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# A relational-theoretic approach to get solution of nonlinear matrix equations

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## Abstract

In this study, we consider a nonlinear matrix equation of the form  $\mathcal{X} = Q + \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i$ , where  $Q$  is a Hermitian positive definite matrix,  $\mathcal{A}_i^*$  stands for the conjugate transpose of an  $n \times n$  matrix  $\mathcal{A}_i$ , and  $\mathcal{G}$  is an order-preserving continuous mapping from the set of all Hermitian matrices to the set of all positive definite matrices such that  $\mathcal{G}(O) = O$ . We discuss sufficient conditions that ensure the existence of a unique positive definite solution of the given matrix equation. For this, we derive some fixed point results for Suzuki-FG contractive mappings on metric spaces (not necessarily complete) endowed with arbitrary binary relation (not necessarily a partial order). We provide adequate examples to validate the fixed-point results and the importance of related work, and the convergence analysis of nonlinear matrix equations through an illustration with graphical representations.

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**Keywords:** Fixed point; Suzuki type contraction; Positive definite matrix; Nonlinear matrix equation; Convergence analysis; Binary relation

## 1 Introduction

The study of nonlinear matrix equations (NME) appeared first in the literature concerned with an algebraic Riccati equation. These equations occur in a large number of problems in control theory, dynamical programming, ladder network, stochastic filtering, queuing theory, statistics, and many other applicable areas.

Let  $\mathcal{H}(n)$  (resp.  $\mathcal{K}(n)$ ,  $\mathcal{P}(n)$ ) denote the set of all  $n \times n$  Hermitian (resp. positive semi-definite, positive definite) matrices over  $\mathbb{C}$  and  $\mathcal{M}(n)$  the set of all  $n \times n$  matrices over  $\mathbb{C}$ . In [1], Ran and Reurings discussed the existence of solutions of the following equation:

$$\mathcal{X} + \mathcal{B}^* F(\mathcal{X}) \mathcal{B} = Q \tag{1}$$

in  $\mathcal{K}(n)$ , where  $\mathcal{B} \in \mathcal{M}(n)$ ,  $Q$  is positive definite, and  $F$  is a mapping from  $\mathcal{K}(n)$  into  $\mathcal{M}(n)$ . Note that  $\mathcal{X}$  is a solution of (1) if and only if it is a fixed point of the mapping  $\mathcal{G}(\mathcal{X}) = Q - \mathcal{B}^* F(\mathcal{X}) \mathcal{B}$ . In [2], authors used the notion of partial ordering and established a modification of the Banach contraction principle, which they applied for solving a class of NMEs of the form  $\mathcal{X} = Q + \sum_{i=1}^m \mathcal{B}_i^* F(\mathcal{X}) \mathcal{B}_i$  using the Ky Fan norm in  $\mathcal{M}(n)$ .

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**Theorem 1.1** ([2]) *Let  $F : \mathcal{H}(n) \rightarrow \mathcal{H}(n)$  be an order-preserving, continuous mapping which maps  $\mathcal{P}(n)$  into itself and  $Q \in \mathcal{P}(n)$ . If  $B_i, B_i^* \in \mathcal{P}(n)$  and  $\sum_{i=1}^m B_i B_i^* < M \cdot \mathcal{I}_n$  for some  $M > 0$  ( $\mathcal{I}_n$  – the unit matrix in  $\mathcal{M}(n)$ ) and if  $|\text{tr}(F(\mathcal{Y}) - F(\mathcal{X}))| \leq \frac{1}{M} |\text{tr}(\mathcal{Y} - \mathcal{X})|$  for all  $\mathcal{X}, \mathcal{Y} \in \mathcal{H}(n)$  with  $\mathcal{X} \leq \mathcal{Y}$ , then the equation  $\mathcal{X} = Q + \sum_{i=1}^m B_i^* F(\mathcal{X}) B_i$  has a unique positive definite solution (PDS).*

In recent years, a number of mathematicians have obtained fixed point results for contraction type mappings in metric spaces equipped with partial order. Some early results in this direction were established by Turinici in [3, 4]; one may note that their starting points were “amorphous” contributions in the area due to Matkowski [5, 6]. These types of results have been reinvestigated by Ran and Reurings [1] and also by Nieto and Rodríguez-López [7, 8]. Samet and Turinici [9] established fixed point theorem for nonlinear contraction under symmetric closure of an arbitrary relation. Ahmadullah et al. [10–12] and Alam and Imdad [13] employed an amorphous relation to prove a relation-theoretic analogue of the Banach contraction principle which in turn unifies a lot of well-known relevant order-theoretic fixed point theorems. Recently, Hasanuzzaman and Imdad [14] used the concept of simulation function and proved the relation theoretic metrical fixed point results for Suzuki type  $\mathcal{Z}_R$ -contraction and discussed application in solving nonlinear matrix equations.

Motivated by the above reference work, we introduce the notion of Suzuki-FG contractive mapping on metric spaces endowed with an arbitrary binary relation (not necessarily partial order), and then we prove existence and uniqueness fixed point results under weaker conditions. We justify our work by some illustrative examples and demonstrate the genuineness of Suzuki-FG contraction over Suzuki contraction, generalized Suzuki contraction, and implicit type contraction mapping. Further, we apply this result to NMEs and discuss its convergence behavior with respect to three different initial values with graphical representations and solutions by the surface plot. The experiment was run on a macOS Mojave version 10.14.6 CPU @1.6 GHz intel core i5 8GB with MATLAB R2020b as the programming language (Online).

## 2 Preliminaries

Throughout this article, the notations  $\mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{R}^+$  have their usual meanings, and  $\mathbb{N}^* = \mathbb{N} \cup \{0\}$ .

We call  $(\mathcal{E}, \mathfrak{R})$  a relational set if (i)  $\mathcal{E} \neq \emptyset$  is a set and (ii)  $\mathfrak{R}$  is a binary relation on  $\mathcal{E}$ .

In addition, if  $(\mathcal{E}, d)$  is a metric space, we call  $(\mathcal{E}, d, \mathfrak{R})$  a relational metric space (RMS, for short).

The following are some standard terms used in the theory of relational sets (see, e.g., [9, 13, 15–17]).

Let  $(\mathcal{E}, \mathfrak{R})$  be a relational set,  $(\mathcal{E}, d, \mathfrak{R})$  be an RMS, and let  $\mathfrak{S}$  be a self-mapping on  $\mathcal{E}$ . Then:

1.  $v \in \mathcal{E}$  is  $\mathfrak{R}$ -related to  $\vartheta \in \mathcal{E}$  if and only if  $(v, \vartheta) \in \mathfrak{R}$ .
2. The set  $(\mathcal{E}, \mathfrak{R})$  is said to be comparable if for all  $v, \vartheta \in \mathcal{E}$ ,  $[v, \vartheta] \in \mathfrak{R}$ , where  $[v, \vartheta] \in \mathfrak{R}$  means that either  $(v, \vartheta) \in \mathfrak{R}$  or  $(\vartheta, v) \in \mathfrak{R}$ .
3. A sequence  $(v_n)$  in  $\mathcal{E}$  is said to be  $\mathfrak{R}$ -preserving if  $(v_n, v_{n+1}) \in \mathfrak{R}, \forall n \in \mathbb{N} \cup \{0\}$ .
4.  $(\mathcal{E}, d, \mathfrak{R})$  is said to be  $\mathfrak{R}$ -complete if every  $\mathfrak{R}$ -preserving Cauchy sequence converges in  $\mathcal{E}$ .

5.  $\mathfrak{R}$  is said to be  $\mathfrak{S}$ -closed if  $(\nu, \vartheta) \in \mathfrak{R} \Rightarrow (\mathfrak{S}\nu, \mathfrak{S}\vartheta) \in \mathfrak{R}$ . It is said to be weakly  $\mathfrak{S}$ -closed if  $(\nu, \vartheta) \in \mathfrak{R} \Rightarrow [\mathfrak{S}\nu, \mathfrak{S}\vartheta] \in \mathfrak{R}$ .
6.  $\mathfrak{R}$  is said to be  $d$ -self-closed if for every  $\mathfrak{R}$ -preserving sequence with  $\nu_n \rightarrow \nu$ , there is a subsequence  $(\nu_{n_k})$  of  $(\nu_n)$  such that  $[\nu_{n_k}, \nu] \in \mathfrak{R}$  for all  $k \in \mathbb{N} \cup \{0\}$ .
7. A subset  $\mathfrak{Z}$  of  $\mathcal{E}$  is called  $\mathfrak{R}$ -directed if for each  $\nu, \vartheta \in \mathfrak{Z}$ , there exists  $\mu \in \mathcal{E}$  such that  $(\nu, \mu) \in \mathfrak{R}$  and  $(\vartheta, \mu) \in \mathfrak{R}$ . It is called  $(\mathfrak{S}, \mathfrak{R})$ -directed if for each  $\nu, \vartheta \in \mathfrak{Z}$  there exists  $\mu \in \mathcal{E}$  such that  $(\nu, \mathfrak{S}\mu) \in \mathfrak{R}$  and  $(\vartheta, \mathfrak{S}\mu) \in \mathfrak{R}$ .
8.  $\mathfrak{S}$  is said to be  $\mathfrak{R}$ -continuous at  $\nu$  if for every  $\mathfrak{R}$ -preserving sequence  $(\nu_n)$  converging to  $\nu$ , we get  $\mathfrak{S}(\nu_n) \rightarrow \mathfrak{S}(\nu)$  as  $n \rightarrow \infty$ . Moreover,  $\mathfrak{S}$  is said to be  $\mathfrak{R}$ -continuous if it is  $\mathfrak{R}$ -continuous at every point of  $\mathcal{E}$ .
9. For  $\nu, \vartheta \in \mathcal{E}$ , a path of length  $k$  (where  $k$  is a natural number) in  $\mathfrak{R}$  from  $\nu$  to  $\vartheta$  is a finite sequence  $\{\mu_0, \mu_1, \mu_2, \dots, \mu_k\} \subset \mathcal{E}$  satisfying the following conditions:
  - (i)  $\mu_0 = \nu$  and  $\mu_k = \vartheta$ ,
  - (ii)  $(\mu_i, \mu_{i+1}) \in \mathfrak{R}$  for each  $i$  ( $0 \leq i \leq k - 1$ ),
 then this finite sequence is called a path of length  $k$  joining  $\nu$  to  $\vartheta$  in  $\mathfrak{R}$ .
10. If, for a pair of  $\nu, \vartheta \in \mathcal{E}$ , there is a finite sequence  $\{\mu_0, \mu_1, \mu_2, \dots, \mu_k\} \subset \mathcal{E}$  satisfying the following conditions:
  - (i)  $\mathfrak{S}\mu_0 = \nu$  and  $\mathfrak{S}\mu_k = \vartheta$ ,
  - (ii)  $(\mathfrak{S}\mu_i, \mathfrak{S}\mu_{i+1}) \in \mathfrak{R}$  for each  $i$  ( $0 \leq i \leq k - 1$ ),
 then this finite sequence is called a  $\mathfrak{S}$ -path of length  $k$  joining  $\nu$  to  $\vartheta$  in  $\mathfrak{R}$ .

Notice that a path of length  $k$  involves  $k + 1$  elements of  $\mathcal{E}$  although they are not necessarily distinct.

We fix the following notation for a relational metric space  $(\mathcal{E}, d, \mathfrak{R})$ , a self-mapping  $\mathfrak{S}$  on  $\mathcal{E}$ , and an  $\mathfrak{R}$ -directed subset  $\mathfrak{D}$  of  $\mathcal{E}$ :

- (i)  $\text{Fix}(\mathfrak{S}) :=$  the set of all fixed points of  $\mathfrak{S}$ ,
- (ii)  $\mathfrak{X}(\mathfrak{S}, \mathfrak{R}) := \{\nu \in \mathcal{E} : (\nu, \mathfrak{S}\nu) \in \mathfrak{R}\}$ ,
- (iii)  $\mathfrak{P}(\nu, \vartheta, \mathfrak{R}) :=$  the class of all paths in  $\mathfrak{R}$  from  $\nu$  to  $\vartheta$  in  $\mathfrak{R}$ , where  $\nu, \vartheta \in \mathcal{E}$ .

### 3 Results on Suzuki-FG contractive mappings

**Definition 3.1** ([18]) The collection of all functions  $\mathcal{F} : \mathbb{R}_+ \rightarrow \mathbb{R}$  satisfying:

- (F<sub>1</sub>)  $\mathcal{F}$  is continuous and strictly increasing;
- (F<sub>2</sub>) for each  $\{\xi_n\} \subseteq \mathbb{R}_+$ ,  $\lim_{n \rightarrow \infty} \xi_n = 0$  iff  $\lim_{n \rightarrow \infty} \mathcal{F}(\xi_n) = -\infty$ ,

will be denoted by  $\mathbb{F}$ .

The collection of all pairs of mappings  $(\mathcal{G}, \beta)$ , where  $\mathcal{G} : \mathbb{R}_+ \rightarrow \mathbb{R}$ ,  $\beta : \mathbb{R}_+ \rightarrow [0, 1)$ , satisfying:

- (F<sub>3</sub>) for each  $\{\xi_n\} \subseteq \mathbb{R}_+$ ,  $\limsup_{n \rightarrow \infty} \mathcal{G}(\xi_n) \geq 0$  iff  $\limsup_{n \rightarrow \infty} \xi_n \geq 1$ ;
- (F<sub>4</sub>) for each  $\{\xi_n\} \subseteq \mathbb{R}_+$ ,  $\limsup_{n \rightarrow \infty} \beta(\xi_n) = 1$  implies  $\lim_{n \rightarrow \infty} \xi_n = 0$ ;
- (F<sub>5</sub>) for each  $\{\xi_n\} \subseteq \mathbb{R}_+$ ,  $\sum_{n=1}^{\infty} \mathcal{G}(\beta(\xi_n)) = -\infty$ ,

will be denoted by  $\mathbb{G}_\beta$ .

**Definition 3.2** Let  $(\mathcal{E}, d, \mathfrak{R})$  be an RMS and  $\mathcal{P} : \mathcal{E} \rightarrow \mathcal{E}$  be a given mapping. A mapping  $\mathcal{P}$  is said to be a Suzuki-FG contractive mapping if there exist  $\mathcal{F} \in \mathbb{F}$  and  $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$  such that, for  $(\nu, \vartheta) \in \mathcal{E}$  with  $(\nu, \vartheta) \in \mathfrak{R}^*$ ,

$$\begin{cases} \frac{1}{2}d(\nu, \mathcal{P}\nu) \leq d(\nu, \vartheta) & \text{implies} \\ \mathcal{F}(d(\mathcal{P}\nu, \mathcal{P}\vartheta)) \leq \mathcal{F}(\mathfrak{R}(\nu, \vartheta)) + \mathcal{G}(\beta(\mathfrak{R}(\nu, \vartheta))), \end{cases} \tag{2}$$

where

$$\begin{aligned} \mathfrak{N}(v, \vartheta) &= \max \left\{ d(v, \vartheta), d(v, \mathcal{P}v), d(\vartheta, \mathcal{P}\vartheta), \frac{d(v, \mathcal{P}\vartheta) + d(\vartheta, \mathcal{P}v)}{2} \right\}, \\ \mathfrak{N}^* &= \{(v, \vartheta) \in \mathfrak{R} \mid \mathcal{P}v \neq \mathcal{P}\vartheta\}. \end{aligned} \tag{3}$$

We denote by  $(\text{SFG})_{\mathfrak{R}}$  the collection of all Suzuki-FG contractive mappings on  $(\mathcal{E}, d, \mathfrak{R})$ .

Now, we are equipped to state and prove our first main result as follows.

**Theorem 3.3** *Let  $(\mathcal{E}, d, \mathfrak{R})$  be an RMS and  $\mathcal{P}: \mathcal{E} \rightarrow \mathcal{E}$ . Suppose that the following conditions hold:*

- (C<sub>1</sub>)  $\mathfrak{X}(\mathcal{P}, \mathfrak{R}) \neq \emptyset$ ;
- (C<sub>2</sub>)  $\mathfrak{R}$  is  $\mathcal{P}$ -closed and  $\mathcal{P}$ -transitive;
- (C<sub>3</sub>)  $\mathcal{E}$  is  $\mathfrak{R}$ -complete;
- (C<sub>4</sub>)  $\mathcal{P} \in (\text{SFG})_{\mathfrak{R}}$ ;
- (C<sub>5</sub>)  $\mathfrak{S}$  is  $\mathfrak{R}$ -continuous or
- (C'<sub>5</sub>)  $\mathfrak{R}$  is  $d$ -self-closed.

Then there exists a point  $v_* \in \text{Fix}(\mathcal{P})$ .

*Proof* Starting with  $v_0 \in \mathcal{E}$  given by (C<sub>1</sub>), we construct a sequence  $\{v_n\}$  of Picard iterates  $v_{n+1} = \mathcal{P}^n(v_0)$  for all  $n \in \mathbb{N}^*$ .

Using (C<sub>1</sub>)–(C<sub>2</sub>), we have that  $(\mathcal{P}v_0, \mathcal{P}^2v_0) \in \mathfrak{R}$ . Continuing this process inductively, we obtain

$$(\mathcal{P}^n v_0, \mathcal{P}^{n+1} v_0) \in \mathfrak{R} \tag{4}$$

for any  $n \in \mathbb{N}^*$ . Hence,  $\{v_n\}$  is an  $\mathfrak{R}$ -preserving sequence.

Now, if there exists some  $n_0 \in \mathbb{N}^*$  such that  $d(v_{n_0}, \mathcal{P}v_{n_0}) = 0$ , then the result follows immediately. Otherwise, for all  $n \in \mathbb{N}^*$ ,  $d(v_n, \mathcal{P}v_n) > 0$  so that  $\mathcal{P}v_n \neq \mathcal{P}v_{n+1}$  which implies that  $(v_n, v_{n+1}) \in \mathfrak{R}^*$  and  $\frac{1}{2}d(v_n, \mathcal{P}v_n) < d(v_n, \mathcal{P}v_n)$ . Therefore, using (C<sub>4</sub>) for  $v = v_n, \vartheta = v_{n+1}$ , we have

$$\mathcal{F}(d(\mathcal{P}v_n, \mathcal{P}v_{n+1})) \leq \mathcal{F}(\mathfrak{N}(v_n, v_{n+1})) + \mathcal{G}(\beta(\mathfrak{N}(v_n, v_{n+1}))),$$

where

$$\begin{aligned} \mathfrak{N}(v_n, v_{n+1}) &= \max \left\{ d(v_n, v_{n+1}), d(v_n, \mathcal{P}v_n), d(v_{n+1}, \mathcal{P}v_{n+1}), \frac{d(v_n, \mathcal{P}v_{n+1}) + d(v_{n+1}, \mathcal{P}v_n)}{2} \right\} \\ &= \max \left\{ d(v_n, v_{n+1}), d(v_n, v_{n+1}), d(v_{n+1}, v_{n+2}), \frac{d(v_n, v_{n+2})}{2} \right\} \\ &\leq \max \left\{ d(v_n, v_{n+1}), d(v_n, v_{n+1}), d(v_{n+1}, v_{n+2}), \frac{d(v_n, v_{n+1}) + d(v_{n+1}, v_{n+2})}{2} \right\} \\ &= \max \{ d(v_n, v_{n+1}), d(v_{n+1}, v_{n+2}) \}. \end{aligned}$$

If  $\mathfrak{N}(v_n, v_{n+1}) = d(v_{n+1}, v_{n+2})$ , then

$$\mathcal{F}(d(v_{n+1}, v_{n+2})) \leq \mathcal{F}(d(v_{n+1}, v_{n+2})) + \mathcal{G}(\beta(d(v_{n+1}, v_{n+2}))),$$

which implies  $\mathcal{G}(\beta(d(v_{n+1}, v_{n+2}))) \geq 0$ , i.e.,  $\beta(d(v_{n+1}, v_{n+2})) \geq 1$ , a contradiction. Therefore

$$d(v_{n+1}, v_{n+2}) \leq d(v_n, v_{n+1}) \quad \text{for all } n \in \mathbb{N}, \tag{5}$$

and so

$$\mathcal{F}(d(v_{n+1}, v_{n+2})) \leq \mathcal{F}(d(v_n, v_{n+1})) + \mathcal{G}(\beta(d(v_n, v_{n+1})))$$

for all  $n \in \mathbb{N}$ . Consequently,

$$\begin{aligned} \mathcal{F}(d(v_n, v_{n+1})) &\leq \mathcal{F}(d(v_{n-1}, v_n)) + \mathcal{G}(\beta(d(v_{n-1}, v_n))) \\ &\vdots \\ &\leq \mathcal{F}(d(v_0, v_1)) + \sum_{i=1}^{i=n} \mathcal{G}(\beta(d(v_i, v_{i-1}))). \end{aligned} \tag{6}$$

Letting  $n \rightarrow \infty$  gives  $\lim_{n \rightarrow \infty} \mathcal{F}(d(v_n, v_{n+1})) = -\infty$  and  $\mathcal{F} \in \mathbb{F}$  gives

$$\lim_{n \rightarrow \infty} d(v_n, v_{n+1}) = 0. \tag{7}$$

We will now show that the sequence  $\{v_n\}$  is an  $\mathfrak{N}$ -preserving Cauchy sequence in  $(\mathcal{E}, d)$ . On the contrary, we suppose that there exist  $\zeta > 0$  and two subsequences  $\{v_{n(j)}\}$  and  $\{v_{m(j)}\}$  of  $\{v_n\}$  such that  $m(j)$  is the smallest index for which  $m(j) > n(j) > j$  and

$$d(v_{m(j)}, v_{n(j)}) \geq \zeta. \tag{8}$$

This means that  $m(j) > n(j) > j$  and

$$d(v_{m(j)-1}, v_{n(j)}) < \zeta. \tag{9}$$

On the other hand,

$$\zeta \leq d(v_{m(j)}, v_{n(j)}) \leq d(v_{m(j)}, v_{m(j)-1}) + d(v_{m(j)-1}, v_{n(j)}) \leq d(v_{m(j)}, v_{m(j)-1}) + \zeta.$$

Taking  $j \rightarrow \infty$  and using (7), we get

$$\lim_{j \rightarrow \infty} d(v_{m(j)}, v_{n(j)}) = \zeta, \tag{10}$$

and hence

$$\lim_{j \rightarrow \infty} d(v_{m(j)+1}, v_{n(j)+1}) = \zeta. \tag{11}$$

Then, from (7) and (10), one can select a positive integer  $N \in \mathbb{N}$  such that

$$\frac{1}{2}d(v_{m(j)}, \mathcal{P}v_{m(j)}) < \frac{1}{2}\xi < d(v_{m(j)}, v_{n(j)}) \quad \text{for all } j \geq N.$$

As the sequence  $\{v_n\}$  is  $\mathfrak{R}$ -preserving and  $\mathfrak{R}$  is  $\mathcal{P}$ -transitive, therefore  $(v_{m(j)}, v_{n(j)}) \in \mathfrak{R}^*$ , and we get

$$\begin{aligned} & \mathcal{F}\left(\limsup_{j \rightarrow \infty} d(v_{m(j)+1}, v_{n(j)+1})\right) \\ & \leq \mathcal{F}\left(\limsup_{j \rightarrow \infty} \mathfrak{N}(v_{m(j)}, v_{n(j)})\right) + \limsup_{j \rightarrow \infty} \mathcal{G}\left(\beta(\mathfrak{N}(v_{m(j)}, v_{n(j)}))\right), \end{aligned} \tag{12}$$

where

$$\begin{aligned} & \mathfrak{N}(v_{m(j)}, v_{n(j)}) \\ & = \max \left\{ \begin{aligned} & d(v_{m(j)}, v_{n(j)}), d(v_{m(j)}, \mathcal{P}v_{m(j)}), d(v_{n(j)}, \mathcal{P}v_{n(j)}), \\ & \frac{d(v_{n(j)}, \mathcal{P}v_{m(j)}) + d(v_{m(j)}, \mathcal{P}v_{n(j)})}{2} \end{aligned} \right\} \\ & = \max \left\{ \begin{aligned} & d(v_{m(j)}, v_{n(j)}), d(v_{m(j)}, v_{m(j)+1}), d(v_{n(j)}, v_{n(j)+1}), \\ & \frac{d(v_{n(j)}, v_{m(j)+1}) + d(v_{m(j)}, v_{n(j)+1})}{2} \end{aligned} \right\} \\ & \leq \max \left\{ \begin{aligned} & d(v_{m(j)}, v_{n(j)}), d(v_{m(j)}, v_{m(j)+1}), d(v_{n(j)}, v_{n(j)+1}), \\ & \frac{d(v_{n(j)}, v_{m(j)}) + d(v_{m(j)}, v_{m(j)+1}) + d(v_{m(j)}, v_{n(j)}) + d(v_{n(j)}, v_{n(j)+1})}{2} \end{aligned} \right\}. \end{aligned}$$

Taking upper limit as  $j \rightarrow \infty$  and making use of (7), (10), and (11), we get

$$\limsup_{j \rightarrow \infty} \mathfrak{N}(v_{m(j)}, v_{n(j)}) = \limsup_{j \rightarrow \infty} d(v_{m(j)}, v_{n(j)}). \tag{13}$$

Therefore, from (12), (11), and (13), we have

$$\begin{aligned} \mathcal{F}(\xi) & = \mathcal{F}\left(\limsup_{j \rightarrow \infty} d(v_{m(j)+1}, v_{n(j)+1})\right) \\ & \leq \mathcal{F}\left(\limsup_{j \rightarrow \infty} d(v_{m(j)}, v_{n(j)})\right) + \limsup_{j \rightarrow \infty} \mathcal{G}\left(\beta(d(v_{m(j)}, v_{n(j)}))\right) \\ & = \mathcal{F}(\xi) + \limsup_{j \rightarrow \infty} \mathcal{G}\left(\beta(d(v_{m(j)}, v_{n(j)}))\right), \end{aligned}$$

which implies that  $\limsup_{j \rightarrow \infty} \mathcal{G}(\beta(d(v_{m(j)}, v_{n(j)}))) \geq 0$ , which gives  $\limsup_{j \rightarrow \infty} \beta(d(v_{m(j)}, v_{n(j)})) \geq 1$ , and taking into account that  $\beta(\xi) < 1$  for all  $\xi \geq 0$ , we have  $\limsup_{j \rightarrow \infty} \beta(d(v_{m(j)}, v_{n(j)})) = 1$ . Therefore,  $\limsup_{j \rightarrow \infty} d(v_{m(j)}, v_{n(j)}) = 0$ , a contradiction. Hence,  $\{v_n\}$  is an  $\mathfrak{R}$ -preserving Cauchy sequence in  $\mathcal{E}$ .

The  $\mathfrak{R}$ -completeness of  $\mathcal{E}$  implies that there exists  $v^* \in \mathcal{E}$  such that  $\lim_{n \rightarrow \infty} v_n = v^*$ . Now, first by (C5), we have

$$v^* = \lim_{n \rightarrow \infty} v_{n+1} = \lim_{n \rightarrow \infty} \mathcal{P}v_n = \mathcal{P}v^*, \tag{14}$$

and hence  $v^*$  is a fixed point of  $\mathcal{P}$ .

Alternatively, suppose that  $\mathfrak{N}$  is  $d$ -self-closed. Then there exists a subsequence  $\{v_{n_k}\}$  of  $\{v_n\}$  with  $[v_{n_k}, v_*] \in \mathfrak{N}$  for all  $k \in \mathbb{N}^*$ . Now, we assert that

$$\frac{1}{2}d(v_{n_k}, \mathcal{P}v_{n_k}) < d(v_{n_k}, v_*) \quad \text{or} \quad \frac{1}{2}d(\mathcal{P}v_{n_k}, \mathcal{P}^2v_{n_k}) < d(\mathcal{P}v_{n_k}, v_*) \tag{15}$$

for all  $k \in \mathbb{N}^*$ .

Let, to the contrary, there exist  $\zeta \in \mathbb{N}$  such that

$$\frac{1}{2}d(v_{n(\zeta)}, \mathcal{P}v_{n(\zeta)}) \geq d(v_{n(\zeta)}, v_*) \quad \text{and} \quad \frac{1}{2}d(\mathcal{P}v_{n(\zeta)}, \mathcal{P}^2v_{n(\zeta)}) \geq d(\mathcal{P}v_{n(\zeta)}, v_*), \tag{16}$$

so that

$$2d(v_{n(\zeta)}, v_*) \leq d(v_{n(\zeta)}, \mathcal{P}v_{n(\zeta)}) \leq d(v_{n(\zeta)}, v_*) + d(v_*, \mathcal{P}v_{n(\zeta)}),$$

and

$$d(v_{n(\zeta)}, v_*) \leq d(v_*, \mathcal{P}v_{n(\zeta)}) \leq \frac{1}{2}d(\mathcal{P}v_{n(\zeta)}, \mathcal{P}^2v_{n(\zeta)}). \tag{17}$$

Now, from (5) and using (16), (17), we have

$$\begin{aligned} d(\mathcal{P}v_{n(\zeta)}, \mathcal{P}^2v_{n(\zeta)}) &< d(v_{n(\zeta)}, \mathcal{P}v_{n(\zeta)}) \\ &\leq d(v_{n(\zeta)}, v_*) + d(v_*, \mathcal{P}v_{n(\zeta)}) \\ &\leq \frac{1}{2}d(\mathcal{P}v_{n(\zeta)}, \mathcal{P}^2v_{n(\zeta)}) + \frac{1}{2}d(\mathcal{P}v_{n(\zeta)}, \mathcal{P}^2v_{n(k)}) \\ &= d(\mathcal{P}v_{n(\zeta)}, \mathcal{P}^2v_{n(\zeta)}), \end{aligned}$$

a contradiction, and therefore (15) remains true.

Now, we distinguish two cases for  $\Lambda = \{k \in \mathbb{N} : \mathcal{P}v_{n_k} = \mathcal{P}v_*\}$ . If  $\Lambda$  is finite, then there exists  $k_0 \in \mathbb{N}$  such that  $\mathcal{P}v_{n_k} \neq \mathcal{P}v_*$  for all  $k > k_0$ . It follows from (15) (for all  $k > k_0$ ) that either

$$\mathcal{F}(d(\mathcal{P}v_{n_k}, \mathcal{P}v_*)) \leq \mathcal{F}(\mathfrak{N}(v_{n_k}, v_*)) + \mathcal{G}(\beta(\mathfrak{N}(v_{n_k}, v_*))),$$

where

$$\mathfrak{N}(v_{n_k}, v_*) = \max \left\{ d(v_{n_k}, v_*), d(v_{n_k}, \mathcal{P}v_{n_k}), d(v_*, \mathcal{P}v_*), \frac{d(v_{n_k}, \mathcal{P}v_*) + d(v_*, \mathcal{P}v_{n_k})}{2} \right\}.$$

Applying limit as  $n \rightarrow \infty$ , we get  $\lim_{n \rightarrow \infty} \mathfrak{N}(v_{n_k}, v_*) = d(v_*, \mathcal{P}v_*)$ , which implies that  $\limsup_{n \rightarrow \infty} \mathcal{G}(\beta(\mathfrak{N}(v_{n_k}, v_*))) \geq 0$ , which gives  $\limsup_{n \rightarrow \infty} \beta(\mathfrak{N}(v_{n_k}, v_*)) \geq 1$ , and taking into account that  $\beta(\xi) < 1$  for all  $\xi \geq 0$ , we have  $\limsup_{n \rightarrow \infty} \beta(\mathfrak{N}(v_{n_k}, v_*)) = 1$ . Therefore,  $\limsup_{n \rightarrow \infty} \mathfrak{N}(v_{n_k}, v_*) = 0$ . Hence,  $d(v_*, \mathcal{P}v_*) = 0$ , we get  $v_* = \mathcal{P}v_*$ .

Otherwise, if  $\Lambda$  is not finite, then there is a subsequence  $\{v_{n(k(\zeta))}\}$  of  $\{v_{n_k}\}$  such that

$$v_{n(k(\zeta))+1} = \mathcal{P}v_{n(k(\zeta))} = \mathcal{P}v_*, \quad \forall \zeta \in \mathbb{N}.$$

As  $v_{n_k} \xrightarrow{d} v_*$ , therefore  $\mathcal{P}v_* = v_*$ . □

**Theorem 3.4** *In addition to the assumptions of Theorem 3.3, let  $\mathfrak{A}(v, \vartheta; \mathfrak{A}|_{\mathcal{P}(\mathcal{E})}) \neq \emptyset$  for all  $v, \vartheta \in \mathcal{P}(\mathcal{E})$ . Then  $\mathcal{P}$  has a unique fixed point.*

*Proof* In view of Theorem 3.3,  $\text{Fix}(\mathcal{P}) \neq \emptyset$ . If  $\text{Fix}(\mathcal{P})$  is a singleton, then we concluded the proof. Otherwise, let  $v_* \neq w \in \text{Fix}(\mathcal{P})$ . Since  $\mathfrak{A}(v, \vartheta; \mathfrak{A}|_{\mathcal{P}(\mathcal{E})}) \neq \emptyset$  for all  $\vartheta, v \in \mathcal{P}(\mathcal{E})$ , there exists a path  $\{\mathcal{P}z_0, \mathcal{P}z_1, \dots, \mathcal{P}z_k\}$  of some length  $k$  in  $\mathfrak{A}|_{\mathcal{P}(\mathcal{E})}$  such that  $\mathcal{P}z_0 = v_*, \mathcal{P}z_k = w$  and  $(\mathcal{P}z_j, \mathcal{P}z_{j+1}) \in \mathfrak{A}|_{\mathcal{P}(\mathcal{E})}$  for each  $j = 0, 1, 2, \dots, k - 1$ . Since  $\mathfrak{A}$  is  $\mathcal{P}$ -transitive, we have

$$(v_*, \mathcal{P}z_1) \in \mathfrak{A}, (\mathcal{P}z_1, \mathcal{P}z_2) \in \mathfrak{A}, \dots, (\mathcal{P}z_{k-1}, w) \in \mathfrak{A} \Rightarrow (v_*, w) \in \mathfrak{A}.$$

Also, due to the fact  $\frac{1}{2}d(v_*, \mathcal{P}v_*) < d(v_*, w)$  and  $(v_*, w) \in \mathfrak{A}^*$ , we have

$$\mathcal{F}(d(\mathcal{P}v_*, \mathcal{P}w)) \leq \mathcal{F}(\mathfrak{N}(w, v_*)) + \mathcal{G}(\beta(\mathfrak{N}(w, v_*))), \tag{18}$$

where

$$\begin{aligned} \mathfrak{N}(w, v_*) &= \max \left\{ d(v_*, w), d(v_*, \mathcal{P}v_*), d(w, \mathcal{P}w), \frac{d(v_*, \mathcal{P}w) + d(w, \mathcal{P}v_*)}{2} \right\} \\ &= d(v_*, w), \end{aligned}$$

which on substituting in (18) gives

$$\mathcal{F}(d(v_*, w)) \leq \mathcal{F}(d(v_*, w)) + \mathcal{G}(\beta(d(v_*, w))),$$

which gives  $\mathcal{G}(\beta(d(v_*, w))) \geq 0$  implies  $\beta(d(v_*, w)) \geq 1$ , a contradiction. Therefore  $d(v_*, w) = 0$ . □

**Theorem 3.5** *In addition to the hypotheses of Theorem 3.3 (or Theorem 3.4), if any of the following conditions is fulfilled:*

- (I) *for all  $u, v \in \mathcal{E}$ , there exists  $z \in \mathcal{E}$  such that*

$$\{(z, \mathcal{P}z), (z, u), (z, v)\} \subseteq \mathfrak{A}; \tag{19}$$

- (II) *the set  $\mathcal{P}(\mathcal{E})$  is  $\mathfrak{A}$ -directed;*
- (III)  *$\mathfrak{A}|_{\mathcal{P}(\mathcal{E})}$  is complete;*
- (IV)  *$\mathcal{Y}(u, v, \text{Fix}(\mathcal{P}), \mathfrak{A}^s)$  is nonempty for each  $u, v \in \text{Fix}(\mathcal{P})$ ,*

*then  $\mathcal{P}$  has a unique fixed point.*

*Proof* In view of Theorem 3.3 (or Theorem 3.4),  $\text{Fix}(\mathcal{P}) \neq \emptyset$ .

- *Assume (I).* Suppose that there exist distinct fixed points  $u$  and  $v$  of  $\mathcal{P}$ . We will consider the following two cases.

*Case (A):* We have  $(u, v) \in \mathfrak{A}$ , then  $\mathcal{P}^n u = u$  and  $\mathcal{P}^n v = v$  such that  $(\mathcal{P}^n u, \mathcal{P}^n v) \in \mathfrak{R}^*$  for  $n = 0, 1, \dots$ . Now, we assert that

$$\frac{1}{2}d(\mathcal{P}^n u, \mathcal{P}^{n+1} u) < d(\mathcal{P}^n u, \mathcal{P}^n v) \quad \text{or} \quad \frac{1}{2}d(\mathcal{P}^{n+1} u, \mathcal{P}^{n+2} u) < d(\mathcal{P}^{n+1} u, \mathcal{P}^n v). \tag{20}$$

Let, to the contrary, there exist  $\varsigma \in \mathbb{N}$  such that

$$\frac{1}{2}d(\mathcal{P}^n u_\varsigma, \mathcal{P}^{n+1} u_\varsigma) \geq d(\mathcal{P}^n u_\varsigma, \mathcal{P}^n v_\varsigma) \tag{21}$$

and

$$\frac{1}{2}d(\mathcal{P}^{n+1} u_\varsigma, \mathcal{P}^{n+2} u_\varsigma) \geq d(\mathcal{P}^{n+1} u_\varsigma, \mathcal{P}^n v_\varsigma). \tag{22}$$

These imply that

$$2d(\mathcal{P}^n u_\varsigma, \mathcal{P}^n v_\varsigma) \leq d(\mathcal{P}^n u_\varsigma, \mathcal{P}^{n+1} u_\varsigma) \leq d(\mathcal{P}^n u_\varsigma, \mathcal{P}^n v_\varsigma) + d(\mathcal{P}^n v_\varsigma, \mathcal{P}^{n+1} u_\varsigma),$$

and so

$$d(\mathcal{P}^n u_\varsigma, \mathcal{P}^n v_\varsigma) \leq d(\mathcal{P}^n v_\varsigma, \mathcal{P}^{n+1} u_\varsigma) \leq \frac{1}{2}d(\mathcal{P}^{n+1} u_\varsigma, \mathcal{P}^{n+2} u_\varsigma). \tag{23}$$

Now, from (5) and using (21)–(23), we have

$$\begin{aligned} d(\mathcal{P}^{n+1} u_\varsigma, \mathcal{P}^{n+2} u_\varsigma) &< d(\mathcal{P}^n u_\varsigma, \mathcal{P}^{n+1} u_\varsigma) \\ &\leq d(\mathcal{P}^n u_\varsigma, \mathcal{P}^n v_\varsigma) + d(\mathcal{P}^n v_\varsigma, \mathcal{P}^{n+1} u_\varsigma) \\ &\leq \frac{1}{2}d(\mathcal{P}^{n+1} u_\varsigma, \mathcal{P}^{n+2} u_\varsigma) + \frac{1}{2}d(\mathcal{P}^{n+1} u_\varsigma, \mathcal{P}^{n+2} u_\varsigma) \\ &= d(\mathcal{P}^{n+1} u_\varsigma, \mathcal{P}^{n+2} u_\varsigma), \end{aligned}$$

a contradiction, and therefore (20) remains true. Therefore, using condition (2),

$$\mathcal{F}(d(\mathcal{P}^{n+1} u, \mathcal{P}^{n+1} v)) \leq \mathcal{F}(\mathfrak{N}(\mathcal{P}^n u, \mathcal{P}^n v)) + \mathcal{G}(\beta(\mathfrak{N}(\mathcal{P}^n u, \mathcal{P}^n v))),$$

where

$$\mathfrak{N}(\mathcal{P}^n u, \mathcal{P}^n v) = \max \left\{ d(\mathcal{P}^n u, \mathcal{P}^n v), d(\mathcal{P}^n u, \mathcal{P}^{n+1} u), d(\mathcal{P}^n v, \mathcal{P}^{n+1} v), \frac{d(\mathcal{P}^n u, \mathcal{P}^{n+1} v) + d(\mathcal{P}^n v, \mathcal{P}^{n+1} u)}{2} \right\}.$$

Since  $u$  and  $v$  are fixed points of  $\mathcal{P}$ , we have

$$\mathfrak{N}(\mathcal{P}^n u, \mathcal{P}^n v) = d(u, v),$$

and so we get

$$\mathcal{F}(d(u, v)) \leq \mathcal{F}(d(u, v)) + \mathcal{G}(\beta(d(u, v))),$$

which gives  $\mathcal{G}(\beta(d(u, v))) \geq 0$ , and so  $\beta(d(u, v)) \geq 1$ , a contradiction. Therefore the fixed point is unique.

*Case (B):* By assumption (I), there exists  $z \in \mathcal{E}$  satisfying condition (19). Due to the  $\mathcal{P}$ -closedness of  $\mathfrak{A}$ , we get

$$(\mathcal{P}^{n-1} z, u) \in \mathfrak{A}, \quad (\mathcal{P}^{n-1} z, v) \in \mathfrak{A}.$$

Now, we assert that

$$\frac{1}{2}d(\mathcal{P}^{n-1}z, \mathcal{P}^nz) < d(\mathcal{P}^{n-1}z, u) \quad \text{or} \quad \frac{1}{2}d(\mathcal{P}^nz, \mathcal{P}^{n+1}z) < d(\mathcal{P}^nz, u). \tag{24}$$

Let, to the contrary, there exist  $\varsigma \in \mathbb{N}$  such that

$$\frac{1}{2}d(\mathcal{P}^{n-1}z_\varsigma, \mathcal{P}^nz_\varsigma) \geq d(\mathcal{P}^{n-1}z_\varsigma, u_\varsigma) \tag{25}$$

and

$$\frac{1}{2}d(\mathcal{P}^nz_\varsigma, \mathcal{P}^{n+1}z_\varsigma) \geq d(\mathcal{P}^nz_\varsigma, u_\varsigma). \tag{26}$$

These imply that

$$2d(\mathcal{P}^{n-1}z_\varsigma, u_\varsigma) \leq d(\mathcal{P}^{n-1}z_\varsigma, \mathcal{P}^nz_\varsigma) \leq d(\mathcal{P}^{n-1}z_\varsigma, u_\varsigma) + d(u_\varsigma, \mathcal{P}^nz_\varsigma),$$

which implies that (using (26))

$$d(\mathcal{P}^{n-1}z_\varsigma, u_\varsigma) \leq d(u_\varsigma, \mathcal{P}^nz_\varsigma) \leq \frac{1}{2}d(\mathcal{P}^nz_\varsigma, \mathcal{P}^{n+1}z_\varsigma). \tag{27}$$

Now, from (5) and using (25)–(27), we have

$$\begin{aligned} d(\mathcal{P}^nz_\varsigma, \mathcal{P}^{n+1}z_\varsigma) &< d(\mathcal{P}^{n-1}z_\varsigma, \mathcal{P}^nz_\varsigma) \\ &\leq d(\mathcal{P}^{n-1}z_\varsigma, u_\varsigma) + d(u_\varsigma, \mathcal{P}^nz_\varsigma) \\ &\leq \frac{1}{2}d(\mathcal{P}^nz_\varsigma, \mathcal{P}^{n+1}z_\varsigma) + \frac{1}{2}d(\mathcal{P}^nz_\varsigma, \mathcal{P}^{n+1}z_\varsigma) \\ &= d(\mathcal{P}^nz_\varsigma, \mathcal{P}^{n+1}z_\varsigma), \end{aligned}$$

a contradiction, and therefore (24) remains true. Therefore, using condition (2),

$$\mathcal{F}(d(\mathcal{P}^nz, u)) \leq \mathcal{F}(\mathfrak{N}(\mathcal{P}^{n-1}z, u)) + \mathcal{G}(\beta(\mathfrak{N}(\mathcal{P}^{n-1}z, u))), \tag{28}$$

where

$$\begin{aligned} \mathfrak{N}(\mathcal{P}^{n-1}z, u) &= \max \left\{ d(\mathcal{P}^{n-1}z, u), d(\mathcal{P}^{n-1}z, \mathcal{P}^nz), d(u, \mathcal{P}u), \frac{d(\mathcal{P}^{n-1}z, \mathcal{P}u) + d(u, \mathcal{P}^nz)}{2} \right\} \\ &\leq \max \left\{ d(\mathcal{P}^{n-1}z, u), d(\mathcal{P}^{n-1}z, \mathcal{P}^nz), d(u, \mathcal{P}u), \frac{2d(\mathcal{P}^{n-1}z, u) + d(\mathcal{P}^{n-1}z, \mathcal{P}^nz)}{2} \right\} \\ &\leq \max \{ d(\mathcal{P}^{n-1}z, u), d(\mathcal{P}^{n-1}z, \mathcal{P}^nz), d(u, \mathcal{P}u) \}. \end{aligned}$$

Using  $(z, \mathcal{P}z) \in \mathfrak{A}$ , similarly as in the proof of Theorem 3.3, it can be shown that  $d(\mathcal{P}^{n-1}z, \mathcal{P}^nz) \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, for  $n$  sufficiently large,

$$\max \{ d(\mathcal{P}^{n-1}z, u), d(\mathcal{P}^{n-1}z, \mathcal{P}^nz), d(u, \mathcal{P}u) \} = d(\mathcal{P}^{n-1}z, u)$$

and from (28) we have

$$\mathcal{F}(d(\mathcal{P}^n z, u)) \leq \mathcal{F}(d(\mathcal{P}^{n-1} z, u)) + \mathcal{G}(\beta(d(\mathcal{P}^{n-1} z, u))).$$

As in the proof of Theorem 3.3, it can be shown that  $d(\mathcal{P}^n z, u) \leq d(\mathcal{P}^{n-1} z, u)$ . It follows that the sequence  $\{d(\mathcal{P}^n z, u)\}$  is nonincreasing. As earlier, we have

$$\lim_{n \rightarrow \infty} d(\mathcal{P}^n z, u) = 0.$$

Also, since  $(z, v) \in \mathfrak{R}$ , proceeding as earlier, we can prove that

$$\lim_{n \rightarrow \infty} d(\mathcal{P}^n z, v) = 0,$$

and by using limit uniqueness, we infer that  $u = v$ ; i.e., the fixed point of  $\mathcal{P}$  is unique.

- *Assume (II).* For any two fixed points  $u, v$  of  $\mathcal{P}$ , there must be an element  $z \in \mathcal{P}(\mathcal{E})$  such that

$$(z, u) \in \mathfrak{R} \quad \text{and} \quad (z, v) \in \mathfrak{R}.$$

As  $\mathfrak{R}$  is  $\mathcal{P}$ -closed, so for all  $n \in \mathbb{N} \cup \{0\}$ ,

$$(\mathcal{P}^n z, u) \in \mathfrak{R} \quad \text{and} \quad (\mathcal{P}^n z, v) \in \mathfrak{R}.$$

In the line of proof of Case(B) (I), we obtain  $u = v$ , i.e.,  $\mathcal{P}$  has a unique fixed point.

- *Assume (III).* Suppose that  $u, v$  are two fixed points of  $\mathcal{P}$ . Then we must have  $(u, v) \in \mathfrak{R}$ , and since  $u \neq \mathcal{P}v$ , we have  $(v, u) \in \mathcal{R}^*$ . Also we can get  $\frac{1}{2}d(u, \mathcal{P}u) \leq d(u, v)$  following the lines of the proof of Case A (I). Therefore, using condition (2),

$$\mathcal{F}(d(\mathcal{P}u, \mathcal{P}v)) \leq \mathcal{F}(\mathfrak{N}(u, v) + \mathcal{G}(\beta(\mathfrak{N}(u, v))),$$

where

$$\begin{aligned} \mathfrak{N}(u, v) &= \max \left\{ d(u, v), d(u, \mathcal{P}u), d(v, \mathcal{P}v), \frac{d(u, \mathcal{P}v) + d(v, \mathcal{P}u)}{2} \right\} \\ &= d(u, v), \end{aligned}$$

which gives  $\mathcal{G}(\beta(d(u, v))) \geq 0$ , and so  $\beta(d(u, v)) \geq 1$ , a contradiction. Therefore the fixed point is unique. In a similar way, if  $(v, u) \in \mathfrak{R}$ , we have  $u = v$ .

- *Assume (IV).* Suppose that  $u, v$  are two fixed points of  $\mathcal{P}$ . Let  $\{z_0, z_1, \dots, z_k\}$  be an  $\mathfrak{R}^s$ -path in  $\text{Fix}(\mathcal{P})$  connecting  $u$  and  $v$ . As in Case(A) (I), it must be  $z_{i-1} = z_i$  for each  $i = 1, 2, \dots, k$ , and it follows that  $u = v$ .

If we take  $\mathfrak{R} = \{(v, v) \in \mathcal{E} \times \mathcal{E} \mid v \preceq v\}$ , then we have more new results as consequences of Theorem 3.3.

**Corollary 3.6** *Let  $(\mathcal{E}, d, \preceq)$  be an ordered complete metric space. Let  $\mathcal{P}: \mathcal{E} \rightarrow \mathcal{E}$  be increasing and  $(\text{SFG})_{\mathfrak{R}}$  on  $\mathcal{E}_{\preceq}$ . Suppose that there exists  $v_0 \in \mathcal{E}$  such that  $v_0 \preceq \mathcal{P}v_0$ . If  $\mathcal{P}$  is  $\mathcal{E}_{\preceq}$ -continuous or  $\mathcal{E}_{\preceq}$  is  $d$ -self-closed, then  $v_* \in \text{Fix}(\mathcal{P})$ . Moreover, for each  $v_0 \in \mathcal{E}$  with  $v_0 \preceq \mathcal{P}v_0$ , the Picard sequence  $\mathcal{P}^n(v_0)$  for all  $n \in \mathbb{N}$  converges to a  $v_* \in \text{Fix}(\mathcal{P})$ .*

**Corollary 3.7** Let  $(\mathcal{E}, d, \mathfrak{R})$  be an RMS and  $\mathcal{P}: \mathcal{E} \rightarrow \mathcal{E}$ . Suppose that the following conditions hold:

- (I)  $\mathfrak{X}(\mathcal{P}, \mathfrak{R}) \neq \emptyset$ ;
- (II)  $\mathfrak{R}$  is  $\mathcal{P}$ -closed and  $\mathcal{P}$ -transitive;
- (III)  $\mathcal{E}$  is  $\mathfrak{R}$ -complete;
- (IV)  $\mathcal{P}$  is FG-contraction, that is, there exists  $\mathcal{G} \in \mathfrak{G}$  such that, for  $(v, \vartheta) \in \mathcal{E}$  with  $(v, \vartheta) \in \mathfrak{R}$ ,

$$\mathcal{F}(d(\mathcal{P}v, \mathcal{P}\vartheta)) \leq \mathcal{F}(\mathfrak{N}(v, \vartheta)) + \mathcal{G}(\beta(\mathfrak{N}(v, \vartheta))), \tag{29}$$

where  $\mathfrak{N}(v, \vartheta)$  is defined in (3);

- (V)  $\mathcal{P}$  is  $\mathfrak{R}$ -continuous, or
- (V')  $\mathfrak{R}$  is  $d$ -self-closed.

Then there exists a point  $v_* \in \text{Fix}(\mathcal{P})$ .

### 4 Illustrations

**Example 4.1** Let  $\mathcal{E} = [0, 8)$  be equipped with usual metric  $d$ . Consider the binary relation on  $\mathcal{E}$  as follows:

$$\mathfrak{R} = \{(0, 1), (1, 3), (2, 1), (2, 2), (2, 5), (3, 1), (3, 2), (3, 3), (3, 5), (5, 1), (5, 2), (5, 5)\}.$$

Define a mapping  $\mathcal{P}: \mathcal{E} \rightarrow \mathcal{E}$  by

$$\mathcal{P}v = \begin{cases} 1, & 0 \leq v < 1; \\ 3, & v = 1; \\ 5, & 1 < v < 8. \end{cases}$$

Then  $\mathcal{P}$  is not continuous while  $\mathcal{P}$  is  $\mathfrak{R}$ -continuous,  $\mathfrak{R}$  is  $\mathcal{P}$ -closed, and  $\mathcal{P}$ -transitive;  $\mathcal{E}$  is  $\mathfrak{R}$ -complete. Also  $\mathfrak{R}^* = \{(0, 1), (1, 3), (5, 1)\}$  and  $\mathfrak{X}(\mathcal{P}; \mathfrak{R}) \neq \emptyset$  as  $(5, \mathcal{P}5) = (5, 5) \in \mathfrak{R}$ .

Now we take  $F(t) = -\frac{1}{\sqrt{t}}$ ,  $G(t) = \ln t$  ( $t > 0$ ) and  $\beta(t) = \lambda \in (0, 1)$ ,  $\tau = -\ln \lambda > 0$ , then (2) converted to

$$\frac{1}{2}d(v, \mathcal{P}v) \leq d(v, \vartheta) \text{ implies } d(\mathcal{P}v, \mathcal{P}\vartheta) \leq \frac{\mathfrak{N}(v, \vartheta)}{(1 + \tau\sqrt{\mathfrak{N}(v, \vartheta)})^2}, \tag{30}$$

where  $\mathfrak{N}(v, \vartheta)$  given in (3).

Consider  $(v, \vartheta) = (5, 1) \in \mathfrak{R}^*$  with  $\frac{1}{2}d(v, \mathcal{P}v) = 0 < 4 = d(v, \vartheta)$ . Then  $d(\mathcal{P}v, \mathcal{P}\vartheta) = 2$  and  $\mathfrak{N}(v, \vartheta) = 4$ . Therefore, condition (30) reduces to  $2 \leq \frac{4}{(1 + \tau\sqrt{4})^2}$ , which is true for  $\tau = 0.1$ . Similarly, we can check for  $(v, \vartheta) = (1, 3) \in \mathfrak{R}^*$ . Thus all the conditions of Theorem 3.3 are satisfied, hence  $\mathcal{P}$  has a fixed point. Moreover,  $\mathfrak{R}|_{\mathcal{P}(\mathcal{E})}$  is transitive, while  $\mathfrak{R}$  is not, and for all  $v, \vartheta \in \mathcal{P}(\mathcal{E})$ , we have  $(v, \vartheta) \in \mathfrak{R}$ , so  $\mathfrak{P}(v, \vartheta, \mathfrak{R})|_{\mathcal{P}(\mathcal{E})}$  is nonempty for all  $v, \vartheta \in \mathcal{P}(\mathcal{E})$ . Following Theorem 3.4,  $\mathcal{P}$  has a unique fixed point which is  $v^* = 5$ .

Now, for  $(0, 1) \in \mathfrak{R}$ ,

$$d(\mathcal{P}v, \mathcal{P}\vartheta) = 2 \not\leq 2k = k \max \left\{ d(v, \vartheta), d(v, \mathcal{P}v), d(\vartheta, \mathcal{P}\vartheta), \frac{1}{2}[d(v, \mathcal{P}\vartheta) + d(\vartheta, \mathcal{P}v)] \right\},$$

which is not true for any  $k \in (0, 1)$ , and hence  $\mathcal{P}$  is not an implicit type mapping on  $(\mathcal{E}, d, \mathfrak{R})$ . Hence [10, Theorem 1 and Theorem 2] cannot be applied to the present example.

Also, as  $1, 0 \in \mathcal{E}$ ,  $(1, 0) \notin \mathfrak{R}$  with  $\mathcal{P}1 = 3 \neq 1 = \mathcal{P}0$  such that  $\frac{1}{2}d(1, \mathcal{P}1) = d(1, 0)$  but  $d(\mathcal{P}1, \mathcal{P}0) \not\leq k d(1, 0)$  and

$$d(\mathcal{P}v, \mathcal{P}\vartheta) = 2 \not\leq 2k = k \max \left\{ d(v, \vartheta), d(v, \mathcal{P}v), d(\vartheta, \mathcal{P}\vartheta), \frac{1}{2}[d(v, \mathcal{P}\vartheta) + d(\vartheta, \mathcal{P}v)] \right\},$$

which shows that  $\mathcal{P}$  is neither Suzuki-contraction nor generalized Suzuki-contraction for any  $k \in [0, 1)$ . Hence the results of Suzuki [19] and Popescu [20] cannot be applied to the present example, while our Theorem 3.3 and Theorem 3.4 are applicable. This shows that our results are genuine improvements over the corresponding results contained in Suzuki [19], Popescu [20], and Ahmadullah et al. [10, Theorem 1 and Theorem 2].

*Example 4.2* Consider the set  $\mathcal{E} = [0, 1]$  with the usual metric  $d$ . Define a binary relation  $\mathfrak{R}$  by

$$\mathfrak{R} = \left\{ (0, 0), (0, 1), \left(\frac{1}{5}, 1\right), \left(\frac{1}{5}, 0\right), \left(0, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{5}\right) \right\}.$$

Consider the self-mapping  $\mathcal{P}$  on  $\mathcal{E}$  given by

$$\mathcal{P}(\vartheta) = \begin{cases} 0, & 0 \leq \vartheta \leq \frac{1}{5}, \\ \frac{1}{5}, & \frac{1}{5} < \vartheta \leq 1. \end{cases}$$

It is clear that  $\mathcal{E}$  is  $\mathfrak{R}$  is complete and  $\mathfrak{R}$  is  $\mathcal{P}$ -closed. Also  $\mathfrak{R}^* = \{(0, 1), (\frac{1}{5}, 1)\}$  and  $\mathfrak{X}(\mathcal{P}; \mathfrak{R}) \neq \emptyset$  as  $(0, \mathcal{P}0) = (0, 0) \in \mathfrak{R}$ .

We consider (30) of previous Example 4.1 to verify  $\mathcal{P} \in (\text{SFG})_{\mathfrak{R}}$ .

- Let  $(\vartheta, v) = (0, 1)$  with  $\frac{1}{2}d(v, \mathcal{P}v) = 0 < 1 = d(v, \vartheta)$ . Then  $d(\mathcal{P}\vartheta, \mathcal{P}v) = \frac{1}{5}$  and  $\mathfrak{R}(v, \vartheta) = 1$ . Therefore, condition (30) reduces to  $1/5 \leq \frac{1}{(1+\tau)^2}$ .
- Let  $(\vartheta, v) = (\frac{1}{5}, 1)$  with  $\frac{1}{2}d(v, \mathcal{P}v) = 0 < \frac{4}{5} = d(v, \vartheta)$ . Then  $d(\mathcal{P}\vartheta, \mathcal{P}v) = \frac{1}{5}$  and  $\mathfrak{R}(v, \vartheta) = 4/5$ . Therefore, condition (30) reduces to  $1/5 \leq \frac{4/5}{(1+\tau\sqrt{4/5})^2}$ .

It can be easily checked that the above cases hold true for  $\tau > 0$  (in particular  $\tau = 0.1$ ).

Thus  $\mathcal{P} \in (\text{SFG})_{\mathfrak{R}}$ .

Let  $(v_n)$  be an  $\mathfrak{R}$ -preserving sequence converging to  $v$  as  $n \rightarrow \infty$ . Then we must have

$$(v_n, v_{n+1}) \in \left\{ (0, 0), \left(\frac{1}{5}, 0\right), \left(0, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{1}{5}\right) \right\}$$

implies that

$$v_n \in \left\{ 0, \frac{1}{5} \right\}.$$

This implies that either  $v_n \rightarrow 0$  or  $v_n \rightarrow \frac{1}{5}$  as  $n \rightarrow \infty$ , and clearly we have  $[v_n, v] \in \mathfrak{R}$  for all  $n \in \mathbb{N}$ , where  $v = 0$  and  $\frac{1}{5}$ . This shows that  $\mathfrak{R}$  is  $d$ -self-closed. Thus all the conditions of Theorem 3.3 are satisfied, hence  $\mathcal{P}$  has a fixed point  $(\vartheta^* = 1/5)$ .

### 5 Application to nonlinear matrix equations

For a matrix  $\mathcal{B} \in \mathcal{H}(n)$ , we will denote by  $s(\mathcal{B})$  any of its singular values and by  $s^+(\mathcal{B})$  the sum of all of its singular values, that is, the trace norm  $\|\mathcal{B}\|_{\text{tr}} = s^+(\mathcal{B})$ . For  $\mathcal{C}, \mathcal{D} \in \mathcal{H}(n)$ ,  $\mathcal{C} \succeq \mathcal{D}$  (resp.  $\mathcal{C} \succ \mathcal{D}$ ) will mean that the matrix  $\mathcal{C} - \mathcal{D}$  is positive semi-definite (resp. positive definite).

The following lemmas are needed in the subsequent discussion.

**Lemma 5.1** ([1]) *If  $\mathcal{A} \succeq O$  and  $\mathcal{B} \succeq O$  are  $n \times n$  matrices, then*

$$0 \leq \text{tr}(\mathcal{A}\mathcal{B}) \leq \|\mathcal{A}\| \text{tr}(\mathcal{B}).$$

**Lemma 5.2** ([1]) *If  $\mathcal{A} \in \mathcal{H}(n)$  such that  $\mathcal{A} \prec I_n$ , then  $\|\mathcal{A}\| < 1$ .*

We establish the existence and uniqueness of the solution of the nonlinear matrix equation (NME)

$$\mathcal{X} = \mathcal{Q} + \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i, \tag{31}$$

where  $\mathcal{Q}$  is a Hermitian positive definite matrix,  $\mathcal{A}_i^*$  stands for the conjugate transpose of an  $n \times n$  matrix  $\mathcal{A}_i$ , and  $\mathcal{G}$  is an order-preserving continuous mapping from the set of all Hermitian matrices to the set of all positive definite matrices such that  $\mathcal{G}(O) = O$ .

**Theorem 5.3** *Consider NME (31). Assume that there exists a positive real number  $\eta$  such that*

- (H<sub>1</sub>) *There exists  $\mathcal{Q} \in \mathcal{P}(n)$  such that  $\sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{Q}) \mathcal{A}_i \succ 0$ ;*
- (H<sub>2</sub>)  *$\sum_{i=1}^m \mathcal{A}_i \mathcal{A}_i^* \prec \eta I_n$ .*
- (H<sub>3</sub>) *For every  $\mathcal{X}, \mathcal{Y} \in \mathcal{P}(n)$  such that  $\mathcal{X} \preceq \mathcal{Y}$  with*

$$\sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i \neq \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{Y}) \mathcal{A}_i$$

and if

$$\left| s^+ \left( \mathcal{X} - \mathcal{Q} - \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i \right) \right| < 2 |s^+(\mathcal{X} - \mathcal{Y})|$$

holds, then for  $\tau > 0$  we have

$$|s^+(\mathcal{G}(\mathcal{X}) - \mathcal{G}(\mathcal{Y}))| \leq \frac{1}{\eta} \times \max \left\{ \begin{array}{l} \frac{|s^+(\mathcal{X} - \mathcal{Y})|}{[1 + \tau |s^+(\mathcal{X} - \mathcal{Y})|^{1/2}]^2}, \frac{|s^+(\mathcal{X} - \mathcal{Q} - \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i)|}{[1 + \tau |s^+(\mathcal{X} - \mathcal{Q} - \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i)|^{1/2}]^2}, \\ \frac{|s^+(\mathcal{Y} - \mathcal{Q} - \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{Y}) \mathcal{A}_i)|}{[1 + \tau |s^+(\mathcal{Y} - \mathcal{Q} - \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{Y}) \mathcal{A}_i)|^{1/2}]^2}, \\ \frac{|s^+(\mathcal{X} - \mathcal{Q} - \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{Y}) \mathcal{A}_i)|}{[1 + \tau |s^+(\mathcal{X} - \mathcal{Q} - \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{Y}) \mathcal{A}_i)|^{1/2}]^2}, \\ \frac{|s^+(\mathcal{Y} - \mathcal{Q} - \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i)|}{[1 + \tau |s^+(\mathcal{Y} - \mathcal{Q} - \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i)|^{1/2}]^2}. \end{array} \right.$$

Then NME (31) has a unique solution. Moreover, the iteration

$$\mathcal{X}_n = \mathcal{Q} + \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}_{n-1}) \mathcal{A}_i, \tag{32}$$

where  $\mathcal{X}_0 \in \mathcal{P}(n)$  satisfies

$$\mathcal{X}_0 \leq \mathcal{Q} + \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}_0) \mathcal{A}_i,$$

converges in the sense of trace norm  $\| \cdot \|_{\text{tr}}$  to the solution of matrix equation (31).

*Proof* Define a mapping  $\mathcal{T} : \mathcal{P}(n) \rightarrow \mathcal{P}(n)$  by

$$\mathcal{T}(\mathcal{X}) = \mathcal{Q} + \sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{X}) \mathcal{A}_i \quad \text{for all } \mathcal{X} \in \mathcal{P}(n),$$

and a binary relation

$$\mathfrak{R} = \{(\mathcal{X}, \mathcal{Y}) \in \mathcal{P}(n) \times \mathcal{P}(n) : \mathcal{X} \leq \mathcal{Y}\}.$$

Then a fixed point of the mapping  $\mathcal{T}$  is a solution of matrix equation (31). Notice that  $\mathcal{T}$  is well defined,  $\mathfrak{R}$ -continuous, and  $\mathfrak{R}$  is  $\mathcal{T}$ -closed. Since

$$\sum_{i=1}^m \mathcal{A}_i^* \mathcal{G}(\mathcal{Q}) \mathcal{A}_i > 0,$$

for some  $\mathcal{Q} \in \mathcal{P}(n)$ , we have  $(\mathcal{Q}, \mathcal{T}(\mathcal{Q})) \in \mathfrak{R}$ , and hence  $\mathcal{P}(n)(\mathcal{T}; \mathfrak{R}) \neq \emptyset$ .

Now, let  $(\mathcal{X}, \mathcal{Y}) \in \mathfrak{R}^* = \{(\mathcal{X}, \mathcal{Y}) \in \mathfrak{R} : \mathcal{T}(\mathcal{X}) \neq \mathcal{T}(\mathcal{Y})\}$  such that

$$\frac{1}{2} \|\mathcal{X} - \mathcal{T}(\mathcal{X})\|_{\text{tr}} < \|\mathcal{X} - \mathcal{Y}\|_{\text{tr}}.$$

Then

$$\begin{aligned} & \|\mathcal{T}(\mathcal{X}) - \mathcal{T}(\mathcal{Y})\|_{\text{tr}} \\ &= s^+(\mathcal{T}(\mathcal{X}) - \mathcal{T}(\mathcal{Y})) \\ &= s^+\left(\sum_{i=1}^m \mathcal{A}_i^* (\mathcal{G}(\mathcal{X}) - \mathcal{G}(\mathcal{Y})) \mathcal{A}_i\right) \\ &= \sum_{i=1}^m s^+(\mathcal{A}_i^* (\mathcal{G}(\mathcal{X}) - \mathcal{G}(\mathcal{Y})) \mathcal{A}_i) \\ &= \sum_{i=1}^m s^+(\mathcal{A}_i \mathcal{A}_i^* (\mathcal{G}(\mathcal{X}) - \mathcal{G}(\mathcal{Y}))) \\ &= s^+\left(\sum_{i=1}^m \mathcal{A}_i \mathcal{A}_i^*\right) s^+(\mathcal{G}(\mathcal{X}) - \mathcal{G}(\mathcal{Y})) \end{aligned}$$

$$\begin{aligned} &\leq \frac{\|\sum_{i=1}^m \mathcal{A}_i \mathcal{A}_i^*\|}{\eta} \times \max \left\{ \frac{\frac{\|\mathcal{X}-\mathcal{Y}\|_{\text{tr}}}{[1+\tau\|\mathcal{X}-\mathcal{Y}\|_{\text{tr}}^{1/2}]^2}, \frac{\|\mathcal{X}-\mathcal{T}\mathcal{X}\|_{\text{tr}}}{[1+\tau\|\mathcal{X}-\mathcal{T}\mathcal{X}\|_{\text{tr}}^{1/2}]^2}, \frac{\|\mathcal{Y}-\mathcal{T}\mathcal{Y}\|_{\text{tr}}}{[1+\tau\|\mathcal{Y}-\mathcal{T}\mathcal{Y}\|_{\text{tr}}^{1/2}]^2}}{\frac{\|\mathcal{X}-\mathcal{T}\mathcal{Y}\|_{\text{tr}}}{[1+\tau\|\mathcal{X}-\mathcal{T}\mathcal{Y}\|_{\text{tr}}^{1/2}]^2}, \frac{\|\mathcal{Y}-\mathcal{T}\mathcal{X}\|_{\text{tr}}}{[1+\tau\|\mathcal{Y}-\mathcal{T}\mathcal{X}\|_{\text{tr}}^{1/2}]^2}} \right\} \\ &\leq \frac{\Theta(\mathcal{X}, \mathcal{Y})}{[1 + \tau(\Theta(\mathcal{X}, \mathcal{Y}))^{1/2}]^2}, \end{aligned} \tag{33}$$

where

$$\Theta(\mathcal{X}, \mathcal{Y}) = \max \left\{ \frac{\frac{\|\mathcal{X}-\mathcal{Y}\|_{\text{tr}}}{[1+\tau\|\mathcal{X}-\mathcal{Y}\|_{\text{tr}}^{1/2}]^2}, \frac{\|\mathcal{X}-\mathcal{T}\mathcal{X}\|_{\text{tr}}}{[1+\tau\|\mathcal{X}-\mathcal{T}\mathcal{X}\|_{\text{tr}}^{1/2}]^2}, \frac{\|\mathcal{Y}-\mathcal{T}\mathcal{Y}\|_{\text{tr}}}{[1+\tau\|\mathcal{Y}-\mathcal{T}\mathcal{Y}\|_{\text{tr}}^{1/2}]^2}}{\frac{\|\mathcal{X}-\mathcal{T}\mathcal{Y}\|_{\text{tr}}}{[1+\tau\|\mathcal{X}-\mathcal{T}\mathcal{Y}\|_{\text{tr}}^{1/2}]^2}, \frac{\|\mathcal{Y}-\mathcal{T}\mathcal{X}\|_{\text{tr}}}{[1+\tau\|\mathcal{Y}-\mathcal{T}\mathcal{X}\|_{\text{tr}}^{1/2}]^2}} \right\}. \tag{34}$$

Consider  $\mathcal{F}(t) = -\frac{1}{\sqrt{t}}$ ,  $\mathcal{G}(t) = \ln t$  ( $t > 0$ ) and  $\beta(t) = \lambda \in (0, 1)$ ,  $\tau = -\ln \lambda > 0$ , then (33) converted to

$$\begin{aligned} &\frac{1}{2} \|\mathcal{X} - \mathcal{T}(\mathcal{X})\|_{\text{tr}} < \|\mathcal{X} - \mathcal{Y}\|_{\text{tr}} \quad \text{implies} \\ &\mathcal{F}(\|\mathcal{T}(\mathcal{X}) - \mathcal{T}(\mathcal{Y})\|_{\text{tr}}) \leq \mathcal{F}(\Theta(\mathcal{X}, \mathcal{Y})) + \mathcal{G}(\beta(\Theta(\mathcal{X}, \mathcal{Y}))), \end{aligned}$$

where  $\Theta(\mathcal{X}, \mathcal{Y})$  is given in (34). Thus all the hypotheses of Theorem 3.3 are satisfied, therefore there exists  $\hat{\mathcal{X}} \in \mathcal{P}(n)$  such that  $\mathcal{T}(\hat{\mathcal{X}}) = \hat{\mathcal{X}}$ , and hence matrix equation (31) has a solution in  $\mathcal{P}(n)$ . Furthermore, due to the existence of least upper bound and greatest lower bound for each  $\mathcal{X}, \mathcal{Y} \in \mathcal{T}(\mathcal{P}(n))$ , we have  $\mathfrak{F}(\mathcal{X}, \mathcal{Y}; \mathfrak{R}|_{\mathcal{T}(\mathcal{P}(n))}) \neq \emptyset$  for all  $\mathcal{X}, \mathcal{Y} \in \mathcal{T}(\mathcal{P}(n))$ . Hence, on using Theorem 3.4,  $\mathcal{T}$  has a unique fixed point, and hence we conclude that matrix equation (31) has a unique solution in  $\mathcal{P}(n)$ . □

*Example 5.4* Consider NME (31) for  $m = 3$ ,  $\eta = 4.5$ ,  $n = 3$  with  $\mathcal{G}(\mathcal{X}) = \mathcal{X}^{1/5}$ , i.e.,

$$\mathcal{X} = \mathcal{Q} + \mathcal{A}_1^* \mathcal{X}^{1/5} \mathcal{A}_1 + \mathcal{A}_2^* \mathcal{X}^{1/5} \mathcal{A}_2 + \mathcal{A}_3^* \mathcal{X}^{1/5} \mathcal{A}_3, \tag{35}$$

where

$$\begin{aligned} \mathcal{Q} &= \begin{bmatrix} 11.699540782825979 & 0.914622941324684 & 1.507188535497828 \\ 0.914622941324684 & 10.833657911203609 & 1.249452950221198 \\ 1.507188535497828 & 1.249452950221198 & 12.080319343374171 \end{bmatrix}, \\ \mathcal{A}_1 &= \begin{bmatrix} 0.082250000000000 & 0.110600000000000 & 0.218400000000000 \\ 0.088900000000000 & 0.053900000000000 & 0.223300000000000 \\ 0.228900000000000 & 0.090300000000000 & 0.042700000000000 \end{bmatrix}, \\ \mathcal{A}_2 &= \begin{bmatrix} 0.028000000000000 & 0.036250000000000 & 0.041250000000000 \\ 0.058750000000000 & 0.039250000000000 & 0.046000000000000 \\ 0.061250000000000 & 0.059750000000000 & 0.039750000000000 \end{bmatrix}, \\ \mathcal{A}_3 &= \begin{bmatrix} 0.679012345679012 & 1.061728395061728 & 0.333333333333333 \\ 0.567901234567901 & 0.296296296296296 & 0.641975308641975 \\ 1.185185185185185 & 0.444444444444444 & 0.691358024691358 \end{bmatrix}. \end{aligned}$$

The conditions of Theorem 5.3 can be checked numerically, taking various special values for matrices involved. For example, they can be tested (and verified to be true) for

$$\mathcal{X} = \begin{bmatrix} 1.699436061575979 & 0.914189910074684 & 1.507087334247828 \\ 0.914189910074684 & 0.822435604328608 & 1.248590153939948 \\ 1.507087334247828 & 1.248590153939948 & 2.080170705685109 \end{bmatrix},$$

$$\mathcal{Y} = \begin{bmatrix} 10.000104721250000 & 0.000433031250000 & 0.000101201250000 \\ 0.000433031250000 & 10.011222306875000 & 0.000862796281250 \\ 0.000101201250000 & 0.000862796281250 & 10.000148637689062 \end{bmatrix}.$$

To see the convergence of the sequence  $\{\mathcal{X}_n\}$  defined in (32), we start with three different initial values

$$\mathcal{U}_0 = \begin{bmatrix} 0.015970559290683 & 0.014219828729812 & 0.004760641350592 \\ 0.014219828729812 & 0.045823355744100 & 0.011986278815522 \\ 0.004760641350592 & 0.011986278815522 & 0.014342909184651 \end{bmatrix}$$

with  $\|\mathcal{U}_0\| = 0.076136824219434$ ,

$$\mathcal{V}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with  $\|\mathcal{V}_0\| = 1$ ,

$$\mathcal{W}_0 = \begin{bmatrix} 64.303848221681193 & 14.585212879712167 & 16.765822087028965 \\ 14.585212879712167 & 54.844490660932415 & 11.815345676105265 \\ 16.765822087028969 & 11.815345676105263 & 57.346307431417692 \end{bmatrix}$$

with  $\|\mathcal{W}_0\| = 1.764946463140313 \times 10^2$ .

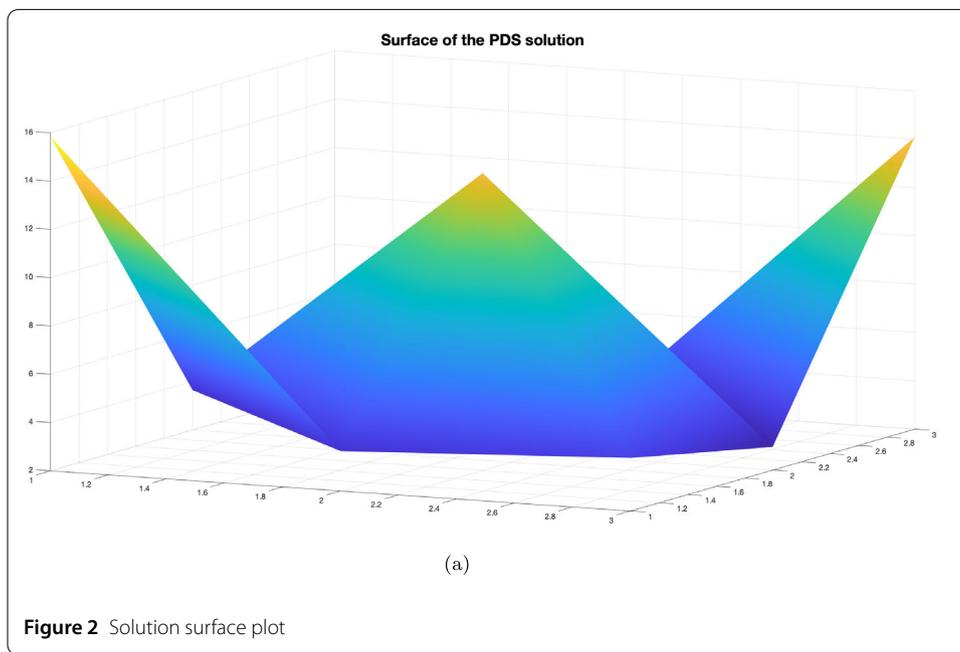
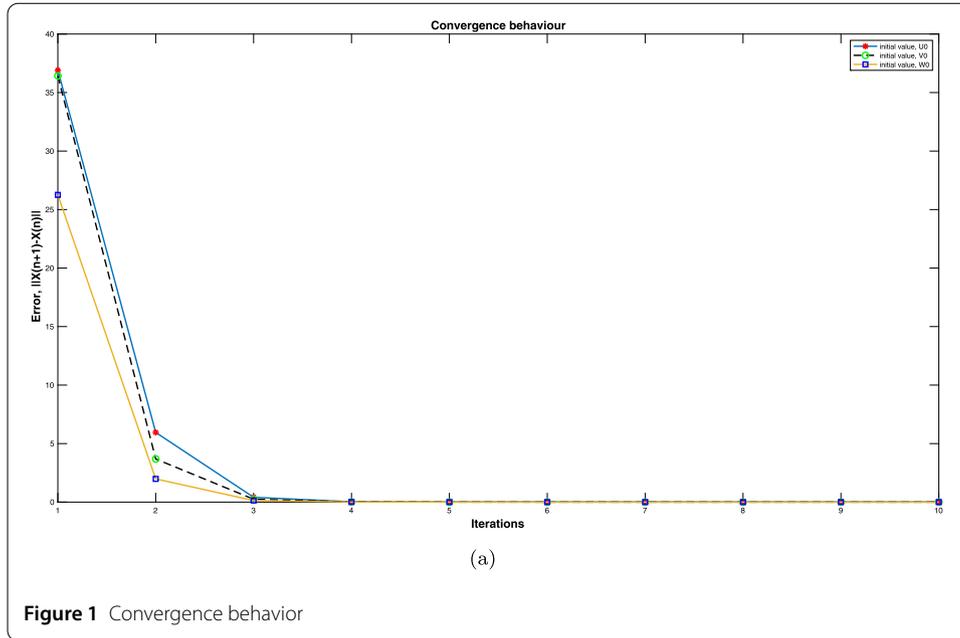
After 10 iterations, we have the following approximation of the unique positive definite solution of system (31):

$$\widehat{\mathcal{U}} \approx \mathcal{U}_{10} = \begin{bmatrix} 15.825962055386070 & 3.646303219900028 & 4.191455521733169 \\ 3.646303219900028 & 13.461122665210109 & 2.953836419006667 \\ 4.191455521733170 & 2.953836419006668 & 14.086576857837056 \end{bmatrix}$$

$$\widehat{\mathcal{V}} \approx \mathcal{V}_{10} = \begin{bmatrix} 15.825962055420298 & 3.646303219928042 & 4.191455521757241 \\ 3.646303219928042 & 13.461122665233104 & 2.953836419026316 \\ 4.191455521757242 & 2.953836419026316 & 14.086576857854423 \end{bmatrix}$$

$$\widehat{\mathcal{W}} \approx \mathcal{W}_{10} = \begin{bmatrix} 15.825962055444425 & 3.646303219947785 & 4.191455521774207 \\ 3.646303219947785 & 13.461122665249309 & 2.953836419040164 \\ 4.191455521774207 & 2.953836419040163 & 14.086576857866664 \end{bmatrix}$$

Also, the elements of each sequence are order preserving. The graphical representation of convergence of a sequence and a surface plot of solution are shown in Figs. 1 and 2, respectively.



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### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

Investigation, HKN, RJ, and VP; Methodology, RJ and HKN; Supervision, HKN and VP; Writing—original draft, RJ and HKN; Writing—review and editing, HKN and VP; and Software, RJ and HKN; All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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