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On a nonlinear second-order difference equation

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Abstract
We study a nonlinear second-order difference equation which considerably extends some equations in the literature. Our main result shows that the difference equation is solvable in closed form. Some applications of the main result are also given.

MSC: 39A20

Keywords: Difference equation, Solvable equation, Closed-form formula, Equilibrium

1 Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ be the sets of all natural, whole, rational, and real numbers respectively, and $\mathbb{R}_+ = [0,\infty)$. For fixed $k \in \mathbb{Z}$, we use the notation $\mathbb{N}_k = \{n \in \mathbb{Z} : n \geq k\}$. Throughout the paper we will also use the following standard convention:

$$\prod_{j=k}^{k-1} a_j = 1,$$

where $k \in \mathbb{Z}$.

There has been a huge interest in difference equations and systems of difference equations (see, for example, [1–48] and the references therein), because they naturally appear in many branches of mathematics and science, where they model real and abstract phenomena (see, for example, [6, 8, 15, 20, 25, 27, 31, 37, 48]). Of many topics in the area, there has been a growing renewed interest in their solvability, invariants, and their applications (see, for example, [3–6, 14, 28–30, 32–35, 38, 39, 41–47] and the references therein), although nowadays mainstream of investigations is on the long-term behavior of their solutions (see, e.g., [3, 4, 6–8, 16–19, 36, 40]).

A typical situation is that many solvable difference equations and systems of difference equations are transformed by suitable changes of variables to well-known solvable ones such as linear difference equations and systems with constant coefficients and their close relatives (see, for example, [5, 32, 41–47] and many related references therein). For some classical solvable difference equations and systems, see, for example, original sources [9, 11, 13, 23, 24], as well as some of the oldest presentations of the topic in [21] and [22] (for some later presentations see, for example, [10] and [26]). It should be pointed out that the original sources have been also motivated by some practical problems, usually from combinatorics and probability, but also in economics. Generally speaking, investigation of difference equations and systems of difference equations has always had some direct or potential applications.

Special cases of the difference equation

$$x_{n+1} = ax_n + \frac{bx_n^2}{cx_n + dx_{n-1}}, \quad n \in \mathbb{N}_0, \tag{1}$$

where parameters a, b, c, d and initial values x_{-1} and x_0 are real or positive real numbers, are some of the difference equations that appear from time to time in the literature (see, e.g., [12]).

Our aim here it to show solvability of an extension of equation (1), considerably extending some results on solvability of difference equations in the literature. We also give some applications of our main result, as well as some comments related to some results in [12] on equation (1).

2 Main results

In this section we study solvability of the difference equation

$$x_{n+1} = f^{-1}\left(f(x_n)\frac{\alpha f(x_n) + \beta f(x_{n-1})}{\gamma f(x_n) + \delta f(x_{n-1})}\right), \quad n \in \mathbb{N}_0, \tag{2}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\gamma^2 + \delta^2 \neq 0$, $f : \mathbb{R} \to \mathbb{R}$ is a strictly monotone (increasing or decreasing) continuous function, $f(\mathbb{R}) = \mathbb{R}$ and $f(0) = 0$. We use some methods and ideas related to the ones in [14, 41, 43–45, 47].

First note that if $x_{n_0} = 0$ for some $n_0 \in \mathbb{N}_0$, then from equation (2) we easily obtain $x_{n_0+1} = 0$, from which together with equation (2) it follows that x_{n_0+2} is not defined. Hence, from now on we consider only well-defined solutions to equation (2) such that $x_n \neq 0$, $n \in \mathbb{N}_0$. We may also assume that $x_{-1} \neq 0$. Namely, if we assume that $x_{-1} = 0$, the fact/assumption $x_0 \neq 0 \neq x_1$ enables us to consider the solutions on the domain $\mathbb{N}_0$, that is, we can discard the member x_{-1}. Hence, we may assume

$$x_n \neq 0, \quad n \in \mathbb{N}_{-1}. \tag{3}$$

The following result is the main in this paper.

Theorem 1 *Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $\gamma^2 + \delta^2 \neq 0$, $f : \mathbb{R} \to \mathbb{R}$ be a strictly monotone continuous function such that $f(\mathbb{R}) = \mathbb{R}$ and $f(0) = 0$. Then equation (2) is solvable in closed form.*

Proof First note that since $f(\mathbb{R}) = \mathbb{R}$, $f(0) = 0$ and f is strictly monotone and continuous, then it is one-to-one, point 0 is a unique root of the function, and f is a homeomorphism of the real line $\mathbb{R}$ (see, e.g., [49]).

There are several cases to be considered.

Case $\alpha\delta = \beta\gamma$, $\alpha = \beta = 0$. Since $\alpha = \beta = 0$, from (2) we have

$$x_{n+1} = 0, \quad n \in \mathbb{N}_0, \tag{4}$$

so we get a trivial equation, with an obvious solution.

Case $\alpha\delta = \beta\gamma$, $\alpha = 0$, $\beta \neq 0$. From these conditions we immediately obtain $\gamma = 0$ and $\delta \neq 0$, which implies

$$x_{n+1} = f^{-1}\left(\frac{\beta}{\delta} f(x_n)\right), \quad n \in \mathbb{N}_0.$$

Hence

$$f(x_{n+1}) = \frac{\beta}{\delta} f(x_n), \quad n \in \mathbb{N}_0.$$

From this we get

$$f(x_n) = \left(\frac{\beta}{\delta}\right)^n f(x_0), \quad n \in \mathbb{N}_0,$$

which implies

$$x_n = f^{-1}\left(\left(\frac{\beta}{\delta}\right)^n f(x_0)\right), \quad n \in \mathbb{N}_0. \tag{5}$$

Case $\alpha\delta = \beta\gamma$, $\alpha \neq 0$, $\beta = 0$. From these conditions we immediately obtain $\delta = 0$, and consequently $\gamma \neq 0$, which implies

$$x_{n+1} = f^{-1}\left(\frac{\alpha}{\gamma} f(x_n)\right), \quad n \in \mathbb{N}_0,$$

and consequently

$$f(x_{n+1}) = \frac{\alpha}{\gamma} f(x_n), \quad n \in \mathbb{N}_0.$$

From this we get

$$f(x_n) = \left(\frac{\alpha}{\gamma}\right)^n f(x_0), \quad n \in \mathbb{N}_0,$$

which implies

$$x_n = f^{-1}\left(\left(\frac{\alpha}{\gamma}\right)^n f(x_0)\right), \quad n \in \mathbb{N}_0. \tag{6}$$

Case $\alpha\delta = \beta\gamma$, $\delta = 0$. From these conditions we have $\gamma \neq 0$, and consequently $\beta = 0$. Hence, we have two cases $\alpha = 0$ and $\alpha \neq 0$, which have been considered above.

Case $\alpha\delta = \beta\gamma$, $\gamma = 0$. From these conditions we have $\delta \neq 0$, and consequently $\alpha = 0$. Hence, we have two cases $\beta = 0$ and $\beta \neq 0$, which have been also considered above.

Case $\alpha\delta = \beta\gamma$, $\alpha\beta\gamma\delta \neq 0$. Since $\alpha\beta\gamma\delta \neq 0$, we have $\alpha = \beta\gamma/\delta$, from which it follows that

$$x_{n+1} = f^{-1}\left(\frac{\beta}{\delta} f(x_n)\right),$$

so formula (5) holds in this case.

Case $\alpha\delta \neq \beta\gamma$. We have

$$f(x_{n+1}) = f(x_n)\frac{\alpha f(x_n) + \beta f(x_{n-1})}{\gamma f(x_n) + \delta f(x_{n-1})}, \quad n \in \mathbb{N}_0. \tag{7}$$

Since we assume that (3) holds, then we have

$$f(x_n) \neq 0, \quad n \in \mathbb{N}_{-1},$$

so we can use the change of variables

$$y_n = \frac{f(x_n)}{f(x_{n-1})}, \quad n \in \mathbb{N}_0, \tag{8}$$

in (7) and obtain the equation

$$y_{n+1} = \frac{\alpha y_n + \beta}{\gamma y_n + \delta}, \quad n \in \mathbb{N}_0. \tag{9}$$

Case $\alpha\delta \neq \beta\gamma$, $\gamma = 0$. First note that it must be $\delta \neq 0$, and we get the linear equation

$$y_{n+1} = \frac{\alpha}{\delta} y_n + \frac{\beta}{\delta} \tag{10}$$

for $n \in \mathbb{N}_0$.

Case $\alpha\delta \neq \beta\gamma$, $\gamma = 0$, $\alpha = \delta$. Since $\alpha = \delta$, from equation (10) it follows that

$$y_n = \frac{\beta}{\delta} n + y_0$$

for $n \in \mathbb{N}_0$, from which along with (8) it follows that

$$f(x_n) = \left(\frac{\beta}{\delta} n + \frac{f(x_0)}{f(x_{-1})}\right) f(x_{n-1})$$

for $n \in \mathbb{N}_0$, and consequently

$$f(x_n) = f(x_{-1}) \prod_{j=0}^{n} \left(\frac{\beta}{\delta} j + \frac{f(x_0)}{f(x_{-1})}\right)$$

for $n \in \mathbb{N}_{-1}$, from which it follows that

$$x_n = f^{-1}\left(f(x_{-1}) \prod_{j=0}^{n} \left(\frac{\beta}{\delta} j + \frac{f(x_0)}{f(x_{-1})}\right)\right) \tag{11}$$

for $n \in \mathbb{N}_{-1}$.

Case $\alpha\delta \neq \beta\gamma$, $\gamma = 0$, $\alpha \neq \delta$. Since $\alpha \neq \delta$, then from (10) we have

$$y_n = \beta \frac{(\alpha/\delta)^n - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^n y_0$$

for $n \in \mathbb{N}_0$, from which along with (8) it follows that

$$f(x_n) = \left(\beta \frac{(\alpha/\delta)^n - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^n \frac{f(x_0)}{f(x_{-1})}\right) f(x_{n-1})$$

for $n \in \mathbb{N}_0$, and consequently

$$f(x_n) = f(x_{-1}) \prod_{j=0}^{n} \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{f(x_0)}{f(x_{-1})}\right)$$

for $n \in \mathbb{N}_{-1}$, from which we obtain

$$x_n = f^{-1}\left(f(x_{-1}) \prod_{j=0}^{n} \left(\beta \frac{(\alpha/\delta)^j - 1}{\alpha - \delta} + \left(\frac{\alpha}{\delta}\right)^j \frac{f(x_0)}{f(x_{-1})}\right)\right) \tag{12}$$

for $n \in \mathbb{N}_{-1}$.

Case $\alpha\delta \neq \beta\gamma$, $\gamma \neq 0$. In this case equation (9) is a bilinear/fractional linear difference equation [1, 2, 22, 27, 45], hence we can use the following change of variables:

$$y_n = \frac{z_{n+1}}{z_n} + f, \quad n \in \mathbb{N}_0, \tag{13}$$

where f is a constant which should be suitably chosen so that equation (9) is transformed to a known solvable one.

We have

$$\left(\frac{z_{n+2}}{z_{n+1}} + f\right)\left(\gamma \frac{z_{n+1}}{z_n} + \gamma f + \delta\right) - \left(\alpha \frac{z_{n+1}}{z_n} + \alpha f + \beta\right) = 0 \tag{14}$$

for $n \in \mathbb{N}_0$.

By choosing

$$f = -\frac{\delta}{\gamma}$$

in (14), after some calculation, we get that it must be

$$\gamma^2 z_{n+2} - \gamma(\alpha + \delta) z_{n+1} + (\alpha\delta - \beta\gamma) z_n = 0 \tag{15}$$

for $n \in \mathbb{N}_0$.

If $\Delta := (\alpha+\delta)^2 - 4(\alpha\delta - \beta\gamma) \neq 0$, then the roots of the characteristic polynomial associated with equation (15) are different and given by

$$\lambda_1 = \frac{\alpha + \delta + \sqrt{\Delta}}{2\gamma} \quad \text{and} \quad \lambda_2 = \frac{\alpha + \delta - \sqrt{\Delta}}{2\gamma}.$$

A general solution to equation (15) is

$$z_n = \frac{(z_1 - \lambda_2 z_0)\lambda_1^n - (z_1 - \lambda_1 z_0)\lambda_2^n}{\lambda_1 - \lambda_2} \tag{16}$$

for $n \in \mathbb{N}_0$ (see [11, p. 84]).

Using (16) in (13), we have

$$y_n = \frac{(z_1-\lambda_2 z_0)\lambda_1^{n+1}-(z_1-\lambda_1 z_0)\lambda_2^{n+1}}{(z_1-\lambda_2 z_0)\lambda_1^{n}-(z_1-\lambda_1 z_0)\lambda_2^{n}}-\frac{\delta}{\gamma}$$
$$= \frac{(y_0-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{n+1}-(y_0-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{n+1}}{(y_0-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{n}-(y_0-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{n}}-\frac{\delta}{\gamma}$$

for $n\in\mathbb{N}_0$, from which along with (8) it follows that

$$f(x_n)=\left(\frac{(\frac{f(x_0)}{f(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{n+1}-(\frac{f(x_0)}{f(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{n+1}}{(\frac{f(x_0)}{f(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{n}-(\frac{f(x_0)}{f(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{n}}-\frac{\delta}{\gamma}\right)f(x_{n-1})$$

for $n\in\mathbb{N}_0$, and consequently

$$f(x_n)=f(x_{-1})\prod_{j=0}^{n}\left(\frac{(\frac{f(x_0)}{f(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{j+1}-(\frac{f(x_0)}{f(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{j+1}}{(\frac{f(x_0)}{f(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{j}-(\frac{f(x_0)}{f(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{j}}-\frac{\delta}{\gamma}\right)$$

for $n\in\mathbb{N}_{-1}$.

Hence

$$x_n=f^{-1}\left(f(x_{-1})\prod_{j=0}^{n}\left(\frac{(\frac{f(x_0)}{f(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{j+1}-(\frac{f(x_0)}{f(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{j+1}}{(\frac{f(x_0)}{f(x_{-1})}-\lambda_2+\frac{\delta}{\gamma})\lambda_1^{j}-(\frac{f(x_0)}{f(x_{-1})}-\lambda_1+\frac{\delta}{\gamma})\lambda_2^{j}}-\frac{\delta}{\gamma}\right)\right) \quad (17)$$

for $n\in\mathbb{N}_{-1}$.

If $\Delta = 0$, then the roots of the characteristic polynomial associated with equation (15) are

$$\lambda_{1,2}=\frac{\alpha+\delta}{2\gamma}\neq 0,$$

and the general solution to equation (15) is

$$z_n=\big((z_1-\lambda_1 z_0)n+\lambda_1 z_0\big)\lambda_1^{n-1} \quad (18)$$

for $n\in\mathbb{N}_0$ (see, e.g., [42]).

Using (18) in (13), we have

$$y_n=\frac{((z_1-\lambda_1 z_0)(n+1)+\lambda_1 z_0)\lambda_1}{(z_1-\lambda_1 z_0)n+\lambda_1 z_0}-\frac{\delta}{\gamma}$$
$$=\frac{((y_0+\frac{\delta}{\gamma}-\lambda_1)(n+1)+\lambda_1)\lambda_1}{(y_0+\frac{\delta}{\gamma}-\lambda_1)n+\lambda_1}-\frac{\delta}{\gamma}$$

for $n\in\mathbb{N}_0$, from which along with (8) it follows that

$$f(x_n)=\left(\frac{((f(x_0)+(\frac{\delta}{\gamma}-\lambda_1)f(x_{-1}))(n+1)+\lambda_1 f(x_{-1}))\lambda_1}{(f(x_0)+(\frac{\delta}{\gamma}-\lambda_1)f(x_{-1}))n+\lambda_1 f(x_{-1})}-\frac{\delta}{\gamma}\right)f(x_{n-1})$$

for $n \in \mathbb{N}_0$, and consequently

$$f(x_n) = f(x_{-1}) \prod_{j=0}^{n} \left(\frac{((f(x_0) + (\frac{\delta}{\gamma} - \lambda_1) f(x_{-1}))(j+1) + \lambda_1 f(x_{-1}))\lambda_1}{(f(x_0) + (\frac{\delta}{\gamma} - \lambda_1) f(x_{-1}))j + \lambda_1 f(x_{-1})} - \frac{\delta}{\gamma} \right)$$

for $n \in \mathbb{N}_{-1}$.

Hence

$$x_n = f^{-1}\left(f(x_{-1}) \prod_{j=0}^{n} \left(\frac{((f(x_0) + (\frac{\delta}{\gamma} - \lambda_1) f(x_{-1}))(j+1) + \lambda_1 f(x_{-1}))\lambda_1}{(f(x_0) + (\frac{\delta}{\gamma} - \lambda_1) f(x_{-1}))j + \lambda_1 f(x_{-1})} - \frac{\delta}{\gamma} \right) \right) \tag{19}$$

for $n \in \mathbb{N}_{-1}$.

The formulas in (4), (5), (6), (11), (12), (17), and (19) imply the claim of the theorem. □

From the proof of Theorem 1 and since the bilinear function

$$b(x) = \frac{\alpha x + \beta}{\gamma x + \delta}$$

maps $\mathbb{R}_+$ into itself, when $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$, $\gamma^2 + \delta^2 \neq 0$, we see that the following result also holds.

Theorem 2 *Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$, $\gamma^2 + \delta^2 \neq 0$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly monotone continuous function such that $f(\mathbb{R}_+) = \mathbb{R}_+$ and $f(0) = 0$. Then equation (2) is solvable in closed form.*

3 Applications

In this section we present some applications of the main result concerning solutions to equation (1). We also give some comments related to some of the results in [12] to equation (1). First, we present a corollary of Theorem 1 related to solvability of equation (1).

Corollary 1 *Assume that $a, b, c, d \in \mathbb{R}$, $c^2 + d^2 \neq 0$. Then equation (1) is solvable in closed form.*

Proof First note that equation (1) can be written in the following form:

$$x_{n+1} = x_n \frac{(ac+b)x_n + adx_{n-1}}{cx_n + dx_{n-1}}$$

for $n \in \mathbb{N}_0$.

From this observation we see that equation (1) is obtained from equation (2) with

$$\begin{aligned} & f(x) = x, \quad \text{for } x \in \mathbb{R}, \\ & \alpha = ac + b, \qquad \beta = ad, \qquad \gamma = c, \quad \text{and} \quad \delta = d. \end{aligned} \tag{20}$$

Hence, the result follows from Theorem 1. □

Many recent papers on difference equations and systems of difference equations present some results and closed form formulas for their solutions with no or minor theoretical explanations related to them, as well as with some incomplete arguments. For some previous discussions on related issues of this type see, for example, [42, 43, 45, 46].

Related to equation (1), in [12] it is claimed that $\bar{x} = 0$ is a unique equilibrium point of the equation, when

$$(1 - a)(c + d) \neq b. \tag{21}$$

However, if $\bar{x}$ is an equilibrium of the equation, then it must be

$$\bar{x} = a\bar{x} + \frac{b\bar{x}^2}{(c + d)\bar{x}}, \tag{22}$$

and consequently

$$\bar{x} \neq 0 \quad \text{and} \quad c + d \neq 0.$$

So, $\bar{x}$ cannot be equal to zero.

On the other hand, if we assume that $\bar{x} \neq 0$, then from equation (22) it follows that

$$\bar{x}\left(1 - a - \frac{b}{c + d}\right) = 0,$$

which implies

$$1 - a - \frac{b}{c + d} = 0,$$

from which it immediately follows that each $\bar{x} \neq 0$ is an equilibrium of equation (1) in this case.

In [12, Theorem 1] it is claimed that, under a specified condition, the equilibrium point of (1) is locally asymptotically stable. But, as the simple analysis shows, equation (1) does not have an equilibrium or it has infinitely many, so the formulation of [12, Theorem 1] is obscure. Further, Theorem 2 in [12] claims that the following result holds.

Theorem 3 *The equilibrium point $\bar{x}$ of equation* (1) *is a global attractor if*

$$c(1 - a) \neq b. \tag{23}$$

A simple result on boundedness is also given therein as well as closed-form formulas for solutions to four special cases of equation (1) without any explanation how they are obtained.

Remark 1 Equation (1) should be folklore. For example, Problem 1572 in Mathematics Magazine 72 (2) 1999 is on the equation with

$$a = \frac{2}{3}, \qquad b = -\frac{1}{3}, \qquad c = 2, \quad \text{and} \quad d = -3.$$

The present study is based on our original idea for solving the difference equation back in 1999.

Example 1 Here we give a counterexample to the claim in Theorem 3. Consider equation (1) with

$$a = b = c = d = 1,$$

that is, the equation

$$x_{n+1} = x_n \frac{2x_n + x_{n-1}}{x_n + x_{n-1}} \tag{24}$$

for $n \in \mathbb{N}_0$.

Since in this case

$$c(1-a) - b = -1 \neq 0,$$

we see that condition (23) posed in the formulation of the claim in Theorem 3 is satisfied.

Employing formula (17), where f is given by (20) and

$$\alpha = 2, \qquad \beta = \gamma = \delta = 1,$$

we have

$$x_n = x_{-1} \prod_{j=0}^{n} \left(\frac{(x_0 + (1-\widehat{\lambda}_2)x_{-1})\widehat{\lambda}_1^{j+1} - (x_0 + (1-\widehat{\lambda}_1)x_{-1})\widehat{\lambda}_2^{j+1}}{(x_0 + (1-\widehat{\lambda}_2)x_{-1})\widehat{\lambda}_1^{j} - (x_0 + (1-\widehat{\lambda}_1)x_{-1})\widehat{\lambda}_2^{j}} - 1 \right), \tag{25}$$

where

$$\widehat{\lambda}_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \widehat{\lambda}_2 = \frac{3-\sqrt{5}}{2}.$$

Now note that

$$\lim_{n\to+\infty} \frac{(x_0 + (1-\widehat{\lambda}_2)x_{-1})\widehat{\lambda}_1^{n+1} - (x_0 + (1-\widehat{\lambda}_1)x_{-1})\widehat{\lambda}_2^{n+1}}{(x_0 + (1-\widehat{\lambda}_2)x_{-1})\widehat{\lambda}_1^{n} - (x_0 + (1-\widehat{\lambda}_1)x_{-1})\widehat{\lambda}_2^{n}} - 1 = \widehat{\lambda}_1 - 1, \tag{26}$$

when

$$\frac{x_0}{x_{-1}} \neq \widehat{\lambda}_2 - 1 = \frac{1-\sqrt{5}}{2}.$$

This obviously holds if, for example, we choose the initial values $x_{-1}, x_0 \in \mathbb{Q} \cap (0, +\infty)$, since in this case the quotient x_0/x_{-1} is a rational number, whereas $\frac{1-\sqrt{5}}{2}$ is an irrational number.

From (25), (26) and since

$$\widehat{\lambda}_1 - 1 = \frac{1+\sqrt{5}}{2} > 1,$$

we get

$$\lim_{n\to+\infty} x_n = +\infty.$$

Hence, such solutions, as unbounded, are not convergent. Since some of them are positive ones, the claim in Theorem 3 is not correct.

Remark 2 By the closed-form formulas for solutions to equation (2) obtained in the proof of Theorem 1, after some calculations, the closed-form formulas in [12] are easily obtained in terms of the Fibonacci sequence [42]. Many facts on the Fibonacci sequence can be found in [48] (see also [20] and [27]).

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Competing interests
The authors declare no competing interests.

Authors' contributions
SS initiated the investigation, proposed some preliminary ideas, and conducted some detailed investigations. BI, WK and ZŠ analyzed the proposed ideas, made some calculations, and gave many ideas and comments. All authors read and approved the final manuscript.

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