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Qualitative analysis of very weak solutions to Dirac-harmonic equations

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Abstract

In this paper, we introduce a definition of very weak solutions to the homogenous Dirac-harmonic equations for differential forms. In this setting, applying the Gehring lemma and interpolation theorems, we establish a higher integrability of the Dirac operator based on the very weak solutions and explore the relation between weak solutions and very weak solutions.

Keywords: Differential forms; Dirac operator; Higher integrability; Very weak solutions

1 Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. In the paper, we focus on the homogeneous harmonic equation for differential forms driven by the Hodge–Dirac operator

$$d^*A(x, Du) = 0, \tag{1.1}$$

where the natural space we consider in equation (1.1) is the Sobolev space $W^{1,p}(\Omega, \Lambda)$, $D = d + d^*$ is the Hodge–Dirac operator, and the operator $A : \Omega \times \Lambda(\mathbb{R}^n) \rightarrow \Lambda(\mathbb{R}^n)$ satisfies the following conditions:

- (i) the mapping $x \rightarrow A(x, \xi)$ is measurable for all $\xi \in \Lambda(\Omega)$;
- (ii) the Lipschitz-type inequality

$$|A(x, \xi) - A(x, \eta)| \leq L_1(|\xi| + |\eta|)^{p-2}|\xi - \eta|;$$

- (iii) the monotonicity inequality

$$|\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle| \geq L_2(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2;$$

- (iv) the homogeneity

$$A(x, \lambda\xi) = |\lambda|^{p-2}\lambda A(x, \xi)$$

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for almost all $x \in \Omega$, all $\xi, \eta \in \Lambda^l(\mathbb{R}^n)$, and some $\lambda \in \mathbb{R}$. Here the constants $L_1, L_2 > 0$, the fixed exponent $p > 1$ is associated with the operator A , $\Lambda = \Lambda(\mathbb{R}^n) = \bigoplus_{k=0}^n \Lambda^k(\mathbb{R}^n)$ is a graded algebra with respect to the exterior products, and $\Lambda^k = \Lambda^k(\mathbb{R}^n)$ denotes the set of all k -forms

$$u(x) = \sum_{1 \leq i_1 < \dots < i_k \leq n} u_{i_1 \dots i_k}(x) dx_{i_1} \wedge \dots \wedge dx_{i_k}, \quad x \in \mathbb{R}^n.$$

A k -form $u(x)$ is said to be differentiable if its coefficients $u_{i_1 \dots i_k}$ are differentiable functions in \mathbb{R}^n . Moreover, we say that a differential form $u \in W_{loc}^{1,p}(\Omega, \Lambda)$ is a weak Dirac-harmonic tensor if it is a weak solution to equation (1.1), that is, it satisfies

$$\int_{\Omega} \langle A(x, Du), d\phi \rangle = 0 \tag{1.2}$$

for every $\phi \in W_0^{1,p}(\Omega, \Lambda^{l-1})$ such that $\int_{\Omega} \phi \, dx = 1$.

It is worth mentioning that Dirac-harmonic equation (1.1), proposed by Ding and Liu in [1], is a classical counterpart of the A -harmonic equation via differential forms; in some sense, it can be viewed as a particular case of equation (1.1). For example, if $d^*u = 0$, then due to the fact that $D(u) = du + d^*u$, equation (1.1) can be rewritten as

$$d^*A(x, du) = 0, \tag{1.3}$$

which is called the A -harmonic equation. In particular, if $A(x, \xi) = \xi |\xi|^{p-2}$ with $p > 1$, then the A -harmonic equation becomes the p -harmonic equation for differential forms

$$d^*(du|du|^{p-2}) = 0. \tag{1.4}$$

The research on A -harmonic equations for differential forms has a long history. Kodaira [2] in 1949 presented the original homogenous A -harmonic equation for differential forms, where the operator A is defined by $A(x, \xi) = \xi$. Based on equation (1.4), Sibner [3] gave a systematic study of the p -harmonic tensor for $p > 1$ and established the nonlinear Hodge–De Rham theorems. Afterward, many authors paid great attention on A -harmonic equations and showed many powerful results by using different techniques; for instance, see [4–8] and the references therein. Particularly, to explore the properties of weak and very weak solutions to equation (1.3) rigorously, some investigations are mainly devoted to the regularity and higher integrability of weak and very weak solutions to the A -harmonic equations for differential forms. More precisely, Stroffolini [9] introduced the notion of a very weak solution to equation (1.3) with some restriction on the operator A and performed a quantitative analysis of very weak solutions. In spirit of [9], Giannetti [6] established a regularity result for very weak solutions of degenerate p -harmonic equations. Also, Beck and Stroffolini [10] considered the degenerate systems

$$d^*A(\cdot, \omega) = 0 \quad \text{and} \quad d\omega = 0$$

in the weak sense and proved a partial Hölder regularity result in the case of bounded domains. However, until now, there is no literature on very weak solutions to the homogenous Dirac-harmonic equation. This motivated us to study very weak solutions to equation (1.1) for differential forms.

On the other hand, since the L^p integrability of operators involving the function spaces and differential forms has a significant and active role in analysis (see, for instance, [11–13]), the solutions of nonlinear partial differential equations are also extensively applied to the operators for differential forms. This is due to the fact that differential forms are coordinate-system independent. We refer to [14–16] for details. For example, according to the properties of weak solutions to A -harmonic equations, Agarwal et al. [14] and Ding [17] gave a complete investigation on the estimates for the operators in terms of various norms, such as the L^p , Lipschitz, and BMO norms, and compared these norms with the same integral exponent $1 < p < \infty$. In addition, Bi et al. [18] stated the higher embedding inequality of the operator T applied to the differential l -form u satisfying the homogenous A -harmonic equation (1.3) with the operator A satisfying the monotonicity condition

$$|A(x, \xi)| \leq a|\xi|^{p-1} \quad \text{and} \quad \langle A(x, \xi), \xi \rangle \geq |\xi|^p, \tag{1.5}$$

where $a > 0$ is a constant, and $1 < p < \infty$. Note that the higher regularity of the Hodge–Dirac operator D applied to the very weak solution to equation (1.1) has not been previously established. So we present an exhaustive study of very weak and weak solutions to equation (1.1).

Before going further, observing formulation (1.2), we notice that the integral degree of the weak harmonic tensor u is the same as the exponent p appearing in the structural assumption. In contrast with the weak solutions, our concern arises from the question of what is the minimal integral degree of the solution to equation (1.1). To answer this question, in this paper, we introduce a generalized definition of the solution to the homogenous Dirac-harmonic equation (1.1), develop new techniques, and combine them with methods previously given by others to explore the properties of very weak solutions. Precisely, in Sect. 2, we first give the definition of a very weak solution to equation (1.1) and give some basic discussion for a further proof of the main results. Then, using the Hodge decomposition for differential forms and the interpolation theorem, in Sect. 3, we establish the higher integrability of the Hodge–Dirac operator D applied to the very weak solutions; see Theorems 3.2 and 3.3. Finally, as an application of our main results, we present a relation between the very weak solutions and weak solutions to the homogenous Dirac-harmonic equation (1.1).

Throughout this paper, we use the following notation. Let $B = B(x, \rho)$ be the ball in \mathbb{R}^n with radius ρ centered at x , which satisfies $\text{diam}(\sigma B) = \sigma \text{diam}(B)$. For a bounded convex domain Ω , the homotopy operator T is the bounded linear operator in L^p with values in $W^{1,p}(\Omega)$, $1 \leq p \leq \infty$, defined by

$$Tu(x; \xi) = \int_0^1 t^{l-1} \int_{\Omega} \phi(y)u(tx + y - ty; x - y, \xi) dy dt \tag{1.6}$$

for $x \in \Omega$ and vectors $\xi = (\xi_0, \dots, \xi_l)$, $\xi_i \in \mathbb{R}^n, i = 0, \dots, l$, where the function ϕ from $C_0^\infty(\Omega, \Lambda^l)$ is normalized so that $\int_{\Omega} \phi(y) dy = 1$; see [15] and [19] for details about T . The definition of T can be extended to any bounded domain [14]. This definition of T allows

us to construct the new notation

$$u_\Omega = \begin{cases} \frac{1}{|\Omega|} \int_\Omega u(y) dy, & l = 0, \\ u_\Omega = dTu, & l = 1, 2, \dots, n, \end{cases} \tag{1.7}$$

for $u \in L^p(\Omega, \Lambda^l)$, $1 \leq p \leq +\infty$. We denote by f_Ω the integral mean over Ω , that is, for $u \in L^p(\Omega, \Lambda^l)$, we have

$$f_\Omega |u(y)| dy = \frac{1}{|\Omega|} \int_\Omega |u(y)| dy.$$

Moreover, when $1 < p < \infty$, we have the estimate [19]

$$\|u_\Omega\|_{p,\Omega} \leq A_n(p)\mu(\Omega)\|u\|_{p,\Omega}. \tag{1.8}$$

We denote by $\mathbb{D}'(\Omega, \Lambda^k)$ the set of all differentiable k -forms defined in M . We use the symbol d to denote the exterior differential operator from $\mathbb{D}'(\Omega, \Lambda^k)$ to $\mathbb{D}'(\Omega, \Lambda^{k+1})$, and $d^* = (-1)^{nk+1} \star d \star : \mathbb{D}'(\Omega, \Lambda^{k+1}) \rightarrow \mathbb{D}'(\Omega, \Lambda^k)$ is the formal adjoint of d , $0 \leq k \leq n - 1$; see [20] and [21] for more descriptions. We denote by $L^p(\Omega, \Lambda^k)$ the classical L^p -space for differential forms, $1 < p < \infty$, equipped with the norm

$$\|u\|_{p,\Omega} = \left(\int_\Omega |u|^p dx \right)^{\frac{1}{p}} = \left(\int_\Omega \left(\sum_I |u_I|^2 \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}}$$

and by $W^{1,p}(\Omega, \Lambda^k)$ the classical Sobolev space for differential forms, equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = (\text{diam}(\Omega))^{-1} \|u\|_{p,\Omega} + \|\nabla u\|_{p,\Omega}. \tag{1.9}$$

Analogously to the L^p -space, $W^{1,p}(\Omega, \Lambda^k)$ is a Banach space when $1 < p < n$. For appropriate properties, see [22] and [23].

2 Preliminaries

In the preparation for the main results, in this section, we give some useful lemmas and basic discussion. We start with the following definition.

Definition 2.1 Suppose there exists an exponent $s > 1$ with $\max\{1, p - 1\} < s < p$ such that, in the distributional sense, a differential form $u \in W_{\text{loc}}^{1,s}(\Omega, \Lambda)$ satisfies the identity

$$\int_\Omega \langle A(x, Du), d\psi \rangle = 0 \tag{2.1}$$

for every $\psi \in W_0^{1,s/(s-p+1)}(\Omega, \Lambda)$ with $\int_\Omega \psi dx = 1$. Then such a differential form u is called a very weak solution (or a very weak Dirac-harmonic tensor).

For any differential form $u \in \mathbb{D}'(\Omega, \Lambda^l)$, $1 \leq l \leq n$, according to the expression of a differential form, du and d^*u can be written as

$$du(x) = \sum_I \xi_I dx_{i_1} \wedge \dots \wedge dx_{i_{l+1}}$$

and

$$d^*u = \sum_J \eta_J dx_{j_1} \wedge \cdots \wedge dx_{j_{l-1}},$$

where $I = \{1 \leq i_1 < \cdots < i_{l+1} \leq n\}$, $J = \{1 \leq j_1 < \cdots < j_{l-1}\}$, and all coefficients ξ_I, η_J are differentiable functions on Ω . Then by simple calculation we derive that

$$|du| = \left(\sum_I |\xi_I|^2 \right)^{1/2} \leq \left(\sum_I |\xi|^2 + \sum_J |\eta|^2 \right)^{1/2} = |du + d^*u| = |Du|.$$

Similarly, we have that

$$|d^*u| = \left(\sum_J |\eta_J|^2 \right)^{1/2} \leq \left(\sum_I |\xi|^2 + \sum_J |\eta|^2 \right)^{1/2} = |du + d^*u| = |Du|,$$

which implies that

$$\left(\int_{\Omega} |du|^p dx \right)^{1/p} \leq \left(\int_{\Omega} |Du|^p dx \right)^{1/p} \tag{2.2}$$

and

$$\left(\int_{\Omega} |d^*u|^p dx \right)^{1/p} \leq \left(\int_{\Omega} |Du|^p dx \right)^{1/p}. \tag{2.3}$$

Also, from Corollaries 3 and 4 in [9] it follows that for any $u \in L^s(\Omega, \Lambda^l)$, $s > 1$, if $du \in L^s(\Omega, \Lambda^l)$, then there is a constant $C > 0$ such that

$$\frac{1}{\text{diam}(\Omega)} \left(\int_{\Omega} |u - u_{\Omega}|^s dx \right)^{1/s} \leq C(n, s) \left(\int_{\Omega} |du|^{ns/(n+s-1)} dx \right)^{(n+s-1)/ns}, \tag{2.4}$$

where u_{Ω} is a closed form; if $d^*u \in L^s(\Omega, \Lambda^l)$, then there is a constant $C > 0$ such that

$$\frac{1}{\text{diam}(\Omega)} \left(\int_{\Omega} |u - u_{\Omega}^*|^s dx \right)^{1/s} \leq C(n, s) \left(\int_{\Omega} |d^*u|^{ns/(n+s-1)} dx \right)^{(n+s-1)/ns}, \tag{2.5}$$

where u_{Ω}^* is a coclosed form.

Consequently, by (2.2) and (2.4) we derive the following result.

Lemma 2.2 *Let $u \in W^{1,s}(\Omega, \Lambda^l)$, $l = 1, 2, \dots, s > 1$, where $\Omega \subset \mathbb{R}^n$ is a cube or a ball. Then there exists a constant $C(n, s) > 0$, independent of u and Du , such that*

$$\frac{1}{\text{diam}(\Omega)} \left(\int_{\Omega} |u - u_{\Omega}|^s dx \right)^{1/s} \leq C(n, s) \left(\int_{\Omega} |Du|^{ns/(n+s-1)} dx \right)^{(n+s-1)/ns}, \tag{2.6}$$

where u_{Ω} is a closed form of u .

Similarly, combining (2.3) with (2.5), we get the following higher-order Poincaré-type inequality.

Lemma 2.3 *Let Ω be a cube or a ball, and let $u \in W^{1,s}(\Omega, \Lambda^l)$. Then there exists a constant $C(n, s) > 0$, independent of u and Du , such that*

$$\frac{1}{\text{diam}(\Omega)} \left(\int_{\Omega} |u - u_{\Omega}^*|^s dx \right)^{1/s} \leq C(n, s) \left(\int_{\Omega} |Du|^{ns/(n+s-1)} dx \right)^{(n+s-1)/ns}, \tag{2.7}$$

where u_{Ω}^* is a coclosed form.

In particular, if the closed form u_{Ω} in (2.6) and coclosed form u_{Ω}^* in (2.7) are both harmonic forms, denoted by u_{Ω}^b , then we immediately establish the following result.

Lemma 2.4 *Let $u \in W^{1,s}(\Omega, \Lambda^l)$, $l = 1, 2, \dots, s > 1$, where $\Omega \subset \mathbb{R}^n$ is a cube or a ball. Then, there exists a constant $C(n, s) > 0$, independent of u and Du , such that*

$$\frac{1}{\text{diam}(\Omega)} \left(\int_{\Omega} |u - u_{\Omega}^b|^s dx \right)^{1/s} \leq C(n, s) \left(\int_{\Omega} |Du|^{ns/(n+s-1)} dx \right)^{(n+s-1)/ns}, \tag{2.8}$$

where u_{Ω}^b is harmonic form of u .

Morrey [23] extended the Hodge decomposition into Sobolev space $W^{1,s}(\Omega, \Lambda)$, $1 < s < n$, where $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain. Namely, given the differential form $u \in L^s(\Omega, \Lambda)$, there exist $\alpha \in W_T^{1,s}(\Omega, \Lambda^{l-1})$, $\beta \in W_N^{1,s}(\Omega, \Lambda^{l+1})$, and $h \in \mathbb{H}_s^l$ such that

$$u = d\alpha + d^* \beta + h. \tag{2.9}$$

Then, for the differential forms α, β , and h , we derive the following bounded estimate in terms of the norm of u :

$$\|d\alpha\|_{s,\Omega} + \|d^* \beta\|_{s,\Omega} + \|h\|_{s,\Omega} \leq \|u\|_{s,\Omega}. \tag{2.10}$$

In addition, to facilitate the upcoming theorem, we need some lemmas.

Lemma 2.5 ([24]) *Let (X, μ) be a measure space, and let E be a separable complex Hilbert space. For all $r \in [r_1, r_2]$, if $T : L^r(X, E) \rightarrow L^r(X, E)$ is a linear bounded operator and $r/r_2 \leq 1 + \varepsilon \leq r/r_1$, $1 \leq r_1 \leq r_2 \leq \infty$, then we have*

$$\|T(|f|^\varepsilon f)\|_{r/(1+\varepsilon)} \leq K|\varepsilon| \|f\|_r^{1+\varepsilon}$$

for all $f \in L^r(X, E) \cap \text{Ker } T$, where

$$K = \frac{2r(r_2 - r_1)}{(r - r_1)(r_2 - r)} (\|T\|_{r_1} + \|T\|_{r_2}). \tag{2.11}$$

Lemma 2.6 ([1]) *Let $u = \sum_I u_I dx_I \in \mathbb{D}'(\Omega, \Lambda)$ be a l -form, and let η be a differentiable function in Ω . Then*

$$D(u\eta) = (Du)\eta + (-1)^l u(d\eta) + (-1)^m \sum_{I-k} u_I \frac{\partial \eta}{\partial x_k} dx_{I-k}, \tag{2.12}$$

where m and k are integers, $1 \leq k \leq n$, and $I - k$ is an abusive notation to represent an $(l - 1)$ -tuple with \hat{i}_k missing in $(i_1, \dots, \hat{i}_k, \dots, i_k)$ and $k \in J$ meaning that $k \neq j_s$ for any j_s in an $(n - l)$ -tuple J . Also, \sum_I means the sum of all possible l -tuples.

Lemma 2.7 *Let s, r, σ , and C be positive numbers such that $0 < r < s < \infty$ and $\sigma > 1$. If*

$$\left(\frac{1}{|B|} \int_B |f|^s dx\right)^{1/s} \leq C \left(\frac{1}{|B|} \int_{\sigma B} |f|^r dx\right)^{1/r} \tag{2.13}$$

for any ball B with $\sigma B \subset \Omega$, then there exists $\varepsilon > 0$ such that

$$\left(\frac{1}{|B|} \int_B |f|^q dx\right)^{\frac{1}{(s+\varepsilon)}} \leq C \left(\frac{1}{|B|} \int_B |f|^s dx\right)^{\frac{1}{s}} \tag{2.14}$$

for all $q \in [s, s + \varepsilon]$ and all balls B with $\sigma B \subset \Omega$.

It should be pointed out that Lemma 2.7 from [14] is the modified Gehring lemma.

3 Main results and proofs

Now, given a differential form $g \in L^s(\Omega, \Lambda)$ and $f \in L^{s/(p-1)}(\Omega, \Lambda)$, we consider the nonhomogenous Dirac-harmonic equation of the form

$$d^*A(x, g + Du) = d^*f(u) \tag{3.1}$$

for $u \in W_0^{1,s}(\Omega, \Lambda)$. We say that $u \in W_0^{1,s}(\Omega, \Lambda)$ is a very weak solution to the nonhomogeneous Dirac-harmonic equation (3.1) if

$$\int_{\Omega} \langle A(x, g + Du), d\psi \rangle = \int_{\Omega} \langle f(u), d\psi \rangle \tag{3.2}$$

for all $\psi \in W_0^{1,s/(s-p+1)}(\Omega, \Lambda)$ with $\int_{\Omega} \psi dx = 1$, where $s \geq \max\{1, p - 1\}$.

Theorem 3.1 *Suppose that $u \in W_0^{1,s}(\Omega, \Lambda)$ is a very weak solution to the nonhomogeneous Dirac-harmonic equation (3.1). Then there exists $\varepsilon = \varepsilon(n, p, L_1, L_2) \in (0, p - 1)$ such that*

$$\int_{\Omega} |Du|^s dx \leq C \int_{\Omega} (|g|^s + |f|^{s/(p-1)}) dx \tag{3.3}$$

for $p - \varepsilon \leq s \leq p + \varepsilon$.

Proof First, for the differential form $u \in W_0^{1,s}(\Omega, \Lambda)$ in equation (3.1), we take a nonlinear perturbation of Du , that is, $\omega = |Du|^{s-p}Du$. Since $\omega \in L^{\frac{s}{s-p+1}}(\Omega, \Lambda)$, by the Hodge decomposition we obtain that

$$|Du|^{s-p}Du = d\phi + d^*\beta + h. \tag{3.4}$$

In the meantime, we define the linear operator

$$T : L^r(\Omega, \Lambda) \rightarrow L^r(\Omega, \Lambda)$$

such that $T(v) = h$ for every $v \in L^r(\Omega, \Lambda)$. Then, according to (2.10), we easily see that T is a bounded linear operator for every $1 < r < \infty$. Moreover, the element in the kernel of the operator is of the class $dW^{1,s}(\Omega, \Lambda) \oplus d^*W^{1,s}(\Omega, \Lambda)$. Thus letting $f = Du$ and $\varepsilon = s - p$ in Lemma 2.5 yields that

$$\begin{aligned} \|h\|_{\frac{s}{s-p+1}} &= \|T(|Du|^{s-p}Du)\|_{\frac{s}{s-p+1}} \\ &\leq K|s - p|\|Du\|_s^{s-p+1}, \end{aligned} \tag{3.5}$$

where the constant K is determined by (2.11). By the definition of a very weak solution to equation (3.1), choosing the test function ψ with

$$d\psi = |Du|^{s-p}Du - d^*\beta - h$$

infers that

$$\int_{\Omega} \langle A(x, g + Du), d\psi \rangle = \int_{\Omega} \langle f, d\psi \rangle. \tag{3.6}$$

On the other side, note that

$$\begin{aligned} &\int_{\Omega} \langle A(x, Du), |Du|^{s-p}Du \rangle \\ &= \int_{\Omega} \langle A(x, Du) - A(x, g + Du), |Du|^{s-p}Du \rangle \\ &\quad + \int_{\Omega} \langle A(x, g + Du), d^*\beta \rangle + \int_{\Omega} \langle A(x, g + Du), h \rangle + \int_{\Omega} \langle f, d\psi \rangle. \end{aligned} \tag{3.7}$$

Thus by the homogeneity of the operator A we have

$$\begin{aligned} L_2 \int_{\Omega} |Du|^s dx &\leq \int_{\Omega} \langle A(x, Du), |Du|^{s-p}Du \rangle \\ &= \int_{\Omega} \langle A(x, Du) - A(x, g + Du), |Du|^{s-p}Du \rangle \\ &\quad + \int_{\Omega} \langle A(x, g + Du), d^*\beta \rangle \\ &\quad + \int_{\Omega} \langle A(x, g + Du), h \rangle + \int_{\Omega} \langle f, d\psi \rangle \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.8}$$

Now we divide our work into four parts. For the term I_1 , by the Lipschitz continuity of the operator A we get

$$\begin{aligned} I_1 &= \left| \int_{\Omega} \langle A(x, Du) - A(x, g + Du), |Du|^{s-p}Du \rangle \right| \\ &\leq \int_{\Omega} |A(x, Du) - A(x, g + Du)| \cdot ||Du|^{s-p}Du| dx \\ &\leq \int_{\Omega} L_1 |g| (|Du| + |g + Du|)^{p-2} |Du|^{s-p+1} dx \end{aligned}$$

$$\begin{aligned} &\leq L_1 \int_{\Omega} |g| (|Du| + |g + Du|)^{p-2} (|Du| + |g + Du|)^{s-p+1} dx \\ &\leq L_1 \int_{\Omega} |g| (|Du| + |g + Du|)^{s-1} dx. \end{aligned}$$

Since $1/s + (s - 1)/s = 1$, by the Hölder inequality it follows that

$$I_1 \leq C_1 \|g\|_{s,\Omega} \| |Du| + |g + Du| \|_{s,\Omega}^{s-1} \tag{3.9}$$

Then, repeating this argument for I_2, I_3 , and I_4 , we obtain that

$$\begin{aligned} I_2 &\leq \int_{\Omega} |A(x, g + Du)| \cdot |d^* \beta| dx \\ &\leq L_1 \int_{\Omega} |g + Du|^{p-1} |d^* \beta| dx \\ &\leq C_2 \|g + Du\|_{s,\Omega}^{p-1} \|d^* \beta\|_{s/(s-p+1),\Omega}, \end{aligned} \tag{3.10}$$

$$\begin{aligned} I_3 &\leq \int_{\Omega} |A(x, g + Du)| \cdot |h| dx \\ &\leq L_1 \int_{\Omega} |g + Du|^{p-1} |h| dx \\ &\leq C_3 \|g + Du\|_{s,\Omega}^{p-1} \|h\|_{s/(s-p+1),\Omega}, \end{aligned} \tag{3.11}$$

and

$$I_4 \leq \int_{\Omega} |f| |d\psi| dx \leq C_3 \|f\|_{\frac{s}{p-1},\Omega} \|d\psi\|_{s/(s-p+1),\Omega}. \tag{3.12}$$

Then by substituting inequalities (3.9), (3.10), (3.11), and (3.12) into (3.8) it follows that

$$\begin{aligned} L_2 \int_{\Omega} |Du|^s dx &\leq C_1 \|g\|_{s,\Omega} \| |Du| + |g + Du| \|_{s,\Omega}^{s-1} \\ &\quad + C_2 \|g + Du\|_{s,\Omega}^{p-1} \|h\|_{s/(s-p+1),\Omega} \\ &\quad + C_3 \|g + Du\|_{s,\Omega}^{p-1} \|d^* \beta\|_{s/(s-p+1),\Omega} \\ &\quad + C_4 \|f\|_{s/(p-1),\Omega} \|d\psi\|_{s/(s-p+1),\Omega}. \end{aligned} \tag{3.13}$$

Observe that $|Du|^{s-p} Du \in L^{\frac{s}{s-p+1}}(\Omega, \Lambda)$, so applying inequality (2.10) gives that

$$\begin{aligned} \|d^* \beta\|_{s/(s-p+1),\Omega} &\leq C_5 \|Du\|_{s,\Omega}^{s-p+1}, \\ \|d\psi\|_{s/(s-p+1),\Omega} &\leq C_6 \|Du\|_{s,\Omega}^{s-p+1}. \end{aligned} \tag{3.14}$$

Plugging formula (3.14) and (3.5) into (3.11), by the Minkowski inequality we have that

$$\begin{aligned} L_2 \int_{\Omega} |Du|^s dx &\leq C_1 \|g\|_s (\|Du\|_s + \|g + Du\|_s)^{s-1} \\ &\quad + C_7 K |s - p| (\|g\|_s + \|Du\|_s)^{p-1} \|Du\|_s^{s-p+1} \end{aligned}$$

$$+ C_8 \|g\|_s^{p-1} \|Du\|_s^{s-p+1} + C_9 \|f\|_{s/(p-1)} \|Du\|_s^{s-p+1}. \tag{3.15}$$

Applying the inequality $(k_1 + k_2)^n \leq 2^n(k_1^n + k_2^n)$ for positive numbers $k_1, k_2 \in \mathbb{R}$ and integer $n \geq 0$ to (3.15), we get that

$$\begin{aligned} \int_{\Omega} |Du|^s dx &\leq C_{10} (\|g\|_s^s + \|g\|_s \|Du\|_s^{s-1}) \\ &\quad + C_{11} K |s - p| (\|g\|_s^{p-1} + \|Du\|_s^{p-1}) \|Du\|_s^{s-p+1} \\ &\quad + C_{12} \|g\|_s^{p-1} \|Du\|_s^{s-p+1} + C_9 \|f\|_{s/(p-1)} \|Du\|_s^{s-p+1}. \end{aligned} \tag{3.16}$$

By a simple integration we get that

$$\begin{aligned} \int_{\Omega} |Du|^s dx &\leq C_{10} \|g\|_s^s + C_{10} \|g\|_s \|Du\|_s^{s-1} + (C_{12} \\ &\quad + C_{11} K |s - p|) \|g\|_s^{p-1} \|Du\|_s^{s-p+1} \\ &\quad + C_{11} K |s - p| \|Du\|_s^s + C_9 \|f\|_{\frac{s}{p-1}} \|Du\|_s^{s-p+1}. \end{aligned} \tag{3.17}$$

Then by Lemma 2.5 there exists $\varepsilon > 0$ small enough such that

$$C_{11} K |s - p| < \frac{1}{2}$$

for all $p - \varepsilon < s < p + \varepsilon$. Recall the δ -Young's inequality: for any $1 < p < q < \infty$,

$$ab \leq \delta a^p + c \delta^{-\frac{q}{p}} b^q,$$

where $c = \frac{1}{p-1/pq}$ is a constant, and $\delta > 0$ is an arbitrary number. Applying it and the interpolation inequalities, we get that

$$\begin{aligned} \int_{\Omega} |Du|^s dx &\leq C_{10} \|g\|_s^s + C_{10} \tau \|g\|_s^s + C_{10} \tau \|Du\|_s^s + C_{13} \zeta \|g\|_s^s \\ &\quad + C_{13} \zeta \|Du\|_s^s + C_9 \gamma \|f\|_{s/(p-1)}^{s/p-1} + C_9 \gamma \|Du\|_s^s, \end{aligned} \tag{3.18}$$

where γ, η, τ are positive numbers associated with δ and ε . So we see that it is easy to guarantee that

$$C_9 \gamma + C_{10} \tau + C_{13} \zeta \leq \frac{1}{2}.$$

Therefore putting the terms involving $\|Du\|_s^s$ into the left side, by a simple calculation we obtain the desired result. □

Theorem 3.2 *Let ε be as in Theorem 3.1. If $u \in W_{loc}^{1,s}(\Omega, \Lambda)$ is a very weak solution to the homogenous Dirac-harmonic equation (1.1), $s \in (p - \varepsilon, p)$, then there exists a constant $C > 0$, independent of u , such that*

$$\left(\int_B |Du|^s dx \right)^{1/s} \leq C(n, p) \left(\int_{\sigma B} |Du|^r dx \right)^{1/r} \tag{3.19}$$

for any ball $B \subset \sigma B \subset \mathbb{R}^n$ with $\sigma > 1$, where $r < s$, and

$$r = \max \left\{ \frac{ns}{n+s-1}, \frac{ns(p-1)}{np-n+s-p+1} \right\}.$$

Proof Observing that u is a very weak solution, we have that

$$\int_{\Omega} \langle A(x, Du), d\phi \rangle = 0 \tag{3.20}$$

for all $\phi \in W^{1,s/(s-p+1)}(\Omega, \Lambda)$ with compact support. By the homogeneity of the operator A , replacing $\lambda\xi$ with $\eta^{p/(p-1)}Du$, it follows that

$$d^*A(x, \eta^{p/(p-1)}Du) = d^*f,$$

where $f = \eta^p A(x, Du)$, and $\eta \in C_0^\infty(\sigma B)$ is a nonnegative cutoff function such that $\eta = 1$ in $B \subset \sigma B \subset \Omega$ with $|\nabla \eta| < \frac{c}{\text{diam}(B)}$.

Moreover, let $v = \eta^q(u - u_{\sigma B}^b)$, where $q = p/(p-1)$, and $u_{\sigma B}^b$ is a harmonic form of u . In view of Lemma 2.6, we easily derive that

$$D(v) = \eta^q(Du) + (-1)^l(u - u_{\sigma B}^b)d(\eta^q) + (-1)^m \sum_{I-k} (u - u_{\sigma B}^b)_I \frac{\partial(\eta^q)}{\partial x_k} dx_{I-k}$$

or, shortly,

$$\eta^q Du = Dv + g, \tag{3.21}$$

where

$$g = (-1)^l(u - u_{\sigma B}^b)d(\eta^q) + (-1)^m \sum_{I-k} (u - u_{\sigma B}^b)_I \frac{\partial(\eta^q)}{\partial x_k} dx_{I-k}.$$

Then we obtain a nonhomogeneous Dirac-harmonic equation of the form

$$d^*A(x, g + Dv) = d^*f.$$

Applying Theorem 3.1 to the domain $\Omega = \sigma B$ yields

$$\int_{\sigma B} |Dv|^s dx \leq C_1 \left(\int_{\sigma B} |g|^s dx + |f|^{s/(p-1)} dx \right), \tag{3.22}$$

which, taking the integral means of the both sides, can be simplified to

$$\left(\int_{\sigma B} |Dv|^s dx \right)^{\frac{1}{s}} \leq C_3 \left(\int_{\sigma B} |g|^s dx + |f|^{s/(p-1)} dx \right)^{\frac{1}{s}}. \tag{3.23}$$

By the inequality $(a + b)^s \leq 2^s(a^s + b^s)$ this results in

$$\left(\int_{\sigma B} |Dv|^s dx \right)^{\frac{1}{s}} \leq C_3 \left(\int_{\sigma B} |g|^s dx \right)^{\frac{1}{s}} + C_3 \left(\int_{\sigma B} |f|^{s/(p-1)} dx \right)^{\frac{1}{s}}. \tag{3.24}$$

Next, we focus on the estimates for the right terms of equality (3.24).

First, to estimate the term $\int_{\sigma_B} |g|^s dx$, since the bounded function η satisfies $|d\eta| \leq C_4|\nabla\eta|$, we get that

$$\begin{aligned} & \left(\int_{\sigma_B} |g|^s dx \right)^{\frac{1}{s}} \\ &= \left(\int_{\sigma_B} \left| (-1)^l (u - u_{\sigma_B}^b) d(\eta^q) + (-1)^m \sum_{I-k} (u - u_{\sigma_B}^b)_I \frac{\partial(\eta^q)}{\partial x_k} \right|^s dx \right)^{\frac{1}{s}} \\ &= \left(\int_{\sigma_B} \left| (-1)^l q(u - u_{\sigma_B}^b) \eta^{q-1} d(\eta) + (-1)^m \sum_{I-k} q(u - u_{\sigma_B}^b)_I \eta^{q-1} \frac{\partial(\eta)}{\partial x_k} \right|^s dx \right)^{\frac{1}{s}} \\ &\leq C_5 \left(\int_{\sigma_B} |u - u_{\sigma_B}^b|^s |\nabla\eta|^s dx + \int_{\sigma_B} |u - u_{\sigma_B}^b|^s |\nabla\eta|^s dx \right)^{\frac{1}{s}} \\ &\leq \frac{C_6}{\text{diam}(B)} \left(\int_{\sigma_B} |u - u_{\sigma_B}^b|^s dx \right)^{\frac{1}{s}}. \end{aligned} \tag{3.25}$$

By Lemma 2.4 this gives that

$$\left(\int_{\sigma_B} |g|^s dx \right)^{\frac{1}{s}} \leq C_7 \left(\int_{\sigma_B} |Du|^{ns/(n+s-1)} dx \right)^{(n+s-1)/n}. \tag{3.26}$$

On the other side, to estimate the second term on the right side of inequality (3.24), by Lemma 2.3 we get that

$$\left(\int_{\sigma_B} |f|^{s/(p-1)} dx \right)^{\frac{p-1}{s}} \leq C_8 \text{diam}(\sigma_B) \left(\int_{\sigma_B} |d^*f|^{ns/(np-n+s-p+1)} dx \right)^{\frac{np-n+s-p+1}{ns}}. \tag{3.27}$$

Since $d^*f = d^*A(x, \eta^q Du)$, we derive that

$$\begin{aligned} d^*f &= d^*(\eta^p A(x, Du)) \\ &= \eta^p d^*A(x, Du) + (-1)^{n(l-1)} \star (d\eta^p \wedge \star A(x, Du)). \end{aligned}$$

Since u is a very weak solution to the Dirac-harmonic equation (1.1), that is, $d^*A(x, Du) = 0$, by the Lipschitz continuity we get that

$$|d^*f| = |d\eta^p \wedge \star A(x, Du)| \leq b |\nabla(\eta^p)| |Du|^{p-1}. \tag{3.28}$$

Substituting (3.28) into (3.27) gives

$$\left(\int_{\sigma_B} |f|^{s/(p-1)} dx \right)^{\frac{p-1}{s}} \leq C_9 \left(\int_{\sigma_B} |Du|^{\frac{n(p-1)s}{np-n+s-p+1}} dx \right)^{\frac{np-n+s-p+1}{ns}}. \tag{3.29}$$

Then, substituting (3.29) and (3.26) into (3.23), we have that

$$\left(\int_{\sigma_B} |Dv|^s dx \right)^{\frac{1}{s}} \leq C_7 \left(\int_{\sigma_B} |Du|^{\frac{ns}{n+s-1}} dx \right)^{\frac{n+s-1}{ns}}$$

$$+ C_9 \left(\int_{\sigma B} |Du|^{\frac{n(p-1)s}{np-n+s-p+1}} dx \right)^{\frac{np-n+s-p+1}{ns(p-1)}}. \tag{3.30}$$

Noting that $dv = du$ on B and $\mu(\sigma B) = \sigma^n \mu(B)$, we obtain that

$$\begin{aligned} \left(\int_B |Du|^s dx \right)^{\frac{1}{s}} &\leq \frac{\mu(\sigma B)^s}{\mu(B)^s} \left(\int_{\sigma B} |Du|^s dx \right)^{\frac{1}{s}} \\ &\leq \sigma^{n+s} \left(\int_{\sigma B} |Du|^s dx \right)^{\frac{1}{s}}. \end{aligned} \tag{3.31}$$

Choosing $r = \max\left\{\frac{ns}{n+s-1}, \frac{n(p-1)s}{np-n+s-p+1}\right\}$ and substituting (3.31) into (3.30) give

$$\left(\int_B |Du|^s dx \right)^{\frac{1}{s}} \leq C_{10} \left(\int_{\sigma B} |Du|^r dx \right)^{\frac{1}{r}},$$

where $\sigma > 1$ is some expansion factor. So we have the desired result. □

Note that inequality (3.19) is the classical reverse Hölder inequality since the exponent s in the left side is larger than the exponent r in the right one. In fact, due to this nice result, it provides us a powerful technique for the latter discussion on the locally higher integrability of Du .

Theorem 3.3 *Suppose that $u \in W^{1,s}(\Omega, \Lambda)$ is a very weak solution to the homogeneous Dirac-harmonic equation (1.1) and Ω is a regular bounded domain, where $s \in (p - \varepsilon, p)$ and $\varepsilon = \varepsilon(n, p, L_1, L_2)$, Then, for any real number $t \in (1, \infty)$, there exists a constant $C > 0$, independent of u and Du , such that*

$$\left(\int_B |Du|^t dx \right)^{1/t} \leq C \left(\int_{\sigma B} |Du|^s dx \right)^{1/s}, \tag{3.32}$$

where $B \subset \sigma B \subset \Omega$ is any ball with $\sigma > 1$ and $1 < p < \infty$.

Proof Initially, to estimate inequality (3.32), we consider two cases. In the case $1 < t < s$, by the monotonic property of the L^p -space it is obvious that

$$\left(\int_B |Du|^t dx \right)^{1/t} \leq C_1 \left(\int_{\sigma B} |Du|^s dx \right)^{1/s} \tag{3.33}$$

for all balls $B \subset \sigma B \subset \Omega$ with $\sigma > 1$.

Now let us turn to the case $s \leq t \leq \infty$. First, by Theorem 3.2 we can easily find $C > 0$ such that

$$\left(\int_B |Du|^s dx \right)^{1/s} \leq C_2(n, p) \left(\int_{\sigma B} |Du|^r dx \right)^{1/r} \tag{3.34}$$

for any $r < s$ and

$$r = \max \left\{ \frac{ns}{n+s-1}, \frac{ns(p-1)}{np-n+s-p+1} \right\}.$$

Applying Lemma 2.7 with (3.34), we get that

$$\left(\int_B |Du|^{s+\delta}\right)^{1/(s+\delta)} \leq C_3(n, p) \left(\int_{\sigma B} |Du|^s dx\right)^{1/s} \tag{3.35}$$

for all $\delta > 0$. Then repeating this process k times, from (3.35) we have that

$$\left(\int_B |Du|^{s+k\delta} dx\right)^{1/(s+k\delta)} \leq C_4(n, p) \left(\int_{\sigma B} |Du|^s dx\right)^{1/s}, \tag{3.36}$$

for all integer $k = 0, 1, \dots$ and $\delta > 0$. Let $t = s + k\delta$, Since δ is arbitrary, for all $s \leq t < \infty$, inequality (3.36) can be rewritten as

$$\left(\int_B |Du|^t dx\right)^{1/t} \leq C_4(n, p) \left(\int_{\sigma B} |Du|^s dx\right)^{1/s}. \tag{3.37}$$

Therefore, combining (3.33) and (3.37), we have that the desired result (3.32) holds for all $1 < t < \infty$ and $s \in (p - \varepsilon, p)$.

We point out that for any very weak tensor $u \in W^{1,s}(\Omega, \Lambda)$, if s is closed enough to the natural exponent p , then Theorem 3.3 gives us the best possible integrability in terms of the norm of Du . Moreover, recalling the Poincaré inequality, we have that

$$\|u\|_{p,\Omega} \leq C(n, p) \|du\|_{p,\Omega} \tag{3.38}$$

for all $1 < p < \infty$ and $u \in W^{1,p}(\Omega, \Lambda)$. Applying (2.2) into (3.38), it follows that

$$\|u\|_{p,\Omega} \leq C(n, p) \|Du\|_{p,\Omega}. \tag{3.39}$$

In particular, if $u \in W^{1,s}(\Omega, \Lambda)$ is defined as in Theorem 3.3, then combining (3.32) with (3.39), we easily obtain the following result. □

Corollary 3.4 *Let $1 < p < \infty$, and let Ω be a regular bounded domain. If $u \in W^{1,s}(\Omega, \Lambda)$ is a very weak tensor for $s \in (p - \varepsilon, p)$, where ε is given in Theorem 3.1, then $u \in W^{1,t}(\Omega, \Lambda)$ for any $1 < t < \infty$. In particular, when $t = p$, we have that $u \in W^{1,p}$ is also a weak tensor.*

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Declarations

Competing interests

The authors declare no competing interests.

Authors' contributions

GS and YZ contributed their efforts jointly to this manuscript. In precise, GS proposed the idea of this manuscript, finished the proofs of the main results in Sect. 3 and drafted the manuscript. YZ collected all lemmas in Sect. 2 and other supporting materials, including the references and definitions, and improved the final version. All authors read and approved the final manuscript.

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