl(\sigma|Q|\bigr)^{\frac{2}{n}}\bigr\}.
\end{aligned} \tag{3.2}$$

Proof Denote $u(x) - u_0 - p_0(x - x_0)$ by $v(x)$ and $0 < |Q|^{\frac{1}{n}} < \sigma|Q|^{\frac{1}{n}} < \min\{1, \operatorname{dist}(x_0, \partial\Omega)\}$ for $\sigma > 1$, consider a standard cut-off function $\eta \in C_0^\infty(\sigma Q(x_0))$ satisfying $0 \le \eta \le 1$, $|\nabla\eta| < \frac{1}{|Q|^{1/n}}$, $\eta \equiv 1$ on $Q(x_0)$. Then $\varphi = \eta^2 v$ is admissible as a test-function, and we obtain

$$\begin{aligned}
&\int_{\sigma Q} A(x,u,Du)\cdot(Du-p_0)\eta^2\,dx \\
&\quad= -2\int_{\sigma Q} A(x,u,Du)\eta v\cdot\nabla\eta\,dx+\int_{\sigma Q} f(x,u,Du)\cdot\varphi\,dx.
\end{aligned}$$

We further have

$$\begin{aligned}
&-\int_{\sigma Q} A(x,u,p_0)\cdot(Du-p_0)\eta^2\,dx \\
&\quad= 2\int_{\sigma Q} A(x,u,p_0)\eta v\cdot\nabla\eta\,dx-\int_{\sigma Q} A(x,u,p_0)\cdot D\varphi\,dx,
\end{aligned}$$

and

$$\int_{\sigma Q} A(x_0,u_0,p_0)\cdot D\varphi\,dx = 0.$$

Adding these equations yields

$$\begin{aligned}
&\int_{\sigma Q}\bigl(A(x,u,Du)-A(x,u,p_0)\bigr)(Du-p_0)\eta^2\,dx \\
&\quad= -2\int_{\sigma Q}\bigl(A(x,u,Du)-A(x,u,p_0)\bigr)(Du-p_0)\eta v\cdot\nabla\eta\,dx \\
&\qquad-\int_{\sigma Q}\bigl(A(x,u,p_0)-A\bigl(x,u_0+p_0(x-x_0),p_0\bigr)\bigr)\cdot D\varphi\,dx \\
&\qquad-\int_{\sigma Q}\bigl(A\bigl(x,u_0+p_0(x-x_0),p_0\bigr)-A(x_0,u_0,p_0)\bigr)\cdot D\varphi\,dx+\int_{\sigma Q} f(x,u,Du)\cdot\varphi\,dx \\
&\quad\le I+II+III+IV+V,
\end{aligned} \tag{3.3}$$

where

$$I = 2C \int_{\sigma Q} \left(1 + |Du|^2 + |p_0|^2\right)^{\frac{m-2}{2}} |Du - p_0| \eta |v| |\nabla \eta| \, dx;$$

$$II = K\left(|u_0| + |p_0|\right) \int_{\sigma Q} |v|^{\beta} |Du - p_0| \left(1 + |p_0|\right)^{\frac{m}{2}} \eta^2 \, dx;$$

$$III = 2K\left(|u_0| + |p_0|\right) \int_{\sigma Q} |v|^{\beta+1} |\nabla \eta| \left(1 + |p_0|\right)^{\frac{m}{2}} \eta \, dx;$$

$$IV = K\left(|u_0| + |p_0|\right) \int_{\sigma Q} \left(|x - x_0|^m + \left|p_0(x - x_0)\right|^m\right)^{\frac{\beta}{m}} \left(1 + |p_0|\right)^{\frac{m}{2}} \cdot \left(2\eta |\nabla \eta| |v| + \eta^2 |Du - p_0|\right) dx;$$

$$V = \int_{\sigma Q} \left(C_1 |Du|^m + C_2\right) |v| \eta^2 \, dx,$$

after using (1.3), (H3), (H4).

For positive ε, to be fixed later, using Young's inequality, we have

$$\begin{aligned}
I &\le 2C \int_{\sigma Q} \left(1 + 2|Du - p_0|^2 + 3|p_0|^2\right)^{\frac{m-2}{2}} |Du - p_0| \eta |v| |\nabla \eta| \, dx \\
&\le C\left[\int_{\sigma Q} \left(1 + |p_0|^2\right)^{\frac{m-2}{2}} |Du - p_0| \eta |v| |\nabla \eta| \, dx + \int_{\sigma Q} |Du - p_0|^{m-1} \eta |v| |\nabla \eta| \, dx\right] \\
&\le C\varepsilon \int_{\sigma Q} \left(1 + |p_0|^2\right)^{\frac{m-2}{2}} |Du - p_0|^2 \eta^2 \, dx + C\frac{1}{\varepsilon} \int_{\sigma Q} \left(1 + |p_0|^2\right)^{\frac{m-2}{2}} |v|^2 |\nabla \eta|^2 \, dx \\
&\quad + C\varepsilon \int_{\sigma Q} |Du - p_0|^m \eta^2 \, dx + C(\varepsilon) \int_{\sigma Q} |v|^m |\nabla \eta|^m \, dx.
\end{aligned}$$

Using Young's inequality twice in II, we have

$$\begin{aligned}
II &\le \varepsilon \int_{\sigma Q} |Du - p_0|^2 \eta^2 \, dx + \frac{1}{\varepsilon} K^2\left(|u_0| + |p_0|\right)\left(1 + |p_0|\right)^m \int_{\sigma Q} |v|^{2\beta} \, dx \\
&\le \varepsilon \int_{\sigma Q} |Du - p_0|^2 \eta^2 \, dx + \frac{1}{\varepsilon} \int_{\sigma Q} \left(\frac{1}{(\sigma |Q|)^{1/n}} |v|\right)^2 dx \\
&\quad + \frac{1}{\varepsilon}\left[K\left(|u_0| + |p_0|\right)\left(1 + |p_0|\right)^{\frac{m}{2}}\right]^{\frac{2}{1-\beta}} (\sigma |Q|)^{(\frac{2\beta}{1-\beta} + n)/n} \\
&\le \varepsilon \int_{\sigma Q} \left(1 + |p_0|^2\right)^{\frac{m-2}{2}} |Du - p_0|^2 \eta^2 \, dx + \frac{1}{\varepsilon} \int_{\sigma Q} \left(1 + |p_0|^2\right)^{\frac{m-2}{2}} \left(\frac{1}{(\sigma |Q|)^{1/n}}\right)^2 |v|^2 \, dx \\
&\quad + \frac{1}{\varepsilon}\left[K\left(|u_0| + |p_0|\right)\left(1 + |p_0|\right)^{\frac{m}{2}}\right]^{\frac{2}{1-\beta}} (\sigma |Q|)^{(\frac{2\beta}{1-\beta} + n)/n},
\end{aligned}$$

and similarly we see

$$\begin{aligned}
III &\le \frac{1}{2} \int_{\sigma Q} |v|^2 |\nabla \eta|^2 \, dx \\
&\quad + 4K^2\left(|u_0| + |p_0|\right)\left(1 + |p_0|\right)^m \int_{\sigma Q} (\sigma |Q|)^{\frac{2\beta}{n}} \left(\frac{|v|}{(\sigma |Q|)^{1/n}}\right)^{2\beta} \eta^2 \, dx
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{\sigma Q}\left(\frac{1}{(\sigma|Q|)^{1/n}}\right)^2|v|^2\,dx + \left[4K\big(|u_0|+|p_0|\big)\big(1+|p_0|\big)^{\frac{m}{2}}\right]^{\frac{2}{1-\beta}}\big(\sigma|Q|\big)^{(\frac{2\beta}{1-\beta}+n)/n} \\
&\le \big(1+|p_0|^2\big)^{\frac{m-2}{2}}\int_{\sigma Q}\left(\frac{1}{(\sigma|Q|)^{1/n}}\right)^2|v|^2\,dx \\
&\quad + \left[4K\big(|u_0|+|p_0|\big)\big(1+|p_0|\big)^{\frac{m}{2}}\right]^{\frac{2}{1-\beta}}\big(\sigma|Q|\big)^{(\frac{2\beta}{1-\beta}+n)/n}
\end{aligned}
$$

and

$$
\begin{aligned}
IV &\le \int_{\sigma Q} K\big(|u_0|+|p_0|\big)\big(1+|p_0|\big)^{\frac{m}{2}}\big(\sigma|Q|\big)^{\frac{\beta}{n}}\big(1+|p_0|^m\big)^{\frac{\beta}{m}}\big(\eta|Du-p_0|+2\eta|\nabla\eta||v|\big)\,dx \\
&\le \int_{\sigma Q} K\big(|u_0|+|p_0|\big)\big(1+|p_0|\big)^{\frac{m}{2}}\big(\sigma|Q|\big)^{\frac{\beta}{n}}\big(1+|p_0|\big)^{\beta}\big(\eta|Du-p_0|+2\eta|\nabla\eta||v|\big)\,dx \\
&\le \varepsilon\int_{\sigma Q}\big(1+|p_0|^2\big)^{\frac{m-2}{2}}|Du-p_0|^2\eta^2\,dx + \int_{\sigma Q}\big(1+|p_0|^2\big)^{\frac{m-2}{2}}|\nabla\eta|^2|v|^2\,dx \\
&\quad + \left(4+\frac{1}{\varepsilon}\right)K^2\big(|u_0|+|p_0|\big)\big(1+|p_0|\big)^{2(\frac{m}{2}+\beta)}\big(\sigma|Q|\big)^{\frac{n+2\beta}{n}},
\end{aligned}
$$

and for positive μ, to be fixed later, this yields

$$
\begin{aligned}
V &= \int_{\sigma Q} C_1|Du|^m\big|u-u_0-p_0(x-x_0)\big|\eta^2\,dx + \int_{\sigma Q}\left(\frac{1}{(\sigma|Q|)^{1/n}}|v|\eta\right)\left(C_2\big(\sigma|Q|\big)^{\frac{1}{n}}\eta\right)dx \\
&\le \int_{\sigma Q} C_1\left[(1+\mu)|Du-p_0|^m + \left(1+\frac{1}{\mu}\right)|p_0|^m\right]\big|u-u_0-p_0(x-x_0)\big|\eta^2\,dx \\
&\quad + \frac{1}{2}\varepsilon C_2^2\big(\sigma|Q|\big)^{\frac{n+2}{n}} + \frac{1}{2\varepsilon\big(\sigma|Q|\big)^{2/n}}\int_{\sigma Q}|v|^2\,dx \\
&\le C_1(1+\mu)\big(2M+p_0\big(\sigma|Q|\big)^{\frac{1}{n}}\big)\int_{\sigma Q}|Du-p_0|^m\eta^2\,dx + C_1\left(1+\frac{1}{\mu}\right)|p_0|^m\int_{\sigma Q}|v|\eta^2\,dx \\
&\quad + \frac{1}{2}\varepsilon C_2^2\big(\sigma|Q|\big)^{\frac{n+2}{n}} + \frac{1}{2\varepsilon}\int_{\sigma Q}\frac{1}{(\sigma|Q|)^{2/n}}|v|^2\,dx \\
&\le C_1(1+\mu)\big(2M+p_0\big(\sigma|Q|\big)^{\frac{1}{n}}\big)\int_{\sigma Q}|Du-p_0|^m\eta^2\,dx + \frac{1}{\varepsilon}\int_{\sigma Q}\frac{1}{(\sigma|Q|)^{2/n}}|v|^2\,dx \\
&\quad + \frac{\varepsilon}{2}\left[C_2^2 + C_1^2\left(1+\frac{1}{\mu}\right)^2|p_0|^{2m}\right]\big(\sigma|Q|\big)^{\frac{n+2}{n}}.
\end{aligned}
$$

By (1.4), we obtain

$$
\begin{aligned}
&\int_{\sigma Q}\big(A(x,u,Du)-A(x,u,p_0)\big)(Du-p_0)\eta^2\,dx \\
&\quad \ge \lambda\int_{\sigma Q}\big(1+|Du|^2+|p_0|^2\big)^{\frac{m-2}{2}}|Du-p_0|^2\eta^2\,dx \\
&\quad \ge \lambda\int_{\sigma Q}\big(1+|Du-p_0|^2+|p_0|^2\big)^{\frac{m-2}{2}}|Du-p_0|^2\eta^2\,dx \\
&\quad \ge \lambda\left\{\int_{\sigma Q}\big(1+|p_0|^2\big)^{\frac{m-2}{2}}|Du-p_0|^2\eta^2\,dx + \int_{\sigma Q}|Du-p_0|^m\eta^2\,dx\right\}.
\end{aligned}
$$

Combining these estimates in (3.3) and noting that $K^2 \le K^{\frac{2}{1-\beta}}$ (as $K \ge 1$), $(\sigma|Q|)^{\frac{2\beta}{(1-\beta)n}} \le (\sigma|Q|)^{\frac{2\beta}{n}}$ for $\sigma > 1$, $[(1+|p_0|)^{\frac{m}{2}}]^{\frac{2}{1-\beta}} \ge (1+|p_0|)^{2(\frac{m}{2}+\beta)}$, and $4 \le 4^{\frac{2}{1-\beta}}$, we can estimate

$$\begin{aligned}
&\bigl[\lambda - 2C\varepsilon - 2\varepsilon - C_1(1+\mu)\bigl(2M + p_0(\sigma|Q|)^{\frac{1}{n}}\bigr)\bigr] \\
&\quad\cdot\left\{\int_{\sigma Q}\bigl(1+|p_0|^2\bigr)^{\frac{m-2}{2}}|Du-p_0|^2\eta^2\,dx + \int_{\sigma Q}|Du-p_0|^m\eta^2\,dx\right\} \\
&\quad\le\left(\frac{C}{\varepsilon}+\frac{2}{\varepsilon}+C(\varepsilon)+2\right)\left\{\frac{1}{(\sigma|Q|)^{2/n}}\int_{\sigma Q}\bigl(1+|p_0|^2\bigr)^{\frac{m-2}{2}}|u-P|^2\,dx\right. \\
&\quad\left.+\frac{1}{(\sigma|Q|)^{m/n}}\int_{\sigma Q}|u-P|^m\,dx\right\} + 2\left(\frac{1}{\varepsilon}+4^{\frac{2}{1-\beta}}\right) \\
&\quad\cdot\bigl[K\bigl(|u_0|+|p_0|\bigr)\bigl(1+|p_0|\bigr)^{\frac{m}{2}}\bigr]^{\frac{2}{1-\beta}}(\sigma|Q|)^{\frac{n+2\beta}{n}} \\
&\quad+\frac{\varepsilon}{2}\left[C_2^2 + C_1^2\left(1+\frac{1}{\mu}\right)^2|p_0|^{2m}\right]\bigl(\sigma|Q|\bigr)^{\frac{n+2}{n}}.
\end{aligned}$$

Define $\varepsilon = \varepsilon(\lambda, m)$, $\mu = \mu(C_1, M, m, \lambda)$ small enough, we obtain

$$\begin{aligned}
&\int_{\sigma Q}\bigl(1+|p_0|^2\bigr)^{\frac{m-2}{2}}|Du-p_0|^2\eta^2\,dx + \int_{\sigma Q}|Du-p_0|^m\eta^2\,dx \\
&\quad\le C\left\{\frac{1}{(\sigma|Q|)^{2/n}}\int_{\sigma Q}\bigl(1+|p_0|^2\bigr)^{\frac{m-2}{2}}|u-P|^2\,dx\right. \\
&\quad\left.+\frac{1}{(\sigma|Q|)^{m/n}}\int_{\sigma Q}|u-P|^m\,dx + \int_{\sigma Q}G^2\,dx\right\},
\end{aligned}$$

where $C = C(m, \lambda, \beta, M)$ and

$$\int_{\sigma Q}G^2\,dx = \sigma|Q|\bigl\{\bigl[K\bigl(|u_0|+|p_0|\bigr)\bigl(1+|p_0|\bigr)^{\frac{m}{2}}\bigr]^{\frac{2}{1-\beta}}(\sigma|Q|)^{\frac{2\beta}{n}} + \bigl(C_2^2 + C_1^2|p_0|^{2m}\bigr)\bigl(\sigma|Q|\bigr)^{\frac{2}{n}}\bigr\}.$$

Now let the domain of the left-hand side be Q, then we can get the right inequality immediately. □

In order to remove singularity of solutions to A-Dirac system, we also need the fact that real-valued functions satisfying various regularity properties. Thus we have the following.

Definition 3.2 [7] Assume that $u \in L^1_{\mathrm{loc}}(\Omega, \mathcal{U}_n)$, $q > 0$, and that $-\infty < k < 1$. We say that u is of the type of a q, k-oscillation in Ω when

$$\sup_{2Q\subset\Omega}|Q|^{-(qk+n)/qn}\inf_{u_Q\in\mathcal{M}_Q}\left(\int_Q|u-u_Q|^q\right)^{1/q} < \infty. \tag{3.4}$$

If $q = 1$ and $k = 0$, then the inequality (3.4) is equivalent to the usual definition of the bounded mean oscillation; when $q = 1$ and $0 < k \le 1$, then the inequality (3.4) is equivalent to the usual local Lipschitz condition [16]. Further discussion of the inequality (3.4) can be found in [8, 17]. In these cases, the supremum is finite if we choose u_Q to be the average value of the function u over the cube Q.

We remark that it follows from Hölder's inequality that if $s \le q$ and if u is of the type of an q, k-oscillation, then u is of the type of an s, k-oscillation.

The following lemma shows that Definition 3.2 is independent of the expansion factor of the sphere.

Lemma 3.3 [7] *Suppose that* $F \in L^1_{\mathrm{loc}}(\Omega, R)$, $F > 0$ *a.e.*, $r \in R$ *and* $\sigma_1, \sigma_2 > 1$. *If*

$$\sup_{\sigma_1 Q \subset Q} |Q|^r \int_Q F < \infty,$$

then

$$\sup_{\sigma_2 Q \subset Q} |Q|^r \int_Q F < \infty. \tag{3.5}$$

Then we proceed to prove the main result, Theorem 1.2.

Proof of Theorem 1.2 Let Q be a cube in the Whitney decomposition of $\Omega \setminus E$. The decomposition consists of closed dyadic cubes with disjoint interiors which satisfy

(a) $\Omega \setminus E = \bigcup_{Q \in \mathcal{W}} Q$,
(b) $|Q|^{1/n} \le d(Q, \partial\Omega) \le 4|Q|^{1/n}$,
(c) $(1/4)|Q_1|^{1/n} \le |Q_2|^{1/n} \le 4|Q_1|^{1/n}$ when $Q_1 \cap Q_2$ is not empty.

Here $d(Q, \partial\Omega)$ is the Euclidean distance between Q and the boundary of Ω [18].

If $A \subset R^n$ and $r > 0$, then we define the r-inflation of A as

$$A(r) = \bigcup B(x, r). \tag{3.6}$$

Let Q be a cube in the Whitney decomposition of $\Omega \setminus E$. Using the Caccioppoli estimate (3.1), we have

$$\begin{aligned} &\int_Q \bigl[\bigl(1 + |p_0|^2\bigr)^{\frac{m-2}{2}} |Du - p_0|^2 + |Du - p_0|^m\bigr]\,dx \\ &\quad \le C\biggl\{\frac{1}{(\sigma Q)^{2/n}} \int_{\sigma Q} \bigl(1 + |p_0|^2\bigr)^{\frac{m-2}{2}} |u - P|^2\,dx \\ &\qquad + \frac{1}{(\sigma Q)^{m/n}} \int_{\sigma Q} |u - P|^m\,dx + \int_{\sigma Q} G^2\,dx\biggr\}, \end{aligned}$$

with (3.2)

$$\int_{\sigma Q} G^2\,dx \le C|\sigma Q|^{\frac{n+2\beta}{n}} H^2\bigl(1 + |u_Q| + |p_0|\bigr), \tag{3.7}$$

where

$$H(t) = \bigl[\tilde{K}(t)(1+t)^{\frac{m}{2}}\bigr]^{\frac{2}{1-\beta}}, \qquad \tilde{K}(t) = \max\bigl\{K(t), C_1, C_2\bigr\},$$

and choose $|Q|$ small enough such that

$$|Q|^{\frac{\beta}{n}} H\bigl(1 + |u_Q| + |p_0|\bigr) \le 1.$$

By the definition of the q,k-oscillation condition, we have

$$\begin{aligned}
&\int_Q \big[\big(1+|p_0|^2\big)^{\frac{m-2}{2}}|Du-p_0|^2+|Du-p_0|^m\big]\,dx\\
&\quad\le C_1|Q|^{-\frac{2}{n}}|Q|^{\frac{2k+n}{n}}+C_2|Q|^{-\frac{m}{n}}|Q|^{(mk+n)/n}+C_3|Q|\\
&\quad\le C|Q|^a.
\end{aligned}\tag{3.8}$$

Here $a=(n+mk-m)/n$. Since the problem is local (use a partition of unity), we show that (2.10) holds whenever $\phi\in W_0^{1,m}(B(x_0,r))$ with $x_0\in E$ and $r>0$ sufficiently small. Choose $r=(1/5\sqrt{n})\min\{1,d(x_0,\partial\Omega)\}$ and let $K=E\cap\bar{B}(x_0,4r)$. Then K is a compact subset of E. Also let W_0 be those cubes in the Whitney decomposition of $\Omega\setminus E$ which meet $B=B(x_0,r)$. Notice that each cube $Q\in W_0$ lies in $\Omega\setminus K$. Let $\gamma=m(k-1)-k$. First, since $\gamma\ge -1$, from [12] we have $m(K)=m(E)=0$. Also since $na-n\ge\gamma$, using (1.6) and (3.8), we obtain

$$\begin{aligned}
&\int_{B(x_0,r)} \big[\big(1+|p_0|^2\big)^{\frac{m-2}{2}}|Du-p_0|^2+|Du-p_0|^m\big]\,dx\\
&\quad\le C\sum_{Q\in W_0}|Q|^a\le C\sum_{Q\in W_0}d(Q,K)^{na}\\
&\quad\le C\sum_{Q\in W_0}\int_Q d(x,K)^{na-n}\,dx\le C\int_{K(1)\setminus K}d(x,K)^{na-n}\,dx\\
&\quad\le C\int_{K(1)\setminus K}d(x,K)^{\gamma}\,dx<\infty.
\end{aligned}\tag{3.9}$$

Hence $u\in W^{D,m}_{\mathrm{loc}}(\Omega)$.

Next let $B=B(x_0,r)$ and assume that $\psi\in C_0^\infty(B)$. Also let W_j, $j=1,2,\ldots$, be those cubes $Q\in W_0$ with $l(Q)\le 2^{-j}$.

Consider the scalar functions

$$\phi_j=\max\big\{\big(2^{-j}-d(x,K)\big)2^j,0\big\}.\tag{3.10}$$

Thus each ϕ_j, $j=1,2,\ldots$, is Lipschitz, equal to 1 on K and as such $\psi(1-\phi_j)\in W^{1,m}(B\setminus E)$ with compact support. Hence

$$\begin{aligned}
&\int_B\big[\overline{A(x,u,Du)}D\psi-\overline{f(x,u,Du)}\psi\big]\,dx\\
&\quad=\int_{B\setminus E}\big[\overline{A(x,u,Du)}D\big(\psi(1-\phi_j)\big)-\overline{f(x,u,Du)}\psi(1-\phi_j)\big]\,dx\\
&\qquad+\int_B\big[\overline{A(x,u,Du)}D(\psi\phi_j)-\overline{f(x,u,Du)}\psi\phi_j\big]\,dx.
\end{aligned}\tag{3.11}$$

Let

$$\begin{aligned}
J_1&=\int_{B\setminus E}\big[\overline{A(x,u,Du)}D\big(\psi(1-\phi_j)\big)-\overline{f(x,u,Du)}\psi(1-\phi_j)\big]\,dx,\\
J_2&=\int_B\big[\overline{A(x,u,Du)}D(\psi\phi_j)-\overline{f(x,u,Du)}\psi\phi_j\big]\,dx.
\end{aligned}$$

Since u is a solution in $B\setminus E$, $J_1=0$.

Next we estimate J_2 as

$$\begin{aligned} J_2 &= \int_B A(x,u,Du)\psi D\phi_j\,dx + \int_B \phi_j A(x,u,Du)D\psi\,dx - \int_B \overline{f(x,u,Du)}\psi\phi_j\,dx \\ &= J_2' + J_2'' + J_2'''. \end{aligned} \tag{3.12}$$

Noting that there exists a constant C such that $|\psi| \le C < \infty$,

$$\left|J_2'\right| \le C \sum_{Q\in W_j} \int_B \left|A(x,u,Du)\right| |D\phi_j|\,dx.$$

Recalling that $|Q|^{\frac{\beta}{n}} K(t) \le 1$, we have

$$\begin{aligned} &\int_B \left|A(x,u,Du)\right| |D\phi_j|\,dx \\ &\quad \le \int_B \left|A(x,u,Du) - A(x,u,p_0)\right| |D\phi_j|\,dx \\ &\qquad + \int_B \left|A(x,u,p_0) - A(x_0,u_0,p_0)\right| |D\phi_j|\,dx \\ &\quad \le C \int_B \left(1 + |Du|^2 + |p_0|^2\right)^{\frac{m-2}{2}} |Du - p_0||D\phi_j|\,dx \\ &\qquad + C \int_B K\left(|u|\right)\left(|x-x_0|^m + |u-u_0|^m\right)^{\frac{\beta}{m}} \left(1 + |p_0|\right)^{\frac{m}{2}} |D\phi_j|\,dx \\ &\quad \le C \int_B \left(\left(1 + |p_0|^2\right)^{\frac{m-2}{2}} + |Du-p_0|^{m-2}\right) |Du - p_0||D\phi_j|\,dx \\ &\qquad + C \int_B K\left(|u|\right)\left(|x-x_0|^\beta + |u-u_0|^\beta\right) \left(1 + |p_0|\right)^{\frac{m}{2}} |D\phi_j|\,dx \\ &\quad \le C \int_B \left(1 + |p_0|^2\right)^{\frac{m-2}{2}} |Du - p_0||D\phi_j|\,dx + C \int_B |Du-p_0|^{m-1}|D\phi_j|\,dx \\ &\qquad + C \int_B K\left(|u|\right)|x-x_0|^\beta \left(1 + |p_0|\right)^{\frac{m}{2}} |D\phi_j|\,dx \\ &\qquad + C \int_B K\left(|u|\right)|u-u_0|^\beta \left(1 + |p_0|\right)^{\frac{m}{2}} |D\phi_j|\,dx \\ &\quad \le C \int_B \left[\left(1 + |p_0|^2\right)^{\frac{m-2}{2}} |Du - p_0|^2 + |Du-p_0|^m\right] dx \\ &\qquad + C \int_B \left[\left(1 + |p_0|^2\right)^{\frac{m-2}{2}} |D\phi_j|^2 + |D\phi_j|^m\right] dx \\ &\qquad + C \int_B K\left(|u|\right)|Q|^{\frac{\beta}{n}} \left(1 + |p_0|\right)^{\frac{m}{2}} |D\phi_j|\,dx \\ &\qquad + C \int_B K\left(|u|\right)|Q|^{\frac{\beta}{n}} \left(1 + |p_0|\right)^{\frac{m}{2}} |D\phi_j|\,dx \\ &\quad \le C|Q|^a + C \int_B \left[|D\phi_j|^2 + |D\phi_j|^m\right] dx + C \int_B |D\phi_j|\,dx + C \int_B |D\phi_j|\,dx \\ &\quad \le C|Q|^a + C \int_B \left(2^{2j} + 2^{mj}\right) dx + C \int_B 2^j\,dx. \end{aligned} \tag{3.13}$$

Now for $x \in Q \in W_j$, $d(x,K)$ is bounded above and below by a multiple of $|Q|^{1/n}$ and for $Q \in W_j$, $|Q|^{1/n} \le 2^{-j}$. Hence

$$\begin{aligned} |J_2'| &\le C \sum_{Q \in W_j} \bigl(|Q|^a + |Q|^{-\frac{m}{n}}|Q| + C|Q|^{-\frac{2}{n}}|Q| + |Q|^{-\frac{1}{n}}|Q|^n\bigr) \\ &\le C \sum_{Q\in W_j} |Q|^a \le C \int_{\bigcup W_j} d(x,K)^{m(k-1)-k}. \end{aligned} \tag{3.14}$$

Since $\bigcup W_j \subset \Omega \setminus K$ and $|\bigcup W_j| \to 0$ as $j \to \infty$, it follows that $J_2' \to 0$ as $j \to \infty$.
For

$$|J_2''| \le C \sum_{Q\in W_j} \int_B \phi_j A(x,u,Du) D\psi\, dx.$$

Similarly, we get

$$\begin{aligned} &\int_B \phi_j A(x,u,Du) D\psi\, dx \\ &\quad\le \int_B \bigl(A(x,u,Du) - A(x,u,p_0)\bigr) D\psi\, dx + \int_B \bigl(A(x,u,p_0) - A(x_0,u_0,p_0)\bigr) D\psi\, dx \\ &\quad\le C\int_B \bigl(1 + |Du|^2 + |p_0|^2\bigr)^{\frac{m-2}{2}} |Du - p_0|\, dx \\ &\qquad + C\int_B K\bigl(|u|\bigr)\bigl(|x-x_0|^m + |u-u_0|^m\bigr)^{\frac{\beta}{m}} \bigl(1+|p_0|\bigr)^{\frac{m}{2}} |D\psi|\, dx \\ &\quad\le C\int_B \bigl(\bigl(1+|p_0|^2\bigr)^{\frac{m-2}{2}} + |Du-p_0|^{m-2}\bigr)|Du-p_0||D\psi|\, dx \\ &\qquad + C\int_B K\bigl(|u|\bigr)\bigl(|x-x_0|^\beta + |u-u_0|^\beta\bigr)\bigl(1+|p_0|\bigr)^{\frac{m}{2}}|D\psi|\, dx \\ &\quad\le C\int_B \bigl(1+|p_0|^2\bigr)^{\frac{m-2}{2}} |Du-p_0||D\psi|\, dx + C\int_B |Du-p_0|^{m-1}|D\psi|\, dx \\ &\qquad + C\int_B K\bigl(|u|\bigr)|x-x_0|^\beta \bigl(1+|p_0|\bigr)^{\frac{m}{2}}|D\psi|\, dx \\ &\qquad + C\int_B K\bigl(|u|\bigr)|u-u_0|^\beta \bigl(1+|p_0|\bigr)^{\frac{m}{2}}|D\psi|\, dx \\ &\quad\le C\int_B \bigl[\bigl(1+|p_0|^2\bigr)^{\frac{m-2}{2}}|Du-p_0|^2 + |Du-p_0|^m\bigr]\, dx \\ &\qquad + C\int_B \bigl[\bigl(1+|p_0|^2\bigr)^{\frac{m-2}{2}}|D\psi|^2 + |D\psi|^m\bigr]\, dx \\ &\qquad + C\int_B |D\psi|\, dx + C\int_B K\bigl(|u|\bigr)|Q|^{\frac{\beta}{n}}|D\psi|\, dx \\ &\quad\le C|Q|^a + C\int_B \bigl(|D\psi|^2 + |D\psi|^m\bigr)\, dx + C\int_B |D\psi|\, dx \\ &\quad\le C|Q|^a + C\int_B dx. \end{aligned}$$

Thus,

$$\left|J_2''\right| \le C \sum_{Q\in W_j} \left(|Q|^a + |Q|\right) \le C \sum_{Q\in W_j} |Q|^a \le C \int_{\bigcup W_j} d(x,K)^{m(k-1)-k}. \tag{3.15}$$

Since $u \in W^{1,D}_{\mathrm{loc}}(\Omega)$ and $|\bigcup W_j| \to 0$ as $j \to \infty$, we have $J_2'' \to 0$ as $j \to \infty$. In order to estimate J_2''', we should use (H4):

$$J_2''' = \int_B \overline{f(x,u,Du)}\psi\phi_j\,dx \le C \int_B |Du - p_0|^m\,dx + C \int_B |p_0|^m\,dx = J_3' + J_3''. \tag{3.16}$$

Similar to the estimate of (3.14), using the Caccioppoli inequality (3.1) and the inequality (3.8), we get

$$\begin{aligned} J_3' &\le C \sum_{Q\in W_j} \int_Q |Du - p_0|^m\,dx \le C \sum_{Q\in W_j} |Q|^{\frac{(n+mk-m)}{n}} \\ &\le C \int_{\bigcup W_j} d(x,K)^{n+mk-m}\,dx \le C \int_{\bigcup W_j} d(x,K)^{m(k-1)-k}\,dx. \\ &\to 0 \quad (j \to \infty), \end{aligned}$$

and

$$\begin{aligned} J_3'' &\le C \sum_{Q\in W_j} \int_Q dx = C \sum_{Q\in W_j} |Q| \\ &\le C \int_{\bigcup W_j} d(x,K)^n\,dx \le C \int_{\bigcup W_j} d(x,K)^{n+mk-m}\,dx \\ &\le C \int_{\bigcup W_j} d(x,K)^{m(k-1)-k}\,dx \\ &\to 0 \quad (j \to \infty). \end{aligned}$$

Hence $J_2 \to 0$.

Combining estimates J_1 and J_2 in (3.11), we prove Theorem 1.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
FS participated in design of the study and drafted the manuscript. SC participated in and conceived of the study and the amendment of the paper. All authors read and approved the final manuscript.

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