Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On a cubic–quadratic equation relative to elliptic curves

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Abstract

We investigate the Hyers–Ulam stability of the following cubic–quadratic functional equation relative to elliptic curves
$f(x+y+z,u+v+w)+f(x+y-z,u+v+w)+2f(x,u-w)+2f(y,v-w)=f(x+y,u+w)+f(x+y,v+w)+f(x+z,u+w)+f(x-z,u+v-w)+f(y+z,v+w)+f(y-z,u+v-w)$.
The function

$$f(x,y)=x^3+ax+b-y^2$$

having level curves as elliptic curves is a solution of the above functional equation.

Keywords: 2-Banach space; Elliptic curve; Hyers–Ulam stability

1 Introduction

A graph of an equation of the form $y^2 = x^3 + ax + b$ is called an *elliptic curve* [13], where a and b are constants. Since $f(x,y) = x^3 + ax + b - y^2$ has level curves as elliptic curves, functional equations having the mapping $f(x,y) = x^3 + ax + b - y^2$ as a solution are helpful to study cryptography and its applications.

We need the following definitions on 2-Banach spaces [4, 5].

Definition 1 Let $\mathcal{X}$ be a real linear space with $\dim \mathcal{X} \geq 2$ and $\|\cdot,\cdot\| : \mathcal{X}^2 \to \mathbb{R}$ be a function. Then, we say $(\mathcal{X}, \|\cdot,\cdot\|)$ is a *linear* 2*-normed space* if

(a) $\|x,y\| = 0$ if and only if x and y are linearly dependent,

(b) $\|x,y\| = \|y,x\|$,

(c) $\|\alpha x,y\| = |\alpha|\,\|x,y\|$,

(d) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$

for all $\alpha \in \mathbb{R}$ and $x,y,z \in \mathcal{X}$. In this case, the function $\|\cdot,\cdot\|$ is called a 2*-norm* on $\mathcal{X}$.

Definition 2 Let $\{x_n\}$ be a sequence in a linear 2-normed space $\mathcal{X}$. The sequence $\{x_n\}$ is said to *convergent* in $\mathcal{X}$ if there exists an element $x \in \mathcal{X}$ such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for all $y \in \mathcal{X}$. In this case, we say that a sequence $\{x_n\}$ converges to the limit x, simply, denoted by $\lim_{n\to\infty} x_n = x$.

Definition 3 A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *Cauchy sequence* if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|x_m - x_n, y\| < \varepsilon$ for all $y \in \mathcal{X}$. For convenience, we will write $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ for a Cauchy sequence $\{x_n\}$. A 2-*Banach space* is defined to be a linear 2-normed space in which every Cauchy sequence is convergent.

In the following lemma, we obtain some basic properties in a linear 2-normed space.

Lemma 1 ([9]) *Let* $(\mathcal{X}, \|\cdot,\cdot\|)$ *be a linear* 2*-normed space and* $x \in \mathcal{X}$.

(a) *If* $\|x, y\| = 0$ *for all* $y \in \mathcal{X}$, *then* $x = 0$.
(b) $|\|x, z\| - \|y, z\|| \leq \|x - y, z\|$ *for all* $x, y, z \in \mathcal{X}$.
(c) *If a sequence* $\{x_n\}$ *is convergent in* $\mathcal{X}$, *then* $\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$ *for all* $y \in \mathcal{X}$.

The stability of a functional equation means, roughly speaking, that an approximate solution of the equation (i.e., a mapping that only approximately satisfies the equation) is not far from an exact solution of the equation. It may happen that all approximate solutions are in fact exact solutions. Considering the Cauchy equation $f(x + y) = f(x) + f(y)$ one may deal with the class of its approximate solution defined by the functional inequality introduced by Rassias [11].

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon\left(\|x\|^p + \|y\|^p\right).$$

It turns out that for $p \neq 1$ each solution of the above inequality can be approximated by an additive function A in such a way that the inequality

$$\|f(x) - A(x)\| \leq k\varepsilon\|x\|^p$$

holds, with a suitable real number k, on the whole domain (for $p = 0$ it coincides with the classical Hyers–Ulam result [6, 12]).

A graph of an equation of the form $y^2 = x^3 + ax + b$ is called an *elliptic curve* [13], where a and b are constants. Since $f(x, y) = x^3 + ax + b - y^2$ has level curves as elliptic curves, functional equations having the mapping $f(x, y) = x^3 + ax + b - y^2$ as a solution are helpful to study cryptography and its applications. The stability of functional equations has been studied by some authors [1–3, 7, 8].

Jun and Kim [7] introduced the cubic functional equation

$$\begin{aligned} &g(x+y+z) + g(x+y-z) + 2g(x) + 2g(y) \\ &\quad = 2g(x+y) + g(x+z) + g(x-z) + g(y+z) + g(y-z). \end{aligned} \tag{1}$$

In 2008, Park and Bae [10] introduced the following functional equations and investigated their stability problems in Banach spaces:

$$g(x+y+z) + g(x-z) + g(y-z) = g(x+y-z) + g(x+z) + g(y+z) \tag{2}$$

and

$$\begin{aligned}
&f(x+y+z,u+v+w)+f(x+y-z,u+v+w)\\
&\qquad+2f(x,u-w)+2f(y,v-w)\\
&\quad=f(x+y,u+w)+f(x+y,v+w)+f(x+z,u+w)\\
&\qquad+f(x-z,u+v-w)+f(y+z,v+w)+f(y-z,u+v-w). \qquad (3)
\end{aligned}$$

The function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by $f(x,y):=x^3+ax+b-y^2$ is a particular solution of (3). In 2011, Park [9] investigated approximate additive, Jensen, and quadratic mappings in 2-Banach spaces.

In this paper, we investigate the stability of functional equations (2) and (3) in 2-Banach spaces and quasi-Banach spaces.

2 Stability in 2-Banach spaces

In this section, let $\mathcal{X}$ be a normed space and $\mathcal{Y}$ a 2-Banach space. We will first prove the following lemma.

Lemma 2 *Let $h:\mathcal{X}\to\mathcal{Y}$ be a mapping. Then,*

$$\begin{aligned}
&h(x)-\frac{4^{n+1}-1}{3\cdot 8^n}h\bigl(2^nx\bigr)+\frac{4^n-1}{6\cdot 8^n}h\bigl(2^{n+1}x\bigr)\\
&\quad=\sum_{j=1}^{n}\biggl[\frac{4^j-1}{3\cdot 8^{j-1}}h\bigl(2^{j-1}x\bigr)-\frac{5(4^j-1)}{3\cdot 8^j}h\bigl(2^jx\bigr)+\frac{4^j-1}{6\cdot 8^j}h\bigl(2^{j+1}x\bigr)\biggr]
\end{aligned}$$

for all $x\in\mathcal{X}$ and all n.

Proof The given equality is obviously true for all $x\in\mathcal{X}$ and $n=1,2$. For an integer $k\geq 2$, assume that it holds for all $x\in\mathcal{X}$ and $n=k-1,k$. Then,

$$\begin{aligned}
&\sum_{j=1}^{k+1}\biggl[\frac{4^j-1}{3\cdot 8^{j-1}}h\bigl(2^{j-1}x\bigr)-\frac{5(4^j-1)}{3\cdot 8^j}h\bigl(2^jx\bigr)+\frac{4^j-1}{6\cdot 8^j}h\bigl(2^{j+1}x\bigr)\biggr]\\
&\quad=\sum_{j=1}^{k-1}\biggl[\frac{4^j-1}{3\cdot 8^{j-1}}h\bigl(2^{j-1}x\bigr)-\frac{5(4^j-1)}{3\cdot 8^j}h\bigl(2^jx\bigr)+\frac{4^j-1}{6\cdot 8^j}h\bigl(2^{j+1}x\bigr)\biggr]\\
&\qquad+\frac{4^k-1}{3\cdot 8^{k-1}}h\bigl(2^{k-1}x\bigr)-\frac{5(4^k-1)}{3\cdot 8^k}h\bigl(2^kx\bigr)+\frac{4^k-1}{6\cdot 8^k}h\bigl(2^{k+1}x\bigr)\\
&\qquad+\frac{4^{k+1}-1}{3\cdot 8^k}h\bigl(2^kx\bigr)-\frac{5(4^{k+1}-1)}{3\cdot 8^{k+1}}h\bigl(2^{k+1}x\bigr)+\frac{4^{k+1}-1}{6\cdot 8^{k+1}}h\bigl(2^{k+2}x\bigr)\\
&\quad=h(x)-\frac{4^k-1}{3\cdot 8^{k-1}}h\bigl(2^{k-1}x\bigr)+\frac{4^{k-1}-1}{6\cdot 8^{k-1}}h\bigl(2^kx\bigr)\\
&\qquad+\frac{4^k-1}{3\cdot 8^{k-1}}h\bigl(2^{k-1}x\bigr)-\frac{5(4^k-1)}{3\cdot 8^k}h\bigl(2^kx\bigr)+\frac{4^k-1}{6\cdot 8^k}h\bigl(2^{k+1}x\bigr)\\
&\qquad+\frac{4^{k+1}-1}{3\cdot 8^k}h\bigl(2^kx\bigr)-\frac{5(4^{k+1}-1)}{3\cdot 8^{k+1}}h\bigl(2^{k+1}x\bigr)+\frac{4^{k+1}-1}{6\cdot 8^{k+1}}h\bigl(2^{k+2}x\bigr)
\end{aligned}$$

$$= h(x) - \frac{4^{k+2}-1}{3\cdot 8^{k+1}} h\bigl(2^{k+1}x\bigr) + \frac{4^{k+1}-1}{6\cdot 8^{k+1}} h\bigl(2^{k+2}x\bigr)$$

for all $x \in \mathcal{X}$. By the induction on n, we obtain that it holds for all $x \in \mathcal{X}$ and all n. □

The following theorem proves the stability of equation (1) in 2-Banach spaces.

Theorem 1 *Let $p \in (0,1)$, $\varepsilon \geq 0$, $\delta, \eta \geq 0$ and let $g : \mathcal{X} \to \mathcal{Y}$ be a surjection such that*

$$\begin{aligned}&\bigl\|g(x+y+z) + g(x+y-z) + 2g(x) + 2g(y)\\&\quad - 2g(x+y) - g(x+z) - g(x-z) - g(y+z) - g(y-z), g(t)\bigr\|\\&\leq \varepsilon + \delta\bigl(\|x\|^p + \|y\|^p + \|z\|^p\bigr) + \eta\|t\|\end{aligned} \tag{4}$$

for all $x,y,z,t \in \mathcal{X}$. Then, there are a cubic mapping $T : \mathcal{X} \to \mathcal{Y}$ and an additive mapping $A : \mathcal{X} \to \mathcal{Y}$ that satisfy equation (1) and the inequality

$$\bigl\|g(x) - g(0) - A(x) - T(x), g(t)\bigr\| \leq \frac{(11+2^p)\delta}{(2-2^p)(8-2^p)}\|x\|^p + \frac{4}{7}\bigl(\varepsilon + \eta\|t\|\bigr)$$

for all $x,t \in \mathcal{X}$.

Proof Define $h : \mathcal{X} \to \mathcal{Y}$ by $h(x) := g(x) - g(0)$ for all $x,t \in \mathcal{X}$. Then, h satisfies also the functional inequality

$$\begin{aligned}&\bigl\|h(x+y+z) + h(x+y-z) + 2h(x) + 2h(y)\\&\quad - 2h(x+y) - h(x+z) - h(x-z) - h(y+z) - h(y-z), g(t)\bigr\|\\&\leq \varepsilon + \delta\bigl(\|x\|^p + \|y\|^p + \|z\|^p\bigr) + \eta\|t\|\end{aligned} \tag{5}$$

for all $x,y,z,t \in \mathcal{X}$. If we replace (x,y,z) by $(2x,x,x)$ in (5), we gain

$$\bigl\|h(4x) + 2h(2x) + h(x) - 3h(3x), g(t)\bigr\| \leq \varepsilon + \bigl(2^p+2\bigr)\delta\|x\|^p + \eta\|t\|$$

for all $x,t \in \mathcal{X}$. Replacing (x,y,z) by (x,x,x) in (5), we obtain

$$\bigl\|h(3x) + 5h(x) - 4h(2x), g(t)\bigr\| \leq \varepsilon + 3\delta\|x\|^p + \eta\|t\|$$

for all $x,t \in \mathcal{X}$. By the above two inequalities, we have

$$\bigl\|h(4x) + 16h(x) - 10h(2x), g(t)\bigr\| \leq 4\varepsilon + \bigl(2^p+11\bigr)\delta\|x\|^p + 4\eta\|t\| \tag{6}$$

for all $x,t \in \mathcal{X}$. By Lemma 2 and (6), we obtain that

$$\begin{aligned}&\left\|h(x) - \frac{4^{n+1}-1}{3\cdot 8^n}h\bigl(2^n x\bigr) + \frac{4^n-1}{6\cdot 8^n}h\bigl(2^{n+1}x\bigr), g(t)\right\|\\&= \left\|\sum_{j=1}^{n}\left[\frac{4^j-1}{3\cdot 8^{j-1}}h\bigl(2^{j-1}x\bigr) - \frac{5(4^j-1)}{3\cdot 8^j}h\bigl(2^j x\bigr) + \frac{4^j-1}{6\cdot 8^j}h\bigl(2^{j+1}x\bigr)\right], g(t)\right\|\end{aligned}$$

$$
\begin{aligned}
&= \left\| \sum_{j=1}^{n} \frac{4^j-1}{3\cdot 8^{j-1}} \left[h\left(2^{j-1}x\right) - \frac{5}{8} h\left(2^j x\right) + \frac{1}{16} h\left(2^{j+1}x\right) \right], g(t) \right\| \\
&\le \sum_{j=1}^{n} \frac{4^j-1}{3\cdot 8^{j-1}} \left\| h\left(2^{j-1}x\right) - \frac{5}{8} h\left(2^j x\right) + \frac{1}{16} h\left(2^{j+1}x\right), g(t) \right\| \\
&= \frac{1}{16} \sum_{j=1}^{n} \frac{4^j-1}{3\cdot 8^{j-1}} \left\| 16h\left(2^{j-1}x\right) - 10h\left(2^j x\right) + h\left(2^{j+1}x\right), g(t) \right\| \\
&\le \frac{1}{16} \sum_{j=1}^{n} \frac{4^j-1}{3\cdot 8^{j-1}} \left[4\varepsilon + \left(2^p+11\right) 2^{p(j-1)} \delta \|x\|^p + 4\eta \|t\| \right] \qquad (7)
\end{aligned}
$$

for all $x, t \in \mathcal{X}$ and all n. We set a sequence $\{h_n(x)\}$ given by

$$
h_n(x) := \frac{4^{n+1}-1}{3\cdot 8^n} h\left(2^n x\right) - \frac{4^n-1}{6\cdot 8^n} h\left(2^{n+1}x\right)
$$

for all $x \in \mathcal{X}$ and all n, and prove the convergence of the sequence. By (6), we see that

$$
\begin{aligned}
&\left\| h_{n+1}(x) - h_n(x), g(t) \right\| \\
&\quad = \left\| \frac{4^{n+2}-1}{3\cdot 8^{n+1}} h\left(2^{n+1}x\right) - \frac{4^{n+1}-1}{6\cdot 8^{n+1}} h\left(2^{n+2}x\right) - \frac{4^{n+1}-1}{3\cdot 8^n} h\left(2^n x\right) \right. \\
&\qquad \left. + \frac{4^n-1}{6\cdot 8^n} h\left(2^{n+1}x\right), g(t) \right\| \\
&\quad = \frac{1}{6\cdot 8^{n+1}} \left\| 2\left(4^{n+2}-1\right) h\left(2^{n+1}x\right) + 8\left(4^n-1\right) h\left(2^{n+1}x\right) \right. \\
&\qquad \left. - 16\left(4^{n+1}-1\right) h\left(2^n x\right) - \left(4^{n+1}-1\right) h\left(2^{n+2}x\right), g(t) \right\| \\
&\quad = \frac{1}{6\cdot 8^{n+1}} \left\| 4^{n+1}\left[10h\left(2^{n+1}x\right) - 16h\left(2^n x\right) - h\left(2^{n+2}x\right)\right] \right. \\
&\qquad \left. - \left[10h\left(2^{n+1}x\right) - 16h\left(2^n x\right) - h\left(2^{n+2}x\right)\right], g(t) \right\| \\
&\quad \le \frac{4^{n+1}-1}{6\cdot 8^{n+1}} \left[4\varepsilon + \left(2^p+11\right) 2^{pn} \delta \|x\|^p + 4\eta \|t\| \right]
\end{aligned}
$$

for all $x, t \in \mathcal{X}$ and all n. Hence, it follows from the last inequality that for any positive integers m, n with $m > n > 0$, we have

$$
\begin{aligned}
\left\| h_n(x) - h_m(x), g(t) \right\| &\le \sum_{k=n}^{m-1} \left\| h_{k+1}(x) - h_k(x), g(t) \right\| \\
&\le \sum_{k=n}^{m-1} \frac{4^{k+1}-1}{6\cdot 8^{k+1}} \left[4\varepsilon + \left(2^p+11\right) 2^{pk} \delta \|x\|^p + 4\eta \|t\| \right]
\end{aligned}
$$

for all $x, t \in \mathcal{X}$. Since $p < 1$, the right-hand side of the above inequality tends to 0 as n tends to infinity and thus the sequence $\{h_n(x)\}$ is a Cauchy sequence in $\mathcal{Y}$. Therefore, we may define a mapping $G : \mathcal{X} \to \mathcal{Y}$ by

$$
G(x) := \lim_{n\to\infty} h_n(x) = \lim_{n\to\infty} \left[\frac{4^{n+1}-1}{3\cdot 8^n} h\left(2^n x\right) - \frac{4^n-1}{6\cdot 8^n} h\left(2^{n+1}x\right) \right]
$$

$$= \lim_{n\to\infty}\left[\frac{4^{n+1}-1}{3\cdot 8^n}g\left(2^n x\right) - \frac{4^n-1}{6\cdot 8^n}g\left(2^{n+1}x\right)\right]$$

for all $x \in \mathcal{X}$ and hence we arrive at the formula

$$\begin{aligned}
&\left\|g(x) - g(0) - G(x), g(t)\right\| \\
&\quad\le \frac{1}{16}\sum_{j=1}^{\infty}\frac{4^j-1}{3\cdot 8^{j-1}}\left[4\varepsilon + \left(2^p+11\right)2^{p(j-1)}\delta\|x\|^p + 4\eta\|t\|\right] \\
&\quad= \frac{(11+2^p)\delta}{(2-2^p)(8-2^p)}\|x\|^p + \frac{4}{7}\left(\varepsilon + \eta\|t\|\right)
\end{aligned}$$

by letting $n \to \infty$ in (7).

By (4), we have

$$\begin{aligned}
&\frac{4^{n+1}-1}{3\cdot 8^n}\left\|g\left(2^n x + 2^n y + 2^n z\right) + g\left(2^n x + 2^n y - 2^n z\right) + 2g\left(2^n x\right) + 2g\left(2^n y\right)\right. \\
&\quad - 2g\left(2^n x + 2^n y\right) - g\left(2^n x + 2^n z\right) - g\left(2^n x - 2^n z\right) - g\left(2^n y + 2^n z\right) \\
&\quad \left.- g\left(2^n y - 2^n z\right), g(t)\right\| \\
&\quad\le \frac{4^{n+1}-1}{3\cdot 8^n}\left[\varepsilon + 2^{pn}\delta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) + \eta\|t\|\right]
\end{aligned}$$

and

$$\begin{aligned}
&\frac{4^n-1}{6\cdot 8^n}\left\|g\left(2^{n+1}x + 2^{n+1}y + 2^{n+1}z\right) + g\left(2^{n+1}x + 2^{n+1}y - 2^{n+1}z\right) + 2g\left(2^{n+1}x\right)\right. \\
&\quad + 2g\left(2^{n+1}y\right) - 2g\left(2^{n+1}x + 2^{n+1}y\right) - g\left(2^{n+1}x + 2^{n+1}z\right) - g\left(2^{n+1}x - 2^{n+1}z\right) \\
&\quad \left.- g\left(2^{n+1}y + 2^{n+1}z\right) - g\left(2^{n+1}y - 2^{n+1}z\right), g(t)\right\| \\
&\quad\le \frac{4^n-1}{6\cdot 8^n}\left[\varepsilon + 2^{p(n+1)}\delta\left(\|x\|^p + \|y\|^p + \|z\|^p\right) + \eta\|t\|\right]
\end{aligned}$$

for all $x, y, z, t \in \mathcal{X}$. Thus, it follows from the above two inequalities that

$$\begin{aligned}
&\left\|h_n(x+y+z) + h_n(x+y-z) + 2h_n(x) + 2h_n(y) - 2h_n(x+y) - h_n(x+z)\right. \\
&\quad \left.- h_n(x-z) - h_n(y+z) - h_n(y-z), g(t)\right\| \\
&\quad= \left\|\frac{4^{n+1}-1}{3\cdot 8^n}\left[g\left(2^n x + 2^n y + 2^n z\right) + g\left(2^n x + 2^n y - 2^n z\right) + 2g\left(2^n x\right) + 2g\left(2^n y\right)\right.\right. \\
&\quad - 2g\left(2^n x + 2^n y\right) - g\left(2^n x + 2^n z\right) - g\left(2^n x - 2^n z\right) \\
&\quad \left.- g\left(2^n y + 2^n z\right) - g\left(2^n y - 2^n z\right)\right] \\
&\quad - \frac{4^n-1}{6\cdot 8^n}\left[g\left(2^{n+1}x + 2^{n+1}y + 2^{n+1}z\right) + g\left(2^{n+1}x + 2^{n+1}y - 2^{n+1}z\right)\right. \\
&\quad + 2g\left(2^{n+1}x\right) + 2g\left(2^{n+1}y\right) - 2g\left(2^{n+1}x + 2^{n+1}y\right) - g\left(2^{n+1}x + 2^{n+1}z\right) \\
&\quad \left.\left.- g\left(2^{n+1}x - 2^{n+1}z\right) - g\left(2^{n+1}y + 2^{n+1}z\right) - g\left(2^{n+1}y - 2^{n+1}z\right)\right], g(t)\right\|
\end{aligned}$$

$$\le \frac{4^{n+1}-1}{3\cdot 8^n}\bigl[\varepsilon + 2^{pn}\delta\bigl(\|x\|^p+\|y\|^p+\|z\|^p\bigr)+\eta\|t\|\bigr]$$
$$+\frac{4^n-1}{6\cdot 8^n}\bigl[\varepsilon + 2^{p(n+1)}\delta\bigl(\|x\|^p+\|y\|^p+\|z\|^p\bigr)+\eta\|t\|\bigr]$$

for all $x,y,z,t\in\mathcal{X}$. Taking $n\to\infty$, since $0<p<1$, we obtain that G is a solution of (1). By Theorem 2.1 in [7], there exist a cubic mapping T, an additive mapping A, and a constant c such that $G(x)=T(x)+A(x)+c$ for all $x\in\mathcal{X}$. Since $h(0)=0$, we obtain $h_n(0)=0$ for all n. Hence, $c=G(0)=\lim_{n\to\infty}h_n(0)=0$. □

In the case $p>1$ in Theorem 1, one can also obtain a similar result.

The following theorem proves the stability of equation (2) in 2-Banach spaces.

Theorem 2 *Let $p\in(0,2)$, $\varepsilon\ge 0$, $\delta,\eta\ge 0$ and let $g:\mathcal{X}\to\mathcal{Y}$ be a surjection such that*

$$\begin{aligned}&\bigl\|g(x+y+z)+g(x-z)+g(y-z)-g(x+y-z)-g(x+z)-g(y+z),g(t)\bigr\|\\&\quad\le\varepsilon+\delta\bigl(\|x\|^p+\|y\|^p+\|z\|^p\bigr)+\eta\|t\|\end{aligned}\tag{8}$$

for all $x,y,z,t\in\mathcal{X}$. Then, there is a unique quadratic mapping $G:\mathcal{X}\to\mathcal{Y}$ satisfying (2) such that

$$\bigl\|g(x)-g(0)-G(x),g(t)\bigr\|\le\frac{(5+2^p)\delta}{2^p(4-2^p)}\|x\|^p+\frac{2}{3}\bigl(\varepsilon+\eta\|t\|\bigr)\tag{9}$$

for all $x,t\in\mathcal{X}$.

Proof Let $h:\mathcal{X}\to\mathcal{Y}$ be a mapping given by $h(x):=g(x)-g(0)$ for all $x\in\mathcal{X}$, then $h(0)=0$. Letting $x=y=z$ in (8), we have

$$\begin{aligned}\bigl\|h(3x)-2h(2x)-h(x),g(t)\bigr\|&=\bigl\|g(3x)-2g(2x)-g(x)+2g(0),g(t)\bigr\|\\&\le\varepsilon+3\delta\|x\|^p+\eta\|t\|\end{aligned}$$

for all $x,t\in\mathcal{X}$. Replacing x, y, z by $2x$, x, x in (8), respectively, we have

$$\begin{aligned}&\bigl\|h(4x)-h(3x)-2h(2x)+h(x),g(t)\bigr\|\\&\quad=\bigl\|g(4x)-g(3x)-2g(2x)+g(x)+g(0),g(t)\bigr\|\le\varepsilon+\bigl(2^p+2\bigr)\delta\|x\|^p+\eta\|t\|\end{aligned}$$

for all $x,t\in\mathcal{X}$. By the above two inequalities, we obtain

$$\bigl\|h(4x)-4h(2x),g(t)\bigr\|\le 2\varepsilon+\bigl(2^p+5\bigr)\delta\|x\|^p+2\eta\|t\|$$

for all $x,t\in\mathcal{X}$. Replacing x by $\frac{x}{2}$, we have

$$\left\|h(x)-\frac{1}{4}h(2x),g(t)\right\|\le\frac{1}{4}\left(1+\frac{5}{2^p}\right)\delta\|x\|^p+\frac{1}{2}\bigl(\varepsilon+\eta\|t\|\bigr)$$

for all $x, t \in \mathcal{X}$. Thus, we obtain

$$\begin{aligned}
&\left\| \frac{1}{4^j} h(2^j x) - \frac{1}{4^{j+1}} h(2^{j+1} x), g(t) \right\| \\
&\quad \le \frac{1}{4}\left(1 + \frac{5}{2^p}\right) \delta \|x\|^p \left(\frac{2^p}{4}\right)^j + \frac{1}{2}\left(\varepsilon + \eta \|t\|\right)\left(\frac{1}{4}\right)^j
\end{aligned}$$

for all $x, t \in \mathcal{X}$ and all j. For given integers l, m $(0 \le l < m)$, we obtain

$$\begin{aligned}
&\left\| \frac{1}{4^l} h(2^l x) - \frac{1}{4^m} h(2^m x), g(t) \right\| \\
&\quad \le \sum_{j=l}^{m-1} \left[\frac{1}{4}\left(1 + \frac{5}{2^p}\right) \delta \|x\|^p \left(\frac{2^p}{4}\right)^j + \frac{1}{2}\left(\varepsilon + \eta \|t\|\right)\left(\frac{1}{4}\right)^j \right]
\end{aligned} \tag{10}$$

for all $x, t \in \mathcal{X}$. Since $p \in (0, 2)$, the sequence $\{\frac{1}{4^j} h(2^j x)\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is a 2-Banach space, the sequence $\{\frac{1}{4^j} h(2^j x)\}$ converges for all $x \in \mathcal{X}$. Define $G : \mathcal{X} \to \mathcal{Y}$ by

$$G(x) := \lim_{j\to\infty} \frac{1}{4^j} h(2^j x)$$

for all $x \in \mathcal{X}$. Then,

$$G(2x) = \lim_{j\to\infty} \frac{1}{4^j} h(2^{j+1} x) = 4 \lim_{j\to\infty} \frac{1}{4^{j+1}} h(2^{j+1} x) = 4G(x)$$

for all $x \in \mathcal{X}$. By (8), we have

$$\begin{aligned}
&\left\| \frac{1}{4^j} \left[h(2^j(x+y+z)) + h(2^j(x-z)) + h(2^j(y-z)) - h(2^j(x+y-z)) \right.\right. \\
&\qquad \left.\left. - h(2^j(x+z)) - h(2^j(y+z)) \right], g(t) \right\| \\
&\quad = \left\| \frac{1}{4^j} \left[g(2^j(x+y+z)) + g(2^j(x-z)) + g(2^j(y-z)) - g(2^j(x+y-z)) \right.\right. \\
&\qquad \left.\left. - g(2^j(x+z)) - g(2^j(y+z)) \right], g(t) \right\| \\
&\quad \le \frac{1}{4^j} \left[\varepsilon + 2^{jp} \delta \left(\|x\|^p + \|y\|^p + \|z\|^p \right) + \eta \|t\| \right]
\end{aligned}$$

for all $x, y, z, t \in \mathcal{X}$ and all j. Letting $j \to \infty$, we see that G satisfies (2). Setting $l = 0$ and taking $m \to \infty$ in (10), one can obtain the inequality (9).

Let $H : \mathcal{X} \to \mathcal{Y}$ be another quadratic mapping satisfying (2) and (9). By the proof of Lemma 2.1 in [10], $H(2x) = 4H(x) - 3H(0)$ for all $x \in \mathcal{X}$. Thus, we obtain

$$\begin{aligned}
&\left\| G(x) - H(x), g(t) \right\| \\
&\quad = \frac{1}{4^n} \left\| G(2^n x) - H(2^n x) - \left(1 - \frac{1}{4^n}\right) H(0), g(t) \right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4^n}\big\|G\big(2^n x\big) - g\big(2^n x\big) + g(0), g(t)\big\| \\
&\quad + \frac{1}{4^n}\big\|-g(0) + g\big(2^n x\big) - H\big(2^n x\big), g(t)\big\| + \frac{1}{4^n}\left(1 - \frac{1}{4^n}\right)\big\|H(0), g(t)\big\| \\
&\leq \frac{2}{4^n}\left[\frac{2^{(n-1)p}(5+2^p)\delta}{4-2^p}\|x\|^p + \frac{2}{3}\big(\epsilon + \eta\|t\|\big)\right] + \frac{1}{4^n}\left(1 - \frac{1}{4^n}\right)\big\|H(0), g(t)\big\| \\
&\to 0 \quad \text{as } n \to \infty
\end{aligned}$$

for all $x, t \in \mathcal{X}$. Hence, G is a unique quadratic mapping, as desired. □

In the case $p > 2$ in Theorem 2, one can also obtain a similar result.

Theorem 2 leaves the case $p = 2$ undecided. This is not a mere coincidence. It turns out that 2 is the only critical value of $p > 0$ to which Theorem 2 can not be extended. Note that $\mathbb{R}^2$ is a 2-Banach space with the 2-norm given by $|A, B|$ = the area of the triangle OAB, where O is the origin. In fact, we shall show that, for $\delta, \eta > 0$ one can find a mapping $g : \mathbb{R} \to \mathbb{R}^2$ such that

$$\begin{aligned}
&\big|g(x+y+z) + g(x-z) + g(y-z) - g(x+y-z) - g(x+z) - g(y+z), g(t)\big| \\
&\quad \leq \delta\big(|x| + |y| + |z|\big) + \eta|t|
\end{aligned}$$

for all $x, y, z, t \in \mathbb{R}$, but, at the same time, there is no constant $\lambda, \mu \in [0, \infty)$ and no quadratic mapping $G : \mathbb{R} \to \mathbb{R}^2$ satisfying the condition

$$\big|g(x) - g(0) - G(x), g(t)\big| \leq \lambda|x| + \mu|t|$$

for all $x, y, z, t \in \mathbb{R}$.

The following theorem proves the stability of equation (3) in 2-Banach spaces.

Theorem 3 *Let $p \in (0, 1)$, $\varepsilon > 0$, $\delta, \eta \geq 0$ and let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a surjection such that*

$$\begin{aligned}
&\big\|f(x+y+z, u+v+w) + f(x+y-z, u+v+w) + 2f(x, u-w) \\
&\qquad + 2f(y, v-w) - f(x+y, u+w) - f(x+y, v+w) - f(x+z, u+w) \\
&\qquad - f(x-z, u+v-w) - f(y+z, v+w) - f(y-z, u+v-w), f(s,t)\big\| \\
&\quad \leq \varepsilon + \delta\big(\|x\|^p + \|y\|^p + \|z\|^p + \|u\|^p + \|v\|^p + \|w\|^p\big) + \eta\big(\|s\| + \|t\|\big) \qquad (11)
\end{aligned}$$

for all $x, y, z, u, v, w, s, t \in \mathcal{X}$. Then, there exists a mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfying (3) such that

$$\begin{aligned}
&\big\|f(x,y) - f(0,0) - F(x,y), f(s,t)\big\| \\
&\quad \leq \frac{5 + 3\cdot 2^p - 4^p}{2^p(2-2^p)}\delta\|x\|^p + \frac{5 + 9\cdot 2^p - 2\cdot 4^p}{2^p(4-2^p)}\delta\|y\|^p + \frac{28}{9}\big[\varepsilon + \eta\big(\|s\| + \|t\|\big)\big]
\end{aligned}$$

for all $x, y, s, t \in \mathcal{X}$.

Proof Define $f_1, f_2 : \mathcal{X} \to \mathcal{Y}$ by $f_1(x) := f(x, 0)$ and $f_2(y) := f(0, y)$ for all $x, y \in \mathcal{X}$. Putting $u = v = w = 0$ in (11), we have

$$\begin{aligned}&\big\|f_1(x+y+z) + f_1(x+y-z) + 2f_1(x) + 2f_1(y) - 2f_1(x+y) - f_1(x+z)\\&\quad - f_1(x-z) - f_1(y+z) - f_1(y-z), f(s,t)\big\|\\&\le \varepsilon + \delta\big(\|x\|^p + \|y\|^p + \|z\|^p\big) + \eta\big(\|s\| + \|t\|\big)\end{aligned}$$

for all $x, y, z, s, t \in \mathcal{X}$. By Theorem 1, there exist a cubic mapping $F_1 : \mathcal{X} \to \mathcal{Y}$ and an additive mapping A_1 which satisfy equation (1) and the inequality

$$\big\|f_1(x) - f_1(0) - A_1(x) - F_1(x), f(s,t)\big\| \le \frac{5 + 2^p}{2^p(2-2^p)}\delta\|x\|^p + \frac{16}{9}\big[(\varepsilon + \eta\big(\|s\| + \|t\|\big)\big]$$

for all $x, s, t \in \mathcal{X}$. Setting $x = y = z = 0$ in (11), we have

$$\begin{aligned}&\big\|2f_2(u+v+w) + 2f_2(u-w) + 2f_2(v-w) - 2f_2(u+v-w)\\&\quad - 2f_2(u+w) - 2f_2(v+w), f(s,t)\big\|\\&\le \varepsilon + \delta\big(\|u\|^p + \|v\|^p + \|w\|^p\big) + \eta\big(\|s\| + \|t\|\big)\end{aligned}$$

for all $u, v, w, s, t \in \mathcal{X}$. By Theorem 2, there exists a quadratic mapping $F_2 : \mathcal{X} \to \mathcal{Y}$ satisfying (2) such that

$$\big\|f_2(y) - f_2(0) - F_2(y), f(s,t)\big\| \le \frac{5 + 2^p}{2^{p+1}(4-2^p)}\delta\|y\|^p + \frac{1}{3}\big[\varepsilon + \eta\big(\|s\| + \|t\|\big)\big]$$

for all $y, s, t \in \mathcal{X}$. Setting $y = z = v = w = 0$ in (11), we have

$$\big\|f(x,u) - f_1(x) - f_2(u) + f(0,0), f(s,t)\big\| \le \varepsilon + \delta\big(\|x\|^p + \|u\|^p\big) + \eta\big(\|s\| + \|t\|\big)$$

for all $x, u, s, t \in \mathcal{X}$.

If we define

$$F(x,y) := A_1(x) + F_1(x) + F_2(y)$$

for all $x, y \in \mathcal{X}$, then we conclude

$$\begin{aligned}&\big\|f(x,y) - f(0,0) - F(x,y), f(s,t)\big\|\\&= \big\|f(x,y) - A_1(x) - F_1(x) - F_2(y) - f(0,0), f(s,t)\big\|\\&\le \big\|f(x,y) - f_1(x) - f_2(y) + f(0,0), f(s,t)\big\|\\&\quad + \big\|f_1(x) - f_1(0) - A_1(x) - F_1(x), f(s,t)\big\| + \big\|f_2(y) - f_2(0) - F_2(y), f(s,t)\big\|\\&\le \frac{5 + 3\cdot 2^p - 4^p}{2^p(2-2^p)}\delta\|x\|^p + \frac{5 + 9\cdot 2^p - 2\cdot 4^p}{2^p(4-2^p)}\delta\|y\|^p + \frac{28}{9}\big[\varepsilon + \eta\big(\|s\| + \|t\|\big)\big]\end{aligned}$$

for all $x, y, s, t \in \mathcal{X}$. □

In the case $p \in (1,2) \cup (2,\infty)$ in Theorem 3, one can also obtain a similar result. Taking $\delta = \eta = 0$ in Theorem 3, we solve the Ulam stability problem for the functional equation (3).

Corollary 1 *Let* $\varepsilon > 0$ *and let* $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ *be a surjection such that*

$$\begin{aligned}
&\| f(x+y+z, u+v+w) + f(x+y-z, u+v+w) + 2f(x, u-w) \\
&\quad + 2f(y, v-w) - f(x+y, u+w) - f(x+y, v+w) - f(x+z, u+w) \\
&\quad - f(x-z, u+v-w) - f(y+z, v+w) - f(y-z, u+v-w), f(s,t) \| \leq \varepsilon
\end{aligned}$$

for all $x, y, z, u, v, w, s, t \in \mathcal{X}$. *Then, there exists a mapping* $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ *satisfying* (3) *such that*

$$\| f(x,y) - f(0,0) - F(x,y), f(s,t) \| \leq \frac{28}{9}\varepsilon$$

for all $x, y, s, t \in \mathcal{X}$.

3 Stability in quasi-Banach spaces

Definition 4 Let $\mathcal{X}$ be a real linear space. A *quasinorm* is a real-valued function on $\mathcal{X}$ satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in \mathcal{X}$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in \mathcal{X}$.
(iii) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in \mathcal{X}$.

The pair $(\mathcal{X}, \|\cdot\|)$ is called a *quasinormed space* if $\|\cdot\|$ is a quasinorm on $\mathcal{X}$. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasinormed space. A quasinorm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in \mathcal{X}$. In this case, a quasi-Banach space is called a *p-Banach space*.

A quasinorm gives rise to a linear topology on $\mathcal{X}$, namely the least linear topology for which the unit ball $B = \{x \in \mathcal{X} : \|x\| \leq 1\}$ is a neighborhood of zero. This topology is locally bounded, that is, it has a bounded neighborhood of zero. Actually, every locally bounded topology arises in this way.

From now on, assume that $\mathcal{X}$ is a quasinormed space with quasinorm $\|\cdot\|$ and that $\mathcal{Y}$ is a p-Banach space with p-norm $\|\cdot\|_{\mathcal{Y}}$. Let K be the modulus of concavity of $\|\cdot\|_{\mathcal{Y}}$.

We will use the following lemma in this section.

Lemma 3 ([8]) *Let* $0 \leq p \leq 1$ *and let* $x_1, x_2, \ldots, x_n$ *be nonnegative real numbers. Then,*

$$(x_1 + x_2 + \cdots + x_n)^p \leq x_1^p + x_2^p + \cdots + x_n^p.$$

The following theorem proves the stability of equation (1) in quasi-Banach spaces.

Theorem 4 *Let $q \in (0,1)$, $\varepsilon > 0$ and $\delta \geq 0$ and let $g : \mathcal{X} \to \mathcal{Y}$ be a mapping such that*

$$\begin{aligned}
&\big\| g(x+y+z) + g(x+y-z) + 2g(x) + 2g(y) \\
&\quad - 2g(x+y) - g(x+z) - g(x-z) - g(y+z) - g(y-z) \big\|_{\mathcal{Y}} \\
&\leq \varepsilon + \delta\big(\|x\|^q + \|y\|^q + \|z\|^q\big)
\end{aligned} \tag{12}$$

for all $x, y, z \in \mathcal{X}$. Then, there are a cubic mapping $T : \mathcal{X} \to \mathcal{Y}$ and an additive mapping $A : \mathcal{X} \to \mathcal{Y}$ that satisfy equation (1) and the inequality

$$\big\| g(x) - g(0) - A(x) - T(x) \big\|_{\mathcal{Y}} \leq \frac{1}{16} \left(\frac{8(2\varepsilon^p + 3\delta^p \|x\|^{pq})}{2^p - 1} \right)^{\frac{1}{p}} \tag{13}$$

for all $x \in \mathcal{X}$.

Proof Define $h : \mathcal{X} \to \mathcal{Y}$ by $h(x) := g(x) - g(0)$ for all $x \in \mathcal{X}$. Then, h satisfies also the functional inequality

$$\begin{aligned}
&\big\| h(x+y+z) + h(x+y-z) + 2h(x) + 2h(y) \\
&\quad - 2h(x+y) - h(x+z) - h(x-z) - h(y+z) - h(y-z) \big\|_{\mathcal{Y}} \\
&\leq \varepsilon + \delta\big(\|x\|^q + \|y\|^q + \|z\|^q\big)
\end{aligned} \tag{14}$$

for all $x, y, z \in \mathcal{X}$. If we replace (x, y, z) by $(2x, x, x)$ in (14), we have

$$\big\| h(4x) + 2h(2x) + h(x) - 3h(3x) \big\|_{\mathcal{Y}} \leq \varepsilon + \big(2 + 2^q\big)\delta \|x\|^q$$

for all $x \in \mathcal{X}$. Replacing (x, y, z) by (x, x, x) in (14), we obtain

$$\big\| h(3x) + 5h(x) - 4h(2x) \big\|_{\mathcal{Y}} \leq \varepsilon + 3\delta \|x\|^q$$

for all $x \in \mathcal{X}$. By the above two inequalities, we obtain

$$\begin{aligned}
\big\| h(4x) + 16h(x) - 10h(2x) \big\|_{\mathcal{Y}}^p &\leq \big[\varepsilon + \delta\big(2 + 2^q\big)\|x\|^q\big]^p + 3^p\big(\varepsilon + 3\delta\|x\|^q\big)^p \\
&\leq \big(1 + 3^p\big)\varepsilon^p + \big(2^p + 3^p + 2^{pq}\big)\delta^p \|x\|^{pq}
\end{aligned} \tag{15}$$

for all $x \in \mathcal{X}$.

By Lemma 2 and (15), we obtain that

$$\begin{aligned}
&\left\| h(x) - \frac{4^{n+1} - 1}{3 \cdot 8^n} h\big(2^n x\big) + \frac{4^n - 1}{6 \cdot 8^n} h\big(2^{n+1} x\big) \right\|_{\mathcal{Y}}^p \\
&\quad = \left\| \sum_{j=1}^{n} \left[\frac{4^j - 1}{3 \cdot 8^{j-1}} h\big(2^{j-1} x\big) - \frac{5(4^j - 1)}{3 \cdot 8^j} h\big(2^j x\big) + \frac{4^j - 1}{6 \cdot 8^j} h\big(2^{j+1} x\big) \right] \right\|_{\mathcal{Y}}^p \\
&\quad = \left\| \sum_{j=1}^{n} \frac{4^j - 1}{3 \cdot 8^{j-1}} \left[h\big(2^{j-1} x\big) - \frac{5}{8} h\big(2^j x\big) + \frac{1}{16} h\big(2^{j+1} x\big) \right] \right\|_{\mathcal{Y}}^p
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{n}\left(\frac{4^j-1}{3\cdot 8^{j-1}}\right)^p \left\| h\left(2^{j-1}x\right)-\frac{5}{8}h\left(2^jx\right)+\frac{1}{16}h\left(2^{j+1}x\right)\right\|_{\mathcal{Y}}^p \\
&= \frac{1}{16^p}\sum_{j=1}^{n}\left(\frac{4^j-1}{3\cdot 8^{j-1}}\right)^p \left\|16h\left(2^{j-1}x\right)-10h\left(2^jx\right)+h\left(2^{j+1}x\right)\right\|_{\mathcal{Y}}^p \\
&\le \frac{1}{16^p}\sum_{j=1}^{n}\left(\frac{4^j-1}{3\cdot 8^{j-1}}\right)^p \left[\left(1+3^p\right)\varepsilon^p+\left(2^p+3^p+2^{pq}\right)2^{pq(j-1)}\delta^p\|x\|^{pq}\right] \qquad (16)
\end{aligned}
$$

for all $x\in\mathcal{X}$ and all n. We set a sequence $\{h_n(x)\}$ given by

$$
h_n(x):=\frac{4^{n+1}-1}{3\cdot 8^n}h\left(2^nx\right)-\frac{4^n-1}{6\cdot 8^n}h\left(2^{n+1}x\right)
$$

for all $x\in\mathcal{X}$ and all n, and prove the convergence of the sequence. By (15), using Lemma 3, we see that

$$
\begin{aligned}
&\left\|h_{n+1}(x)-h_n(x)\right\|_{\mathcal{Y}}^p \\
&= \left\|\frac{4^{n+2}-1}{3\cdot 8^{n+1}}h\left(2^{n+1}x\right)-\frac{4^{n+1}-1}{6\cdot 8^{n+1}}h\left(2^{n+2}x\right)\right. \\
&\quad \left. -\frac{4^{n+1}-1}{3\cdot 8^n}h\left(2^nx\right)+\frac{4^n-1}{6\cdot 8^n}h\left(2^{n+1}x\right)\right\|_{\mathcal{Y}}^p \\
&= \left(\frac{1}{6\cdot 8^{n+1}}\right)^p \left\|2\left(4^{n+2}-1\right)h\left(2^{n+1}x\right)+8\left(4^n-1\right)h\left(2^{n+1}x\right)\right. \\
&\quad \left. -16\left(4^{n+1}-1\right)h\left(2^nx\right)-\left(4^{n+1}-1\right)h\left(2^{n+2}x\right)\right\|_{\mathcal{Y}}^p \\
&= \left(\frac{1}{6\cdot 8^{n+1}}\right)^p \left\|4^{n+1}\left[10h\left(2^{n+1}x\right)-16h\left(2^nx\right)-h\left(2^{n+2}x\right)\right]\right. \\
&\quad \left. -\left[10h\left(2^{n+1}x\right)-16h\left(2^nx\right)-h\left(2^{n+2}x\right)\right]\right\|_{\mathcal{Y}}^p \\
&\le \left(\frac{4^{n+1}-1}{6\cdot 8^{n+1}}\right)^p\left[\left(1+3^p\right)\varepsilon^p+\left(2^p+3^p+2^{pq}\right)2^{pqn}\delta^p\|x\|^{pq}\right]
\end{aligned}
$$

for all $x\in\mathcal{X}$ and all n. Hence, it follows from the last inequality that for any positive integers m, n with $m>n>0$, we have

$$
\begin{aligned}
\left\|h_n(x)-h_m(x)\right\|_{\mathcal{Y}}^p &\le \sum_{k=n}^{m-1}\left\|h_{k+1}(x)-h_k(x)\right\|_{\mathcal{Y}}^p \\
&\le \sum_{k=n}^{m-1}\left(\frac{4^{k+1}-1}{6\cdot 8^{k+1}}\right)^p\left[\left(1+3^p\right)\varepsilon^p+\left(2^p+3^p+2^{pq}\right)2^{pqk}\delta^p\|x\|^{pq}\right]
\end{aligned}
$$

for all $x\in\mathcal{X}$. Since $0<p\le 1$ and $0<q<1$, the right-hand side of the above inequality tends to 0 as n tends to infinity and thus the sequence $\{h_n(x)\}$ is a Cauchy sequence in $\mathcal{Y}$. Therefore, we may define a mapping $G:\mathcal{X}\to\mathcal{Y}$ by

$$
G(x):=\lim_{n\to\infty}h_n(x)=\lim_{n\to\infty}\left[\frac{4^{n+1}-1}{3\cdot 8^n}h\left(2^nx\right)-\frac{4^n-1}{6\cdot 8^n}h\left(2^{n+1}x\right)\right]
$$

$$= \lim_{n\to\infty}\left[\frac{4^{n+1}-1}{3\cdot 8^n}g(2^n x)-\frac{4^n-1}{6\cdot 8^n}g(2^{n+1}x)\right]$$

for all $x \in \mathcal{X}$. By letting $n \to \infty$ in (16), we have

$$\begin{aligned}
&\|g(x)-g(0)-G(x)\|_{\mathcal{Y}}\\
&\leq \frac{1}{16}\left(\sum_{j=1}^{\infty}\left(\frac{4^j-1}{3\cdot 8^{j-1}}\right)^p\left[(1+3^p)\varepsilon^p+(2^p+3^p+2^{pq})\delta^p\|2^{j-1}x\|^{pq}\right]\right)^{\frac{1}{p}}\\
&\leq \frac{1}{16}\left(\sum_{j=1}^{\infty}\left(\frac{4^j}{3\cdot 8^{j-1}}\right)^p\left[(1+3^p)\varepsilon^p+(2^p+3^p+2^{pq})\delta^p\|2^{j-1}x\|^{pq}\right]\right)^{\frac{1}{p}}\\
&= \frac{1}{16}\left(\frac{8^p(3^{-p}+1)}{2^p-1}\varepsilon^p+\frac{2^{p(3-q)}(2^p+3^p+2^{pq})}{3^p(2^{p(1-q)-1})}\delta^p\|x\|^{pq}\right)^{\frac{1}{p}}\\
&\leq \frac{1}{16}\left(\frac{8}{2^p-1}(2\varepsilon^p+7\cdot 3^{-p}\delta^p\|x\|^{pq})\right)^{\frac{1}{p}}\leq \frac{1}{16}\left(\frac{8}{2^p-1}\left[2\varepsilon^p+\frac{7}{3}\delta^p\|x\|^{pq}\right]\right)^{\frac{1}{p}}\\
&\leq \frac{1}{16}\left(\frac{8(2\varepsilon^p+3\delta^p\|x\|^{pq})}{2^p-1}\right)^{\frac{1}{p}}
\end{aligned}$$

for all $x \in \mathcal{X}$. Thus, we arrive at the formula (13).

By Lemma 3 and (12), we have

$$\begin{aligned}
&\left(\frac{4^{n+1}-1}{3\cdot 8^n}\right)^p\|g(2^nx+2^ny+2^nz)+g(2^nx+2^ny-2^nz)+2g(2^nx)+2g(2^ny)\\
&\quad -2g(2^nx+2^ny)-g(2^nx+2^nz)-g(2^nx-2^nz)-g(2^ny+2^nz)\\
&\quad -g(2^ny-2^nz)\|_{\mathcal{Y}}^p\\
&\leq \left(\frac{4^{n+1}-1}{3\cdot 8^n}\right)^p\left[\varepsilon^p+\delta^p 2^{pqn}(\|x\|^q+\|y\|^q+\|z\|^q)^p\right]
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{4^n-1}{6\cdot 8^n}\right)^p\|g(2^{n+1}x+2^{n+1}y+2^{n+1}z)+g(2^{n+1}x+2^{n+1}y-2^{n+1}z)\\
&\quad +2g(2^{n+1}x)+2g(2^{n+1}y)-2g(2^{n+1}x+2^{n+1}y)-g(2^{n+1}x+2^{n+1}z)\\
&\quad -g(2^{n+1}x-2^{n+1}z)-g(2^{n+1}y+2^{n+1}z)-g(2^{n+1}y-2^{n+1}z)\|_{\mathcal{Y}}^p\\
&\leq \left(\frac{4^n-1}{6\cdot 8^n}\right)^p\left[\varepsilon^p+\delta^p 2^{pq(n+1)}(\|x\|^q+\|y\|^q+\|z\|^q)^p\right]
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$ and all n. Thus, it follows from the above two inequalities that

$$\begin{aligned}
&\|h_n(x+y+z)+h_n(x+y-z)+2h_n(x)+2h_n(y)-2h_n(x+y)\\
&\quad -h_n(x+z)-h_n(x-z)-h_n(y+z)-h_n(y-z)\|_{\mathcal{Y}}^p\\
&= \left\|\frac{4^{n+1}-1}{3\cdot 8^n}\left[g(2^nx+2^ny+2^nz)+g(2^nx+2^ny-2^nz)+2g(2^nx)\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&+2g(2^n y)-2g(2^n x+2^n y)-g(2^n x+2^n z)-g(2^n x-2^n z)\\
&-g(2^n y+2^n z)-g(2^n y-2^n z)\big]\\
&-\frac{4^n-1}{6\cdot 8^n}\big[g(2^{n+1}x+2^{n+1}y+2^{n+1}z)+g(2^{n+1}x+2^{n+1}y-2^{n+1}z)\\
&+2g(2^{n+1}x)+2g(2^{n+1}y)-2g(2^{n+1}x+2^{n+1}y)-g(2^{n+1}x+2^{n+1}z)\\
&-g(2^{n+1}x-2^{n+1}z)-g(2^{n+1}y+2^{n+1}z)-g(2^{n+1}y-2^{n+1}z)\big]\Big\|_{\mathcal{Y}}^{p}\\
\le{}& \left(\frac{4^{n+1}-1}{3\cdot 8^n}\right)^p\big[\varepsilon^p+\delta^p 2^{pqn}\big(\|x\|^q+\|y\|^q+\|z\|^q\big)^p\big]\\
&+\left(\frac{4^n-1}{6\cdot 8^n}\right)^p\big[\varepsilon^p+\delta^p 2^{pq(n+1)}\big(\|x\|^q+\|y\|^q+\|z\|^q\big)^p\big]
\end{aligned}
$$

for all $x,y,z\in\mathcal{X}$ and all n. Taking $n\to\infty$, since $0<p<1$, we obtain that G is a solution of (1). By Theorem 2.1 in [7], there exist a cubic mapping T, an additive mapping A, and a constant c such that $G(x)=T(x)+A(x)+c$ for all $x\in\mathcal{X}$. Since $h(0)=0$, we obtain $h_n(0)=0$ for all n. So $c=G(0)=\lim_{n\to\infty}h_n(0)=0$. □

The following theorem proves the stability of equation (2) in quasi-Banach spaces.

Theorem 5 *Let $q\in(0,1)$, $\varepsilon>0$ and $\delta\ge 0$ and let $g:\mathcal{X}\to\mathcal{Y}$ be a mapping such that*

$$
\begin{aligned}
&\big\|g(x+y+z)+g(x-z)+g(y-z)-g(x+y-z)-g(x+z)-g(y+z)\big\|_{\mathcal{Y}}\\
&\quad\le\varepsilon+\delta\big(\|x\|^q+\|y\|^q+\|z\|^q\big)
\end{aligned}
\tag{17}
$$

for all $x,y,z\in\mathcal{X}$. Then, there exists a unique quadratic mapping $G:\mathcal{X}\to\mathcal{Y}$ satisfying (2) such that

$$
\big\|g(x)-g(0)-G(x)\big\|_{\mathcal{Y}}\le\left(\frac{5\varepsilon^p}{4^p-1}+\frac{\delta^p 2^{pq}[2(3+2^p)+2^{pq}]}{4^p-2^{pq}}\|x\|^{pq}\right)^{\frac{1}{p}}
\tag{18}
$$

for all $x\in\mathcal{X}$.

Proof Let $h:\mathcal{X}\to\mathcal{Y}$ be a mapping given by $h(x):=g(x)-g(0)$ for all $x\in\mathcal{X}$, then $h(0)=0$. Letting $x=y=0$ in (17), we have

$$
\big\|h(-z)-h(z)\big\|_{\mathcal{Y}}\le\varepsilon+\delta\|z\|^q
$$

for all $z\in\mathcal{X}$. Replacing z by $x+y$ in (17), we obtain

$$
\begin{aligned}
&\big\|h(2x+2y)+h(-y)+h(-x)-h(2x+y)-h(x+2y)\big\|_{\mathcal{Y}}\\
&\quad\le\varepsilon+\delta\big(\|x\|^q+\|y\|^q+\|x+y\|^q\big)
\end{aligned}
$$

for all $x,y\in\mathcal{X}$. Taking $z=x$ in (17), we see that

$$
\big\|h(2x+y)+h(y-x)-h(y)-h(2x)-h(x+y)\big\|_{\mathcal{Y}}\le\varepsilon+\delta\big(2\|x\|^q+\|y\|^q\big)
$$

for all $x, y \in \mathcal{X}$. Interchanging x and y in the above inequality, we see that

$$\|h(x+2y)+h(x-y)-h(x)-h(2y)-h(x+y)\|_{\mathcal{Y}} \le \varepsilon+\delta\left(\|x\|^q+2\|y\|^q\right)$$

for all $x, y \in \mathcal{X}$. By the above four inequalities and Lemma 3, we have

$$\begin{aligned}
&\|h(2x+2y)+h(x-y)+h(y-x)-2h(x+y)-h(2x)-h(2y)\|_{\mathcal{Y}}^p \\
&\quad\le 3\varepsilon^p+\delta^p\left[\left(2+2^p\right)\left(\|x\|^{pq}+\|y\|^{pq}\right)+\|x+y\|^{pq}\right] \\
&\qquad+\|h(x)-h(-x)\|_{\mathcal{Y}}^p+\|h(y)-h(-y)\|_{\mathcal{Y}}^p \\
&\quad\le 5\varepsilon^p+\delta^p\left[\left(3+2^p\right)\left(\|x\|^{pq}+\|y\|^{pq}\right)+\|x+y\|^{pq}\right]
\end{aligned}$$

for all $x, y \in \mathcal{X}$. Putting $y=x$ in the above inequality, we obtain

$$\|h(4x)-4h(2x)\|_{\mathcal{Y}}^p \le 5\varepsilon^p+\delta^p\left[2\left(3+2^p\right)+2^{pq}\right]\|x\|^{pq}$$

for all $x \in \mathcal{X}$. Hence,

$$\left\|h(x)-\frac{1}{4}h(2x)\right\|_{\mathcal{Y}}^p \le \frac{1}{4^p}\left(5\varepsilon^p+\delta^p 2^{pq}\left[2\left(3+2^p\right)+2^{pq}\right]\|x\|^{pq}\right)$$

for all $x \in \mathcal{X}$. Thus, we obtain

$$\left\|\frac{1}{4^j}h\left(2^j x\right)-\frac{1}{4^{j+1}}h\left(2^{j+1}x\right)\right\|_{\mathcal{Y}}^p \le \frac{1}{4^{p(j+1)}}\left(5\varepsilon^p+\delta^p 2^{pq}\left[2\left(3+2^p\right)+2^{pq}\right]2^{pqj}\|x\|^{pq}\right) \tag{19}$$

for all $x \in \mathcal{X}$ and all j. For given integers l, m $(0 \le l < m)$, we obtain

$$\left\|\frac{1}{4^l}h\left(2^l x\right)-\frac{1}{4^m}h\left(2^m x\right)\right\|_{\mathcal{Y}}^p \le \sum_{j=l}^{m-1}\frac{1}{4^{p(j+1)}}\left(5\varepsilon^p+\delta^p 2^{pq}\left[2\left(3+2^p\right)+2^{pq}\right]2^{pqj}\|x\|^{pq}\right) \tag{20}$$

for all $x \in \mathcal{X}$. By (20), the sequence $\{\frac{1}{4^j}h(2^j x)\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{1}{4^j}h(2^j x)\}$ converges for all $x \in \mathcal{X}$. Define $G:\mathcal{X}\to\mathcal{Y}$ by

$$G(x) := \lim_{j\to\infty}\frac{1}{4^j}h\left(2^j x\right)$$

for all $x \in \mathcal{X}$. By (17), we have

$$\begin{aligned}
&\|G(x+y+z)+G(x-z)+G(y-z)-G(x+y-z)-G(x+z)-G(y+z)\|_{\mathcal{Y}}^p \\
&\quad=\lim_{n\to\infty}\frac{1}{4^{pn}}\|\left[h\left(2^n(x+y+z)\right)+h\left(2^n(x-z)\right)+h\left(2^n(y-z)\right)\right. \\
&\qquad\left.-h\left(2^n(x+y-z)\right)-h\left(2^n(x+z)\right)-h\left(2^n(y+z)\right)\right]\|_{\mathcal{Y}}^p \\
&\quad\le \lim_{n\to\infty}\frac{1}{4^{pn}}\left[\varepsilon+2^{qn}\delta\left(\|x\|^q+\|y\|^q+\|z\|^q\right)\right]^p
\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Hence, the mapping G satisfies (2). Setting $l=0$ and taking $m\to\infty$ in (20), one can obtain the inequality (18).

To prove the uniqueness of G, let $H:\mathcal{X}\to\mathcal{Y}$ be another quadratic mapping satisfying (2) and (9). It follows from (19) that

$$\begin{aligned}\left\|G(2x)-4G(x)\right\|_{\mathcal{Y}}&=\lim_{n\to\infty}\left\|\frac{1}{4^n}h\left(2^{n+1}x\right)-\frac{1}{4^{n-1}}h\left(2^nx\right)\right\|_{\mathcal{Y}}\\&=4\lim_{n\to\infty}\left\|\frac{1}{4^n}h\left(2^nx\right)-\frac{1}{4^{n+1}}h\left(2^{n+1}x\right)\right\|_{\mathcal{Y}}\\&\le 4\lim_{n\to\infty}\frac{1}{4^{n+1}}\left(5\varepsilon^p+\delta^p2^{pq}\left[2\left(3+2^p\right)+2^{pq}\right]2^{pqn}\|x\|^{pq}\right)^{\frac{1}{p}}=0\end{aligned}$$

for all $x\in\mathcal{X}$. So $G(2x)=4G(x)$ for all $x\in\mathcal{X}$. It follows from (18) that

$$\begin{aligned}\left\|G(x)-H(x)\right\|_{\mathcal{Y}}^p&=\lim_{n\to\infty}\frac{1}{4^{pn}}\left\|h\left(2^nx\right)-H\left(2^nx\right)\right\|_{\mathcal{Y}}^p\\&\le\lim_{n\to\infty}\frac{1}{4^{pn}}\left(\frac{5\varepsilon^p}{4^p-1}+\frac{\delta^p[2(3+2^p)+2^{pq}]2^{pq(n+1)}}{4^p-2^{pq}}\|x\|^{pq}\right)=0\end{aligned}$$

for all $x\in\mathcal{X}$. So $G=H$. □

The following theorem proves the stability of equation (3) in quasi-Banach spaces.

Theorem 6 *Let $q\in(0,1)$, $\varepsilon>0$ and $\delta\ge 0$ and let $f:\mathcal{X}\times\mathcal{X}\to\mathcal{Y}$ be a mapping such that*

$$\begin{aligned}&\big\|f(x+y+z,u+v+w)+f(x+y-z,u+v+w)\\&\quad+2f(x,u-w)+2f(y,v-w)\\&\quad-f(x+y,u+w)-f(x+y,v+w)-f(x+z,u+w)-f(x-z,u+v-w)\\&\quad-f(y+z,v+w)-f(y-z,u+v-w)\big\|_{\mathcal{Y}}\\&\le\varepsilon+\delta\left(\|x\|^q+\|y\|^q+\|z\|^q+\|u\|^q+\|v\|^q+\|w\|^q\right)\end{aligned}\tag{21}$$

for all $x,y,z,u,v,w\in\mathcal{X}$. Then, there exists a mapping $F:\mathcal{X}\times\mathcal{X}\to\mathcal{Y}$ satisfying (3) such that

$$\begin{aligned}\left\|f(x,y)-F(x,y)\right\|_{\mathcal{Y}}\le\Bigg(&\left[\varepsilon+\delta\left(\|x\|^q+\|y\|^q\right)\right]^p+\frac{1}{16^p}\frac{8(2\varepsilon^p+3\delta^p\|x\|^{pq})}{2^p-1}\\&+\frac{5\varepsilon^p}{4^p-1}+\frac{\delta^p2^{pq}[2(3+2^p)+2^{pq}]}{4^p-2^{pq}}\|y\|^{pq}\Bigg)^{\frac{1}{p}}\end{aligned}$$

for all $x,y\in\mathcal{X}$.

Proof Define $f_1:\mathcal{X}\to\mathcal{Y}$ by $f_1(x):=f(x,0)$ for all $x\in\mathcal{X}$. Also, define $f_2:\mathcal{X}\to\mathcal{Y}$ by $f_2(y):=f(0,y)$ for all $y\in\mathcal{X}$. Letting $y=z=v=w=0$ in (21), we have

$$\left\|f(x,u)-f_1(x)-f_2(u)+f(0,0)\right\|_{\mathcal{Y}}\le\varepsilon+\delta\left(\|x\|^q+\|u\|^q\right)$$

for all $x, u \in \mathcal{X}$. Putting $u = v = w = 0$ in (21), we have

$$\begin{aligned}&\big\| f_1(x+y+z) + f_1(x+y-z) + 2f_1(x) + 2f_1(y)\\&\quad - 2f_1(x+y) - f_1(x+z) - f_1(x-z) - f_1(y+z) - f_1(y-z)\big\|_{\mathcal{Y}}\\&\le \varepsilon + \delta\big(\|x\|^q + \|y\|^q + \|z\|^q\big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Setting $x = y = z = 0$ in (21), we have

$$\begin{aligned}&\big\| f_2(u+v+w) + f_2(u-w) + f_2(v-w) - f_2(u+v-w)\\&\quad - f_2(u+w) - f_2(v+w)\big\|_{\mathcal{Y}}\\&\le \varepsilon + \delta\big(\|u\|^q + \|v\|^q + \|w\|^q\big)\end{aligned}$$

for all $u, v, w \in \mathcal{X}$.

By Theorem 4, there are a cubic mapping $T : \mathcal{X} \to \mathcal{Y}$ and an additive mapping $A : \mathcal{X} \to \mathcal{Y}$ that satisfy equation (1) and the inequality

$$\big\| f_1(x) - f_1(0) - A(x) - T(x)\big\|_{\mathcal{Y}} \le \frac{1}{16}\left(\frac{8(2\varepsilon^p + 3\delta^p\|x\|^{pq})}{2^p - 1}\right)^{\frac{1}{p}}$$

for all $x \in \mathcal{X}$. By Theorem 5, there exists a unique quadratic mapping $G : \mathcal{X} \to \mathcal{Y}$ satisfying (2) such that

$$\big\| f_2(y) - f_2(0) - G(y)\big\|_{\mathcal{Y}} \le \left(\frac{5\varepsilon^p}{4^p - 1} + \frac{\delta^p 2^{pq}[2(3+2^p) + 2^{pq}]}{4^p - 2^{pq}}\|y\|^{pq}\right)^{\frac{1}{p}}$$

for all $y \in \mathcal{X}$.

If we define

$$F(x, y) := A(x) + T(x) + G(y) + f(0, 0)$$

for all $x, y \in \mathcal{X}$, we conclude that

$$\begin{aligned}\big\| f(x,y) - F(x,y)\big\|_{\mathcal{Y}}^p &\le \big[\varepsilon + \delta\big(\|x\|^q + \|y\|^q\big)\big]^p + \frac{1}{16^p}\frac{8(2\varepsilon^p + 3\delta^p\|x\|^{pq})}{2^p - 1}\\&\quad + \frac{5\varepsilon^p}{4^p - 1} + \frac{\delta^p 2^{pq}[2(3+2^p) + 2^{pq}]}{4^p - 2^{pq}}\|y\|^{pq}\end{aligned}$$

for all $x, y \in \mathcal{X}$. □

Taking $\delta = 0$ in Theorem 6, we obtain the Ulam stability problem for functional equation (3).

Corollary 2 *Let $\varepsilon > 0$ and let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a mapping such that*

$$\begin{aligned}&\big\| f(x+y+z, u+v+w) + f(x+y-z, u+v+w)\\&\quad + 2f(x, u-w) + 2f(y, v-w)\end{aligned}$$

$$-f(x+y,u+w)-f(x+y,v+w)-f(x+z,u+w)-f(x-z,u+v-w)$$
$$-f(y+z,v+w)-f(y-z,u+v-w)\big\|_{\mathcal{Y}}\le\varepsilon$$

for all $x,y,z,u,v,w\in\mathcal{X}$. Then, there exists a mapping $F:\mathcal{X}\times\mathcal{X}\to\mathcal{Y}$ satisfying (3) *such that*

$$\big\|f(x,y)-F(x,y)\big\|_{\mathcal{Y}}\le\left(1+\frac{1}{16^{p-1}(2^p-1)}+\frac{5}{4^p-1}\right)\varepsilon$$

for all $x,y\in\mathcal{X}$.

Acknowledgements
Not applicable.

Funding
The authors declare that there is no funding available for this paper.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JH analyzed cubic functional equations, quadratic function equations, and cubic–quadratic functional equations. WG devised functional equations related to elliptic curves and was a major contributor to the manuscript. All authors have read and accepted the final manuscript.

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Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 24 February 2022 Accepted: 1 June 2022 Published online: 13 June 2022

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