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Error-constant estimation under the maximum norm for linear Lagrange interpolation

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Abstract

For the linear Lagrange interpolation over a triangular domain, we propose an efficient algorithm to rigorously evaluate the interpolation error constant under the maximum norm by using the finite-element method (FEM). In solving the optimization problem corresponding to the interpolation error constant, the maximum norm in the constraint condition is the most difficult part to process. To handle this difficulty, a novel method is proposed by combining the orthogonality of the space decomposition using the Fujino–Morley FEM space and the convex-hull property of the Bernstein representation of functions in the FEM space. Numerical results for the lower and upper bounds of the interpolation error constant for triangles of various types are presented to verify the efficiency of the proposed method.

Keywords: Lagrange interpolation; Finite-element method; Fujino–Morley interpolation; Bernstein polynomial

1 Introduction

In this paper, we consider the error estimation for the linear Lagrange interpolation over triangle elements and provide explicit values for the error constant in the error estimation under the L^∞-norm.

Before the detailed discussion of our results, let us introduce the existing literature on the Lagrange interpolation function in a general scope.

- *(1D case)* Given a 1-dimensional interval $I = (0, 1)$, since $H^1(I) \subset C(\overline{I})$, we can define the Lagrange interpolation $\Pi^L u$ such that $\Pi^L u$ is a linear function satisfying $(u - \Pi^L u)(0) = (u - \Pi^L u)(1) = 0$. Then, the following results are well known as optimal estimates if u is regular enough in the sense that the right-hand sides of the inequalities are well defined:

$$\|u - \Pi^L u\|_{0,I} \le \frac{1}{\pi^2}|u|_{2,I}, \qquad |u - \Pi^L u|_{1,I} \le \frac{1}{\pi}|u|_{2,I},$$
$$\|u - \Pi^L u\|_{\infty,I} \le \frac{1}{8}\|u^{(2)}\|_{\infty,I},$$

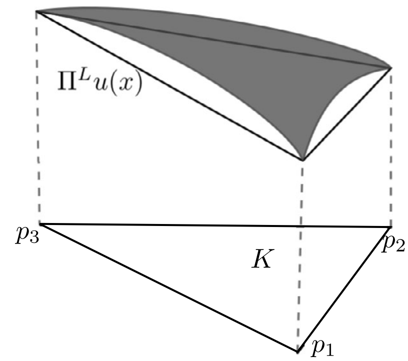


Figure 1 A linear Lagrange interpolation function $\Pi^L u$ defined on a triangle K

where $u^{(2)}$ denotes the second derivative of u, $\|\cdot\|_{0,I}$ and $\|\cdot\|_{\infty,I}$ denote the L^2- and L^∞-norms, respectively, and $|\cdot|_{1,I}$ and $|\cdot|_{2,I}$ denote the H^1- and H^2-seminorms, respectively. The estimations presented above are optimal in the sense that there exist functions for which the equalities hold.

– Let $u(x) := \sin(\pi x)$ on the interval $(0, 1)$. Then, $\Pi^L u(x) = 0$. In this case,

$$\|u - \Pi^L u\|_{0,I} = \frac{1}{\pi^2}|u|_{2,I}, \qquad |u - \Pi^L u|_{1,I} = \frac{1}{\pi}|u|_{2,I}.$$

– Let $u(x) := x^2$ on the interval $(0, 1)$. Then, $\Pi^L u(x) = x$. In this case,

$$\|u - \Pi^L u\|_{\infty,I} = \frac{1}{8}\|u^{(2)}\|_{\infty,I}.$$

- *(2D case)* Over a triangle K with vertices p_i ($i = 1, 2, 3$), the Lagrange interpolation function $\Pi^L u$ is the linear function such that (see Fig. 1)

$$\big(u - \Pi^L u\big)(p_i) = 0, \quad \forall i = 1, 2, 3.$$

In the case of L^2-norm and H^1-seminorm error estimation of Π^L, one needs to estimate the interpolation error constants appearing in the following inequalities:

$$\|u - \Pi^L u\|_{0,K} \le C_0(K)|u|_{2,K}, \qquad |u - \Pi^L u|_{1,K} \le C_1(K)|u|_{2,K}.$$

Let h be the medium edge length of K, θ the maximum angle, and αh ($0 < \alpha \le 1$) the smallest edge length. Kikuchi and Liu [1, 2] obtained a bound of C_0 and C_1 as follows:

$$C_0(K) \le \frac{h}{\pi}\sqrt{1 + |\cos\theta|}, C_1(K) \le 0.493h\frac{1 + \alpha^2 + \sqrt{1 + 2\alpha^2\cos 2\theta + \alpha^4}}{\sqrt{2(1 + \alpha^2 - \sqrt{1 + 2\alpha^2\cos 2\theta + \alpha^4})}}.$$

Also, Kobayashi [3] showed that for a triangle K with edge lengths A, B, C and area S, the following holds:

$$|u - \Pi^L u|_{1,K} \le C_1(K)|u|_{2,K}, \quad \forall u \in H^2(K),$$

where the constant $C_1(K)$ is defined by

$$C_1(K) := \sqrt{\frac{A^2B^2C^2}{16S^2} - \frac{A^2+B^2+C^2}{30} - \frac{S^2}{5}\left(\frac{1}{A^2}+\frac{1}{B^2}+\frac{1}{C^2}\right)}.$$

The optimal estimation of constants $C_0(K)$ and $C_1(K)$ for a concrete K can be obtained by solving the corresponding eigenvalue problems with rigorous lower eigenvalue bounds; see the results of [4, 5].

For L^∞-norm error estimation under the L^∞-norm of objective function, Waldron [6] provides the following sharp inequality:

$$\left\|u - \Pi^L u\right\|_{\infty,K} \le \frac{1}{2}\left(R^2 - d^2\right)\left\|u^{(2)}\right\|_{\infty,K}, \tag{1}$$

where R is the radius of the circumscribed circle of K, d is the distance of the center c of the circumscribed circle from K, and $\|u^{(2)}\|_{\infty,K}$ is defined by

$$\left\|u^{(2)}\right\|_{\infty,K} := \sup_{x\in K} \sup_{\substack{u,v\in\mathbb{R}^2\\ \|u\|=\|v\|=1}} \left|D_u D_v u(x)\right| = \sup_{x\in K} \sup_{\substack{\xi\in\mathbb{R}^2\\ \|\xi\|=1}} \left|D_\xi^2 u(x)\right|.$$

In particular, if $c\in K$,

$$\left\|u - \Pi^L u\right\|_{\infty,K} \le \frac{1}{2}R^2\left\|u^{(2)}\right\|_{\infty,K}.$$

A detailed discussion on the L^∞-norm of interpolation error for a quadratic polynomial f is considered by D'Azevedo and Simpson [7]. In [8], Shewchuk gives a survey of the interpolation error estimation with L^∞-norm for both $f - \Pi^L f$ and $\nabla(f - \Pi^L f)$, along with the discussion on the relation between the interpolation error and the finite-element approximation of error functions. Also, the discussion on the affection of the aspect ratio of a triangle element to the interpolation error can be found in Cao [9].

In this research, we consider the L^∞-norm estimation for the Lagrange interpolation over triangle element K by using the H^2-seminorm of the objective function, that is,

$$\left\|u - \Pi^L u\right\|_{\infty,K} \le C^L(K)|u|_{2,K}, \quad \forall u \in H^2(K). \tag{2}$$

Here, $C^L(K)$ is the interpolation error constant to be evaluated explicitly. Note that since $W^{2,\infty}(K) \subseteq H^2(K)$, the inequality (2) is more general than Waldron's result (1). In this paper, it is aimed to give sharp estimation for the constant $C^L(K)$. For example, for the unit isosceles right triangle element, the following estimation holds for the optimal constant $C^L(K)$ in (2):

$$0.40432 \le C^L(K) \le 0.41596.$$

Estimation of $C^L(K)$ for triangles of general shapes is provided in Theorem 2.3, while sharp bounds for concrete triangles are discussed in Sect. 3. Such a kind of estimation is helpful to provide explicit maximum norm error estimation for the FEM solution to boundary value problems by further applying the point-wise error estimation (see, e.g., [10]), which will be considered in our succeeding work; see also classical qualitative error analysis under the maximum norm in Sects. 19–22 of [11];

The contribution of our paper is summarized as follows.

(1) For triangle element K of general shapes, a formula to give an upper bound of $C^L(K)$ is obtained by theoretical analysis. The bound is raw but works well for triangle elements of arbitrary shapes. In particular, our analysis tells us that the value of $C^L(K)$ can be very large and tends to ∞ if the triangle element tends to degenerate to a 1D segment; see detail in Sect. 2.2.

(2) For a specific triangle element K, the optimal estimation of $C^L(K)$ is obtained by solving the corresponding optimization problem over $H^2(K)$ under the constraint condition involving L^∞-norm. The processing of the constraint condition with L^∞-norm is not an easy task. We develop a novel algorithm to provide efficient and sharp estimation for the solution of the optimization problem. With a light computation, one can obtain the estimation of $C^L(K)$ with relative error less than 1%.

The rest of our paper is structured as follows. At the end of this section, we introduce the preliminary concepts and notations to be used throughout the paper. In Sect. 2, the estimation of the upper bound for $C^L(K)$ is considered using a theoretical approach. The raw upper bound of the interpolation error constant is calculated for a right isosceles triangle. Also, we investigate the asymptotic behavior of the constant as the triangle tends to degenerate. In Sect. 3, using a finite-element method (FEM), an algorithm for the optimal estimation of the constant is proposed. Lower bounds for the constant are calculated to confirm the efficiency of the proposed algorithm. The numerical results are summarized and the conclusion is presented in Sect. 4.

Notation Let us introduce the notation for the function spaces used in this paper. In most cases, the domain Ω of functions is selected as a triangle element K. The standard notation is used for Sobolev function spaces $W^{k,p}(\Omega)$. The associated norms and seminorms are denoted by $\|\cdot\|_{k,p,\Omega}$ and $|\cdot|_{k,p,\Omega}$, respectively (see, e.g., Chap. 1 of [12] and Chap. 1 of [13]). In particular, for special k and p, we use abbreviated notations as $H^k(\Omega) = W^{k,2}(\Omega)$, $|\cdot|_{k,\Omega} = |\cdot|_{k,2,\Omega}$, and $L^p(\Omega) = W^{0,p}(\Omega)$. The set of polynomials over K of up to degree k is denoted by $P_k(K)$. The second-order derivative is given by $D^2u := (u_{xx}, u_{xy}, u_{yx}, u_{yy})$ for $u \in H^2(K)$.

Given a triangle K, denote each vertex by p_i ($i = 1, 2, 3$) and the largest edge length by h_K; see Fig. 2. We follow the notation introduced by Liu and Kikuchi [2] to configure a general triangle with geometric parameters. Let h, α, and θ be positive constants such that

$$h > 0, 0 < \alpha \le 1, \quad \left(\frac{\pi}{3} \le\right) \cos^{-1}\left(\frac{\alpha}{2}\right) \le \theta < \pi.$$

Define a triangle $K_{\alpha,\theta,h}$ with three vertices $p_1(0,0)$, $p_2(h,0)$, and $p_3(\alpha h\cos\theta, \alpha h\sin\theta)$. Note that $h \le h_K$. In the case of $h = 1$, the notation $K_{\alpha,\theta,1}$ is abbreviated as $K_{\alpha,\theta}$.

With the above configuration of the triangle $K_{\alpha,\theta,h}$, the optimal constant $C^L(K)$ in (2) can be defined as follows:

$$C^L(\alpha,\theta,h) := \sup_{u\in H^2(K_{\alpha,\theta,h})} \frac{\|u - \Pi^L u\|_{\infty,K_{\alpha,\theta,h}}}{|u|_{2,K_{\alpha,\theta,h}}}. \tag{3}$$

Figure 2 Configuration of triangle $K_{\alpha,\theta,h}$

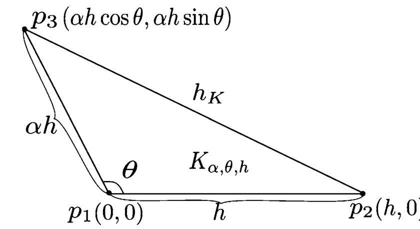
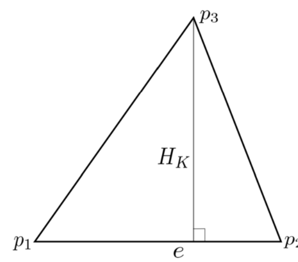


Figure 3 A triangle K with base e and height H_K



By scaling of the triangle element, it is easy to confirm that $C^L(\alpha,\theta,h) = hC^L(\alpha,\theta,1)$.

In the rest of the paper, we show how to obtain explicit bounds for the error constant $C^L(\alpha,\theta,h)$.

2 Raw upper bound of the constant

In this section, a raw upper bound of the constant is obtained through theoretical analysis. Such a bound applies to triangles of arbitrary shapes.

First, let us quote a lemma about the trace theorem, which gives an estimation for the integral over edge of a triangle element. For the reader's convenience, we show the proof in a concise way; refer to, e.g., [14–16] for more detailed discussion.

Lemma 2.1 (Trace theorem) *Let e be one of the edges of triangle K; see Fig. 3. Given $w \in H^1(K)$, we have the following estimation*:

$$\|w\|_{0,e}^2 \le \frac{|e|}{|K|}\left\{\|w\|_{0,K}^2 + h_K\|w\|_{0,K}|w|_{1,K}\right\}.$$

Proof For any $w \in H^1(K)$, the Green theorem leads to

$$\int_K \bigl((x,y)-p_3\bigr)\cdot\nabla\bigl(w^2\bigr)\,\mathrm{d}K = \int_{\partial K}\bigl((x,y)-p_3\bigr)\cdot\overrightarrow{n}\,w^2\,\mathrm{d}s - \int_K 2w^2\,\mathrm{d}K.$$

Here, $\overrightarrow{n}$ is the unit outer normal direction on the boundary of K. For the term $((x,y) - p_3)\cdot\overrightarrow{n}$, we have

$$\bigl((x,y)-p_3\bigr)\cdot\overrightarrow{n} = \begin{cases} 0 & \text{on } p_1p_3, p_2p_3,\\ H_K & \text{on } e.\end{cases}$$

Here, H_K is the height of the triangle with base as e. Thus,

$$H_K\int_e w^2\,\mathrm{d}s = \int_K 2w^2\,\mathrm{d}K + \int_K \bigl((x,y)-p_3\bigr)\cdot\nabla\bigl(w^2\bigr)\,\mathrm{d}K$$

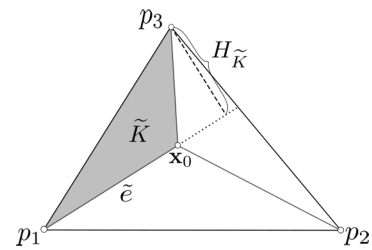


Figure 4 A subtriangle $\widetilde{K}$ in a triangle K

$$\leq \int_K 2w^2 \,\mathrm{d}K + 2h_K \int_K w|\nabla w| \,\mathrm{d}K$$
$$\leq 2\|w\|_{0,K}^2 + 2h_K\|w\|_{0,K}\|\nabla w\|_{0,K}.$$

We can now draw the conclusion by sorting the above inequality. □

Using the trace theorem, the following result provides a pointwise estimation of the interpolation error.

Lemma 2.2 *Given $u \in H^2(K)$, for any point $\mathbf{x}_0 \in K$, we have*

$$\left|(u - \Pi^L u)(\mathbf{x}_0)\right| \leq \frac{\sqrt{2|p_1\mathbf{x}_0|}}{\sqrt{H_{\widetilde{K}}}}\left(h_K\left|u - \Pi^L u\right|_{1,K}|u|_{2,K} + \left|u - \Pi^L u\right|_{1,K}^2\right)^{\frac{1}{2}},$$

where h_K is the longest edge length of K, and $H_{\widetilde{K}}$ is the height of the subtriangle $\widetilde{K} = p_1\mathbf{x}_0p_3$ with respect to the base $\tilde{e} = p_1\mathbf{x}_0$ (see Fig. 4).

Proof Let $g = u - \Pi^L u$ and t be the direction along edge $p_1\mathbf{x}_0$. In Lemma 2.1, by taking $w := \frac{\partial g}{\partial t}$, we have

$$\left\|\frac{\partial g}{\partial t}\right\|_{0,\tilde{e}}^2 \leq \frac{|\tilde{e}|}{|\widetilde{K}|}\left(\|w\|_{0,\widetilde{K}}^2 + h_{\widetilde{K}}\|w\|_{0,\widetilde{K}}|w|_{1,\widetilde{K}}\right) \leq \frac{|\tilde{e}|}{|\widetilde{K}|}\left(|g|_{1,\widetilde{K}}^2 + h_{\widetilde{K}}|g|_{1,\widetilde{K}}|g|_{2,\widetilde{K}}\right).$$

Taking the Taylor expansion of g on the segment $\tilde{e}$ and noting that $g(p_1) = 0$,

$$\left|g(\mathbf{x}_0)\right| = \left|\int_{p_1\mathbf{x}_0}\frac{\partial g}{\partial t}\,\mathrm{d}t + g(p_1)\right| \leq \sqrt{|p_1\mathbf{x}_0|}\cdot\left\|\frac{\partial g}{\partial t}\right\|_{0,\tilde{e}}$$
$$\leq \frac{|p_1\mathbf{x}_0|}{\sqrt{|\widetilde{K}|}}\left(h_K|g|_{1,\widetilde{K}}|g|_{2,\widetilde{K}} + |g|_{1,\widetilde{K}}^2\right)^{\frac{1}{2}}$$
$$\leq \frac{\sqrt{2|p_1\mathbf{x}_0|}}{\sqrt{H_{\widetilde{K}}}}\left(h_K|g|_{1,K}|g|_{2,K} + |g|_{1,K}^2\right)^{\frac{1}{2}}.$$

The conclusion follows. □

Liu and Kikuchi [2] considered the estimation of the constant $C_1(\alpha,\theta)$ for different types of triangles $K = K_{\alpha,\theta}$ such that

$$\left|u - \Pi^L u\right|_{1,K} \leq C_1(\alpha,\theta)h|u|_{2,K}, \quad \forall u \in H^2(K),$$

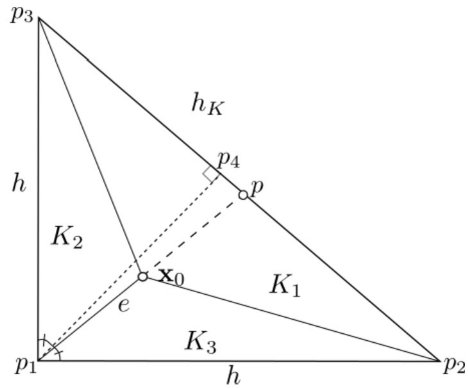


Figure 5 A right isosceles triangle $K_{1,\frac{\pi}{2},h}$

where h is the medium length of K. The constant $C_1(\alpha,\theta)$ is used to give a bound for $C^L(K)$, as shown in the lemma below.

Lemma 2.3 *Given $u \in H^2(K)$, for any point $\mathbf{x}_0 \in K$, we have*

$$\left|\left(u - \Pi^L u\right)(\mathbf{x}_0)\right| \le \frac{\sqrt{2|p_1\mathbf{x}_0|}}{\sqrt{H_{\widetilde{K}}}}\left(C_1(\alpha,\theta)hh_K + C_1^2(\alpha,\theta)h^2\right)^{\frac{1}{2}}|u|_{2,K}. \tag{4}$$

2.1 The case for a right isosceles triangle

Using Lemma 2.3, we obtain the upper bound of the constant for right isosceles triangles: For the right isosceles triangle $K = K_{1,\frac{\pi}{2},h}$,

$$\left\|u - \Pi^L u\right\|_{\infty,K} \le 1.3712h|u|_{2,K}. \tag{5}$$

Suppose a point $\mathbf{x}_0$ subdivides K into K_1, K_2, K_3; see Fig. 5. Let us consider the estimation of the term $|p_1\mathbf{x}_0|/H_{K_2}$, which is required in Lemma 2.3. Let p_1p_4 be the height of K with base as p_2p_3. Due to the symmetry of K, it is enough to only consider the case that $\mathbf{x}_0 \in K$ is below the line p_1p_4. Let p be the intersection of the extended line of $p_1\mathbf{x}_0$ and edge p_2p_3. Note that $|p_1\mathbf{x}_0| \le |p_1p|$. For $p := (\overline{x},\overline{y})$ on p_2p_4, $|p_1p| = \sqrt{\overline{x}^2 + \overline{y}^2}$. The height of K_2 with base $p_1\mathbf{x}_0$ is given by

$$H_{K_2} = \frac{h\overline{x}}{\sqrt{\overline{x}^2 + \overline{y}^2}}.$$

Then, since $\overline{y} = h - \overline{x}$,

$$\frac{|p_1p|}{H_{K_2}} = \frac{2\overline{x}^2 - 2h\overline{x} + h^2}{h\overline{x}}.$$

The above quantity takes its maximum value at $p = (\frac{h}{2}, \frac{h}{2})$ and $p = (h, 0)$, and its maximum value is 1. Thus, for any p on p_2p_4, $|p_1p|/H_{K_2} \le 1$. From [2], $C_1(1, \frac{\pi}{2}) \le 0.49293$. Since

$h_K = \sqrt{2}h$, by inequality (4),

$$\left|\left(u - \Pi^L u\right)(\mathbf{x}_0)\right| \leq \sqrt{2}\left[(0.49293)\sqrt{2} + 0.49293^2\right]^{\frac{1}{2}} h|u|_{2,K} \leq 1.3712h|u|_{2,K}.$$

Hence, we obtain the error estimate for a right isosceles triangle as in (5).

2.2 Dependence of the constant on the shape of K

In this subsection, we consider the variation of the interpolation constant when a reference triangle, i.e., the right isosceles triangle, is transformed to a general triangle.

Theorem 2.1 *For a general element $K_{\alpha,\theta}$, the following estimation for constant $C^L(\alpha,\theta)$ holds*:

$$C^L(\alpha,\theta) \leq \frac{\nu_+(\alpha,\theta)}{2\sqrt{\alpha\sin\theta}} C^L\left(1, \frac{\pi}{2}\right), \tag{6}$$

where $\nu_+(\alpha,\theta) = 1 + \alpha^2 + \sqrt{1 + 2\alpha^2\cos 2\theta + \alpha^4}$.

Proof Let us consider the affine transformation between $x = (x_1, x_2) \in K_{\alpha,\theta}$ and $\xi = (\xi_1, \xi_2) \in K_{1,\frac{\pi}{2}}$:

$$\xi_1 = x_1 - \frac{x_2}{\tan\theta}, \qquad \xi_2 = \frac{x_2}{\alpha\sin\theta} \quad \text{or} \quad x_1 = \xi_1 + \alpha\xi_2\cos\theta, \qquad x_2 = \alpha\xi_2\sin\theta.$$

Given $\tilde{v}(\xi)$ over $K_{1,\frac{\pi}{2}}$, define $v(x)$ over $K_{\alpha,\theta}$ by $v(x_1, x_2) = \tilde{v}(\xi_1, \xi_2)$. Thus,

$$\|v\|_{\infty,K_{\alpha,\theta}} = \|\tilde{v}\|_{\infty,K_{1,\frac{\pi}{2}}}.$$

The estimation for the variation of H^2-seminorm in Theorem 1 of [2] tells us that

$$|v|_{2,K_{\alpha,\theta}} \geq \frac{2\sqrt{\alpha\sin\theta}}{\nu_+(\alpha,\theta)} |\tilde{v}|_{2,K_{1,\frac{\pi}{2}}}.$$

Thus, we draw the conclusion from the definition of constant $C^L(\alpha,\theta)$ in (3). □

Lemma 2.4 *For shape-regular triangles, $C^L(\alpha,\theta)$ is bounded. Here, by "shape-regular triangles" it means that for a certain positive quantity δ, the minimal inner angle of each triangle, denoted by $\theta_{\min}$, the inequality $\theta_{\min} \geq \delta$ holds.*

Proof It is easy to see that for all triangles with $\theta_{\min} \geq \delta$, the term $\nu_+(\alpha,\theta)/\sqrt{\alpha\sin\theta}$ in (6) is uniformly bounded. As $C^L(1, \pi/2)$ has a finite value, we draw the conclusion from the estimation (6). □

Remark 2.1 By using the raw bound of $C^L(1, \frac{\pi}{2}) \leq 1.3712h$ in (5), an explicit but raw bound of $C^L(\alpha,\theta)$ is available. Later, with a sharp and rigorous estimation of $C^L(1, \frac{\pi}{2})$ based on a numerical approach, the bound can be improved as

$$C^L(\alpha,\theta,h) \leq 0.41596h \frac{\nu_+(\alpha,\theta)}{2\sqrt{\alpha\sin\theta}}. \tag{7}$$

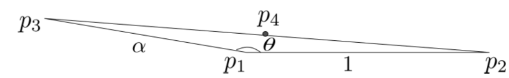


Figure 6 A triangle $K_{\alpha,\theta}$ with angle θ close to π

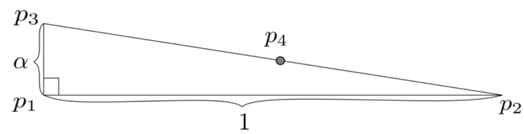


Figure 7 A right triangle $K_{\alpha,\frac{\pi}{2}}$ with one leg length close to 0

Remark 2.2 Here are two remarks on the asymptotic behavior of the constant when the triangle degenerates to a segment.

1. Suppose the maximum inner angle θ of $K_{\alpha,\theta}$ is close to π; see Fig. 6. Let $u(x,y) := x^2 + y^2$. Then, $\Pi^L u(x,y) = x + ((\alpha - \cos\theta)/\sin\theta)y$ and

$$\left\|u - \Pi^L u\right\|_{\infty,K_{\alpha,\theta}} = \left(2\alpha\cos\theta - \alpha^2 - 1\right)/4, \qquad |u|_{2,K_{\alpha,\theta}} = 2\sqrt{\alpha\sin\theta}.$$

   Thus, we have a lower bound of $C^L(\alpha,\theta)$ as follows,

$$C^L(\alpha,\theta) \geq \frac{2\alpha\cos\theta - \alpha^2 - 1}{8\sqrt{\alpha\sin\theta}}.$$

   In this case, $C^L(\alpha,\theta)$ diverges to ∞ as θ tends to π.

2. For triangle $K_{\alpha,\frac{\pi}{2}}$ shown in Fig. 7, let $u(x,y) := |(x,y) - p_4|^2$, where p_4 is the midpoint of the edge p_2p_3. Then, $\Pi^L u = (\alpha^2 + 1)/4$ and

$$\left\|u - \Pi^L u\right\|_{\infty,K_{\alpha,\frac{\pi}{2}}} = \left(\alpha^2 + 1\right)/4, \qquad |u|_{2,K_{\alpha,\frac{\pi}{2}}} = 2\sqrt{\alpha}.$$

   Thus,

$$\frac{\|u - \Pi^L u\|_{\infty,K_{\alpha,\frac{\pi}{2}}}}{|u|_{2,K_{\alpha,\frac{\pi}{2}}}} = \frac{\alpha^2 + 1}{8\sqrt{\alpha}}\left(\leq C^L\left(\alpha, \frac{\pi}{2}\right)\right).$$

   When $\alpha \to 0$, although the maximum inner angle is invariant, the interpolation error constant $C^L(\alpha, \frac{\pi}{2})$ tends to ∞.

3 Optimal estimation of the constant

In the previous section, we obtained explicit bounds for the interpolation constant for triangles of general shape. Basically, such bounds from theoretical analysis only provide a raw bound for the objective constant. In this section, we propose a numerical algorithm to obtain the optimal estimation of the constant $C^L(K)$ for specific triangles.

Let us define the space $V^L(K) := \{u \in H^2(K) \mid u(p_i) = 0\ (i = 1,2,3)\}$. Let $\mathcal{T}^h$ be a triangulation of the domain K and define the space

$$V_h^{FM}(K) := \Big\{v \mid v|_{K_h} \in P_2(K_h), \forall K_h \in \mathcal{T}^h; v(p_i) = 0\ (i = 1,2,3); v \text{ is continuous}$$

$$\text{at the nodes; } \int_e \left(\frac{\partial v}{\partial \overrightarrow{n}}\bigg|_{K_h} - \frac{\partial v}{\partial \overrightarrow{n}}\bigg|_{K'_h} \right) \mathrm{d}s = 0 \text{ for each } e = K_h \cap K_{h'} \bigg\}.$$

For $u_h, v_h \in V_h^{FM}(K)$, define the discretized H^2-inner product and seminorm by

$$\langle u_h, v_h \rangle_h := \sum_{K_h \in \mathcal{T}^h} \int_{K_h} D^2 u_h|_{K_h} \cdot D^2 v_h|_{K_h} \, \mathrm{d}K_h, \quad |u_h|_{2,K} := \sqrt{\langle u_h, u_h \rangle_h}.$$

Let us define the two quantities over the triangle K:

$$\lambda(K) := \inf_{u \in V^L(K)} \frac{|u|^2_{2,K}}{\|u\|^2_{\infty,K}}, \qquad \lambda_h(K) := \min_{u_h \in V_h^{FM}(K)} \frac{|u_h|^2_{2,K}}{\|u_h\|^2_{\infty,K}}. \tag{8}$$

Note that $C^L(K) = \sqrt{\lambda(K)}^{-1}$ holds. In Theorem 3.1, we describe the algorithm to bound λ by using λ_h.

Given $u \in H^2(K)$, the Fujino–Morley interpolation $\Pi_h^{FM} u$ is a function satisfying

$$\Pi_h^{FM} u \in V_h^{FM}(K); \qquad \Pi_h^{FM} u|_{K_h} \in P_2(K_h), \quad \forall K_h \in \mathcal{T}^h,$$

and at the vertices p_i and edges e_i of K,

$$\left(u - \Pi_h^{FM} u\right)(p_i) = 0, \qquad \int_{e_i} \frac{\partial}{\partial n}\left(u - \Pi_h^{FM} u\right) \mathrm{d}s = 0 \quad (i = 1, 2, 3).$$

The Fujino–Morley interpolation has the property that (see, e.g., [4, 5])

$$\left\langle u - \Pi_h^{FM} u, v_h \right\rangle_h = 0, \quad \forall v_h \in V_h^{FM}(K). \tag{9}$$

Let $V(h) := \{u + u_h \mid u \in V^L(K), u_h \in V_h^{FM}(K)\}$. Thus, it is easy to see that the Fujino–Morley interpolation is just the projection $P_h : V(h) \to V_h^{FM}(K)$ with respect to the inner product $\langle \cdot, \cdot \rangle_h$.

Below, let us introduce the theorem that provides an explicit lower bound of λ. Such a result is inspired by the idea of [17] for the lower bounds of eigenvalue problems.

Let C_h^{FM} be a quantity that makes the following inequality hold.

$$\left\| u - \Pi_h^{FM} u \right\|_{\infty,K} \le C_h^{FM} \left| u - \Pi_h^{FM} u \right|_{2,K}, \quad \forall u \in V^L(K). \tag{10}$$

The existence of C_h^{FM} is confirmed by the argument in Sect. 3.1.

Theorem 3.1 *With the quantity C_h^{FM}, we have a lower bound of $\lambda(K)$ as follows:*

$$\lambda(K) \ge \frac{\lambda_h}{1 + (C_h^{FM})^2 \lambda_h}. \tag{11}$$

Proof For any $u \in V^L(K)$, noting that $|\Pi_h^{FM} u|_{2,K} \ge \sqrt{\lambda_h} \|\Pi_h^{FM} u\|_{\infty,K}$ and applying the inequality (10), we have

$$\|u\|_{\infty,K} = \left\| \Pi_h^{FM} u + u - \Pi_h^{FM} u \right\|_{\infty,K}$$

$$
\begin{aligned}
&\le \left\| \Pi_h^{FM} u \right\|_{\infty,K} + \left\| u - \Pi_h^{FM} u \right\|_{\infty,K} \\
&\le \frac{|\Pi_h^{FM} u|_{2,K}}{\sqrt{\lambda_h}} + C_h^{FM} \left| u - \Pi_h^{FM} u \right|_{2,K} \\
&\le \sqrt{\frac{1}{\lambda_h} + \left(C_h^{FM}\right)^2} \sqrt{\left|\Pi_h^{FM} u\right|_{2,K}^2 + \left|u - \Pi_h^{FM} u\right|_{2,K}^2}.
\end{aligned}
$$

From the orthogonality in (9), we have

$$
\left|\Pi_h^{FM} u\right|_{2,K}^2 + \left|u - \Pi_h^{FM} u\right|_{2,K}^2 = |u|_{2,K}^2.
$$

Thus,

$$
\|u\|_{\infty,K} \le \sqrt{\frac{1 + (C_h^{FM})^2 \lambda_h}{\lambda_h}} |u|_{2,K}, \quad \forall u \in V^L(K).
$$

From the definition of λ in (8), we draw the conclusion. □

To apply Theorem 3.1 for bounding λ, an explicit value of C_h^{FM} is needed. Below, let us describe the way to obtain this explicit value by utilizing the raw bound of $C^L(\alpha,\theta)$.

3.1 Explicit estimation of C_h^{FM}

To have an explicit value of C_h^{FM}, we first define the quantity $C_{\text{res}}^{FM}(K_h)$ for each element K_h in the triangulation $\mathcal{T}^h$:

$$
C_{\text{res}}^{FM}(K_h) := \sup_{u \in H^2(K_h)} \frac{\|u - \Pi_h^{FM} u\|_{\infty,K_h}}{|u - \Pi_h^{FM} u|_{2,K_h}} = \sup_{w \in W_1} \frac{\|w\|_{\infty,K_h}}{|w|_{2,K_h}}.
$$

Here, $W_1 := \{w \in H^2(K_h) \mid w(p_i) = 0, \int_{e_i} \frac{\partial w}{\partial n}\,\mathrm{d}s = 0\ (i = 1,2,3)\}$. Noting that $W_1 \subseteq W_2$ for $W_2 := \{w \in H^2(K_h) \mid w(p_i) = 0\ (i = 1,2,3)\}$, from the definition of C^L in (3), we have

$$
C_{\text{res}}^{FM}(K_h) \le \sup_{w \in W_2} \frac{\|w\|_{\infty,K_h}}{|w|_{2,K_h}} = C^L(K_h).
$$

Then, the following C_h^{FM} with an upper bound makes certain (10) holds:

$$
C_h^{FM} := \max_{K_h \in \mathcal{T}^h} C_{\text{res}}^{FM}(K_h) \Bigl(\le \max_{K_h \in \mathcal{T}^h} C^L(K_h) \Bigr). \tag{12}
$$

Remark 3.1 Let $\mathcal{T}^h$ be a uniform triangulation of a right isosceles triangle; see a sample mesh in Fig. 8. We choose an explicit upper bound of C_h^{FM} as $C_h^{FM} \le 1.3712h$, since for each $K_h \in \mathcal{T}^h$, $C_{\text{res}}^{FM} \le C^L(K_h) \le 1.3712h$, where h is the leg length of each right triangle element.

3.2 Estimation of λ_h by solving the finite-dimensional optimization problem

In this subsection, we present a method to estimate λ_h, which is required in Theorem 3.1 for bounding λ. Let $M := \text{Dim}(V_h^{FM})$. The estimation of λ_h is equivalent to finding the

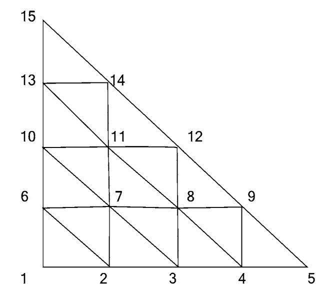


Figure 8 A uniform triangulation of a right isosceles triangle

solution to the optimization problem

$$\lambda_h = \min \mathbf{x}^T \mathbf{A}\mathbf{x}, \quad \text{subject to} \quad \left\| \sum_{i=1}^{M} \mathbf{x}_i \phi_i \right\|_{\infty,K} \geq 1, \tag{13}$$

where the components a_{ij} of $\mathbf{A}$ are given by $a_{ij} = \langle \phi_i, \phi_j \rangle_h$, $\{\phi_i\}_{i=1,\ldots,M}$ are the basis functions for the Fujino–Morley space V_h^{FM}, and $\mathbf{x} \in \mathbb{R}^M$ denotes the Fujino–Morley coefficient vector of $u_h \in V_h^{FM}$.

To solve the optimization problem (13) is not an easy task since the L^∞-norm of the function appears in the constraint. Here, we introduce the technique to apply Bernstein polynomials and their convex-hull property to solve the problem. Strictly speaking, a new optimization problem (14) utilizing the Bernstein polynomials will be formulated to provide a lower bound for the solution of (13).

As preparation, let us introduce the definition of Bernstein polynomials along with the convex-hull property; refer to, e.g., [18, 19] for detailed discussion.

Convex-hull property of Bernstein polynomials Given a triangle K, let (u, v, w) be the barycentric coordinates for a point x in K. A Bernstein polynomial p of degree n over a triangle K is defined by

$$p := \sum_{i+j+k=n} d_{i,j,k} J_{i,j,k}^{(n)}, \qquad J_{i,j,k}^{(n)}(x) := \frac{n!}{i!j!k!} u^i v^j w^k.$$

Here, $J_{i,j,k}^{(n)}(x)$ are the Bernstein basis polynomials; the coefficients $d_{i,j,k}$ are the control points of p. Noting that

$$J_{i,j,k}^{(n)} \geq 0, \qquad \sum_{i+j+k=n} J_{i,j,k}^{(n)} = 1,$$

we can easily obtain the following convex-hull property of Bernstein polynomials:

$$\|p\|_{\infty,K} \leq \max |d_{i,j,k}|.$$

Given $u_h \in V_h^{FM}(K)$, for each $K_h \in \mathcal{T}^h$, $u_h|_{K_h} \in P_2(K_h)$ can be represented by the Bernstein basis polynomials of degree two. Let $\mathbf{B}$ be the $N \times M$ matrix that transforms the Fujino–Morley coefficients $\mathbf{x}$ to the Bernstein coefficients d^B. Note that u_h is regarded as a piecewise Bernstein polynomial so that its Bernstein coefficient vector d^B has the dimension $N = 6 \times \#\{elements\}$. The dimension of d^B can be further reduced considering the continuity of u_h at the vertices of the triangulation. However, it is difficult to utilize the constraints of u_h that cross the edges to reduce the dimension N. From the convex-hull property of the Bernstein polynomials, the following inequality holds:

$$1 \le \left\| \sum_{i=1}^{M} \mathbf{x}_i \phi_i \right\|_{\infty,K} \le \|\mathbf{B}\mathbf{x}\|_\infty.$$

Based on this inequality, we propose a new optimization by relaxing the constraint condition of (13):

$$\lambda_{h,B} = \min \mathbf{x}^T \mathbf{A}\mathbf{x}, \quad \text{subject to} \quad \|\mathbf{B}\mathbf{x}\|_\infty \ge 1. \tag{14}$$

The solution to problem (14) provides a lower bound for (13), i.e., $\lambda_h \ge \lambda_{h,B}$.

Below, we propose an algorithm to solve the problem (14). Since $\mathbf{A}$ is positive-definite, let us consider the Cholesky decomposition of $\mathbf{A}$: $\mathbf{A} = \mathbf{R}^T\mathbf{R}$, where $\mathbf{R}$ is an $M \times M$ upper triangular matrix. Then, by letting $\mathbf{y} := \mathbf{R}\mathbf{x}$ and $\widehat{\mathbf{B}} := \mathbf{B}\mathbf{R}^{-1}$, problem (14) becomes

$$\lambda_{h,B} = \min \mathbf{y}^T \mathbf{y}, \quad \text{subject to} \quad \|\widehat{\mathbf{B}}\mathbf{y}\|_\infty \ge 1. \tag{15}$$

The following lemma shows the solution for problem (15).

Lemma 3.1 *Let b_i^T $(i = 1,\dots,N)$ be the ith row of $\widehat{\mathbf{B}}$ and $b_{\max}^T$ be a row of $\widehat{\mathbf{B}}$ satisfying $\|b_{\max}\|_2 = \max_{i=1,\dots,N} \|b_i\|_2$. Then, the optimal value of problem (15) is given by*[1]

$$\lambda_{h,B} = \frac{1}{\|b_{\max}\|_2^2}.$$

Proof Let $S := \{\mathbf{y} \mid \|\widehat{\mathbf{B}}\mathbf{y}\|_\infty \ge 1\}$ and $\bar{\mathbf{y}} := \|b_{\max}\|_2^{-2} b_{\max}$. Then, we have $\bar{\mathbf{y}} \in S$ because

$$\|\widehat{\mathbf{B}}\bar{\mathbf{y}}\|_\infty = \max_{i=1,\dots,N} \left|b_i^T \bar{\mathbf{y}}\right| \ge \left|b_{\max}^T \bar{\mathbf{y}}\right| = 1.$$

Hence,

$$\min_{\mathbf{y}\in S} \mathbf{y}^T\mathbf{y} \le \bar{\mathbf{y}}^T\bar{\mathbf{y}} = \frac{1}{\|b_{\max}\|_2^2}. \tag{16}$$

For any $\mathbf{y} \in S$, from the Cauchy–Schwarz inequality,

$$1 \le \max_{i=1,\dots,N} \left|{b_i}^T \mathbf{y}\right| \le \max_{i=1,\dots,N} \|b_i\|_2 \|\mathbf{y}\|_2 = \|b_{\max}\|_2 \|\mathbf{y}\|_2.$$

Thus,

$$\frac{1}{\|b_{\max}\|_2^2} \le \min_{\mathbf{y}\in S} \mathbf{y}^T\mathbf{y}. \tag{17}$$

From (16) and (17), we draw the conclusion. □

[1] Appreciation to Tamaki TANAKA and Syuuji YAMADA from the Faculty of Science, Niigata University for their idea of solving this problem in an efficient way.

Note that the diagonal elements of $\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T = \widehat{\mathbf{B}}\widehat{\mathbf{B}}^T$ correspond to $\|b_i\|_2^2$ $(i = 1,\ldots,N)$. Therefore, we can solve problem (14) without performing the Cholesky decomposition of $\mathbf{A}$, as shown by the following lemma.

Lemma 3.2 *Let* $\mathbf{D} := \mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T$. *The optimal value of* (14) *is given by*

$$\lambda_{h,B} = \frac{1}{\max(\mathrm{diag}(\mathbf{D}))},$$

where $\mathrm{diag}(\mathbf{D})$ *is the diagonal elements of* $\mathbf{D}$.

Theorem 3.1 gives a lower bound for λ. Since $C^L(K) = \sqrt{\lambda(K)}^{-1}$, this lower bound is used to obtain an upper bound for $C^L(K)$. Below, let us summarize the procedure to obtain a lower bound for λ.

Algorithm for calculating the lower bound of $\lambda(K)$

a. Set up the FEM space $V_h^{FM}(K) = \mathrm{span}\{\phi_i\}_{i=1}^M$ over a triangulation of the triangle domain K.
b. Assemble the global matrix $\mathbf{A} = (a_{ij})_{M\times M}$ $(a_{ij} = \langle \phi_i, \phi_j\rangle_h)$ and the transformation matrix $\mathbf{B}$ from Fujino–Morley coefficients to Bernstein coefficients.
c. Apply Lemma 2.3 to obtain a raw bound for C_h^{FM}.
d. Apply Lemma 3.1 or Lemma 3.2 to calculate $\lambda_{h,B}(\le \lambda_h)$.
e. The lower bound for λ is obtained through Theorem 3.1 by using $\lambda_{h,B}$ and the upper bound of C_h^{FM}.

Using uniform triangulation of a domain K, a direct estimation of the lower bound for λ without using C_h^{FM} is available.

Corollary 3.1 *For a uniform triangulation of* $K = K_{\alpha,\theta,h}$ *with* N *subdivisions for each side, the following holds*:

$$\lambda(K) \ge \lambda_h\bigl(1 - (1/N)^2\bigr). \tag{18}$$

Proof Since $(C^L(K))^2 = 1/\lambda(K)$ and each $K_h \in \mathcal{T}^h$ is similar to K, we have,

$$\lambda(K) \ge \frac{\lambda_h}{1 + (C_h^{FM})^2\lambda_h} \ge \frac{\lambda_h}{1 + (C^L(K_h))^2\lambda_h} = \frac{\lambda_h}{1 + (1/N)^2\lambda_h/\lambda(K)}.$$

The conclusion is achieved by sorting the inequality. □

Remark 3.2 Theoretically, for a refined uniform triangulation, the lower bound (11) using C_h^{FM} is sharper (i.e., larger) than (18). This can be confirmed by utilizing the following relation:

$$\frac{\lambda_h}{1 + (C_h^{FM})^2\lambda_h} \ge \lambda_h\bigl(1 - (1/N)^2\bigr) \quad \Longleftrightarrow \quad 1 \ge \bigl(N^2 - 1\bigr)\bigl(C_h^{FM}\bigr)^2\lambda_h. \tag{19}$$

For a small value of $h = 1/N$, we have

$$\bigl(N^2 - 1\bigr)\bigl(C_h^{FM}\bigr)^2 \approx \bigl(NC_h^{FM}\bigr)^2 = \bigl(C_{\mathrm{res}}^{FM}(K_h)\bigr)^2, \lambda_h \approx \lambda = \bigl(C^L(K_h)\bigr)^{-2}.$$

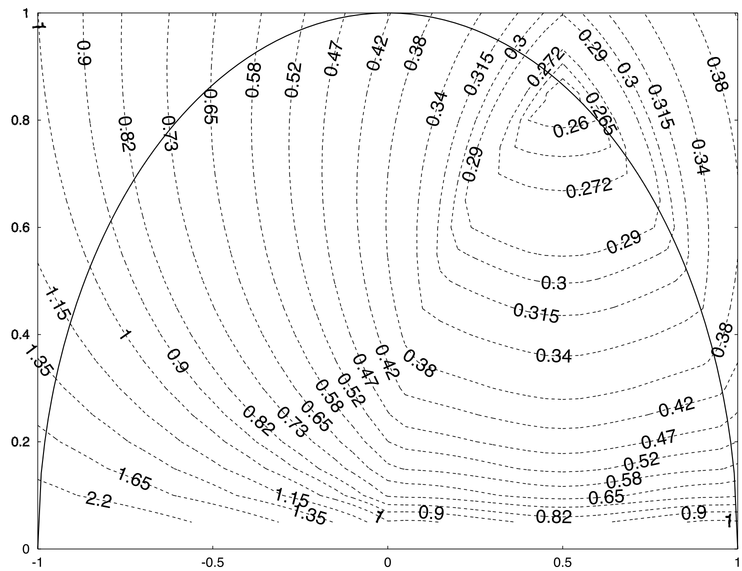


Figure 9 Contour lines of $C^L(\alpha,\theta)$ w.r.t. vertex $p_3(x,y)$

Thus, the second equality of (19) holds due to $C^{FM}_{\text{res}}(K_h) < C^L(K_h)$. However, in practical computation, the raw estimate of $C^{FM}_{\text{res}}(K_h)$ will produce a worse bound of λ than (18).

Using Corollary 3.1, the following steps are modified from the algorithm to obtain a lower bound for λ, without using the quantity of C^{FM}_h:

Revision of algorithm for calculating the lower bound of $\lambda(K)$

c*. Apply Lemma 3.1 or Lemma 3.2 to calculate $\lambda_{h,B}(\leq \lambda_h)$.

d*. Solve the lower bound for λ using Corollary 3.1 along with $\lambda_{h,B}$.

Remark 3.3 To compare the efficiencies of the two formulas (11) and (18), we apply them to estimate λ for a unit right isosceles $K_{1,\pi/2}$. By using uniform triangulation of size $h = 1/64$, the estimate (11) gives $\lambda \geq 5.7659$ and (18) gives a sharper bound as $\lambda \geq 5.7798$. Hence, a sharper upper bound is obtained using (18) and we have the following estimation:

$$\left\| u - \Pi^L u \right\|_{\infty,K_{1,\pi/2,h}} \leq 0.41596h|u|_{2,K_{1,\pi/2,h}}.$$

As a comparison, the result (5) will yield a raw bound as $C^L(1,\pi/2,h) \leq 1.3712h$.

For a triangle $K_{\alpha,\theta}$ with two fixed vertices $p_1(0,0), p_2(1,0)$, let us vary the vertex $p_3(x,y)$ and calculate the approximate value of $C^L(\alpha,\theta)$ for each position of p_3. Note that C^L can be regarded as a function with respect to the coordinate (x,y) of p_3, which is denoted by $C^L(x,y)$. In Fig. 9, we draw the contour lines of $C^L(x,y)$, where the abscissa and the ordinate denote x- and y- coordinates of p_3, respectively.

3.3 Lower bound of the constant

To confirm the precision of the obtained estimation for the Lagrange interpolation constant, the lower bounds of the constants are calculated. Let u_h be the function obtained by numerical computation solving the minimization problem. To obtain the lower bound, an

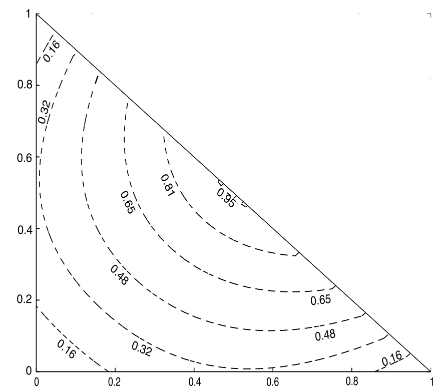


Figure 10 The contour lines of the minimizer u_h of (14) for $K_{1,\pi/2}$

appropriate polynomial f over K of higher degree d is selected by solving the minimization problem below:

$$\min_{f\in P_d(K)} \sum_{i=1}^{n} \left|f(p_i) - u_h(p_i)\right|^2 \quad \bigl(n : \#\{\text{nodes of triangulation}\}\bigr),$$

where p_i denote the nodes of the triangulation of K. From the definition of $\lambda(K)$ in (8) and the relation $C^L(K) = 1/\sqrt{\lambda(K)}$, we have a lower bound of $C^L(K)$ as follows:

$$C^L(K) \geq \frac{\|f\|_{\infty,K}}{|f|_{2,K}}.$$

Remark 3.4 For the unit right isosceles triangle $K_{1,\pi/2}$, the upper bound for the constant is obtained by solving the optimization problem with mesh size 1/64. Meanwhile, the lower bound of the constant is obtained by using a polynomial of degree 9. The two side bounds read:

$$0.40432 \leq C^L\left(1, \frac{\pi}{2}\right) \leq 0.41596.$$

4 Numerical results and conclusion

In this section, we perform numerical computation to obtain the estimation of the interpolation error constant $C^L(K)$ for triangles of various shapes.

First, let us confirm the shape of the function u_h that solves the minimization problem for $\lambda_{h,B}$ in the case of K being the unit isosceles right triangle. The contour lines of u_h are displayed in Fig. 10. The numerical computation tells us that the maximum value of u_h happens on the midpoint of the hypotenuse of K. Note that the maximum value of u_h is around 0.95, while the maximum of its Bernstein coefficients is above 1.

Let us also compare the lower bounds of λ obtained through Theorem 3.1 and Corollary 3.1 for various triangles. Table 1 tells us that the values obtained using Corollary 3.1 give a sharper estimate of λ.

Table 2 summarizes the results for the lower and upper bounds of the constant for different types of triangle $K_{1,\theta}$ with the mesh size as $h = 1/32$ and $h = 1/64$. The upper bounds (denoted by C^L_{ub}) are obtained through Corollary 3.1, while the lower bounds (denoted by C^L_{lb}) are obtained by using high-degree polynomials with degree denoted by d.

Table 1 The lower bounds for λ through Theorem 3.1 and Corollary 3.1

θ	$h = 1/32$			$h = 1/64$		
	$\lambda_{h,B}$	Theorem 3.1	Corollary 3.1	$\lambda_{h,B}$	Theorem 3.1	Corollary 3.1
$\pi/6$	9.8339	8.7356	9.8245	9.8925	9.5892	9.8901
$\pi/4$	13.517	12.263	13.505	13.574	13.234	13.570
$\pi/3$	15.412	14.357	15.397	15.457	15.177	15.454
$\pi/2$	5.5988	5.5418	5.5933	5.7812	5.7660	5.7799
$2\pi/3$	2.3954	2.3683	2.3930	2.5511	2.5433	2.5504
$3\pi/4$	1.5550	1.5369	1.5534	1.6768	1.6715	1.6764
$5\pi/6$	0.93778	0.92669	0.93687	1.0212	1.0179	1.0210

Table 2 The lower and upper bounds of $C^L(1,\theta)$ for triangles of different shapes

θ	$h = 1/32$				$h = 1/64$			
	d	C^L_{lb}	$\lambda_{h,B}$	C^L_{ub}	d	C^L_{lb}	$\lambda_{h,B}$	C^L_{ub}
$\pi/6$	9	0.31511	9.8339	0.31904	9	0.31423	9.8925	0.31799
$\pi/4$	8	0.26777	13.517	0.27212	8	0.26753	13.574	0.27146
$\pi/3$	10	0.25182	15.412	0.25485	10	0.25209	15.457	0.25439
$\pi/2$	9	0.40432	5.5988	0.42283	9	0.40419	5.7812	0.41596
$2\pi/3$	8	0.59964	2.3954	0.64644	8	0.60079	2.5511	0.62618
$3\pi/4$	10	0.72146	1.5550	0.80233	10	0.72420	1.6768	0.77235
$5\pi/6$	8	0.92197	0.93778	1.03314	8	0.92830	1.0212	0.98968

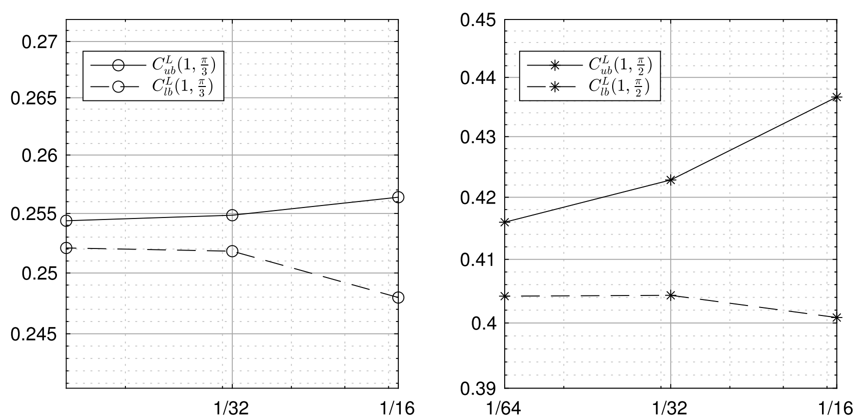


Figure 11 The convergency behavior of the upper and lower bounds of $C^L(1,\theta)$ for $\theta = \pi/3$ and $\pi/2$

Figure 11 demonstrates the convergency of the upper and lower bounds of the interpolation error constant as the mesh is refined. This implies that the convergency order of upper bounds depends on the shape of the triangles. The theoretical analysis on the efficiency and the convergency of the algorithm in solving the optimization problem is beyond the scope of this paper and will be systematically investigated in our succeeding research.

Rigorous result using interval arithmetic Numerical computation with floating-point numbers involves round-off errors. To have rigorous results, we applied the interval arithmetic in assembling the matrices and evaluating the upper bound C^L_{ub} in Table 2. It is observed from the numerical computation results that the accumulation of round-off error in the computation is not so large. For example, for the mesh size being $h = 1/64$, the matrix B has the dimensions 24576×8382 and the rigorous estimation of C^L_{ub} in the case of an isosceles right triangle is given as

$$C^L_{ub}(1,\pi/2) \in [\underline{0.4159516}728, \underline{0.4159516}793].$$

5 Conclusion

In this research, we provide explicit estimates for the L^∞-norm error constant C^L of the linear Lagrange interpolation function over triangular elements. The formula in Theorem 2.1 provides a bound of C^L that holds for triangles of arbitrary shapes. Theorem 3.1 in Sect. 3 proposes a numerical approach to obtain the optimal bounds for the constant C^L over a concrete triangle. The optimization problem corresponding to C^L is solved by utilizing the convex-hull property of Bernstein polynomials in a novel way. In the near future, the convergency of the numerical approach to solve the optimization problems involving the maximum norm will be systematically considered.

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Availability of data and materials
An online demo with source codes of the constant evaluation is available at
https://ganjin.online/shirley/InterpolationErrorEstimate.

Declarations

Competing interests
The authors declare that they have no competing interests.

Author contribution
SG prepared the manuscript and finished the programming code for the computation examples. KI prepared the part on solving the objective optimization problem. XL provided the main idea and advice for this research. All authors read and approved the final manuscript.

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