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Existence and stability results for non-hybrid single-valued and fully hybrid multi-valued problems with multipoint-multistrip conditions

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Abstract

In this paper, we study a new class of non-hybrid single-valued fractional boundary value problems equipped with integro-non-hybrid-multiterm-multipoint-multistrip conditions and a fully hybrid integro-multi-valued fractional boundary value problem by some new methods including the Kuratowski measures based on Sadovskii's theorem, Krasnoselskii–Zabreiko criterion, and Dhage's technique. We generalize the Gronwall inequality in relation to a non-hybrid single-valued fractional boundary value problem, and then we investigate the stability notions in two versions. To examine the correctness of the results, we provide some examples.

MSC: 34A08; 34A12

Keywords: Fully hybrid equation; Gronwall inequality; Stability; Multi-strip; Multi-valued boundary problem; Condensing map

1 Introduction

Fractional differential equations (FDEs) and fractional differential inclusions (FDIs) in the context of a more general field of mathematics, entitled fractional calculus, have attracted the attention of scientific community in recent decades. The scientists and researchers have applied a diversity of new mathematical tools and methods for studying FDEs and FDIs. Some significant books in this field were published in the early years, in which the basic concepts of this field were defined and introduced well for everyone. Some instances are the books of Podlubny [1], Miller *et al.* [2], Deimling [3], Aubin *et al.* [4], and Kilbas *et al.* [5].

The conducted research in different scientific fields can be found in some areas like biomathematics, biophysics, mechanics, biology, control theory, engineering, economics, morphology, rheology, etc. [6, 7]. In the investigation of solutions of different FDEs and FDIs, we can find many valuable scientific articles in which the existence, uniqueness, attractivity and stability, positivity and multiplicity, approximation, and other analytical properties of solutions are studied in the framework of different nonlinear fractional boundary value problems (FBVPs). A large number of mathematicians have constructive contributions in this regard, and in all of them, a trace of fixed point theory and the relevant techniques is observed. Some of their published works can be found in [8–16].

Fractional hybrid differential inclusions (FHDIs) and fractional hybrid differential equations (FHDEs) of the first and second types are the generalizations of standard FDIs and FDEs which were introduced by Dhage *et al.* [17] in 2010 and also by Zhao *et al.* [18] in 2011. Immediately, these hybrid FBVPs found their place in various computations and mathematical modelings so that we highlight some of the strong works in this regard by naming the papers from Baleanu *et al.* [19], Nagajothi *et al.* [20], Matar *et al.* [21], Khan *et al.* [22], Mohammadi *et al.* [23], etc. Along with these, an extended type of boundary conditions (BCs) entitled multi-point and multi-strip conditions were introduced in some models. In these FBVPs equipped with such generalized conditions, the sum of respective multi-strip multi-point values is proportional to the value of unknown function and its derivatives in some points. Examples of these BCs and FBVPs can be seen in some newly-published works. In [24], Alam *et al.* investigated the Caputo multi-point strip FDE with the help of the Laplace transform and Bielecki norm, given as

$$\begin{cases} \mathfrak{u}^{(2)}(\mathfrak{t}) + {}^{c}D_{0}^{q}\mathfrak{u}(\mathfrak{t}) = \hbar(\mathfrak{t},\mathfrak{u}(\mathfrak{t})), \\ \sum_{\jmath=0}^{\ell-2} \varepsilon_{\jmath}\mathfrak{u}(\sigma_{\jmath}) = \mathfrak{u}(0), \qquad I_{0}^{\theta}\phi(\mathfrak{b},\mathfrak{u}(\mathfrak{b})) = \mathfrak{u}(1), \end{cases}$$

where $q \in (1,2)$, $\theta > 1$, $b \in (0,1]$, $\varepsilon_{\jmath}, \sigma_{\jmath} \geq 0$, and $\mathfrak{t} \in [0,1]$. In [25], Ahmad *et al.* generalized a category of FBVPs equipped with the boundary conditions of the multi-point multi-strip non-separated type given by

$$\begin{cases} {}^{c}D_{0}^{q}\mathfrak{u}(\mathfrak{t}) = \hbar(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),{}^{c}D_{0}^{p}\mathfrak{u}(\mathfrak{t})), \\ \sum_{\jmath=1}^{\ell-2} \varepsilon_{\jmath}\mathfrak{u}(\sigma_{\jmath}) + \sum_{k=1}^{j-2} \epsilon_{k}\int_{b_k}^{d_{\jmath}} \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} = l_1\mathfrak{u}(0) + l_2\mathfrak{u}(1), \\ \sum_{\jmath=1}^{\ell-2} \rho_{\jmath}\mathfrak{u}'(\sigma_{\jmath}) + \sum_{k=1}^{j-2} \varrho_{k}\int_{b_k}^{d_{\jmath}} \mathfrak{u}'(\mathfrak{s})\,d\mathfrak{s} = l_3\mathfrak{u}'(0) + l_4\mathfrak{u}'(1), \end{cases}$$

where $q \in (1,2]$, $p \in (0,1)$, $0 < \sigma_{\jmath} < b_k < d_{\jmath} < 1$, $\varepsilon_{\jmath}, \epsilon_k, \rho_{\jmath}, \varrho_k, l_1, l_2, l_3, l_4 > 0$. We see other similar examples in articles by Lv *et al.* [26], by Salem *et al.* [27], by Ahmad *et al.* [28], etc.

Inspired by the above ideas and by [29], in this paper, we investigate the existence of solutions of non-hybrid single-valued FBVP with integro-non-hybrid-multiterm-multipoint-multistrip boundary conditions:

$$\begin{cases} {}^{c}D_{0}^{q}\mathfrak{u}(\mathfrak{t}) = \mathfrak{h}(\mathfrak{t},\mathfrak{u}(\mathfrak{t})) \quad (q \in (2,3], \mathfrak{t} \in \mathbb{I} := [0,1]), \\ \mathfrak{u}(\mathfrak{t})|_{\mathfrak{t}=0} = 0, \qquad \mathfrak{u}'(\mathfrak{t})|_{\mathfrak{t}=0} = 0, \\ \int_0^1 \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} = \sum_{\ell=1}^{r} a_\ell \mathfrak{u}(\mathfrak{t})|_{\mathfrak{t}=\delta_\ell} + \sum_{i=2}^{p} b_{i-1}\int_{m_{i-1}}^{m_i} \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} + \sum_{j=1}^{k} c_j I_0^{\theta_j}\mathfrak{u}(\mathfrak{t})|_{\mathfrak{t}=\eta_j}. \end{cases} \tag{1.1}$$

where for $j = 1,2,\ldots,k$, $\theta_j \geq 0$, $a_1,a_2,\ldots,a_r,b_1,b_2,\ldots,b_{p-1},c_1,c_2,\ldots,c_k \in \mathbb{R}$ are constants and $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_r < m_1 \leq m_2 \leq \cdots \leq m_p < \eta_1 \leq \eta_2 \leq \cdots \leq \eta_k < 1$. Also, $\mathfrak{h} : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ is continuous and ${}^{c}D_0^q$ and $I_0^{\theta_1},\ldots,I_0^{\theta_k}$ denote the fractional operators of Caputo and Riemann–Liouville types.

As a second problem, by applying some notions of functional analysis, we study the existence of solutions for a fully hybrid integro-multi-valued FBVP with integro-hybrid-multiterm-multipoint-multistrip boundary conditions given by

$$\begin{cases} {}^{c}D_{0}^{q}(\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))}) \in \mathcal{G}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}),\int_0^1 \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s}) \quad (q\in(2,3], \mathfrak{t}\in\mathbb{I}:=[0,1]),\\ (\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})|_{\mathfrak{t}=0}=0, \qquad (\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})'|_{\mathfrak{t}=0}=0,\\ \int_0^1(\frac{\mathfrak{u}(\mathfrak{s})}{y(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))})\,d\mathfrak{s} = \sum_{\ell=1}^{r} a_\ell(\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})|_{\mathfrak{t}=\delta_\ell}\\ \qquad + \sum_{i=2}^{p} b_{i-1}\int_{m_{i-1}}^{m_i}(\frac{\mathfrak{u}(\mathfrak{s})}{y(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))})\,d\mathfrak{s} + \sum_{j=1}^{k} c_j I_0^{\theta_j}(\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})|_{\mathfrak{t}=\eta_j}, \end{cases} \tag{1.2}$$

where $y:\mathbb{I}\times\mathbb{R}\to\mathbb{R}\setminus\{0\}$ is continuous and $\mathcal{G}:\mathbb{I}\times\mathbb{R}^2\to\mathbb{P}(\mathbb{R})$ is a multi-valued map with some specific properties.

The main contribution of this work is that we combine some well-known fractional structures in the framework of two generalized FBVPs. In fact, a combination of the non-hybrid equations, fully hybrid equations, integro-differentials, multistrip conditions in multipoint positions, and a generalized inclusion is investigated in the supposed FBVPs (1.1) and (1.2) in this manuscript. Regarding the first novelty of this work, to establish results in relation to the existence criteria for this new abstract model, some pure methods arising in functional analysis will help us in this direction. In other words, with the help of some properties of the Kuratowski measure and by defining the condensing selfmaps on a convex and closed set, we prove our first result by Sadovskii's theorem on FBVP (1.1). We even have tried to derive the required conditions confirming the dependence of solutions via the generalized inequality of Gronwall type. The second novelty of this study is to apply the inclusion type of Dhage's method for generalized fully hybrid integro-multi-valued FBVP with the integro-hybrid-multiterm-multipoint-multistrip boundary conditions (1.2).

We organize the paper as follows: some preliminaries in relation to our methods and techniques are recollected in Sect. 2. We consider a non-hybrid single-valued FBVP (1.1) in Sect. 3, and with the help of Sadovskii's fixed point, we prove our result, and by applying the generalized inequality of Gronwall type, the dependence of solutions is investigated. Also, the Krasnoselskii–Zabreiko criterion gives another existence result for the non-hybrid single-valued case. The stability property in some versions is proved in Sect. 4. For the fully hybrid-multi-valued FBVP (1.2), some results are established in Sect. 5 via Dhage's method. Section 6 is devoted to preparing some examples in the direction of our results. We end our study in Sect. 7 by giving conclusions.

2 Basic preliminaries

In this section, we recall some basic notions which are used in the next sections of the paper. Let $q>0$. The fractional Riemann–Liouville integral (RL-integral) for a real-valued function $\mathfrak{u}$ on $[0,\infty)$ is defined by

$$I_0^q\mathfrak{u}(\mathfrak{t}) = \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{q-1}}{\Gamma(q)}\mathfrak{u}(\mathfrak{s})\,d\mathfrak{s},$$

such that the integral exists [1, 5]. Further, let $n-1<q<n$. The fractional Caputo derivative (C-derivative) of $\mathfrak{u}\in C^{(n)}([a,b],\mathbb{R})$ is defined by

$${}^{c}D_0^q\mathfrak{u}(\mathfrak{t}) = \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{n-q-1}}{\Gamma(n-q)}\mathfrak{u}^{(n)}(\mathfrak{s})\,d\mathfrak{s},$$

so that the integral exists [1, 5]. In [2], it was proved that the solution of ${}^{c}D_0^q\mathrm{u}(\mathrm{t}) = 0$ is $\mathrm{u}(\mathrm{t}) = \alpha_0 + \alpha_1\mathrm{t} + \alpha_2\mathrm{t}^2 + \cdots + \alpha_{n-1}\mathrm{t}^{n-1}$ and

$$I_0^q\left({}^{c}D_0^q\mathrm{u}(\mathrm{t})\right) = \mathrm{u}(\mathrm{t}) + \sum_{k=0}^{n-1}\alpha_k\mathrm{t}^k = \mathrm{u}(\mathrm{t}) + \alpha_0 + \alpha_1\mathrm{t} + \alpha_2\mathrm{t}^2 + \cdots + \alpha_{n-1}\mathrm{t}^{n-1},$$

where $\alpha_0,\dots,\alpha_{n-1} \in \mathbb{R}$ and $n = [q] + 1$.

Let $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be a normed space. $\mathrm{u} \in \mathcal{Y}$ is a fixed point of the set-valued map $\mathcal{G} : \mathcal{Y} \to \mathbb{P}(\mathcal{Y})$ if $\mathrm{u} \in \mathcal{G}(\mathrm{u})$, where $\mathbb{P}(\mathcal{Y})$ is the set of all nonempty subsets of $\mathcal{Y}$ [3]. The Pompeiu–Hausdorff metric $\mathrm{H}_{d_{\mathcal{Y}}} : \mathbb{P}(\mathcal{Y}) \times \mathbb{P}(\mathcal{Y}) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\mathrm{H}_{d_{\mathcal{Y}}}(U_1, U_2) = \max\left\{\sup_{\mathrm{u}_1\in U_1} d_{\mathcal{Y}}(\mathrm{u}_1, U_2), \sup_{\mathrm{u}_2\in U_2} d_{\mathcal{Y}}(U_1, \mathrm{u}_2)\right\},$$

where $d_{\mathcal{Y}}(U_1, \mathrm{u}_2) = \inf_{\mathrm{u}_1\in U_1} d_{\mathcal{Y}}(\mathrm{u}_1, \mathrm{u}_2)$ and $d_{\mathcal{Y}}(\mathrm{u}_1, U_2) = \inf_{\mathrm{u}_2\in U_2} d_{\mathcal{Y}}(\mathrm{u}_1, \mathrm{u}_2)$ [3].

All selections of the multi-function $\mathcal{G}$ at $\mathrm{u} \in C([0,1],\mathbb{R})$ are denoted by

$$S_{\mathcal{G},\mathrm{u}} := \left\{\varkappa \in \mathcal{L}^1\left([0,1],\mathbb{R}\right) : \varkappa(\mathrm{t}) \in \mathcal{G}\left(\mathrm{t}, \mathrm{u}(\mathrm{t})\right)\right\}$$

for almost all $\mathrm{t} \in [0,1]$ [3, 4]. Also, $S_{\mathcal{G},\mathrm{u}} \neq \emptyset$ if $\dim\mathcal{Y} < \infty$ [3].

Theorem 2.1 ([30]) *Let $\mathcal{Y}$ be a separable Banach space, $\mathcal{G} : [0,1] \times \mathcal{Y} \to \mathbb{P}_{cp,cv}(\mathcal{Y})$ be an $\mathcal{L}^1$-Carathéodory, and $F : \mathcal{L}^1([0,1],\mathcal{Y}) \to C([0,1],\mathcal{Y})$ be a continuous linear mapping. Then $F \circ S_{\mathcal{G}} : C([0,1],\mathcal{Y}) \to \mathbb{P}_{cp,cv}(C([0,1],\mathcal{Y}))$ via the action $\mathrm{u} \mapsto (F \circ S_{\mathcal{G}})(\mathrm{u}) = F(S_{\mathcal{G},\mathrm{u}})$ has a closed graph. Here, $\mathbb{P}_{cp,cv}(\mathcal{Y})$ is all compact convex subsets.*

Theorem 2.2 ([31]) *Let $\mathcal{Y}$ be a Banach algebra and $\mathcal{F}_1 : \mathcal{Y} \to \mathcal{Y}$ and $\mathcal{F}_2 : \mathcal{Y} \to \mathbb{P}_{cp,cv}(\mathcal{Y})$ be such that*

(i) *$\mathcal{F}_1$ is Lipschitz with the constant $L^* > 0$,*
(ii) *$\mathcal{F}_2$ is upper semi-continuous and compact,*
(iii) *$2L^*V < 1$ such that $V = \|\mathcal{F}_2(\mathcal{Y})\|$.*

Then either

(a) *there is a solution for $\mathrm{u} \in (\mathcal{F}_1\mathrm{u})(\mathcal{F}_2\mathrm{u})$; or*
(b) *$\mathcal{Q} = \{\mathrm{u}^* \in \mathcal{Y} | c\mathrm{u}^* \in (\mathcal{F}_1\mathrm{u}^*)(\mathcal{F}_2\mathrm{u}^*), c > 1\}$ is unbounded.*

The Kuratowski measure of noncompactness (denoted by $\omega(W)$) is defined by

$$\omega(W) := \inf\left\{\varepsilon > 0 : W = \bigcup_{k=1}^{n} W_k \text{ and } \operatorname{diam}(W_k) \leq \varepsilon \text{ for } k \in N_1^n\right\},$$

where $\operatorname{diam}(W_k) = \sup\{|\mathrm{u} - \mathrm{u}'| : \mathrm{u}, \mathrm{u}' \in W_k\}$ and W is bounded in $\mathcal{Y}$, and $0 \leq \omega(W) \leq \operatorname{diam}(W) < +\infty$ [32].

Lemma 2.3 ([32]) *Consider $\mathcal{Y}$ as a Banach space and $W, W_1, W_2 \subseteq \mathcal{Y}$ as three bounded subsets belonging to $\mathcal{Y}$. Then*

(1) *W is precompact if and only if $\omega(W) = 0$;*

Also, for each $a \in \mathbb{R}$,

(2) *if* $W_1 \subseteq W_2$, *then* $\omega(W_1) \leq \omega(W_2)$;
(3) $\omega(a + W) \leq \omega(W)$;
(4) $\omega(aW) = |a|\omega(W)$;
(5) $\omega(W_1 + W_2) \leq \omega(W_1) + \omega(W_2)$;
(6) $\omega(W_1 \cup W_2) \leq \max\{\omega(W_1), \omega(W_2)\}$.

In the following lemmas, we consider $\mathcal{Y}$ as a Banach space.

Lemma 2.4 ([32]) *For every bounded set* $W \subset \mathcal{Y}$, *there is a countable set* $W_0 \subset W$ *with* $\omega(W) \leq 2\omega(W_0)$.

Lemma 2.5 ([32]) *For a bounded and equi-continuous set* $W \subseteq C([a,b],\mathcal{Y})$, $\omega(W(\mathfrak{t}))$ *is continuous on* $[a,b]$ *and* $\omega(W) = \sup_{\mathfrak{t}\in[a,b]} \omega(W(\mathfrak{t}))$.

Lemma 2.6 ([32]) *If* $W = \{\mathfrak{u}_n\}_{n\geq 1} \subseteq C([a,b],\mathcal{Y})$ *is countable bounded, then* $\omega(W(\mathfrak{t}))$ *is Lebesgue integrable on* $[a,b]$, *and*

$$\omega\left(\left\{\int_0^{\mathfrak{t}} \mathfrak{u}_n(\mathfrak{s})\,d\mathfrak{s}\right\}_{n\geq 1}\right) \leq 2\int_0^{\mathfrak{t}} \omega\left(\left\{\mathfrak{u}_n(\mathfrak{s})\right\}_{n\geq 1}\right)d\mathfrak{s}.$$

Definition 2.7 ([32]) Let $\mathfrak{h} : \mathbb{D} \subset \mathcal{Y} \to \mathcal{Y}$ be bounded and continuous. Then $\mathfrak{h}$ is condensing if $\omega(\mathfrak{h}(W)) < \omega(W)$ for each bounded closed set $W \subseteq \mathbb{D}$.

We recall Sadovskii's fixed point theorem by assuming the same hypothesis on $\mathcal{Y}$ given above.

Theorem 2.8 ([32]) *Let* $W \subseteq \mathcal{Y}$ *be bounded, convex, closed and* $\mathfrak{h} : W \to W$ *be condensing. Then there is* $\mathfrak{u} \in W$ *such that* $\mathfrak{h}\mathfrak{u} = \mathfrak{u}$.

Theorem 2.9 ([33]) *Consider* ϕ *as a nonnegative locally integrable map on* $[0, T \leq \infty]$ *and* ψ *as a nondecreasing nonnegative continuous map on* $[0,T]$ *along with* $\psi(\mathfrak{t}) \leq M$ *so that* M *is assumed to be a nonzero constant. In addition, let* $\breve{\mathfrak{u}} \geq 0$ *be locally integrable on* $[0,T]$ *with*

$$\breve{\mathfrak{u}}(\mathfrak{t}) \leq \phi(\mathfrak{t}) + \psi(\mathfrak{t})\int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1}\breve{\mathfrak{u}}(\mathfrak{s})\,d\mathfrak{s},$$

so that $q > 0$. *Then*

$$\breve{\mathfrak{u}}(\mathfrak{t}) \leq \phi(\mathfrak{t}) + \int_0^{\mathfrak{t}} \sum_{\jmath=1}^{\infty}\left[\frac{(\psi(\mathfrak{t})\Gamma(q))^{\jmath}}{\Gamma(\jmath q)}(\mathfrak{t}-\mathfrak{s})^{\jmath q-1}\phi(\mathfrak{s})\right]d\mathfrak{s} \quad \left(\mathfrak{t} \in [0,T]\right).$$

Note that the above inequality is known as the generalized Gronwall inequality.

Theorem 2.10 ([34], Krasnoselskii–Zabreiko) *Let* $\mathbb{K}$ *be completely continuous on* $\mathcal{Y}$. *If there is a bounded linear function* $\mathbb{L}$ *on* $\mathcal{Y}$ *so that* 1 *is not its eigenvalue and*

$$\lim_{\|\mathfrak{u}\|\to\infty} \frac{\|\mathbb{K}(\mathfrak{u}) - \mathbb{L}(\mathfrak{u})\|}{\|\mathfrak{u}\|} = 0,$$

then there is $\mathfrak{u}^* \in \mathcal{Y}$ *such that* $\mathbb{K}\mathfrak{u}^* = \mathfrak{u}^*$.

3 Results for non-hybrid FBVP (1.1)

In this section, a non-hybrid version of the given FBVP, defined by (1.1), is studied. In other words, in the first step, we aim to investigate the existence of solution for the given non-hybrid single-valued FBVP with integro-non-hybrid-multiterm-multipoint-multistrip boundary conditions (1.1).

Lemma 3.1 *Let, for* $j = 1, 2, \dots, k$, $\theta_j \geq 0$, $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_{p-1}, c_1, c_2, \dots, c_k \in \mathbb{R}$ *be constants and* $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_r < m_1 \leq m_2 \leq \cdots \leq m_p < \eta_1 \leq \eta_2 \leq \cdots \leq \eta_k < 1$ *and* $\mathfrak{f} \in C(\mathbb{I}, \mathbb{R})$. *Then* $\mathfrak{u}^*$ *satisfies the linear non-hybrid FBVP with integro-multiterm-multipoint-multistrip boundary conditions*:

$$\begin{cases} {}^cD_0^q \mathfrak{u}(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}), \quad q \in (2,3], \\ \mathfrak{u}(\mathfrak{t})|_{\mathfrak{t}=0} = 0, \qquad \mathfrak{u}'(\mathfrak{t})|_{\mathfrak{t}=0} = 0, \\ \int_0^1 \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} = \sum_{\ell=1}^r a_\ell \mathfrak{u}(\mathfrak{t})|_{\mathfrak{t}=\delta_\ell} + \sum_{i=2}^p b_{i-1} \int_{m_{i-1}}^{m_i} \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} + \sum_{j=1}^k c_j I_0^{\theta_j} \mathfrak{u}(\mathfrak{t})|_{\mathfrak{t}=\eta_j} \end{cases} \tag{3.1}$$

which is given by

$$\begin{aligned} \mathfrak{u}(\mathfrak{t}) = {}& \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\,d\mathfrak{s} + \frac{\mathfrak{t}^2}{\mu} \Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{f}(\tau)\,d\tau\,d\mathfrak{s} \\ & - \sum_{\ell=1}^r \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\,d\mathfrak{s} - \sum_{i=2}^p \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{f}(\tau)\,d\tau\,d\mathfrak{s} \\ & - \sum_{j=1}^k \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \mathfrak{f}(\mathfrak{s})\,d\mathfrak{s} \Bigg) \quad (\forall \mathfrak{t} \in \mathbb{I}), \end{aligned} \tag{3.2}$$

where μ *is a constant given as follows*:

$$\mu := -\frac{1}{3} + \sum_{\ell=1}^r a_\ell \delta_\ell^2 + \sum_{i=2}^p \frac{b_{i-1}(m_i^3 - m_{i-1}^3)}{3} + \sum_{j=1}^k \frac{2c_j}{\Gamma(\theta_j+3)} \eta_j^{\theta_j+2} \neq 0. \tag{3.3}$$

Proof Let the function $\mathfrak{u}^*$ be a solution of the linear non-hybrid FDE (3.1). Then

$$\big({}^cD_0^q \mathfrak{u}^*\big)(\mathfrak{t}) = \mathfrak{f}(\mathfrak{t}). \tag{3.4}$$

By applying I_0^q on both sides of the non-hybrid differential equation (3.4), we get

$$\mathfrak{u}^*(\mathfrak{t}) = \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\,d\mathfrak{s} + \alpha_0 + \alpha_1 \mathfrak{t} + \alpha_2 \mathfrak{t}^2 \tag{3.5}$$

for some $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$.

In the first step, the first given initial condition $\mathfrak{u}^*(\mathfrak{t})|_{\mathfrak{t}=0} = 0$ yields $\alpha_0 = 0$. So

$$\mathfrak{u}^*(\mathfrak{t}) = \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\,d\mathfrak{s} + \alpha_1 \mathfrak{t} + \alpha_2 \mathfrak{t}^2.$$

Further, since

$$\big(\mathfrak{u}^*\big)'(\mathfrak{t}) = \frac{1}{\Gamma(q-1)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-2} \mathfrak{f}(\mathfrak{s})\,d\mathfrak{s} + \alpha_1 + 2\alpha_2 \mathfrak{t},$$

thus the second given condition $(\mathfrak{u}^*)'(\mathfrak{t})|_{\mathfrak{t}=0} = 0$ gives $\alpha_1 = 0$ immediately. In consequence,

$$\mathfrak{u}^*(\mathfrak{t}) = \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s} + \alpha_2 \mathfrak{t}^2. \tag{3.6}$$

On the other hand, we have

$$\int_0^1 \mathfrak{u}^*(\mathfrak{s})\, d\mathfrak{s} = \frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \mathfrak{f}(\tau)\, d\tau\, d\mathfrak{s} + \frac{1}{3}\alpha_2,$$

and for $j = 1, \dots, k$,

$$\left(I_0^{\theta_j} \mathfrak{u}^*\right)(\mathfrak{t}) = \frac{1}{\Gamma(q + \theta_j)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{q+\theta_j-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s} + \alpha_2 \frac{2}{\Gamma(3 + \theta_j)} \mathfrak{t}^{2+\theta_j},$$

and for $i = 2, \dots, p$,

$$\int_{m_{i-1}}^{m_i} \mathfrak{u}^*(\mathfrak{s})\, d\mathfrak{s} = \frac{1}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \mathfrak{f}(\tau)\, d\tau\, d\mathfrak{s} + \alpha_2 \left(\frac{m_i^3 - m_{i-1}^3}{3} \right).$$

By considering the nonzero constant μ defined in (3.3) and by the third boundary condition, we obtain the following coefficient:

$$\begin{aligned} \alpha_2 = \frac{1}{\mu} \Bigg(& \frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \mathfrak{f}(\tau)\, d\tau\, d\mathfrak{s} - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s} \\ & - \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \mathfrak{f}(\tau)\, d\tau\, d\mathfrak{s} - \sum_{j=1}^{k} \frac{c_j}{\Gamma(q + \theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s} \Bigg). \end{aligned}$$

By substituting the above value α_2 in (3.6), we have

$$\begin{aligned} \mathfrak{u}^*(\mathfrak{t}) = {} & \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s} + \frac{\mathfrak{t}^2}{\mu} \Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \mathfrak{f}(\tau)\, d\tau\, d\mathfrak{s} \\ & - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s} - \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \mathfrak{f}(\tau)\, d\tau\, d\mathfrak{s} \\ & - \sum_{j=1}^{k} \frac{c_j}{\Gamma(q + \theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s} \Bigg) \quad (\forall \mathfrak{t} \in \mathbb{I}). \end{aligned}$$

Thus we see that $\tilde{\mathfrak{u}}$ satisfies (3.2) and it is the solution of the mentioned integral equation, and so the proof is completed. □

Before starting our theorems, we introduce the Banach space $\mathcal{Y} = \{\mathfrak{u}(\mathfrak{t}) : \mathfrak{u}(\mathfrak{t}) \in C(\mathbb{I}, \mathbb{R})\}$ under the norm $\|\mathfrak{u}\|_{\mathcal{Y}} = \sup_{\mathfrak{u} \in \mathbb{I}} |\mathfrak{u}(\mathfrak{t})|$.

The Kuratowski measure of noncompactness will help us to continue our research on the non-hybrid FBVP (1.1).

Theorem 3.2 *Let $\mathfrak{h}$ be a continuous function with the real values defined on $\mathbb{I} \times \mathcal{Y}$. Moreover, assume that there is a continuous function $\varrho : \mathbb{I} \to \mathbb{R}^+$ such that an inequality*

$$\left|\mathfrak{h}\big(\mathfrak{t}, \mathfrak{u}(\mathfrak{t})\big)\right| \le \varrho(\mathfrak{t}) \tag{3.7}$$

holds for any $\mathfrak{t} \in \mathbb{I}$ and $\mathfrak{u} \in \mathcal{Y}$. Furthermore, assume that there is a function $n_{\mathfrak{h}} : \mathbb{I} \to \mathbb{R}^+$ such that

$$\omega\big(\mathfrak{h}(\mathfrak{t}, W)\big) \le n_{\mathfrak{h}}(\mathfrak{t})\omega(W) \tag{3.8}$$

for each bounded set $W \subset \mathcal{Y}$. Then the non-hybrid FBVP with integro-non-hybrid-multiterm-multipoint-multistrip conditions (1.1) has at least one solution on $\mathbb{I}$ if

$$n^*_{\mathfrak{h}} \hat{\Psi} < \frac{1}{4}, \tag{3.9}$$

where

$$\begin{aligned}\hat{\Psi} = {}& \frac{1}{\Gamma(q+1)} + \frac{1}{|\mu|}\left(\frac{1}{\Gamma(q+2)} + \sum_{\ell=1}^{r} \frac{a_\ell \delta_\ell^q}{\Gamma(q+1)} + \sum_{i=2}^{p} \frac{b_{i-1}(m_i^{q+1} - m_{i-1}^{q+1})}{\Gamma(q+2)}\right.\\ &\left. + \sum_{j=1}^{k} \frac{c_j \eta_j^{q+\theta_j}}{\Gamma(q+\theta_j+1)}\right),\end{aligned} \tag{3.10}$$

and $n^*_{\mathfrak{h}} = \sup_{\mathfrak{t}\in\mathbb{I}} |n_{\mathfrak{h}}(\mathfrak{t})|$.

Proof In relation to the non-hybrid-FBVP with integro-non-hybrid-multiterm-multipoint-multistrip boundary conditions (1.1), by Lemma 3.1, we define $\mathfrak{P} : \overline{\mathfrak{B}_\varepsilon} \to \overline{\mathfrak{B}_\varepsilon}$ by

$$\begin{aligned}\mathfrak{P}(\mathfrak{u})(\mathfrak{t}) = {}& \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{q-1} \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}(\mathfrak{s})\big)\, d\mathfrak{s} + \frac{\mathfrak{t}^2}{\mu}\left(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \mathfrak{h}\big(\tau, \mathfrak{u}(\tau)\big)\, d\tau\, d\mathfrak{s}\right.\\ & - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}(\mathfrak{s})\big)\, d\mathfrak{s}\\ & - \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \mathfrak{h}\big(\tau, \mathfrak{u}(\tau)\big)\, d\tau\, d\mathfrak{s}\\ & \left. - \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}(\mathfrak{s})\big)\, d\mathfrak{s}\right) \quad (\forall \mathfrak{t} \in \mathbb{I}),\end{aligned} \tag{3.11}$$

where $\overline{\mathfrak{B}_\varepsilon} := \{\mathfrak{u} \in \mathcal{Y} : \|\mathfrak{u}\|_{\mathcal{Y}} \le \varepsilon, \varepsilon \in \mathbb{R}^+\}$.

To hold Theorem 2.8, we prove the continuity of $\mathfrak{P}$ on $\overline{\mathfrak{B}_\varepsilon}$. Let $\{\mathfrak{u}_n\}_{n\ge1} \subset \overline{\mathfrak{B}_\varepsilon}$ such that $\mathfrak{u}_n \to \mathfrak{u}$ for all $\mathfrak{u} \in \overline{\mathfrak{B}_\varepsilon}$. For the sake of the continuity of the function $\mathfrak{h}$ on $\mathbb{I} \times \mathcal{Y}$, we obtain $\lim_{n\to\infty} \mathfrak{h}(\mathfrak{t}, \mathfrak{u}_n(\mathfrak{t})) = \mathfrak{h}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}))$. By the dominated convergence theorem attributed to Lebesgue, it gives

$$\lim_{n\to\infty} (\mathfrak{P}\mathfrak{u}_n)(\mathfrak{t}) = \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{q-1} \lim_{n\to\infty} \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}_n(\mathfrak{s})\big)\, d\mathfrak{s}$$

$$
\begin{aligned}
&+\frac{\mathfrak{t}^2}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\lim_{n\to\infty}\mathfrak{h}\bigl(\tau,\mathfrak{u}_n(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&-\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\lim_{n\to\infty}\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}_n(\mathfrak{s})\bigr)\,\mathrm{d}\mathfrak{s}\\
&-\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\lim_{n\to\infty}\mathfrak{h}\bigl(\tau,\mathfrak{u}_n(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&-\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\lim_{n\to\infty}\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}_n(\mathfrak{s})\bigr)\,\mathrm{d}\mathfrak{s}\Biggr)\\
=&\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)\,\mathrm{d}\mathfrak{s}\\
&+\frac{\mathfrak{t}^2}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\mathfrak{h}\bigl(\tau,\mathfrak{u}(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&-\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)\,\mathrm{d}\mathfrak{s}\\
&-\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\mathfrak{h}\bigl(\tau,\mathfrak{u}(\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&-\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)\,\mathrm{d}\mathfrak{s}\Biggr)\\
=&(\mathfrak{P}\mathfrak{u})(\mathfrak{t}).
\end{aligned}
$$

Hence, we get $\lim_{n\to\infty}(\mathfrak{P}\mathfrak{u}_n)(\mathfrak{t})=(\mathfrak{P}\mathfrak{u})(\mathfrak{t})$. Now, consider the member $\mathfrak{u}\in\overline{\mathfrak{B}}_\varepsilon$. By (3.7), we estimate

$$
\begin{aligned}
\bigl|\mathfrak{P}(\mathfrak{u})\bigr|(\mathfrak{t})\le&\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\bigl|\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)\bigr|\,\mathrm{d}\mathfrak{s}\\
&+\frac{\mathfrak{t}^2}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\bigl|\mathfrak{h}\bigl(\tau,\mathfrak{u}(\tau)\bigr)\bigr|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\bigl|\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)\bigr|\,\mathrm{d}\mathfrak{s}\\
&+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\bigl|\mathfrak{h}\bigl(\tau,\mathfrak{u}(\tau)\bigr)\bigr|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\bigl|\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)\bigr|\,\mathrm{d}\mathfrak{s}\Biggr)\\
\le&\frac{\mathfrak{t}^q}{\Gamma(q+1)}\varrho(\mathfrak{t})+\frac{\mathfrak{t}^2\varrho(\mathfrak{t})}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\,\mathrm{d}\mathfrak{s}+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\,d\mathfrak{s}\Biggr)\\
&\leq\frac{\varrho^*}{\Gamma(q+1)}+\frac{\varrho^*}{\mu}\Biggl(\frac{1}{\Gamma(q+2)}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell\delta_\ell^q}{\Gamma(q+1)}+\sum_{i=2}^{p}\frac{b_{i-1}(m_i^{q+1}-m_{i-1}^{q+1})}{\Gamma(q+2)}+\sum_{j=1}^{k}\frac{c_j\eta_j^{q+\theta_j}}{\Gamma(q+\theta_j+1)}\Biggr)\\
&=\hat{\Psi}\varrho^*,
\end{aligned}
$$

where $\hat{\Psi}$ is introduced in (3.10). In consequence, the above estimate becomes $\|\mathfrak{P}\mathfrak{u}\|_{\mathcal{Y}}\leq\hat{\Psi}\varrho^*<\infty$. Thus $\mathfrak{P}(\overline{\mathfrak{B}_\varepsilon})$ is uniformly bounded. Let $\mathfrak{t}_1,\mathfrak{t}_2\in\mathbb{I}$ with $\mathfrak{t}_1<\mathfrak{t}_2$ and $\mathfrak{u}\in\overline{\mathfrak{B}_\varepsilon}$. Then, by assuming $\sup_{(\mathfrak{t},\mathfrak{u})\in\mathbb{I}\times\bar{\mathfrak{B}}_\varepsilon}|\mathfrak{h}(\mathfrak{t},\mathfrak{u})|=\widetilde{\mathfrak{h}}_*>0$, it gives

$$
\begin{aligned}
\bigl|(\mathfrak{P}\mathfrak{u})(\mathfrak{t}_2)-(\mathfrak{P}\mathfrak{u})(\mathfrak{t}_1)\bigr|&\leq\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}_1}\bigl[(\mathfrak{t}_2-\mathfrak{s})^{q-1}-(\mathfrak{t}_1-\mathfrak{s})^{q-1}\bigr]\bigl|\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)\bigr|\,d\mathfrak{s}\\
&+\frac{1}{\Gamma(q)}\int_{\mathfrak{t}_1}^{\mathfrak{t}_2}(\mathfrak{t}_2-\mathfrak{s})^{q-1}\bigl|\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)\bigr|\,d\mathfrak{s}\\
&+\frac{(\mathfrak{t}_2^2-\mathfrak{t}_1^2)}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\bigl|\mathfrak{h}\bigl(\tau,\mathfrak{u}(\tau)\bigr)\bigr|\,d\tau\,d\mathfrak{s}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\bigl|\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)\bigr|\,d\mathfrak{s}\\
&+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\bigl|\mathfrak{h}\bigl(\tau,\mathfrak{u}(\tau)\bigr)\bigr|\,d\tau\,d\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)\,d\mathfrak{s}\Biggr)\\
&\leq\frac{\widetilde{\mathfrak{h}}_*}{\Gamma(q+1)}\bigl[(\mathfrak{t}_2-\mathfrak{t}_1)^q+2\bigl(\mathfrak{t}_2^q-\mathfrak{t}_1^q\bigr)\bigr]\\
&+\frac{(\mathfrak{t}_2^2-\mathfrak{t}_1^2)\widetilde{\mathfrak{h}}_*}{\mu}\Biggl(\frac{1}{\Gamma(q+2)}+\sum_{\ell=1}^{r}\frac{a_\ell\delta_\ell^q}{\Gamma(q+1)}\\
&+\sum_{i=2}^{p}\frac{b_{i-1}(m_i^{q+1}-m_{i-1}^{q+1})}{\Gamma(q+2)}+\sum_{j=1}^{k}\frac{c_j\eta_j^{q+\theta_j}}{\Gamma(q+\theta_j+1)}\Biggr)\to 0,
\end{aligned}
$$

as $\mathfrak{t}_1$ tends to $\mathfrak{t}_2$ (independent of $\mathfrak{u}\in\overline{\mathfrak{B}_\varepsilon}$). Hence $\mathfrak{P}$ is equi-continuous. It follows that $\mathfrak{P}$ is completely continuous by the Arzela–Ascoli theorem, and it is compact on $\overline{\mathfrak{B}_\varepsilon}$.

We show that $\mathfrak{P}$ is condensing on $\overline{\mathfrak{B}_\varepsilon}$. Lemma 2.4 gives this fact that there is a countable set $W_0=\{\mathfrak{u}_n\}_{n\geq1}\subset W$ for each bounded set $W\subset\overline{\mathfrak{B}_\varepsilon}$ such that $\omega(\mathfrak{P}(W))\leq2\omega(\mathfrak{P}(W_0))$. Hence, by Lemmas 2.3, 2.5, and 2.6, we get the following inequalities:

$$
\begin{aligned}
\omega\bigl(\mathfrak{P}\bigl(W(\mathfrak{t})\bigr)\bigr)&\leq2\omega\bigl(\mathfrak{P}\bigl(\{\mathfrak{u}_n\}_{n\geq1}\bigr)\bigr)\\
&\leq\frac{2}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\omega\bigl(\mathfrak{h}\bigl(\mathfrak{s},\{\mathfrak{u}_n(\mathfrak{s})\}_{n\geq1}\bigr)\bigr)\,d\mathfrak{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2\mathfrak{t}^2}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\omega\bigl(\mathfrak{h}\bigl(\tau,\{\mathfrak{u}_n(\tau)\}_{n\ge1}\bigr)\bigr)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\omega\bigl(\mathfrak{h}\bigl(\mathfrak{s},\{\mathfrak{u}_n(\mathfrak{s})\}_{n\ge1}\bigr)\bigr)\,\mathrm{d}\mathfrak{s}\\
&+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\omega\bigl(\mathfrak{h}\bigl(\tau,\{\mathfrak{u}_n(\tau)\}_{n\ge1}\bigr)\bigr)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\omega\bigl(\mathfrak{h}\bigl(\mathfrak{s},\{\mathfrak{u}_n(\mathfrak{s})\}_{n\ge1}\bigr)\bigr)\,\mathrm{d}\mathfrak{s}\Biggr)\\
\le{}&\frac{4}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}n_{\mathfrak{h}}(\mathfrak{s})\omega\bigl(\{\mathfrak{u}_n(\mathfrak{s})\}_{n\ge1}\bigr)\,\mathrm{d}\mathfrak{s}\\
&+\frac{4\mathfrak{t}^2}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}n_{\mathfrak{h}}(\tau)\omega\bigl(\{\mathfrak{u}_n(\tau)\}_{n\ge1}\bigr)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}n_{\mathfrak{h}}(\mathfrak{s})\omega\bigl(\{\mathfrak{u}_n(\mathfrak{s})\}_{n\ge1}\bigr)\,\mathrm{d}\mathfrak{s}\\
&+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}n_{\mathfrak{h}}(\tau)\omega\bigl(\{\mathfrak{u}_n(\tau)\}_{n\ge1}\bigr)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}n_{\mathfrak{h}}(\mathfrak{s})\omega\bigl(\{\mathfrak{u}_n(\mathfrak{s})\}_{n\ge1}\bigr)\,\mathrm{d}\mathfrak{s}\Biggr)\\
\le{}&\frac{4n_{\mathfrak{h}}^*\omega(W)}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\,\mathrm{d}\mathfrak{s}+\frac{4n_{\mathfrak{h}}^*\omega(W)}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\,\mathrm{d}\mathfrak{s}+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\,\mathrm{d}\mathfrak{s}\Biggr)\\
\le{}&\frac{4n_{\mathfrak{h}}^*\omega(W)}{\Gamma(q+1)}+\frac{4n_{\mathfrak{h}}^*\omega(W)}{\mu}\Biggl(\frac{1}{\Gamma(q+2)}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell\delta_\ell^q}{\Gamma(q+1)}+\sum_{i=2}^{p}\frac{b_{i-1}(m_i^{q+1}-m_{i-1}^{q+1})}{\Gamma(q+2)}+\sum_{j=1}^{k}\frac{c_j\eta_j^{q+\theta_j}}{\Gamma(q+\theta_j+1)}\Biggr).
\end{aligned}
$$

Hence,

$$\omega\bigl(\mathfrak{P}(W)\bigr)\le 4n_{\mathfrak{h}}^*\hat{\Psi}\omega(W),$$

where $\hat{\Psi}$ is defined in (3.10). Accordingly, by (3.9), we get $\omega(\mathfrak{P}(W))<\omega(W)$. So $\mathfrak{P}$ is a condensing map on $\overline{\mathfrak{B}_\varepsilon}$. By Theorem 2.8, $\mathfrak{P}$ has at least one fixed point in $\overline{\mathfrak{B}_\varepsilon}$, and accordingly, there is a solution for the given non-hybrid single-valued-FBVP with integro-non-hybrid-multiterm-multipoint-multistrip boundary conditions (1.1). □

In the present step, we aim to investigate the dependence of solutions for the non-hybrid single-valued FBVP with integro-non-hybrid-multiterm-multipoint-multistrip boundary conditions (1.1). Indeed, this part of the paper states that the solution of the non-hybrid single-valued FBVP (1.1) depends on some parameters so that the nonlinear map $\mathfrak{h}$ satisfies Theorem 3.2, which ensures the existence of solutions, and the continuous dependence of solutions on the coefficients and orders gives the stability in relation to the solutions of (1.1). We act on the solutions of the non-hybrid single-valued FBVP with integro-non-hybrid-multiterm-multipoint-multistrip boundary conditions (1.1) by changing the order of the non-hybrid single-valued FBVP (1.1) to a small value. The generalized Gronwall inequality will be useful for our purpose.

Theorem 3.3 *Let $q > 0$ such that $2 < q - \alpha < q < 3$. Moreover, assume that $\mathfrak{h} : \mathbb{I} \times \mathcal{Y} \to \mathcal{Y}$ is continuous and there is $\beta > 0$ so that*

$$\left|\mathfrak{h}\big(\mathfrak{t}, \mathfrak{u}(\mathfrak{t})\big) - \mathfrak{h}\big(\mathfrak{t}, \mathfrak{u}'(\mathfrak{t})\big)\right| \leq \beta\left|\mathfrak{u}(\mathfrak{t}) - \mathfrak{u}'(\mathfrak{t})\right| \tag{3.12}$$

for all $\mathfrak{u}, \mathfrak{u}' \in \mathcal{Y}$ and $\mathfrak{t} \in \mathbb{I}$. Moreover, let $\mathfrak{u}$ be the solution of the non-hybrid single-valued FBVP with integro-non-hybrid-multiterm-multipoint-multistrip boundary conditions (1.1) and $\mathfrak{v}$ be the solution of

$$\begin{cases} {}^{c}D_0^{q-\alpha}\mathfrak{v}(\mathfrak{t}) = \mathfrak{h}(\mathfrak{t}, \mathfrak{v}(\mathfrak{t})), \\ \mathfrak{v}(\mathfrak{t})|_{\mathfrak{t}=0} = 0, \qquad \mathfrak{v}'(\mathfrak{t})|_{\mathfrak{t}=0} = 0, \\ \int_0^1 \mathfrak{v}(\mathfrak{s})\,d\mathfrak{s} = \sum_{\ell=1}^{r} a_\ell \mathfrak{v}(\mathfrak{t})|_{\mathfrak{t}=\delta_\ell} + \sum_{i=2}^{p} b_{i-1}\int_{m_{i-1}}^{m_i} \mathfrak{v}(\mathfrak{s})\,d\mathfrak{s} + \sum_{j=1}^{k} c_j I_0^{\theta_j}\mathfrak{v}(\mathfrak{t})|_{\mathfrak{t}=\eta_j}. \end{cases} \tag{3.13}$$

Then the following inequality is valid:

$$\|\mathfrak{u} - \mathfrak{v}\|_{\mathcal{Y}} \leq \frac{\mathcal{F} + \mathcal{F}\sum_{\jmath=1}^{\infty}\frac{\beta^{\kappa}}{\Gamma(\jmath(q-\alpha)+1)}}{1 - B - B\sum_{\jmath=1}^{\infty}\frac{\beta^{\jmath}}{\Gamma(\jmath(q-\alpha)+1)}}, \tag{3.14}$$

provided that $B + B\sum_{\jmath=1}^{\infty}\frac{\beta^{\jmath}}{\Gamma(\jmath(q-\alpha)+1)} < 1$, where

$$\begin{aligned} \mathcal{F} = {}& \|\mathfrak{h}\| \sup_{\mathfrak{t}\in\mathbb{I}}\left|\frac{\mathfrak{t}^q}{\Gamma(q+1)} - \frac{\mathfrak{t}^q}{\Gamma(q-\alpha+1)}\right| \\ & + \sup_{\mathfrak{t}\in\mathbb{I}}\frac{\mathfrak{t}^2}{\mu}\left(\|\mathfrak{h}\|\left|\frac{1}{\Gamma(q+2)} - \frac{1}{\Gamma(q-\alpha+2)}\right|\right. \\ & + \|\mathfrak{h}\|\left|\sum_{\ell=1}^{r}\frac{a_\ell\delta_\ell^q}{\Gamma(q+1)} - \sum_{\ell=1}^{r}\frac{a_\ell\delta_\ell^{q-\alpha}}{\Gamma(q-\alpha+1)}\right| \\ & + \|\mathfrak{h}\|\left|\sum_{i=2}^{p}\frac{b_{i-1}(m_i^{q+1} - m_{i-1}^{q+1})}{\Gamma(q+2)} - \sum_{i=2}^{p}\frac{b_{i-1}(m_i^{q-\alpha+1} - m_{i-1}^{q-\alpha+1})}{\Gamma(q-\alpha+2)}\right| \\ & \left. + \|\mathfrak{h}\|\left|\sum_{j=1}^{k}\frac{c_j\eta_j^{q+\theta_j}}{\Gamma(q+\theta_j+1)} - \sum_{j=1}^{k}\frac{c_j\eta_j^{q-\alpha+\theta_j}}{\Gamma(q-\alpha+\theta_j+1)}\right|\right), \end{aligned} \tag{3.15}$$

and

$$B = \frac{1}{\mu}\left(\frac{1}{\Gamma(q-\alpha+2)} + \sum_{\ell=1}^{r}\frac{a_\ell \delta_\ell^{q-\alpha}}{\Gamma(q-\alpha+1)} + \sum_{i=2}^{p}\frac{b_{i-1}(m_i^{q-\alpha+1} - m_{i-1}^{q-\alpha+1})}{\Gamma(q-\alpha+2)} + \sum_{j=1}^{k}\frac{c_j \eta_j^{q-\alpha+\theta_j}}{\Gamma(q-\alpha+\theta_j+1)}\right|\right)\beta, \tag{3.16}$$

and $\|\mathfrak{h}\|_{\mathcal{Y}} = \sup_{\mathfrak{t}\in\mathbb{I}} |\mathfrak{h}(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))|$.

Proof Prior to proceeding to derive inequality (3.14), we know that the existence of solution for two non-hybrid single-valued-FBVPs (1.1) and (3.13) is guaranteed by the same proof done above, and so the solutions of these two non-hybrid-single-valued-FBVPs are obtained by (3.2) and

$$\begin{aligned} \mathfrak{v}(\mathfrak{t}) = {}& \frac{1}{\Gamma(q-\alpha)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-\alpha-1}\mathfrak{h}\big(\mathfrak{s},\mathfrak{v}(\mathfrak{s})\big)\,d\mathfrak{s} \\ & + \frac{\mathfrak{t}^2}{\mu}\left(\frac{1}{\Gamma(q-\alpha)}\int_0^1\int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-\alpha-1}\mathfrak{h}\big(\tau,\mathfrak{v}(\tau)\big)\,d\tau\,d\mathfrak{s}\right. \\ & - \sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q-\alpha)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-\alpha-1}\mathfrak{h}\big(\mathfrak{s},\mathfrak{v}(\mathfrak{s})\big)\,d\mathfrak{s} \\ & - \sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q-\alpha)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-\alpha-1}\mathfrak{h}\big(\tau,\mathfrak{v}(\tau)\big)\,d\tau\,d\mathfrak{s} \\ & \left. - \sum_{j=1}^{k}\frac{c_j}{\Gamma(q-\alpha+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q-\alpha+\theta_j-1}\mathfrak{h}\big(\mathfrak{s},\mathfrak{v}(\mathfrak{s})\big)\,d\mathfrak{s}\right) \quad (\forall \mathfrak{t}\in\mathbb{I}), \end{aligned} \tag{3.17}$$

respectively. Then, the following estimate for $\mathfrak{u}-\mathfrak{v}$ is calculated as follows:

$$\begin{aligned} \big|\mathfrak{u}(\mathfrak{t}) - \mathfrak{v}(\mathfrak{t})\big| \le {}& \left|\int_0^{\mathfrak{t}}\left(\frac{(\mathfrak{t}-\mathfrak{s})^{q-1}}{\Gamma(q)} - \frac{(\mathfrak{t}-\mathfrak{s})^{q-\alpha-1}}{\Gamma(q-\alpha)}\right)\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big)\,d\mathfrak{s}\right| \\ & + \int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\mathfrak{s})^{q-\alpha-1}}{\Gamma(q-\alpha)}\big|\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big) - \mathfrak{h}\big(\mathfrak{s},\mathfrak{v}(\mathfrak{s})\big)\big|\,d\mathfrak{s} \\ & + \frac{\mathfrak{t}^2}{\mu}\left(\left|\int_0^1\int_0^{\mathfrak{s}}\left(\frac{(\mathfrak{s}-\tau)^{q-1}}{\Gamma(q)} - \frac{(\mathfrak{s}-\tau)^{q-\alpha-1}}{\Gamma(q-\alpha)}\right)\mathfrak{h}\big(\tau,\mathfrak{u}(\tau)\big)\,d\tau\,d\mathfrak{s}\right|\right. \\ & + \int_0^1\int_0^{\mathfrak{s}}\frac{(\mathfrak{s}-\tau)^{q-\alpha-1}}{\Gamma(q-\alpha)}\big|\mathfrak{h}\big(\tau,\mathfrak{u}(\tau)\big) - \mathfrak{h}\big(\tau,\mathfrak{v}(\tau)\big)\big|\,d\tau\,d\mathfrak{s} \\ & + \left|\sum_{\ell=1}^{r}\int_0^{\delta_\ell}\left(\frac{a_\ell(\delta_\ell-\mathfrak{s})^{q-1}}{\Gamma(q)} - \frac{a_\ell(\delta_\ell-\mathfrak{s})^{q-\alpha-1}}{\Gamma(q-\alpha)}\right)\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big)\,d\mathfrak{s}\right| \\ & + \sum_{\ell=1}^{r}\int_0^{\delta_\ell}\frac{a_\ell(\delta_\ell-\mathfrak{s})^{q-\alpha-1}}{\Gamma(q-\alpha)}\big|\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big) - \mathfrak{h}\big(\mathfrak{s},\mathfrak{v}(\mathfrak{s})\big)\big|\,d\mathfrak{s} \\ & + \left|\sum_{i=2}^{p}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}\left(\frac{b_{i-1}(\mathfrak{s}-\tau)^{q-1}}{\Gamma(q)} - \frac{b_{i-1}(\mathfrak{s}-\tau)^{q-\alpha-1}}{\Gamma(q-\alpha)}\right)\mathfrak{h}\big(\tau,\mathfrak{u}(\tau)\big)\,d\tau\,d\mathfrak{s}\right| \end{aligned}$$

$$+\sum_{i=2}^{p}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}\frac{b_{i-1}(\mathfrak{s}-\tau)^{q-\alpha-1}}{\Gamma(q-\alpha)}\left|\mathfrak{h}\big(\tau,\mathfrak{u}(\tau)\big)-\mathfrak{h}\big(\tau,\mathfrak{v}(\tau)\big)\right|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}$$
$$+\left|\sum_{j=1}^{k}\int_0^{\eta_j}\left(\frac{c_j(\eta_j-\mathfrak{s})^{q+\theta_j-1}}{\Gamma(q+\theta_j)}-\frac{c_j(\eta_j-\mathfrak{s})^{q-\alpha+\theta_j-1}}{\Gamma(q-\alpha+\theta_j)}\right)\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big)\,\mathrm{d}\mathfrak{s}\right|$$
$$+\sum_{j=1}^{k}\int_0^{\eta_j}\frac{c_j(\eta_j-\mathfrak{s})^{q-\alpha+\theta_j-1}}{\Gamma(q-\alpha+\theta_j)}\left|\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big)-\mathfrak{h}\big(\mathfrak{s},\mathfrak{v}(\mathfrak{s})\big)\right|\,\mathrm{d}\mathfrak{s}\Bigg)$$
$$\le\mathcal{F}+B\|\mathfrak{u}-\mathfrak{v}\|_{\mathcal{Y}}+\int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\mathfrak{s})^{q-\alpha-1}}{\Gamma(q-\alpha)}\beta\left|\mathfrak{u}(\mathfrak{s})-\mathfrak{v}(\mathfrak{s})\right|\,\mathrm{d}\mathfrak{s},$$

so that $\mathcal{F}$ and B are introduced by (3.15) and (3.16). Thus, by the generalized Gronwall inequality presented in Theorem 2.9 and by assuming $\breve{\mathfrak{u}}(\mathfrak{t})=|\mathfrak{u}(\mathfrak{t})-\mathfrak{v}(\mathfrak{t})|$, $\phi(\mathfrak{t})=\mathcal{F}+B\|\mathfrak{u}-\mathfrak{v}\|_{\mathcal{Y}}$ and $\psi(\mathfrak{t})=\frac{\beta}{\Gamma(q-\alpha)}$, we obtain

$$\left|\mathfrak{u}(\mathfrak{t})-\mathfrak{v}(\mathfrak{t})\right|\le\mathcal{F}+B\|\mathfrak{u}-\mathfrak{v}\|_{\mathcal{Y}}+\int_0^{\mathfrak{t}}\sum_{J=1}^{\infty}\left[\frac{\beta^J(\mathfrak{t}-\mathfrak{s})^{J(q-\alpha)-1}}{\Gamma(J(q-\alpha))}\big(\mathcal{F}+B\|\mathfrak{u}-\mathfrak{v}\|_{\mathcal{Y}}\big)\right]\mathrm{d}\mathfrak{s}.$$

Hence,

$$\|\mathfrak{u}-\mathfrak{v}\|_{\mathcal{Y}}\le\frac{\mathcal{F}+\mathcal{F}\sum_{J=1}^{\infty}\frac{\beta^J}{\Gamma(J(q-\alpha)+1)}}{1-B-B\sum_{J=1}^{\infty}\frac{\beta^J}{\Gamma(J(q-\alpha)+1)}},$$

and the latter inequality completes the proof. □

The next fixed point theorem is due to Krasnoselskii and Zabreiko, and we prove our existence result with the help of it for the non-hybrid single-valued FBVP (1.1).

Theorem 3.4 *Let*

($\mathbb{J}5$) $\mathfrak{h}:\mathbb{I}\times\mathbb{R}\to\mathbb{R}$ *be continuous and for some* $\mathfrak{t}\in\mathbb{I}$, $\mathfrak{h}(\mathfrak{t},0)\neq 0$ *and*

$$\lim_{\|\mathfrak{u}\|\to\infty}\frac{\mathfrak{h}(\mathfrak{t},\mathfrak{u})}{\mathfrak{u}}=\rho(\mathfrak{t});\tag{3.18}$$

($\mathbb{J}6$) *there be* $A\in\mathbb{R}_+$ *such that*

$$\left|\mathfrak{h}(\mathfrak{t},\mathfrak{u})\right|\le A\left|\mathfrak{u}(\mathfrak{t})\right|.$$

Then there exists at least one solution for the non-hybrid single-valued FBVP (1.1) *on* $\mathbb{I}$ *such that*

$$\rho_{\max}:=\max_{\mathfrak{t}\in\mathbb{I}}\left|\rho(\mathfrak{t})\right|<\frac{1}{\hat{\Psi}},\tag{3.19}$$

where $\hat{\Psi}$ *is given by* (3.10).

Proof Assume that $\{\mathfrak{u}_n\}_{n\in\mathbb{N}}$ tends to $\mathfrak{u}$. We know that $\mathfrak{h}$ is continuous. As $n\to\infty$, we get

$$\left|\mathfrak{h}(\mathfrak{t},\mathfrak{u}_n)-\mathfrak{h}(\mathfrak{t},\mathfrak{u})\right|\to 0.$$

Thus, for $\mathfrak{t} \in \mathbb{I}$, and by defining $\mathfrak{P} : \mathcal{Y} \to \mathcal{Y}$ given by (3.11), we write

$$\begin{aligned}
&\left|\mathfrak{P}\mathfrak{u}_n(\mathfrak{t}) - \mathfrak{P}\mathfrak{u}(\mathfrak{t})\right| \\
&\quad \le \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \left|\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}_n(\mathfrak{s})\big) - \mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big)\right| \, d\mathfrak{s} \\
&\qquad + \frac{\mathfrak{t}^2}{\mu}\Big(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \left|\mathfrak{h}\big(\tau,\mathfrak{u}_n(\tau)\big) - \right|\mathfrak{h}\big(\tau,\mathfrak{u}(\tau)\big)\Big| \, d\tau \, d\mathfrak{s} \\
&\qquad + \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \Big|\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}_n(\mathfrak{s})\big) - \mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big)\Big| \, d\mathfrak{s} \\
&\qquad + \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \Big|\mathfrak{h}\big(\tau,\mathfrak{u}_n(\tau)\big) - \mathfrak{h}\big(\tau,\mathfrak{u}(\tau)\big)\Big| \, d\tau \, d\mathfrak{s} \\
&\qquad + \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \Big|\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}_n(\mathfrak{s})\big) - \mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})| \, d\mathfrak{s}\big) \to 0.
\end{aligned} \tag{3.20}$$

Thus (3.20) tends to zero. This yields the continuity of $\mathfrak{P}$. For $r > 0$, we set

$$N = \big\{\mathfrak{u} \in C(J,\mathbb{R}); \|\mathfrak{u}\| \le r\big\}$$

and $\|\mathfrak{h}\| = \sup_{(\mathfrak{t},\mathfrak{u}) \in \mathbb{I} \times N} |\mathfrak{h}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))|$. So

$$\begin{aligned}
\left|\mathfrak{P}(\mathfrak{u})\right|(\mathfrak{t}) &\le \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \left|\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big)\right| \, d\mathfrak{s} \\
&\quad + \frac{\mathfrak{t}^2}{\mu}\Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \left|\mathfrak{h}\big(\tau,\mathfrak{u}(\tau)\big)\right| \, d\tau \, d\mathfrak{s} \\
&\quad + \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \left|\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big)\right| \, d\mathfrak{s} \\
&\quad + \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \left|\mathfrak{h}\big(\tau,\mathfrak{u}(\tau)\big)\right| \, d\tau \, d\mathfrak{s} \\
&\quad + \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \left|\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big)\right| \, d\mathfrak{s}\Bigg) \\
&\le \frac{Ar}{\Gamma(q+1)} + \frac{Ar}{\mu}\Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \, d\tau \, d\mathfrak{s} \\
&\quad + \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \, d\mathfrak{s} + \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \, d\tau \, d\mathfrak{s} \\
&\quad + \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \, d\mathfrak{s}\Bigg) \\
&\le A\hat{\Psi}r,
\end{aligned}$$

which yields $\|\mathfrak{P}u\| \le A\hat{\Psi}r$, where $\hat{\Psi}$ is given by (3.10). This gives the uniform boundedness of $\mathfrak{P}$.

The equicontinuity of $\mathfrak{P}$ is established similar to the proof of Theorem 3.2. Immediately, the Arzelà–Ascoli theorem confirms the compactness of $\mathfrak{P}$ on N.

Now, by considering the non-hybrid single-valued FBVP (1.1), and by taking

$$\mathfrak{h}\big(\mathfrak{t},\mathfrak{u}(\mathfrak{t})\big) = \rho(\mathfrak{t})\mathfrak{u}(\mathfrak{t}),$$

the operator $\mathbb{L}$, by Lemma 3.1, is defined by

$$\begin{aligned}
\mathbb{L}\mathfrak{u}(\mathfrak{t}) = {}& \frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1}\rho(\mathfrak{s})\mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} + \frac{\mathfrak{t}^2}{\mu}\Bigg(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1}\rho(\tau)\mathfrak{u}(\tau)\,d\tau\,d\mathfrak{s} \\
& - \sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\rho(\mathfrak{s})\mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} \\
& - \sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\rho(\tau)\mathfrak{u}(\tau)\,d\tau\,d\mathfrak{s} \\
& - \sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\rho(\mathfrak{s})\mathfrak{u}(\mathfrak{s})\,d\mathfrak{s}\Bigg) \quad (\forall\mathfrak{t}\in\mathbb{I}).
\end{aligned}$$

We further claim that 1 is not an eigenvalue of $\mathbb{L}$. If it is so, then by (3.19) we compute

$$\begin{aligned}
\|\mathfrak{u}\| = {}& \sup_{\mathfrak{t}\in\mathbb{I}}\big|\mathbb{L}\mathfrak{u}(\mathfrak{t})\big| \\
\le {}& \sup_{\mathfrak{t}\in\mathbb{I}}\Bigg\{\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\rho(\mathfrak{s})|\big|\mathfrak{u}(\mathfrak{s})\big|\,d\mathfrak{s} \\
& + \frac{\mathfrak{t}^2}{\mu}\Bigg(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\big|\rho(\tau)\big|\big|\mathfrak{u}(\tau)\big|\,d\tau\,d\mathfrak{s} \\
& + \sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\big|\rho(\mathfrak{s})\big|\big|\mathfrak{u}(\mathfrak{s})\big|\,d\mathfrak{s} \\
& + \sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\big|\rho(\tau)\big|\big|\mathfrak{u}(\tau)\big|\,d\tau\,d\mathfrak{s} \\
& + \sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\big|\rho(\mathfrak{s})\big|\big|\mathfrak{u}(\mathfrak{s})\big|\,d\mathfrak{s}\Bigg)\Bigg\} \\
\le {}& \rho_{\max}\hat{\Psi}\|\mathfrak{u}\| < \|\mathfrak{u}\| \quad (\forall\mathfrak{t}\in\mathbb{I}).
\end{aligned}$$

This is invalid. Hence our claim is correct. To conclude the proof, we claim that $\|\mathfrak{P}(\mathfrak{u}) - \mathbb{L}(\mathfrak{u})\|/\|\mathfrak{u}\|$ vanishes when $\|\mathfrak{u}\| \to \infty$. For $\mathfrak{t}\in\mathbb{I}$, one may write

$$\begin{aligned}
\big|\mathfrak{P}\mathfrak{u}(\mathfrak{t}) - \mathbb{L}\mathfrak{u}(\mathfrak{t})\big| \le {}& \frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\big|\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big)-\rho(\mathfrak{s})\mathfrak{u}(\mathfrak{s})\big|\,d\mathfrak{s} \\
& + \frac{\mathfrak{t}^2}{\mu}\Bigg(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\big|\mathfrak{h}\big(\tau,\mathfrak{u}(\tau)\big)-\rho(\tau)\mathfrak{u}(\tau)\big|\,d\tau\,d\mathfrak{s}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\bigl|\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)-\rho(\mathfrak{s})\mathfrak{u}(\mathfrak{s})\bigr|\,\mathrm{d}\mathfrak{s}\\
&+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\bigl|\mathfrak{h}\bigl(\tau,\mathfrak{u}(\tau)\bigr)-\rho(\tau)\mathfrak{u}(\tau)\bigr|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\bigl|\mathfrak{h}\bigl(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\bigr)-\rho(\mathfrak{s})\mathfrak{u}(\mathfrak{s})\bigr|\,\mathrm{d}\mathfrak{s}\Biggr)\\
\le{}&\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\left|\frac{\mathfrak{h}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))}{\mathfrak{u}(\mathfrak{s})}-\rho(\mathfrak{s})\right|\bigl|\mathfrak{u}(\mathfrak{s})\bigr|\,\mathrm{d}\mathfrak{s}\\
&+\frac{\mathfrak{t}^2}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\left|\frac{\mathfrak{h}(\tau,\mathfrak{u}(\tau))}{\mathfrak{u}(\tau)}-\rho(\tau)\right|\bigl|\mathfrak{u}(\tau)\bigr|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\left|\frac{\mathfrak{h}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))}{\mathfrak{u}(\mathfrak{s})}-\rho(\mathfrak{s})\right|\bigl|\mathfrak{u}(\mathfrak{s})\bigr|\,\mathrm{d}\mathfrak{s}\\
&+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\left|\frac{\mathfrak{h}(\tau,\mathfrak{u}(\tau))}{\mathfrak{u}(\tau)}-\rho(\tau)\right|\bigl|\mathfrak{u}(\tau)\bigr|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\left|\frac{\mathfrak{h}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))}{\mathfrak{u}(\mathfrak{s})}-\rho(\mathfrak{s})\right|\bigl|\mathfrak{u}(\mathfrak{s})\bigr|\,\mathrm{d}\mathfrak{s}\Biggr).
\end{aligned}
$$

This means that

$$
\begin{aligned}
\frac{\|\mathfrak{P}\mathfrak{u}-\mathbb{L}\mathfrak{u}\|}{\|\mathfrak{u}\|}\le{}&\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\left|\frac{\mathfrak{h}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))}{\mathfrak{u}(\mathfrak{s})}-\rho(\mathfrak{s})\right|\,\mathrm{d}\mathfrak{s}\\
&+\frac{\mathfrak{t}^2}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\left|\frac{\mathfrak{h}(\tau,\mathfrak{u}(\tau))}{\mathfrak{u}(\tau)}-\rho(\tau)\right|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\left|\frac{\mathfrak{h}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))}{\mathfrak{u}(\mathfrak{s})}-\rho(\mathfrak{s})\right|\,\mathrm{d}\mathfrak{s}\\
&+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\left|\frac{\mathfrak{h}(\tau,\mathfrak{u}(\tau))}{\mathfrak{u}(\tau)}-\rho(\tau)\right|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\left|\frac{\mathfrak{h}(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))}{\mathfrak{u}(\mathfrak{s})}-\rho(\mathfrak{s})\right|\,\mathrm{d}\mathfrak{s}\Biggr).
\end{aligned}
$$

By (3.18) and letting $\|\mathfrak{u}\|\to\infty$, it is concluded that $|\frac{\mathfrak{h}(\cdot,\mathfrak{u}(\cdot))}{\mathfrak{u}}-\rho(\cdot)|\to 0$. Thus we obtain

$$
\lim_{\|\mathfrak{u}\|\to\infty}\frac{\|\mathfrak{P}(u)-\mathbb{L}(\mathfrak{u})\|}{\|\mathfrak{u}\|}=0.
$$

Consequently, by Theorem 2.10, the supposed non-hybrid single-valued FBVP (1.1) has a solution in $\mathcal{Y}$. The proof is completed. □

4 Stability criteria

We investigate the stability property in the sense of Ulam–Hyers and its generalized version for solutions of the non-hybrid-single-valued-FBVP with integro-non-hybrid-multiterm-multipoint-multistrip boundary conditions (1.1). For simplicity, let $C(\mathbb{I},\mathbb{R}) := \mathfrak{Q}$. For more details, see [35–40].

Definition 4.1 The non-hybrid single-valued FBVP with integro-non-hybrid-multiterm-multipoint-multistrip conditions (1.1) is Ulam–Hyers stable if there is $0 < \sigma_{\mathfrak{h}} \in \mathbb{R}$ such that, for each $\epsilon > 0$ and for each $\mathfrak{u}^*(\mathfrak{t}) \in \mathfrak{Q}$ satisfying

$$\left|{}^{c}D_{0}^{q}\mathfrak{u}^*(\mathfrak{t}) - \mathfrak{h}\big(\mathfrak{t},\mathfrak{u}^*(\mathfrak{t})\big)\right| < \epsilon, \tag{4.1}$$

there is $\mathfrak{u}(\mathfrak{t}) \in \mathfrak{Q}$ satisfying the non-hybrid single-valued FBVP (1.1) with

$$\left|\mathfrak{u}^*(\mathfrak{t}) - \mathfrak{u}(\mathfrak{t})\right| \leq \epsilon\sigma_{\mathfrak{h}}, \quad \forall \mathfrak{t} \in \mathbb{I}.$$

Definition 4.2 The non-hybrid single-valued FBVP via integro-non-hybrid-multiterm-multipoint-multistrip conditions (1.1) is generalized Ulam–Hyers stable if there is $\sigma_{\mathfrak{h}} \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\sigma_{\mathfrak{h}}(0) = 0$ such that, for each $\epsilon > 0$ and for each $\mathfrak{u}^*(\mathfrak{t}) \in \mathfrak{Q}$ satisfying the inequality

$$\left|{}^{c}D_{0}^{q}\mathfrak{u}^*(\mathfrak{t}) - \mathfrak{h}\big(\mathfrak{t},\mathfrak{u}^*(\mathfrak{t})\big)\right| < \epsilon,$$

there is $\mathfrak{u}(\mathfrak{t}) \in \mathfrak{Q}$ satisfying the non-hybrid single-valued FBVP (1.1) with

$$\left|\mathfrak{u}^*(\mathfrak{t}) - \mathfrak{u}(\mathfrak{t})\right| \leq \sigma_{\mathfrak{h}}(\epsilon).$$

Remark 4.3 We have Def. 4.1 $\Rightarrow$ Def. 4.2.

Remark 4.4 Notice that $\mathfrak{u}^*(\mathfrak{t}) \in \mathfrak{Q}$ is a solution for (4.1) if and only if there is $z \in \mathfrak{Q}$ depending on $\mathfrak{u}^*$ such that, for each $\mathfrak{t} \in \mathbb{I}$,

(i) $|z(\mathfrak{t})| < \epsilon$;

(ii) ${}^{c}D_{0}^{q}\mathfrak{u}^*(\mathfrak{t}) = \mathfrak{h}(\mathfrak{t},\mathfrak{u}^*(\mathfrak{t})) + z(\mathfrak{t})$.

The Ulam–Hyers stability is discussed here for the non-hybrid single-valued FBVP (1.1).

Theorem 4.5 *Suppose that there is a constant $\beta > 0$ such that*

$$\left|\mathfrak{h}\big(\mathfrak{t},\mathfrak{u}(\mathfrak{t})\big) - \mathfrak{h}\big(\mathfrak{t},\mathfrak{u}'(\mathfrak{t})\big)\right| \leq \beta\left|\mathfrak{u}(\mathfrak{t}) - \mathfrak{u}'(\mathfrak{t})\right| \tag{4.2}$$

for each $\mathfrak{u},\mathfrak{u}' \in \mathcal{Y}$ and $\mathfrak{t} \in \mathbb{I}$. Then the non-hybrid single-valued FBVP with integro-non-hybrid-multiterm-multipoint-multistrip conditions (1.1) *is Ulam–Hyers stable on $\mathbb{I}$ and is the generalized Ulam–Hyers stable provided that $\beta\hat{\Psi} < 1$, where $\hat{\Psi}$ is given by* (3.10).

Proof For every $\epsilon > 0$ and for each $\mathfrak{u}^*(\mathfrak{t}) \in C(\mathbb{I},\mathbb{R})$ satisfying

$$\left|{}^{c}D_{0}^{q}\mathfrak{u}(\mathfrak{t}) - \mathfrak{h}\big(\mathfrak{t},\mathfrak{u}(\mathfrak{t})\big)\right| < \epsilon,$$

we can find a function $z(\mathfrak{t})$ satisfying

$${}^{c}D_{0}^{q}\mathfrak{u}(\mathfrak{t}) = \mathfrak{h}\big(\mathfrak{t},\mathfrak{u}(\mathfrak{t})\big) + z(\mathfrak{t})$$

with $|z(\mathfrak{t})| \leq \epsilon$. It yields

$$
\begin{aligned}
\mathfrak{u}^*(\mathfrak{t}) = {}& \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}^*(\mathfrak{s})\big)\, \mathrm{d}\mathfrak{s} \\
& + \frac{\mathfrak{t}^2}{\mu} \Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{h}\big(\tau, \mathfrak{u}^*(\tau)\big)\, \mathrm{d}\tau\, \mathrm{d}\mathfrak{s} \\
& - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell-\mathfrak{s})^{q-1} \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}^*(\mathfrak{s})\big)\, \mathrm{d}\mathfrak{s} \\
& - \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{h}\big(\tau, \mathfrak{u}^*(\tau)\big)\, \mathrm{d}\tau\, \mathrm{d}\mathfrak{s} \\
& - \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j-\mathfrak{s})^{q+\theta_j-1} \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}^*(\mathfrak{s})\big)\, \mathrm{d}\mathfrak{s} \Bigg) \\
& + \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} z(\mathfrak{s})\, \mathrm{d}\mathfrak{s} + \frac{\mathfrak{t}^2}{\mu} \Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} z(\tau)\, \mathrm{d}\tau\, \mathrm{d}\mathfrak{s} \\
& - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell-\mathfrak{s})^{q-1} z(\mathfrak{s})\, \mathrm{d}\mathfrak{s} - \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} z(\tau)\, \mathrm{d}\tau\, \mathrm{d}\mathfrak{s} \\
& - \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j-\mathfrak{s})^{q+\theta_j-1} z(\mathfrak{s})\, \mathrm{d}\mathfrak{s} \Bigg).
\end{aligned}
$$

If $\mathfrak{u}$ is the unique solution of the non-hybrid single-valued FBVP (1.1), then $\mathfrak{u}(\mathfrak{t})$ is given by

$$
\begin{aligned}
\mathfrak{u}(\mathfrak{t}) = {}& \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}(\mathfrak{s})\big)\, \mathrm{d}\mathfrak{s} + \frac{\mathfrak{t}^2}{\mu} \Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{h}\big(\tau, \mathfrak{u}(\tau)\big)\, \mathrm{d}\tau\, \mathrm{d}\mathfrak{s} \\
& - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell-\mathfrak{s})^{q-1} \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}(\mathfrak{s})\big)\, \mathrm{d}\mathfrak{s} \\
& - \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{h}\big(\tau, \mathfrak{u}(\tau)\big)\, \mathrm{d}\tau\, \mathrm{d}\mathfrak{s} \\
& - \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j-\mathfrak{s})^{q+\theta_j-1} \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}(\mathfrak{s})\big)\, \mathrm{d}\mathfrak{s} \Bigg) \quad (\forall \mathfrak{t} \in \mathbb{I}).
\end{aligned}
$$

Then

$$
\begin{aligned}
\big|\mathfrak{u}^*(\mathfrak{t}) - \mathfrak{u}(\mathfrak{t})\big| \leq {}& \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \big|\mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}^*(\mathfrak{s})\big) - \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}(\mathfrak{s})\big)\big|\, \mathrm{d}\mathfrak{s} \\
& + \frac{\mathfrak{t}^2}{\mu} \Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \big|\mathfrak{h}\big(\tau, \mathfrak{u}^*(\tau)\big) - \mathfrak{h}\big(\tau, \mathfrak{u}(\tau)\big)\big|\, \mathrm{d}\tau\, \mathrm{d}\mathfrak{s} \\
& + \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell-\mathfrak{s})^{q-1} \big|\mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}^*(\mathfrak{s})\big) - \mathfrak{h}\big(\mathfrak{s}, \mathfrak{u}(\mathfrak{s})\big)\big|\, \mathrm{d}\mathfrak{s}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\big|\mathfrak{h}\big(\tau,\mathfrak{u}^*(\tau)\big)-\mathfrak{h}\big(\tau,\mathfrak{u}(\tau)\big)\big|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\big|\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}^*(\mathfrak{s})\big)-\mathfrak{h}\big(\mathfrak{s},\mathfrak{u}(\mathfrak{s})\big)\big|\,\mathrm{d}\mathfrak{s}\Bigg)\\
&+\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}z(\mathfrak{s})\,\mathrm{d}\mathfrak{s}+\frac{\mathfrak{t}^2}{\mu}\Bigg(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}z(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}z(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\\
&+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}z(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}z(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\Bigg)\\
\le{}& \hat{\Psi}\epsilon+\beta\hat{\Psi}\big\|\mathfrak{u}^*-\mathfrak{u}\big\|.
\end{aligned}
$$

Consequently,

$$
\big\|\mathfrak{u}^*-\mathfrak{u}\big\|\le\hat{\Psi}\epsilon+\beta\hat{\Psi}\big\|\mathfrak{u}^*-\mathfrak{u}\big\|,
$$

where $\hat{\Psi}$ is defined in (3.10). In consequence, it follows that

$$
\big\|\mathfrak{u}^*-\mathfrak{u}\big\|\le\frac{\hat{\Psi}\epsilon}{1-\beta\hat{\Psi}}.
$$

If we let $\sigma_{\mathfrak{h}}=\frac{\hat{\Psi}}{1-\beta\hat{\Psi}}$, then its Ulam–Hyers stability is proved. Further, for

$$
\sigma_{\mathfrak{h}}(\epsilon)=\frac{\hat{\Psi}}{1-\beta\hat{\Psi}}\epsilon,
$$

with $\sigma_{\mathfrak{h}}(0)=0$, the generalized Ulam–Hyers stability will be proved. □

5 Results for fully hybrid multi-valued FBVP (1.2)

Now, in the present step, we aim to investigate the existence of solution for the given fully hybrid integro-multi-valued FBVP with integro-hybrid-multipoint-multistrip boundary conditions (1.2). To reach this purpose, we provide an auxiliary lemma. Before starting it, we introduce the Banach algebra $\mathcal{Y}=\{\mathfrak{u}(\mathfrak{t}):\mathfrak{u}(\mathfrak{t})\in C(\mathbb{I},\mathbb{R})\}$ under the norm $\|\mathfrak{u}\|_{\mathcal{Y}}=\sup_{\mathfrak{u}\in\mathbb{I}}|\mathfrak{u}(\mathfrak{t})|$ and with the multiplication $(\mathfrak{u}\cdot\mathfrak{u}')(\mathfrak{t})=\mathfrak{u}(\mathfrak{t})\mathfrak{u}'(\mathfrak{t})$ for each $\mathfrak{u},\mathfrak{u}'\in\mathcal{Y}$.

Lemma 5.1 *Let, for* $j=1,2,\dots,k$, $\theta_j\ge0$, $a_1,a_2,\dots,a_r,b_1,b_2,\dots,b_{p-1},c_1,c_2,\dots,c_k\in\mathbb{R}$ *be constants and* $0<\delta_1\le\delta_2\le\dots\le\delta_r<m_1\le m_2\le\dots\le m_p<\eta_1\le\eta_2\le\dots\le\eta_k<1$ *and* $\mathfrak{f}\in C(\mathbb{I},\mathbb{R})$. *Then* $\tilde{\mathfrak{u}}$ *satisfies the given linear hybrid-FBVP with integro-hybrid-multiterm-*

multipoint-multistrip boundary conditions:

$$\begin{cases} {}^{c}D_{0}^{q}(\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))}) = \mathfrak{f}(\mathfrak{t}) \quad (q \in (2,3], \mathfrak{t} \in \mathbb{I} := [0,1]), \\ (\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})|_{\mathfrak{t}=0} = 0, \qquad (\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})'|_{\mathfrak{t}=0} = 0, \\ \int_0^1 (\frac{\mathfrak{u}(\mathfrak{s})}{y(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))})\,\mathrm{d}\mathfrak{s} = \sum_{\ell=1}^{r} a_\ell (\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})|_{\mathfrak{t}=\delta_\ell} + \sum_{i=2}^{p} b_{i-1} \int_{m_{i-1}}^{m_i} (\frac{\mathfrak{u}(\mathfrak{s})}{y(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))})\,\mathrm{d}\mathfrak{s} \\ \qquad + \sum_{j=1}^{k} c_j I_0^{\theta_j} (\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})|_{\mathfrak{t}=\eta_j}, \end{cases} \tag{5.1}$$

which is given as

$$\begin{aligned} \mathfrak{u}(\mathfrak{t}) = y\big(\mathfrak{t},\mathfrak{u}(\mathfrak{t})\big)\Bigg[& \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\,\mathrm{d}\mathfrak{s} + \frac{\mathfrak{t}^2}{\mu} \Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{f}(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\ & - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\,\mathrm{d}\mathfrak{s} - \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{f}(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\ & - \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \mathfrak{f}(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \Bigg) \Bigg] \quad (\forall \mathfrak{t} \in \mathbb{I}), \end{aligned} \tag{5.2}$$

where μ is a constant given by (3.3).

Proof Let the function $\tilde{\mathfrak{u}}$ be the solution of the linear hybrid-FDE (5.1). Then

$${}^{c}D_0^q \left(\frac{\tilde{\mathfrak{u}}(\mathfrak{t})}{y(\mathfrak{t},\tilde{\mathfrak{u}}(\mathfrak{t}))} \right) = \mathfrak{f}(\mathfrak{t}). \tag{5.3}$$

By utilizing I_0^q on both sides of the hybrid differential equation (5.3), we get

$$\frac{\tilde{\mathfrak{u}}(\mathfrak{t})}{y(\mathfrak{t},\tilde{\mathfrak{u}}(\mathfrak{t}))} = \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\,\mathrm{d}\mathfrak{s} + \alpha_0 + \alpha_1 \mathfrak{t} + \alpha_2 \mathfrak{t}^2 \tag{5.4}$$

for some $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$.

In the first step, the first given initial condition $(\frac{\tilde{\mathfrak{u}}(\mathfrak{t})}{y(\mathfrak{t},\tilde{\mathfrak{u}}(\mathfrak{t}))})|_{\mathfrak{t}=0} = 0$ yields $\alpha_0 = 0$. So

$$\frac{\tilde{\mathfrak{u}}(\mathfrak{t})}{y(\mathfrak{t},\tilde{\mathfrak{u}}(\mathfrak{t}))} = \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\,\mathrm{d}\mathfrak{s} + \alpha_1 \mathfrak{t} + \alpha_2 \mathfrak{t}^2.$$

Further, since

$$\left(\frac{\tilde{\mathfrak{u}}(\mathfrak{t})}{y(\mathfrak{t},\tilde{\mathfrak{u}}(\mathfrak{t}))} \right)' = \frac{1}{\Gamma(q-1)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-2} \mathfrak{f}(\mathfrak{s})\,\mathrm{d}\mathfrak{s} + \alpha_1 + 2\alpha_2 \mathfrak{t},$$

thus the second given condition $(\frac{\tilde{\mathfrak{u}}(\mathfrak{t})}{y(\mathfrak{t},\tilde{\mathfrak{u}}(\mathfrak{t}))})'|_{\mathfrak{t}=0} = 0$ gives $\alpha_1 = 0$. In consequence,

$$\frac{\tilde{\mathfrak{u}}(\mathfrak{t})}{y(\mathfrak{t},\tilde{\mathfrak{u}}(\mathfrak{t}))} = \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\,\mathrm{d}\mathfrak{s} + \alpha_2 \mathfrak{t}^2. \tag{5.5}$$

On the other side, we have

$$\int_0^1 \left(\frac{\tilde{\mathfrak{u}}(\mathfrak{s})}{y(\mathfrak{s},\tilde{\mathfrak{u}}(\mathfrak{s}))} \right) \mathrm{d}\mathfrak{s} = \frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{f}(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} + \frac{1}{3}\alpha_2,$$

and for $j = 1, \dots, k$,

$$I_0^{\theta_j}\left(\frac{\tilde{\mathfrak{u}}(\mathfrak{t})}{y(\mathfrak{t}, \tilde{\mathfrak{u}}(\mathfrak{t}))}\right) = \frac{1}{\Gamma(q+\theta_j)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q+\theta_j-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s} + \alpha_2 \frac{2}{\Gamma(3+\theta_j)} \mathfrak{t}^{2+\theta_j},$$

and for $i = 2, \dots, p$,

$$\int_{m_{i-1}}^{m_i} \left(\frac{\tilde{\mathfrak{u}}(\mathfrak{s})}{y(\mathfrak{s}, \tilde{\mathfrak{u}}(\mathfrak{s}))}\right) d\mathfrak{s} = \frac{1}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{f}(\tau)\, d\tau\, d\mathfrak{s} + \alpha_2 \left(\frac{m_i^3 - m_{i-1}^3}{3}\right).$$

By assuming the nonzero constant μ given by (3.3) and by the third condition, we obtain the following coefficient:

$$\begin{aligned}\alpha_2 = \frac{1}{\mu}\Bigg(&\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{f}(\tau)\, d\tau\, d\mathfrak{s} - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s} \\ &- \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{f}(\tau)\, d\tau\, d\mathfrak{s} \\ &- \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s}\Bigg).\end{aligned}$$

By substituting the above value α_2 in (5.5),

$$\begin{aligned}\frac{\tilde{\mathfrak{u}}(\mathfrak{t})}{y(\mathfrak{t}, \tilde{\mathfrak{u}}(\mathfrak{t}))} = \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s} + \frac{\mathfrak{t}^2}{\mu}\Bigg(&\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{f}(\tau)\, d\tau\, d\mathfrak{s} \\ &- \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s} \\ &- \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \mathfrak{f}(\tau)\, d\tau\, d\mathfrak{s} \\ &- \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \mathfrak{f}(\mathfrak{s})\, d\mathfrak{s}\Bigg) \quad (\forall \mathfrak{t} \in \mathbb{I}),\end{aligned}$$

from which we see that $\tilde{\mathfrak{u}}$ satisfies (5.2) and it is the solution of the mentioned integral equation, and so the proof is completed. □

Based on the above lemma, we aim to define the solution of supposed fully hybrid integro-multi-valued FBVP (1.2).

Definition 5.2 The absolutely continuous map $\mathfrak{u} : \mathbb{I} \to \mathbb{R}$ is called a solution to the fully hybrid integro-multi-valued FBVP (1.2) if an integrable mapping $\varkappa \in \mathcal{L}^1(\mathbb{I}, \mathbb{R})$ with

$$\varkappa(\mathfrak{t}) \in \mathcal{G}\left(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \int_0^1 \mathfrak{u}(\mathfrak{s})\, d\mathfrak{s}\right)$$

for almost all $\mathfrak{t} \in \mathbb{I}$ satisfies integro-hybrid-multiterm-multipoint-multistrip boundary conditions

$$\begin{cases} (\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})|_{\mathfrak{t}=0} = 0, \qquad (\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})'|_{\mathfrak{t}=0} = 0, \\ \int_0^1 (\frac{\mathfrak{u}(\mathfrak{s})}{y(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))})\,d\mathfrak{s} \\ \quad = \sum_{\ell=1}^{r} a_\ell (\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})|_{\mathfrak{t}=\delta_\ell} + \sum_{i=2}^{p} b_{i-1} \int_{m_{i-1}}^{m_i} (\frac{\mathfrak{u}(\mathfrak{s})}{y(\mathfrak{s},\mathfrak{u}(\mathfrak{s}))})\,d\mathfrak{s} + \sum_{j=1}^{k} c_j I_0^{\theta_j} (\frac{\mathfrak{u}(\mathfrak{t})}{y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))})|_{\mathfrak{t}=\eta_j}, \end{cases}$$

and

$$\begin{aligned} \mathfrak{u}(\mathfrak{t}) = y\big(\mathfrak{t},\mathfrak{u}(\mathfrak{t})\big)\Bigg[& \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \varkappa(\mathfrak{s})\,d\mathfrak{s} + \frac{\mathfrak{t}^2}{\mu} \Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \varkappa(\tau)\,d\tau\,d\mathfrak{s} \\ & - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell-\mathfrak{s})^{q-1} \varkappa(\mathfrak{s})\,d\mathfrak{s} - \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \varkappa(\tau)\,d\tau\,d\mathfrak{s} \\ & - \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j-\mathfrak{s})^{q+\theta_j-1} \varkappa(\mathfrak{s})\,d\mathfrak{s} \Bigg) \Bigg] \quad (\forall \mathfrak{t} \in \mathbb{I}). \end{aligned}$$

The first theorem in relation to the inclusion problem (1.2) is proved here.

Theorem 5.3 *Let $\mathcal{G} : \mathbb{I} \times \mathbb{R}^2 \to \mathbb{P}_{cp,cv}(\mathbb{R})$ and $y : \mathbb{I} \times \mathbb{R} \to \mathbb{R}\backslash\{0\}$ be continuous and:*

($\mathbb{J}$1) *There is $M : \mathbb{I} \to \mathbb{R}^+$ (it is bounded) so that for each $\mathfrak{u}, \mathfrak{v} \in \mathbb{R}$ and for all $\mathfrak{t} \in \mathbb{I}$, we have*

$$\big|y\big(\mathfrak{t},\mathfrak{u}(\mathfrak{t})\big) - y\big(\mathfrak{t},\mathfrak{v}(\mathfrak{t})\big)\big| \le M(t)\big|\mathfrak{u}(\mathfrak{t}) - \mathfrak{v}(\mathfrak{t})\big|;$$

($\mathbb{J}$2) *$\mathcal{G} : \mathbb{I} \times \mathbb{R}^2 \to \mathbb{P}_{cp,cv}(\mathbb{R})$ is $\mathcal{L}^1$-Caratheodory;*

($\mathbb{J}$3) *There is $\Lambda(\mathfrak{t}) \in \mathcal{L}^1(\mathbb{I},\mathbb{R}^+)$ with*

$$\big\|\mathcal{G}(\mathfrak{t},\mathfrak{u})\big\| = \sup\left\{|\varkappa| : \varkappa \in \mathcal{G}\left(\mathfrak{t},\mathfrak{u}(\mathfrak{t}), \int_0^1 \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s}\right)\right\} \le \Lambda(t)$$

for all $\mathfrak{u} \in \mathbb{R}$ and for almost all $\mathfrak{t} \in \mathbb{I}$;

($\mathbb{J}$4) *There is $\tilde{a} \in \mathbb{R}^+$ such that*

$$\tilde{a} > \frac{y^* \hat{\Psi} \|\Lambda\|_{L^1}}{1 - M^* \hat{\Psi} \|\Lambda\|_{L^1}}, \tag{5.6}$$

where

$$\begin{aligned} \hat{\Psi} = \frac{1}{\Gamma(q+1)} + \frac{1}{\mu} \Bigg(& \frac{1}{\Gamma(q+2)} + \sum_{\ell=1}^{r} \frac{a_\ell \delta_\ell^q}{\Gamma(q+1)} + \sum_{i=2}^{p} \frac{b_{i-1}(m_i^{q+1} - m_{i-1}^{q+1})}{\Gamma(q+2)} \\ & + \sum_{j=1}^{k} \frac{c_j \eta_j^{q+\theta_j}}{\Gamma(q+\theta_j+1)} \Bigg), \end{aligned} \tag{5.7}$$

and $\|\Lambda\|_{\mathcal{L}^1} = \int_0^1 |\Lambda(s)|\,ds$, $y^* = \sup_{t\in\mathbb{I}} |y(\mathfrak{t},0)|$, $M^* = \sup_{\mathfrak{t}\in[0,1]} |M(\mathfrak{t})|$. If

$$M^* \hat{\Psi} \|\Lambda\|_{\mathcal{L}^1} < \frac{1}{2},$$

then the fully hybrid integro-multi-valued FBVP with integro-hybrid-multiterm-multipoint-multistrip boundary conditions (1.2) *has a solution.*

Proof For each $\mathfrak{u} \in \mathcal{Y}$, we define

$$S_{\mathcal{G},\mathfrak{u}} = \left\{ \varkappa \in \mathcal{L}^1(\mathbb{I}) : \varkappa(t) \in \mathcal{G}\left(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \int_0^1 \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s}\right)\right\},$$

as the selections of $\mathcal{G}$ for almost all $\mathfrak{t} \in \mathbb{I}$, and define $\mathcal{E} : \mathcal{Y} \to \mathbb{P}(\mathcal{Y})$ by

$$\mathcal{E}(\mathfrak{u}) = \left\{ \begin{array}{c} h \in \mathcal{Y} : \\ h(\mathfrak{t}) = \left\{ \begin{array}{l} y(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}))[\frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \varkappa(\mathfrak{s})\,d\mathfrak{s} \\ \quad + \frac{\mathfrak{t}^2}{\mu} (\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \varkappa(\tau)\,d\tau\,d\mathfrak{s} \\ \quad - \sum_{\ell=1}^r \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \varkappa(\mathfrak{s})\,d\mathfrak{s} \\ \quad - \sum_{i=2}^p \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \varkappa(\tau)\,d\tau\,d\mathfrak{s} \\ \quad - \sum_{j=1}^k \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \varkappa(\mathfrak{s})\,d\mathfrak{s})], \quad \varkappa \in S_{\mathcal{G},\mathfrak{u}} \end{array} \right. \end{array} \right\}$$

for all $\mathfrak{t} \in \mathbb{I}$. By this structure, h_0 satisfies the fully hybrid integro-multi-valued FBVP (1.2) if and only if $\mathcal{E}h_0 = h_0$. Further, define $\mathcal{F}_1 : \mathcal{Y} \to \mathcal{Y}$ by $(\mathcal{F}_1\mathfrak{u})(\mathfrak{t}) = y(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}))$ and $\mathcal{F}_2 : \mathcal{Y} \to \mathbb{P}(\mathcal{Y})$ by

$$\mathcal{F}_2(\mathfrak{u})(\mathfrak{t}) = \left\{ \begin{array}{c} \hbar \in \mathcal{Y} : \\ \hbar(\mathfrak{t}) = \left\{ \begin{array}{l} \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \varkappa(\mathfrak{s})\,d\mathfrak{s} \\ \quad + \frac{\mathfrak{t}^2}{\mu} (\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \varkappa(\tau)\,d\tau\,d\mathfrak{s} \\ \quad - \sum_{\ell=1}^r \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \varkappa(\mathfrak{s})\,d\mathfrak{s} \\ \quad - \sum_{i=2}^p \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \varkappa(\tau)\,d\tau\,d\mathfrak{s} \\ \quad - \sum_{j=1}^k \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \varkappa(\mathfrak{s})\,d\mathfrak{s}), \quad \varkappa \in S_{\mathcal{G},\mathfrak{u}} \end{array} \right. \end{array} \right\}$$

for all $\mathfrak{t} \in \mathbb{I}$. Then, we obtain $\mathcal{E}(\mathfrak{u}) = (\mathcal{F}_1\mathfrak{u})(\mathcal{F}_2\mathfrak{u})$. We show that both operators $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfy Theorem 2.2. We show that $\mathcal{F}_1$ is Lipschitz. Let $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{Y}$. Thus assumption $(\mathbb{J}1)$ implies that

$$\left|(\mathcal{F}_1\mathfrak{u}_1)(t) - (\mathcal{F}_1\mathfrak{u}_2)(t)\right| = \left|y\big(\mathfrak{t}, \mathfrak{u}_1(\mathfrak{t})\big) - y\big(\mathfrak{t}, \mathfrak{u}_2(\mathfrak{t})\big)\right| \le M(t)\left|\mathfrak{u}_1(\mathfrak{t}) - \mathfrak{u}_2(\mathfrak{t})\right|$$

for all $\mathfrak{t} \in \mathbb{I}$. Hence, we get

$$\|\mathcal{F}_1\mathfrak{u}_1 - \mathcal{F}_1\mathfrak{u}_2\|_{\mathcal{Y}} \le M^*\|\mathfrak{u}_1 - \mathfrak{u}_2\|_X$$

for all $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{Y}$. Thus $\mathcal{F}_1$ is Lipschitz with the constant $M^* > 0$. Further, we claim that $\mathcal{F}_2$ is convex-valued. Let $\mathfrak{u}_1, \mathfrak{u}_2 \in \mathcal{F}_2\mathfrak{u}$. Choose $\varkappa_1, \varkappa_2 \in S_{\mathcal{G},\mathfrak{u}}$ such that

$$\begin{aligned} \mathfrak{u}_l(\mathfrak{t}) = {} & \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \varkappa_l(\mathfrak{s})\,d\mathfrak{s} \\ & + \frac{\mathfrak{t}^2}{\mu}\left(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \varkappa_l(\tau)\,d\tau\,d\mathfrak{s} \right. \end{aligned}$$

$$
\begin{aligned}
&- \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \varkappa_l(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \\
&- \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \varkappa_l(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\
&- \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \varkappa_l(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \Bigg) \quad (l = 1,2)
\end{aligned}
$$

for almost all $\mathfrak{t} \in \mathbb{I}$. Let $c \in (0,1)$. Then

$$
\begin{aligned}
c\mathfrak{u}_1(\mathfrak{t}) + (1-c)\mathfrak{u}_2(\mathfrak{t}) &= \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{q-1} \big[c\varkappa_1(\mathfrak{s}) + (1-c)\varkappa_2(\mathfrak{s})\big]\,\mathrm{d}\mathfrak{s} \\
&\quad + \frac{\mathfrak{t}^2}{\mu} \Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \big[c\varkappa_1(\tau) + (1-c)\varkappa_2(\tau)\big]\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\
&\quad - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \big[c\varkappa_1(\mathfrak{s}) + (1-c)\varkappa_2(\mathfrak{s})\big]\,\mathrm{d}\mathfrak{s} \\
&\quad - \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \big[c\varkappa_1(\tau) + (1-c)\varkappa_2(\tau)\big]\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\
&\quad - \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \big[c\varkappa_1(\mathfrak{s}) + (1-c)\varkappa_2(\mathfrak{s})\big]\,\mathrm{d}\mathfrak{s} \Bigg).
\end{aligned}
$$

As $\mathcal{G}$ is convex-valued, $S_{\mathcal{G},\mathfrak{u}}$ is too, and this gives $c\varkappa_1(\mathfrak{t}) + (1-c)\varkappa_2(\mathfrak{t}) \in S_{\mathcal{G},\mathfrak{u}}$, and so $\mathcal{F}_2\mathfrak{u}$ is convex for each $\mathfrak{u} \in \mathcal{Y}$.

We investigate the complete continuity of $\mathcal{F}_2$. For $\varepsilon^* \in \mathbb{R}^+$, set

$$
\mathbb{V}_{\varepsilon^*} = \big\{\mathfrak{u} \in \mathcal{Y} : \|\mathfrak{u}\|_{\mathcal{Y}} \le \varepsilon^*\big\}.
$$

For every $\mathfrak{u} \in \mathbb{V}_{\varepsilon^*}$ and $\hbar \in \mathcal{F}_2\mathfrak{u}$, there is $\varkappa \in S_{\mathcal{G},\mathfrak{u}}$ such that

$$
\begin{aligned}
\hbar(\mathfrak{t}) = {}& \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{q-1} \varkappa(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \\
&+ \frac{\mathfrak{t}^2}{\mu} \Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \varkappa(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\
&- \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \varkappa(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \\
&- \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \varkappa(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\
&- \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \varkappa(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \Bigg).
\end{aligned}
$$

Then

$$
\begin{aligned}
\left|\hbar(\mathfrak{t})\right| \leq{}& \frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\left|\varkappa(\mathfrak{s})\right|\,\mathrm{d}\mathfrak{s}+\frac{\mathfrak{t}^2}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\left|\varkappa(\tau)\right|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\left|\varkappa(\mathfrak{s})\right|\,\mathrm{d}\mathfrak{s}\\
&+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\left|\varkappa(\tau)\right|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\left|\varkappa(\mathfrak{s})\right|\,\mathrm{d}\mathfrak{s}\Biggr)\\
\leq{}& \frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\Lambda(\mathfrak{s})\,\mathrm{d}\mathfrak{s}+\frac{\mathfrak{t}^2}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\Lambda(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\Lambda(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\\
&+\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\Lambda(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&+\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\Lambda(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\Biggr)\\
\leq{}& \frac{\|\Lambda\|_{L^1}}{\Gamma(q+1)}+\frac{\|\Lambda\|_{L^1}}{\mu}\Biggl(\frac{1}{\Gamma(q+2)}\\
&+\sum_{\ell=1}^{r}\frac{a_\ell\delta_\ell^q}{\Gamma(q+1)}+\sum_{i=2}^{p}\frac{b_{i-1}(m_i^{q+1}-m_{i-1}^{q+1})}{\Gamma(q+2)}+\sum_{j=1}^{k}\frac{c_j\eta_j^{q+\theta_j}}{\Gamma(q+\theta_j+1)}\Biggr)\\
\leq{}& \hat{\Psi}\|\Lambda\|_{L^1},
\end{aligned}
$$

where $\hat{\Psi}$ is given in (5.7). Thus, $\|\hbar\|\leq\hat{\Psi}\|\Lambda\|_{L^1}$ and $\mathcal{F}_2(\mathcal{Y})$ is uniformly bounded. Let $\mathfrak{u}\in V_{\varepsilon^*}$ and $\hbar\in\mathcal{F}_2\mathfrak{u}$. Choose $\varkappa\in S_{\mathcal{G},\mathfrak{u}}$ such that

$$
\begin{aligned}
\hbar(\mathfrak{t}) ={}& \frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\varkappa(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\\
&+\frac{\mathfrak{t}^2}{\mu}\Biggl(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\varkappa(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&-\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\varkappa(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\\
&-\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\varkappa(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&-\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\varkappa(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\Biggr)
\end{aligned}
$$

for all $\mathfrak{t} \in \mathbb{I}$. Assume that $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathbb{I}$ with $\mathfrak{t}_1 < \mathfrak{t}_2$. Then we have

$$\begin{aligned}
\bigl|\hbar(\mathfrak{t}_2) - \hbar(\mathfrak{t}_1)\bigr| \le{}& \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}_1} \bigl[(\mathfrak{t}_2 - \mathfrak{s})^{q-1} - (\mathfrak{t}_1 - \mathfrak{s})^{q-1}\bigr] \bigl|\varkappa(\mathfrak{s})\bigr| \,\mathrm{d}\mathfrak{s} \\
&+ \frac{1}{\Gamma(q)} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} (\mathfrak{t}_2 - \mathfrak{s})^{q-1} \bigl|\varkappa(\mathfrak{s})\bigr| \,\mathrm{d}\mathfrak{s} \\
&+ \frac{(\mathfrak{t}_2^2 - \mathfrak{t}_1^2)}{\mu} \Biggl(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \bigl|\varkappa(\tau)\bigr| \,\mathrm{d}\tau \,\mathrm{d}\mathfrak{s} \\
&+ \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \bigl|\varkappa(\mathfrak{s})\bigr| \,\mathrm{d}\mathfrak{s} \\
&+ \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \bigl|\varkappa(\tau)\bigr| \,\mathrm{d}\tau \,\mathrm{d}\mathfrak{s} \\
&+ \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \bigl|\varkappa(\mathfrak{s})\bigr| \,\mathrm{d}\mathfrak{s} \Biggr) \\
\le{}& \frac{\|\Lambda\|_{L^1}}{\Gamma(q+1)} \bigl[(\mathfrak{t}_2 - \mathfrak{t}_1)^q + 2\bigl(\mathfrak{t}_2^q - \mathfrak{t}_1^q\bigr)\bigr] \\
&+ \frac{(\mathfrak{t}_2^2 - \mathfrak{t}_1^2)\|\Lambda\|_{L^1}}{\mu} \Biggl(\frac{1}{\Gamma(q+2)} + \sum_{\ell=1}^{r} \frac{a_\ell \delta_\ell^q}{\Gamma(q+1)} \\
&+ \sum_{i=2}^{p} \frac{b_{i-1}(m_i^{q+1} - m_{i-1}^{q+1})}{\Gamma(q+2)} + \sum_{j=1}^{k} \frac{c_j \eta_j^{q+\theta_j}}{\Gamma(q+\theta_j+1)} \Biggr) \to 0,
\end{aligned}$$

as $\mathfrak{t}_2 \to \mathfrak{t}_1$ (independent of $\mathfrak{u} \in \mathbb{V}_{\varepsilon^*}$). The Arzela–Ascoli theorem gives the complete continuity of $\mathcal{F}_2$. Assume that $\mathfrak{u}_n \in \mathbb{V}_\varepsilon$ and $\hbar_n \in (\mathcal{F}_2 \mathfrak{u}_n)$ with $\mathfrak{u}_n \to \mathfrak{u}^*$ and $\hbar_n \to \hbar^*$. We claim that $\hbar^* \in (\mathcal{F}_2 \mathfrak{u}^*)$. For every $n \ge 1$ and $\hbar_n \in (\mathcal{F}_2 \mathfrak{u}_n)$, choose $\varkappa_n \in S_{\mathcal{G},\mathfrak{u}_n}$ such that

$$\begin{aligned}
\hbar_n(\mathfrak{t}) ={}& \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{q-1} \varkappa_n(\mathfrak{s}) \,\mathrm{d}\mathfrak{s} \\
&+ \frac{\mathfrak{t}^2}{\mu} \Biggl(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \varkappa_n(\tau) \,\mathrm{d}\tau \,\mathrm{d}\mathfrak{s} \\
&- \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell - \mathfrak{s})^{q-1} \varkappa_n(\mathfrak{s}) \,\mathrm{d}\mathfrak{s} \\
&- \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s} - \tau)^{q-1} \varkappa_n(\tau) \,\mathrm{d}\tau \,\mathrm{d}\mathfrak{s} \\
&- \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j - \mathfrak{s})^{q+\theta_j-1} \varkappa_n(\mathfrak{s}) \,\mathrm{d}\mathfrak{s} \Biggr).
\end{aligned}$$

We claim that there is $\varkappa^* \in S_{\mathcal{G},\mathfrak{u}^*}$ such that

$$\hbar^*(\mathfrak{t}) = \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t} - \mathfrak{s})^{q-1} \varkappa^*(\mathfrak{s}) \,\mathrm{d}\mathfrak{s}$$

$$\begin{aligned}
&+\frac{\mathfrak{t}^2}{\mu}\Bigg(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\varkappa^*(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&-\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\varkappa^*(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\\
&-\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\varkappa^*(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&-\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\varkappa^*(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\Bigg)
\end{aligned}$$

for all $\mathfrak{t}\in\mathbb{I}$. Define $F:\mathcal{L}^1(\mathbb{I},\mathbb{R})\to\mathcal{Y}=C(\mathbb{I},\mathbb{R})$ by

$$\begin{aligned}
F\big(\varkappa(\mathfrak{t})\big)&=\mathfrak{u}(\mathfrak{t})\\
&=\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\varkappa(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\\
&\quad+\frac{\mathfrak{t}^2}{\mu}\Bigg(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\varkappa(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&\quad-\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\varkappa(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\\
&\quad-\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\varkappa(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&\quad-\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\varkappa(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\Bigg).
\end{aligned}$$

It is linear and continuous. Hence,

$$\begin{aligned}
\big\|\hbar_n(\mathfrak{t})-\hbar^*(\mathfrak{t})\big\|&=\Bigg\|\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\big(\varkappa_n(\mathfrak{s})-\varkappa^*(\mathfrak{s})\big)\,\mathrm{d}\mathfrak{s}\\
&\quad+\frac{\mathfrak{t}^2}{\mu}\Bigg(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\big(\varkappa_n(\tau)-\varkappa^*(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&\quad-\sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\big(\varkappa_n(\mathfrak{s})-\varkappa^*(\mathfrak{s})\big)\,\mathrm{d}\mathfrak{s}\\
&\quad-\sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\big(\varkappa_n(\tau)-\varkappa^*(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s}\\
&\quad-\sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\big(\varkappa_n(\mathfrak{s})-\varkappa^*(\mathfrak{s})\big)\,\mathrm{d}\mathfrak{s}\Bigg)\Bigg\|\to 0.
\end{aligned}$$

Theorem 2.1 implies that $F \circ S_{\mathcal{G}}$ has a closed graph. On the other hand, $\hbar_n \in F(S_{\mathcal{G},\mathfrak{u}_n})$ and $\mathfrak{u}_n \to \mathfrak{u}^*$. So there exists $\varkappa^* \in S_{\mathcal{G},\mathfrak{u}^*}$ such that

$$\begin{aligned}
\hbar^*(\mathfrak{t}) = {} & \frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \varkappa^*(\mathfrak{s})\,d\mathfrak{s} \\
& + \frac{\mathfrak{t}^2}{\mu}\Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \varkappa^*(\tau)\,d\tau\,d\mathfrak{s} \\
& - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell-\mathfrak{s})^{q-1} \varkappa^*(\mathfrak{s})\,d\mathfrak{s} \\
& - \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \varkappa^*(\tau)\,d\tau\,d\mathfrak{s} \\
& - \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j-\mathfrak{s})^{q+\theta_j-1} \varkappa^*(\mathfrak{s})\,d\mathfrak{s} \Bigg),
\end{aligned}$$

for all $\mathfrak{t} \in \mathbb{I}$. Hence, $\hbar^* \in (\mathcal{F}_2\mathfrak{u}^*)$ and so $\mathcal{F}_2$ has a closed graph, and it is upper semi-continuous. Therefore $\mathcal{F}_2$ is compact and upper semi-continuous. By ($\mathbb{J}1$),

$$\begin{aligned}
\hat{\Sigma} &= \big\| \mathcal{F}_2(\mathcal{Y}) \big\| \\
&= \sup_{t \in \mathbb{I}} \big\{ |\mathcal{F}_2\mathfrak{u}| : \mathfrak{u} \in \mathcal{Y} \big\} \\
&\le \frac{\|\Lambda\|_{L^1}}{\Gamma(q+1)} + \frac{\|\Lambda\|_{L^1}}{\mu} \Bigg(\frac{1}{\Gamma(q+2)} \\
&\quad + \sum_{\ell=1}^{r} \frac{a_\ell \delta_\ell^q}{\Gamma(q+1)} + \sum_{i=2}^{p} \frac{b_{i-1}(m_i^{q+1} - m_{i-1}^{q+1})}{\Gamma(q+2)} + \sum_{j=1}^{k} \frac{c_j \eta_j^{q+\theta_j}}{\Gamma(q+\theta_j+1)} \Bigg) \\
&\le \hat{\Psi} \|\Lambda\|_{L^1}.
\end{aligned}$$

Then $\hat{\Sigma} M^* < \frac{1}{2}$. By Theorem 2.2 in relation to $\mathcal{F}_2$, one of (a) or (b) will be held. By ($\mathbb{J}4$), let $\mathfrak{u} \in \mathcal{Q}^*$ be such that $\|\mathfrak{u}\| = \tilde{a}$. Then $\rho_*\mathfrak{u}(\mathfrak{t}) \in (\mathcal{F}_1\mathfrak{u})(\mathfrak{t})(\mathcal{F}_2\mathfrak{u})(\mathfrak{t})$ for all $\rho_* > 1$. Choose $\varkappa \in S_{\mathcal{G},\mathfrak{u}}$. Then, for each $\rho_* > 1$,

$$\begin{aligned}
\mathfrak{u}(\mathfrak{t}) = {} & \frac{1}{\rho_*} y\big(\mathfrak{t},\mathfrak{u}(\mathfrak{t})\big) \Bigg[\frac{1}{\Gamma(q)} \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})^{q-1} \varkappa(\mathfrak{s})\,d\mathfrak{s} \\
& + \frac{\mathfrak{t}^2}{\mu}\Bigg(\frac{1}{\Gamma(q)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \varkappa(\tau)\,d\tau\,d\mathfrak{s} \\
& - \sum_{\ell=1}^{r} \frac{a_\ell}{\Gamma(q)} \int_0^{\delta_\ell} (\delta_\ell-\mathfrak{s})^{q-1} \varkappa(\mathfrak{s})\,d\mathfrak{s} \\
& - \sum_{i=2}^{p} \frac{b_{i-1}}{\Gamma(q)} \int_{m_{i-1}}^{m_i} \int_0^{\mathfrak{s}} (\mathfrak{s}-\tau)^{q-1} \varkappa(\tau)\,d\tau\,d\mathfrak{s} \\
& - \sum_{j=1}^{k} \frac{c_j}{\Gamma(q+\theta_j)} \int_0^{\eta_j} (\eta_j-\mathfrak{s})^{q+\theta_j-1} \varkappa(\mathfrak{s})\,d\mathfrak{s} \Bigg) \Bigg].
\end{aligned}$$

Thus, one can write

$$
\begin{aligned}
|\mathfrak{u}(\mathfrak{t})| &= \frac{1}{\rho_*}\big|y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))\big|\Bigg[\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\big|\varkappa(\mathfrak{s})\big|\,\mathrm{d}\mathfrak{s} \\
&\quad + \frac{\mathfrak{t}^2}{|\mu|}\Bigg(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\big|\varkappa(\tau)\big|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\
&\quad + \sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\big|\varkappa(\mathfrak{s})\big|\,\mathrm{d}\mathfrak{s} \\
&\quad + \sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\big|\varkappa(\tau)\big|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\
&\quad + \sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\big|\varkappa(\mathfrak{s})\big|\,\mathrm{d}\mathfrak{s}\Bigg)\Bigg] \\
&= \frac{1}{\rho_*}\Big[\big|y(\mathfrak{t},\mathfrak{u}(\mathfrak{t}))-y(\mathfrak{t},0)\big|+\big|y(\mathfrak{t},0)\big|\Big]\Bigg[\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\big|\varkappa(\mathfrak{s})\big|\,\mathrm{d}\mathfrak{s} \\
&\quad + \frac{\mathfrak{t}^2}{|\mu|}\Bigg(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\big|\varkappa(\tau)\big|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\
&\quad + \sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\big|\varkappa(\mathfrak{s})\big|\,\mathrm{d}\mathfrak{s} \\
&\quad + \sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\big|\varkappa(\tau)\big|\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\
&\quad + \sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\big|\varkappa(\mathfrak{s})\big|\,\mathrm{d}\mathfrak{s}\Bigg)\Bigg] \\
&\leq \big[M^*\|\mathfrak{u}\|+y^*\big]\Bigg[\frac{1}{\Gamma(q)}\int_0^{\mathfrak{t}}(\mathfrak{t}-\mathfrak{s})^{q-1}\Lambda(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \\
&\quad + \frac{\mathfrak{t}^2}{|\mu|}\Bigg(\frac{1}{\Gamma(q)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\Lambda(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\
&\quad + \sum_{\ell=1}^{r}\frac{a_\ell}{\Gamma(q)}\int_0^{\delta_\ell}(\delta_\ell-\mathfrak{s})^{q-1}\Lambda(\mathfrak{s})\,\mathrm{d}\mathfrak{s} + \sum_{i=2}^{p}\frac{b_{i-1}}{\Gamma(q)}\int_{m_{i-1}}^{m_i}\int_0^{\mathfrak{s}}(\mathfrak{s}-\tau)^{q-1}\Lambda(\tau)\,\mathrm{d}\tau\,\mathrm{d}\mathfrak{s} \\
&\quad + \sum_{j=1}^{k}\frac{c_j}{\Gamma(q+\theta_j)}\int_0^{\eta_j}(\eta_j-\mathfrak{s})^{q+\theta_j-1}\Lambda(\mathfrak{s})\,\mathrm{d}\mathfrak{s}\Bigg)\Bigg] \\
&\leq \big[M^*\tilde{a}+y^*\big]\hat{\Psi}\|\Lambda\|_{L^1}
\end{aligned}
$$

for all $\mathfrak{t}\in\mathbb{I}$. Hence, we get

$$
\tilde{a} \leq \frac{y^*\hat{\Psi}\|\Lambda\|_{L^1}}{1-M^*\hat{\Psi}\|\Lambda\|_{L^1}}.
$$

According to condition (5.6), we find that (b) is not possible, and so $\mathfrak{u} \in (\mathcal{F}_1\mathfrak{u})(\mathcal{F}_2\mathfrak{u})$. Thus, $\mathcal{E}$ has a fixed point and the fully hybrid integro-multi-valued FBVP (1.2) has a solution. □

6 Examples

Example 6.1 In view of the non-hybrid single-valued FBVP with integro-non-hybrid-multiterm-multipoint-multistrip boundary conditions (1.1), let

$$\begin{cases} {}^cD_0^{2.25}\mathfrak{u}(\mathfrak{t}) = \frac{\exp(-\mathfrak{t})+9}{21008}\sin(\mathfrak{u}(\mathfrak{t})) \quad (\mathfrak{t} \in \mathbb{I} := [0,1]), \\ \mathfrak{u}(\mathfrak{t})|_{\mathfrak{t}=0} = 0, \qquad \mathfrak{u}'(\mathfrak{t})|_{\mathfrak{t}=0} = 0, \\ \int_0^1 \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} = 0.01\mathfrak{u}(0.5) + 0.08\mathfrak{u}(0.75) + 0.09\int_{0.76}^{0.78}\mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} + 0.05\int_{0.78}^{0.80}\mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} \\ \qquad + 0.2I_0^{0.71}\mathfrak{u}(0.85) + 0.3I_0^{0.62}\mathfrak{u}(0.87), \end{cases} \tag{6.1}$$

where $q = 2.25$, $r = k = 2$, $p = 3$, $a_1 = 0.01$, $a_2 = 0.08$, $\delta_1 = 0.5$, $\delta_2 = 0.75$, $b_1 = 0.09$, $b_2 = 0.05$, $m_1 = 0.76$, $m_2 = 0.78$, $m_3 = 0.80$, $c_1 = 0.2$, $c_2 = 0.3$, $\theta_1 = 0.71$, $\theta_2 = 0.62$, $\eta_1 = 0.85$ and $\eta_2 = 0.87$. Also, the function

$$\mathfrak{h}\big(\mathfrak{t},\mathfrak{u}(\mathfrak{t})\big) = \frac{\exp(-\mathfrak{t})+9}{21008}\sin\big(\mathfrak{u}(\mathfrak{t})\big)$$

is a real-valued continuous map on $\mathbb{I} \times \mathbb{R}$. For each $\mathfrak{u}, \mathfrak{u}^* \in \mathbb{R}$, we write

$$\begin{aligned} \big|\mathfrak{h}\big(\mathfrak{t},\mathfrak{u}(\mathfrak{t})\big) - \mathfrak{h}\big(\mathfrak{t},\mathfrak{u}^*(\mathfrak{t})\big)\big| &\le \frac{\exp(-\mathfrak{t})+9}{21008}\big|\sin\big(\mathfrak{u}(\mathfrak{t})\big) - \sin\big(\mathfrak{u}^*(\mathfrak{t})\big)\big| \\ &\le \frac{\exp(-\mathfrak{t})+9}{21008}\big|\mathfrak{u}(\mathfrak{t}) - \mathfrak{u}^*(\mathfrak{t})\big|. \end{aligned} \tag{6.2}$$

Further,

$$\big|\mathfrak{h}\big(\mathfrak{t},\mathfrak{u}(\mathfrak{t})\big)\big| \le \frac{\exp(-\mathfrak{t})+9}{21008}\big|\sin\big(\mathfrak{u}(\mathfrak{t})\big)\big| \le \frac{\exp(-\mathfrak{t})+9}{21008} = \varrho(\mathfrak{t}),$$

where $\varrho : \mathbb{I} \to \mathbb{R}^+$ is a continuous function defined by $\varrho(\mathfrak{t}) = \frac{\exp(-\mathfrak{t})+9}{21008}$.

If $W \subset \mathbb{R}$ is an arbitrary bounded set, then

$$\omega\big(\mathfrak{h}(\mathfrak{t}, W)\big) \le \frac{\exp(-\mathfrak{t})+9}{21008}\omega(W).$$

By taking $n_{\mathfrak{h}}(\mathfrak{t}) := \frac{\exp(-\mathfrak{t})+9}{21008}$, we get

$$\omega\big(\mathfrak{h}(\mathfrak{t}, W)\big) \le n_{\mathfrak{h}}(\mathfrak{t})\omega(W).$$

Clearly, $n_{\mathfrak{h}}^* = \sup_{\mathfrak{t}\in\mathbb{I}} |n_{\mathfrak{h}}(\mathfrak{t})| = 0.004743$. By the above given values for parameters, we find $\hat{\Psi} \simeq 3.89692$. In this case, $n_{\mathfrak{h}}^*\hat{\Psi} \simeq 0.018 < \frac{1}{4}$. By Theorem 3.2, the non-hybrid single-valued FBVP with integro-non-hybrid-multiterm-multipoint-multistrip boundary conditions (6.1) has a solution on $\mathbb{I}$. On the other hand, by (6.2), and assuming $\beta = 0.004743$, we get $\beta\hat{\Psi} \simeq 0.018 < 1$. Then, by taking $\sigma_{\mathfrak{h}} = \frac{\hat{\Psi}}{1-\beta\hat{\Psi}} = 3.96835 > 0$, the conclusion of Theorem 4.5 implies that the non-hybrid single-valued FBVP (6.1) is Ulam–Hyers stable and generalized Ulam–Hyers stable.

Example 6.2 In view of the fully hybrid-FBVP inclusion with integro-hybrid-multiterm-multipoint-multistrip boundary conditions (1.2), we consider the following system:

$$\begin{cases} {}^cD_0^{2.25}\left(\frac{\mathfrak{u}(\mathfrak{t})}{\frac{\exp(-\mathfrak{t})}{5072}\arctan(\mathfrak{u}(\mathfrak{t}))+0.0003}\right) \in [-2, \exp(\mathfrak{t})\cos\mathfrak{u}(\mathfrak{t}) + 3\int_0^1 \cos\mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} + \frac{7}{4}], \\ \left(\frac{\mathfrak{u}(\mathfrak{t})}{\frac{\exp(-\mathfrak{t})}{5072}\arctan(\mathfrak{u}(\mathfrak{t}))+0.0003}\right)\Big|_{\mathfrak{t}=0} = 0, \qquad \left(\frac{\mathfrak{u}(\mathfrak{t})}{\frac{\exp(-\mathfrak{t})}{5072}\arctan(\mathfrak{u}(\mathfrak{t}))+0.0003}\right)'\Big|_{\mathfrak{t}=0} = 0, \\ \int_0^1 \left(\frac{\mathfrak{u}(\mathfrak{s})}{\frac{\exp(-\mathfrak{s})}{5072}\arctan(\mathfrak{u}(\mathfrak{s}))+0.0003}\right) d\mathfrak{s} \\ \quad = 0.01\left(\frac{\mathfrak{u}(0.5)}{\frac{\exp(-0.5)}{5072}\arctan(\mathfrak{u}(0.5))+0.0003}\right) \\ \qquad + 0.08\left(\frac{\mathfrak{u}(0.75)}{\frac{\exp(-0.75)}{5072}\arctan(\mathfrak{u}(0.75))+0.0003}\right) \\ \qquad + 0.09\int_{0.76}^{0.78}\left(\frac{\mathfrak{u}(\mathfrak{s})}{\frac{\exp(-\mathfrak{s})}{5072}\arctan(\mathfrak{u}(\mathfrak{s}))+0.0003}\right) d\mathfrak{s} \\ \qquad + 0.05\int_{0.78}^{0.80}\left(\frac{\mathfrak{u}(\mathfrak{s})}{\frac{\exp(-\mathfrak{s})}{5072}\arctan(\mathfrak{u}(\mathfrak{s}))+0.0003}\right) d\mathfrak{s} \\ \qquad + 0.2I_0^{0.71}\left(\frac{\mathfrak{u}(0.85)}{\frac{\exp(-0.85)}{5072}\arctan(\mathfrak{u}(0.85))+0.0003}\right) \\ \qquad + 0.3I_0^{0.62}\left(\frac{\mathfrak{u}(0.87)}{\frac{\exp(-0.87)}{5072}\arctan(\mathfrak{u}(0.87))+0.0003}\right), \end{cases} \tag{6.3}$$

where $q = 2.25$, $r = k = 2$, $p = 3$, $a_1 = 0.01$, $a_2 = 0.08$, $\delta_1 = 0.5$, $\delta_2 = 0.75$, $b_1 = 0.09$, $b_2 = 0.05$, $m_1 = 0.76$, $m_2 = 0.78$, $m_3 = 0.80$, $c_1 = 0.2$, $c_2 = 0.3$, $\theta_1 = 0.71$, $\theta_2 = 0.62$, $\eta_1 = 0.85$, $\eta_2 = 0.87$, and $\mathbb{I} = [0, 1]$. The continuous function $y : \mathbb{I} \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is defined by

$$y(\mathfrak{t}, \mathfrak{u}(\mathfrak{t})) = \frac{\exp(-\mathfrak{t})}{5072}\arctan(\mathfrak{u}(\mathfrak{t})) + 0.0003.$$

Further, $y^* = \sup_{\mathfrak{t}\in\mathbb{I}} |y(\mathfrak{t}, 0)| = 0.0003$. y is Lipschitz for $\mathfrak{u}, \mathfrak{u}^* \in \mathbb{R}$, and we write

$$\begin{aligned} \left|y(\mathfrak{t}, \mathfrak{u}(\mathfrak{t})) - y(\mathfrak{t}, \mathfrak{u}^*(\mathfrak{t}))\right| &\leq \frac{\exp(-\mathfrak{t})}{5072}\left|\arctan(\mathfrak{u}(\mathfrak{t})) - \arctan(\mathfrak{u}^*(\mathfrak{t}))\right| \\ &\leq \frac{\exp(-\mathfrak{t})}{5072}\left|\mathfrak{u}(\mathfrak{t}) - \mathfrak{u}^*(\mathfrak{t})\right| \\ &:= M(\mathfrak{t})\left|\mathfrak{u}(\mathfrak{t}) - \mathfrak{u}^*(\mathfrak{t})\right|. \end{aligned}$$

Evidently, $M^* = \sup_{\mathfrak{t}\in\mathbb{I}} |M(\mathfrak{t})| \simeq 0.0001971$. We define $\mathcal{G} : \mathbb{I} \times \mathbb{R}^2 \to \mathbb{P}(\mathbb{R})$ by

$$\mathcal{G}\left(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \int_0^1 \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s}\right) = \left[-2, \exp(\mathfrak{t})\cos\mathfrak{u}(\mathfrak{t}) + 3\int_0^1 \cos\mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} + \frac{7}{4}\right].$$

For $\hbar \in \mathcal{G}(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \int_0^1 \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s})$, we have

$$|\hbar| \leq \max\left[-2, \exp(\mathfrak{t})\cos\mathfrak{u}(\mathfrak{t}) + 3\int_0^1 \cos\mathfrak{u}(\mathfrak{s})\,d\mathfrak{s} + \frac{7}{4}\right] \leq \exp(\mathfrak{t}) + \frac{19}{4}.$$

Therefore

$$\left\|\mathcal{G}\left(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \int_0^1 \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s}\right)\right\| = \sup\left\{|\varkappa| : \varkappa \in \mathcal{G}\left(\mathfrak{t}, \mathfrak{u}(\mathfrak{t}), \int_0^1 \mathfrak{u}(\mathfrak{s})\,d\mathfrak{s}\right)\right\} \leq \exp(\mathfrak{t}) + \frac{19}{4}.$$

Set $\Lambda(t) := \exp(t) + \frac{19}{4}$. Thus

$$\|\Lambda\|_{\mathcal{L}^1} = \int_0^1 |\Lambda(\mathfrak{s})|\,d\mathfrak{s} = \int_0^1 \left(\exp(\mathfrak{s}) + \frac{19}{4}\right) d\mathfrak{s} = e - 1 \simeq 1.71.$$

We select $\tilde{a} > 0$ such that $\tilde{a} > 0.0020017$. Also, by the above given values for the parameters, we find $\hat{\Psi} \simeq 3.89692$. Therefore

$$M^* \hat{\Psi} \|\Lambda\|_{\mathcal{L}^1} \simeq 0.0013134 < \frac{1}{2}.$$

The conclusion of Theorem 5.3 gives this fact that there exists a solution for the fully hybrid-FBVP inclusion with integro-hybrid-multiterm-multipoint-multistrip boundary conditions (6.3).

7 Conclusions

In the present manuscript, two novel generalized non-hybrid single-valued FBVP and fully hybrid integro-multi-valued FBVP with integro-hybrid-multiterm-multipoint-multistrip boundary conditions were considered and the qualitative results were proved in relation to its solutions. Precisely, on the non-hybrid-multi-valued FBVP (1.1), we established an existence theorem based on Sadovskii's method, and in the sequel, the Krasnoselskii–Zabreiko theorem was utilized for the second existence result. We got help from the Gronwall inequality in its generalized version to investigate the dependence of solutions of the non-hybrid multi-valued FBVP (1.1). Stability analysis was implemented in the sense of Ulam–Hyers. Further, on the fully hybrid multi-valued FBVP (1.2), we derived the corresponding multipoint integral equation and used Dhage's techniques to establish the third existence theorem. Two numerical examples have been designed to examine the correctness of the theorems. Our boundary conditions are general and cover different simple forms defined in numerous FBVPs. We will continue our study in the context of newly-defined notions of q-calculus and (p, q)-calculus.

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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SR performed the formal analysis, conceptualization, methodology, investigation, validation, and supervision. BA dealt with the formal analysis, validation, and supervision. AB dealt with the methodology, investigation, formal analysis, and validation. KN performed the formal analysis, funding acquisition, validation, edition, original draft preparation, and writing a revised version. SE performed conceptualization, formal analysis, and writing a revised version. All authors read and approved the final manuscript.

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