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Complex symmetric Toeplitz operators on the generalized derivative Hardy space

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Abstract

The generalized derivative Hardy space $S^2_{\alpha,\beta}(\mathbb{D})$ consists of all functions whose derivatives are in the Hardy and Bergman spaces as follows:
for positive integers α, β,

$$S^2_{\alpha,\beta}(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|^2_{S^2_{\alpha,\beta}} = \|f\|^2_{H^2} + \frac{\alpha+\beta}{\alpha\beta}\|f'\|^2_{A^2} + \frac{1}{\alpha\beta}\|f'\|^2_{H^2} < \infty \right\},$$

where $H(\mathbb{D})$ denotes the space of all functions analytic on the open unit disk $\mathbb{D}$. In this paper, we study characterizations for Toeplitz operators to be complex symmetric on the generalized derivative Hardy space $S^2_{\alpha,\beta}(\mathbb{D})$ with respect to some conjugations C_ξ, $C_{\mu,\lambda}$. Moreover, for any conjugation C, we consider the necessary and sufficient conditions for complex symmetric Toeplitz operators with the symbol φ of the form $\varphi(z) = \sum_{n=1}^{\infty} \overline{\hat{\varphi}(-n)}\bar{z}^n + \sum_{n=0}^{\infty} \hat{\varphi}(n)z^n$. Next, we also study complex symmetric Toeplitz operators with non-harmonic symbols on the generalized derivative Hardy space $S^2_{\alpha,\beta}(\mathbb{D})$.

1 Introduction

Let $\mathcal{H}$ be a separable complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. We say that an anti-linear operator C on $\mathcal{H}$ is a *conjugation* if $C^2 = I$ and $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$. If C is a conjugation on $\mathcal{H}$, there exists an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all n (see [5]). We say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *complex symmetric* if $T = CT^*C$ for a conjugation operator C on $\mathcal{H}$. The topic of complex symmetric operators, which includes all truncated Toeplitz operators, Hankel operators, normal operators, and some Volterra integration operators, has been studied by many authors (see [4, 5], and [8] for more details).

For the open unit disk $\mathbb{D}$ in $\mathbb{C}$, let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. Let $L^2(\mathbb{D}, dA)$ be a Hilbert space with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z),$$

where $f,g \in L^2(\mathbb{D}, dA)$ and dA is the area measure of $\mathbb{D}$. The *Hilbert Hardy space* $H^2(\mathbb{D})$ contains all functions f analytic on $\mathbb{D}$ with

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{where } \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

The *Bergman space* $A^2(\mathbb{D})$ consists of the space of analytic functions f in $L^2(\mathbb{D}, dA)$ with

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{where } \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n|^2 < \infty.$$

The *Dirichlet space* $D^2(\mathbb{D})$ is given by

$$D^2(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \|f\|_{D^2}^2 = \|f\|_{H^2}^2 + \|f'\|_{A^2}^2 = \sum_{n=0}^{\infty} (n+1)|f_n|^2 < \infty \right\}.$$

The reproducing kernels of the spaces $H^2(\mathbb{D})$, $A^2(\mathbb{D})$, and $D^2(\mathbb{D})$ have the following forms:

$$K_w^1(z) = \frac{1}{1 - \overline{w}z}, \qquad K_w^2(z) = \frac{1}{(1 - \overline{w}z)^2}, \quad \text{and} \quad K_w^3(z) = \frac{1}{\overline{w}z} \ln \frac{1}{1 - \overline{w}z},$$

respectively. Many authors in [1–3] and [11] studied intensively multiplication and Toeplitz operators on the Hardy space, Bergman space, and Dirichlet space.

In 2019, Gu and Luo [6] introduced the *derivative Hardy space* $S_1^2(\mathbb{D})$ as follows:

$$\begin{aligned} S_1^2(\mathbb{D}) &= \left\{ f \in H(\mathbb{D}) : \|f\|_{S_1^2}^2 = \|f\|_{H^2}^2 + \frac{3}{2} \|f'\|_{A^2}^2 + \frac{1}{2} \|f'\|_{H^2}^2 < \infty \right\} \\ &= \left\{ f \in H(\mathbb{D}) : \|f\|_{S_1^2}^2 = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} |a_n|^2 < \infty \right\}, \end{aligned}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. The reproducing kernel of the derivative Hardy space $S_1^2(\mathbb{D})$ is given by

$$K_w(z) = \frac{2}{(\overline{w}z)^2} \left[\overline{w}z + (\overline{w}z - 1) \ln \frac{1}{1 - \overline{w}z} \right].$$

Recently, the authors in [9] defined the *generalized derivative Hardy space* $S_{\alpha,\beta}^2(\mathbb{D})$ for $\alpha, \beta \in \mathbb{N}$ as

$$\begin{aligned} S_{\alpha,\beta}^2(\mathbb{D}) &= \left\{ f \in H(\mathbb{D}) : \|f\|_{S_{\alpha,\beta}^2}^2 = \|f\|_{H^2}^2 + \frac{\alpha+\beta}{\alpha\beta} \|f'\|_{A^2}^2 + \frac{1}{\alpha\beta} \|f'\|_{H^2}^2 < \infty \right\} \\ &= \left\{ f \in H(\mathbb{D}) : \|f\|_{S_{\alpha,\beta}^2}^2 = \sum_{n=0}^{\infty} \frac{(n+\alpha)(n+\beta)}{\alpha\beta} |f_n|^2 < \infty \right\}, \end{aligned}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since $S_{\alpha,\beta}^2(\mathbb{D}) = S_{\beta,\alpha}^2(\mathbb{D})$ clearly holds, we focus on the space $S_{\alpha,\beta}^2(\mathbb{D})$ for $\alpha < \beta$. Especially, if $\alpha = 1$ and $\beta = 2$, then $S_{\alpha,\beta}^2(\mathbb{D})$ becomes $S_1^2(\mathbb{D})$.

Let $L^\infty(\mathbb{D})$ be the set of all essentially bounded measurable functions in $\mathbb{D}$, and let P be the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto $S^2_{\alpha,\beta}(\mathbb{D})$. For $\varphi \in L^\infty(\mathbb{D})$, the *Toeplitz operator* T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is defined by

$$T_\varphi f := P(\varphi \cdot f) \quad \text{for } f \in S^2_{\alpha,\beta}(\mathbb{D}).$$

Note that, for $\varphi, \psi \in L^\infty(\mathbb{D})$, from the definition of the Toeplitz operator, $T_{\varphi+\psi} = T_\varphi + T_\psi$ and $T^*_\varphi = T_{\overline{\varphi}}$ where $\overline{\varphi}$ is a complex conjugation of φ. The reproducing kernel $K_w(z)$ of the space $S^2_{\alpha,\beta}(\mathbb{D})$ is

$$\begin{aligned} K_w(z) = {} & \frac{\alpha\beta}{\beta-\alpha} \ln \frac{1}{1-\overline{w}z} \left(\frac{1}{(\overline{w}z)^\alpha} - \frac{1}{(\overline{w}z)^\beta} \right) \\ & + \frac{\alpha\beta}{\beta-\alpha} \left\{ \frac{1}{(\overline{w}z)^\alpha} \delta_{\alpha-1}(\overline{w}z) - \frac{1}{(\overline{w}z)^\beta} \delta_{\beta-1}(\overline{w}z) \right\}, \end{aligned}$$

where $\delta_m(z) = \sum_{r=1}^m \frac{{}_mC_r(-1)^r}{r}[(1-z)^r - 1]$ (see [9, Lemma 2.1]). Thus we have

$$(T_\varphi f)(z) = \int_{\mathbb{D}} \varphi(\omega) f(\omega) K_z(w)\, dA(\omega)$$

for $f \in S^2_{\alpha,\beta}(\mathbb{D})$ and $\omega \in \mathbb{D}$.

This paper is organized as follows. First, we study characterizations for Toeplitz operators to be complex symmetric on the generalized derivative Hardy space $S^2_{\alpha,\beta}(\mathbb{D})$ with respect to some conjugations. Moreover, we also focus on complex symmetric Toeplitz operators with non-harmonic symbols on the generalized derivative Hardy space $S^2_{\alpha,\beta}(\mathbb{D})$.

2 Complex symmetric Toeplitz operators

In this section, we study complex symmetry of Toeplitz operators on the space $S^2_{\alpha,\beta}(\mathbb{D})$. For the convenience of readers, we begin with the following lemma which comes from [9]. Let $\mathbb{N}$ be the natural numbers and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 2.1 ([9]) *For $s, t \in \mathbb{N}_0$, the following statements hold*:

(i) $$\langle z^t, z^s \rangle = \begin{cases} \frac{(s+\alpha)(s+\beta)}{\alpha\beta} & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases}$$

(ii) $$P(\overline{z}^t z^s) = \begin{cases} \frac{(s+\alpha)(s+\beta)}{(s-t+\alpha)(s-t+\beta)} z^{s-t} & \text{if } s \geq t, \\ 0 & \text{if } s < t. \end{cases}$$

Remark 2.2 We mentioned in [9] that there is the difference between the Hardy space $H^2(\mathbb{D})$ (or Bergman space $A^2(\mathbb{D})$) and the generalized derivative Hardy space $S^2_{\alpha,\beta}(\mathbb{D})$. Indeed, for $s, t \in \mathbb{N}_0$, the inequality $\|\overline{z}^t z^s\| \geq \|P(\overline{z}^t z^s)\|$ holds on $H^2(\mathbb{D})$ and $A^2(\mathbb{D})$. However, it holds that

$$\|\overline{z}^t z^s\| \leq \|P(\overline{z}^t z^s)\| \quad \text{on } S^2_{\alpha,\beta}(\mathbb{D})$$

because of $\frac{(s+\alpha)(s+\beta)}{(s-t+\alpha)(s-t+\beta)} > 1$.

Theorem 2.3 *For $n \in \mathbb{N}_0$, let $\{e_n\}$ on $S^2_{\alpha,\beta}(\mathbb{D})$ be given by*

$$e_n(z) = \sqrt{\frac{\alpha\beta}{(n+\alpha)(n+\beta)}}z^n \quad \text{for } z \in \mathbb{D}.$$

If C is anti-linear on $S^2_{\alpha,\beta}(\mathbb{D})$ such that $Ce_n = \overline{\delta_n}e_n$ with $|\delta_n| = 1$, then the following statements hold:

(i) *Parseval's identity $\sum_{n=0}^{\infty}|\langle f, Ce_n\rangle|^2 = \sum_{n=0}^{\infty}|\langle f, e_n\rangle|^2 = \|f\|_2^2$ holds for every $f \in S^2_{\alpha,\beta}(\mathbb{D})$.*

(ii) *The set of functions $\{Ce_n(z) := \sqrt{\frac{\alpha\beta}{(n+\alpha)(n+\beta)}}Cz^n\}$ forms an orthonormal basis for $S^2_{\alpha,\beta}(\mathbb{D})$.*

Proof (i) From Lemma 2.1, we have

$$\langle e_n, e_m\rangle = \sqrt{\frac{\alpha\beta}{(n+\alpha)(n+\beta)}}\sqrt{\frac{\alpha\beta}{(m+\alpha)(m+\beta)}}\langle z^n, z^m\rangle = \delta_{nm}, \tag{1}$$

where $\delta_{nm} = 1$ if $n = m$, and $\delta_{nm} = 0$ if $n \neq m$. Thus $\{e_n\}$ is an orthonormal sequence for $S^2_{\alpha,\beta}(\mathbb{D})$. Since C is anti-linear on $S^2_{\alpha,\beta}(\mathbb{D})$ such that $Ce_n = \overline{\delta_n}e_n$ with $|\delta_n| = 1$, it follows that C is a conjugation on $S^2_{\alpha,\beta}(\mathbb{D})$. Hence

$$\begin{aligned}\langle Ce_n, Ce_m\rangle &= \langle e_m, e_n\rangle \\ &= \sqrt{\frac{\alpha\beta}{(m+\alpha)(m+\beta)}}\sqrt{\frac{\alpha\beta}{(n+\alpha)(n+\beta)}}\langle z^m, z^n\rangle = \delta_{mn}.\end{aligned}$$

First, we will show that Parseval's identity $\sum_{n=0}^{\infty}|\langle f, Ce_n\rangle|^2 = \|f\|^2$ holds for every $f \in S^2_{\alpha,\beta}(\mathbb{D})$. Let $Cf(z) = \sum_{k=0}^{\infty}\widetilde{a_k}z^k$. Then

$$\|f\|^2 = \|Cf\|^2 = \sum_{k=0}^{\infty}\frac{(k+\alpha)(k+\beta)}{\alpha\beta}|\widetilde{a_k}|^2$$

and

$$\begin{aligned}\langle f(z), Ce_n(z)\rangle &= \langle e_n(z), Cf(z)\rangle \\ &= \left\langle \sqrt{\frac{\alpha\beta}{(n+\alpha)(n+\beta)}}z^n, \sum_{k=0}^{\infty}\widetilde{a_k}z^k\right\rangle \\ &= \sqrt{\frac{\alpha\beta}{(n+\alpha)(n+\beta)}}\sum_{k=0}^{\infty}\overline{\widetilde{a_k}}\langle z^n, z^k\rangle \\ &= \sqrt{\frac{\alpha\beta}{(n+\alpha)(n+\beta)}}\overline{\widetilde{a_n}}\langle z^n, z^n\rangle \\ &= \sqrt{\frac{(n+\alpha)(n+\beta)}{\alpha\beta}}\overline{\widetilde{a_n}}.\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty}\left|\langle f, Ce_n\rangle\right|^2 = \sum_{n=0}^{\infty}\frac{(n+\alpha)(n+\beta)}{\alpha\beta}|\widetilde{a_n}|^2 = \|f\|^2,$$

and by a similar calculation, we get $\|f\|^2 = \sum_{n=0}^{\infty}|\langle f, e_n\rangle|^2$. Hence Parseval's identity holds.

(ii) Since Parseval's identity holds by (i), $f = \sum_{n=0}^{\infty}\langle f, Ce_n\rangle Ce_n$ for every $f \in S^2_{\alpha,\beta}(\mathbb{D})$. Hence $\{Ce_n\}$ forms an orthonormal basis for $S^2_{\alpha,\beta}(\mathbb{D})$. □

Especially, if $\alpha = 1$ and $\beta = 2$ in Theorem 2.3, then we get the following result.

Corollary 2.4 *For $n \in \mathbb{N}_0$, let $\{e_n\}$ on $S^2_1(\mathbb{D})$ be given by*

$$e_n(z) = \sqrt{\frac{2}{(n+1)(n+2)}}z^n \quad \textit{for } z \in \mathbb{D}.$$

Let C be anti-linear on $S^2_1(\mathbb{D})$ such that $Ce_n = \overline{\delta_n}e_n$ with $|\delta_n| = 1$. Then Parseval's identity $\sum_{n=0}^{\infty}|\langle f, Ce_n\rangle|^2 = \sum_{n=0}^{\infty}|\langle f, e_n\rangle|^2 = \|f\|_2^2$ holds for every $f \in S^2_1(\mathbb{D})$. Moreover, the set of functions $\{Ce_n(z) := \sqrt{\frac{2}{(n+1)(n+2)}}Cz^n\}$ forms an orthonormal basis for $S^2_1(\mathbb{D})$.

In 2016, the authors in [8] introduced the conjugation $C_{\mu,\lambda}$ on the Hardy space H^2 as in (3). Remark that the space $H^2(\mathbb{D})$ has the reproducing kernel $K^1_w(z)$ and the normalized reproducing kernel $k_w(z)$ given by

$$K^1_w(z) = \frac{1}{1-\overline{w}z} \quad \text{and} \quad k_w(z) = \frac{\sqrt{1-|w|^2}}{1-\overline{w}z} \quad \text{for } w \in \mathbb{D},$$

respectively. Recently, the authors in [10] gave the conjugation C_ξ which has the form as in (2) on the Hardy space $H^2(\mathbb{D})$. We can easily show that the following operator as in (2) is the conjugation on $S^2_{\alpha,\beta}(\mathbb{D})$.

Lemma 2.5 (i) *Let $\xi \in \mathbb{R}$ be with $|\xi| < 1$. Assume that the operator C_ξ is defined by*

$$C_\xi f(z) = -k_\xi(z)\overline{f\left(\psi_\xi(\overline{z})\right)} \tag{2}$$

for some $f \in S^2_{\alpha,\beta}(\mathbb{D})$ where $\psi_\xi(z) = \frac{\xi-z}{1-\bar{\xi}z}$. Then C_ξ is a conjugation on $S^2_{\alpha,\beta}(\mathbb{D})$.

(ii) *For every μ and λ in $\mathbb{C}$ with $|\mu| = |\lambda| = 1$, let $C_{\mu,\lambda} : S^2_{\alpha,\beta}(\mathbb{D}) \to S^2_{\alpha,\beta}(\mathbb{D})$ be given by*

$$C_{\mu,\lambda}f(z) = \mu\overline{f(\lambda\overline{z})}. \tag{3}$$

Then $C_{\mu,\lambda}$ is a conjugation on $S^2_{\alpha,\beta}(\mathbb{D})$.

Now, we establish a necessary and sufficient condition for a Toeplitz operator T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ to be complex symmetric with respect to the above conjugations.

Theorem 2.6 *Let $\varphi \in L^\infty(\mathbb{D})$. Then the following statements hold*:

(i) *If $\xi \in \mathbb{R}$ with $|\xi| < 1$, then T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is complex symmetric with the conjugation C_ξ if and only if $\varphi(z) = \varphi(\overline{\psi_\xi(z)})$, where $\psi_\xi(z) = \frac{\xi-z}{1-\bar{\xi}z}$.*

(ii) *T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is complex symmetric with the conjugation $C_{\mu,\lambda}$ if and only if $\varphi(z) = \varphi(\lambda\overline{z})$.*

Proof (i) Let T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ be complex symmetric with the conjugation C_ξ. Since ξ is real, it follows that $|1-\xi\overline{z}| = |1-\xi z|$, and so $|k_\xi(\overline{\psi_\xi(z)})|^2|k_\xi(z)|^2 = 1$. Thus, by Lemma 2.5, we have

$$\begin{aligned}
\langle T_\varphi^* f, g\rangle &= \langle C_\xi T_\varphi C_\xi f, g\rangle \\
&= \langle C_\xi g, T_\varphi C_\xi f\rangle = \langle C_\xi g, P(\varphi C_\xi f)\rangle = \langle C_\xi g, \varphi C_\xi f\rangle \\
&= \int_{\mathbb{D}} k_\xi(z)\overline{g(\psi_\xi(\overline{z}))}\cdot\overline{\varphi(z)k_\xi(z)f(\psi_\xi(\overline{z}))}\,dA(z) \\
&= \int_{\mathbb{D}} |k_\xi(\overline{z})|^2\overline{g(\psi_\xi(z))\varphi(\overline{z})}f(\psi_\xi(z))\,dA(z) \\
&= \int_{\mathbb{D}} |k_\xi(\overline{\psi_\xi(z)})|^2\overline{\varphi(\overline{\psi_\xi(z)})}f(z)\overline{g(z)}|k_\xi(z)|^2\,dA(z) \\
&= \int_{\mathbb{D}} \overline{\varphi(\overline{\psi_\xi(z)})}f(z)\overline{g(z)}\,dA(z) \\
&= \langle \overline{\varphi(\overline{\psi_\xi(z)})}f, g\rangle \\
&= \langle f, P(\varphi(\overline{\psi_\xi(z)}))g\rangle
\end{aligned}$$

for $f, g \in S^2_{\alpha,\beta}(\mathbb{D})$. Hence we get that $T_\varphi g = T_{\varphi(\overline{\psi_\xi(z)})}g$ for all $g \in S^2_{\alpha,\beta}(\mathbb{D})$ and then $\varphi(z) = \varphi(\overline{\psi_\xi(z)})$. The converse implications clearly hold by a similar method.

(ii) We claim that if P denotes the orthogonal projection of L^2 onto $S^2_{\alpha,\beta}(\mathbb{D})$, then the operators $C_{\mu,\lambda}$ and P commute.

Since $e_n(z) = \sqrt{\frac{\alpha\beta}{(n+\alpha)(n+\beta)}}z^n$, it follows that, for $n \geq 0$,

$$\begin{aligned}
PC_{\mu,\lambda}e_n &= P\mu\overline{\lambda}^n e_n \\
&= \sqrt{\frac{\alpha\beta}{(n+\alpha)(n+\beta)}}\mu\overline{\lambda}^n z^n \\
&= \sqrt{\frac{\alpha\beta}{(n+\alpha)(n+\beta)}}C_{\mu,\lambda}z^n = C_{\mu,\lambda}Pe_n,
\end{aligned}$$

and for $n < 0$, we have that $PC_{\mu,\lambda}e_n = P\mu\overline{\lambda}^n e_n = 0 = C_{\mu,\lambda}Pe_n$. Hence

$$C_{\mu,\lambda}P = PC_{\mu,\lambda}. \tag{4}$$

Thus we complete the proof for the claim. By a similar way of the proof of [7, Theorem 2.2], we know from (4) that T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is complex symmetric with the conjugation $C_{\mu,\lambda}$ if and only if T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is complex symmetric with the conjugation $C_{1,\lambda}$ if and only if $\varphi(z) = \varphi(\lambda\overline{z})$. □

As some applications of Theorem 2.6, we get the following corollaries.

Corollary 2.7 *Let $\varphi \in L^\infty(\mathbb{D})$ and let the operator $C_{\frac{1}{2}}$ be defined by*

$$C_{\frac{1}{2}}f(z) = -\left(\frac{\sqrt{3}}{2-z}\right)\cdot \overline{f\left(\frac{1-2\overline{z}}{2-\overline{z}}\right)}$$

for some $f \in S^2_{\alpha,\beta}(\mathbb{D})$. Then T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is complex symmetric with the conjugation $C_{\frac{1}{2}}$ if and only if $\varphi(z) = \varphi(\frac{1-2\overline{z}}{2-\overline{z}})$.

Corollary 2.8 *Let $\varphi(z) = \sum_{n=-\infty}^{\infty} b_n z^n \in L^\infty(\mathbb{D})$. If T_φ is a Toeplitz operator on $S^2_{\alpha,\beta}(\mathbb{D})$, then T_φ is complex symmetric with the conjugation $C_{\mu,\lambda}$ if and only if $\varphi(z) = b_0 + \sum_{n=1}^{\infty} b_n(z^n + \lambda^n\overline{z}^n)$ with $|\lambda| = 1$. Moreover, if T_φ is complex symmetric with the conjugation $C_{\mu,\lambda}$, then T_φ is normal if and only if $\lambda^n b_n = \overline{b_n} = \lambda^n b_{-n}$ for all $n \in \mathbb{N}_0$ with $|\lambda| = 1$.*

Proof The proof follows from Theorem 2.6 and [8]. □

Theorem 2.9 *Let φ be in $L^\infty(\mathbb{D})$ such that $\varphi(z) = \sum_{n=1}^{\infty} \overline{\hat{\varphi}(-n)}\overline{z}^n + \sum_{n=0}^{\infty} \hat{\varphi}(n)z^n$, and let C be a conjugation on $S^2_{\alpha,\beta}(\mathbb{D})$. Then T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is a complex symmetric operator with the conjugation C if and only if $\hat{\varphi}(-k) = C\hat{\varphi}(k)$ for all $k \in \mathbb{N}_0$.*

Proof Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ and $Cf(z) = \sum_{j=0}^{\infty} \widetilde{a}_j z^j$. Denote

$$\varphi_+(z) = \sum_{n=0}^{\infty} \hat{\varphi}(n)z^n \quad \text{and} \quad \varphi_-(z) = \sum_{n=1}^{\infty} \overline{\hat{\varphi}(-n)}\overline{z}^n.$$

By the proof of Theorem 2.3, we obtain that

$$\begin{cases} \varphi_+ f = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \hat{\varphi}(k)a_n z^{k+n}, \\ P(\varphi_- f) = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{(n+\alpha)(n+\beta)}{(n-k+\alpha)(n-k+\beta)}\overline{\hat{\varphi}(-k)}a_n z^{n-k}, \\ P(\overline{\varphi_+}Cf) = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{(n+\alpha)(n+\beta)}{(n-k+\alpha)(n-k+\beta)}\overline{\hat{\varphi}(k)}\widetilde{a}_n z^{n-k}, \\ \overline{\varphi_-}Cf = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \hat{\varphi}(-k)\widetilde{a}_n z^{k+n}. \end{cases} \tag{5}$$

Since T_φ is complex symmetric with the conjugation C if and only if

$$\varphi_+ f + P(\varphi_- f) = CP(\overline{\varphi_+}Cf) + C(\overline{\varphi_-}Cf), \tag{6}$$

thus equation (6) gives that

$$\begin{aligned} &\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \hat{\varphi}(k)a_n z^{k+n} + \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{(n+\alpha)(n+\beta)}{(n-k+\alpha)(n-k+\beta)}\overline{\hat{\varphi}(-k)}a_n z^{n-k}, \\ &\quad = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{(n+\alpha)(n+\beta)}{(n-k+\alpha)(n-k+\beta)}\widetilde{\hat{\varphi}(k)}a_n z^{n-k} + \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \widetilde{\hat{\varphi}(-k)}a_n z^{k+n}. \end{aligned} \tag{7}$$

From the constant term in (7), we have

$$\sum_{k=0}^{\infty} \frac{(n+\alpha)(n+\beta)}{(n-k+\alpha)(n-k+\beta)}\overline{\hat{\varphi}(-k)}a_k = \sum_{k=0}^{\infty} \frac{(n+\alpha)(n+\beta)}{(n-k+\alpha)(n-k+\beta)}\widetilde{\hat{\varphi}(k)}a_k.$$

Since a_k is arbitrary, we have $\hat{\varphi}(-k) = C\hat{\varphi}(k)$ for all $k \in \mathbb{N}_0$. Conversely, if $\hat{\varphi}(-k) = C\hat{\varphi}(k)$ for all $k \in \mathbb{N}_0$, then T_φ is complex symmetric with the conjugation C. □

Corollary 2.10 *Let φ be in $L^\infty(\mathbb{D})$ such that $\varphi(z) = \sum_{n=1}^{\infty} \overline{\hat{\varphi}(-n)}\overline{z}^n + \sum_{n=0}^{\infty} \hat{\varphi}(n)z^n$. If C is anti-linear on $S^2_{\alpha,\beta}(\mathbb{D})$ such that $Ce_n = \overline{\delta_n}e_n$ with $|\delta_n| = 1$, then T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is a complex symmetric operator with the conjugation C if and only if $\hat{\varphi}(-k) = \overline{\delta_k\hat{\varphi}(k)}$ for all $k \in \mathbb{N}_0$.*

3 Complex symmetry with non-harmonic symbols

In this section, we study the complex symmetry with non-harmonic symbols. In the Hardy space $H^2(\mathbb{T})$, $\overline{z}^n z^m$ is equal to z^{m-n}, but in the generalized derivative Hardy $S^2_{\alpha,\beta}(\mathbb{D})$, $\overline{z}^n z^m \neq z^{m-n}$ since $z \in \mathbb{D}$. The following result gives a necessary and sufficient condition for complex symmetric Toeplitz operators with non-harmonic symbols.

Theorem 3.1 *Let $\varphi(z) = \sum_{i=0}^{\infty}(a_i\overline{z}^{n_i}z^{m_i} + b_i\overline{z}^{s_i}z^{t_i})$ for $a_i, b_i \in \mathbb{C}$, and let $n_i - m_i = t_i - s_i$ hold. Then T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is complex symmetric with the conjugation $C_{\mu,\lambda}$ if and only if φ is either*

$$\varphi(z) = \sum_{i=0}^{\infty}\left(a_i|z|^{2m_i} + b_i|z|^{2t_i}\right)$$

or

$$\varphi(z) = \sum_{i=0}^{\infty} a_i\left(\overline{z}^{n_i}z^{m_i} + \lambda^{m_i-n_i}\overline{z}^{m_i}z^{n_i}\right)$$

for $a_i, b_i \in \mathbb{C}$.

Proof Assume that $n_i > m_i$ for $i \in \mathbb{N}$ and T_φ is complex symmetric with the conjugation $C_{\mu,\lambda}$. If $k \geq \max_{i\in\mathbb{N}}\{n_i - m_i\}$, then

$$\begin{aligned}
C_{\mu,\lambda}T_\varphi z^k &= C_{\mu,\lambda}P\left(\sum_{i=0}^{\infty}\left(a_i\overline{z}^{n_i}z^{m_i+k} + b_i\overline{z}^{s_i}z^{t_i+k}\right)\right) \\
&= C_{\mu,\lambda}\left(\sum_{i=0}^{\infty}\left[\frac{(m_i+k+\alpha)(m_i+k+\beta)}{(m_i+k-n_i+\alpha)(m_i+k-n_i+\beta)}a_i z^{m_i+k-n_i}\right.\right. \\
&\quad \left.\left. + \frac{(t_i+k+\alpha)(t_i+k+\beta)}{(t_i+k-s_i+\alpha)(t_i+k-s_i+\beta)}b_i z^{t_i+k-s_i}\right]\right) \\
&= \sum_{i=0}^{\infty}\left[\mu\frac{(m_i+k+\alpha)(m_i+k+\beta)}{(m_i+k-n_i+\alpha)(m_i+k-n_i+\beta)}\overline{a_i\lambda}^{m_i+k-n_i}z^{m_i+k-n_i}\right. \\
&\quad \left. + \mu\frac{(t_i+k+\alpha)(t_i+k+\beta)}{(t_i+k-s_i+\alpha)(t_i+k-s_i+\beta)}\overline{b_i\lambda}^{t_i+k-s_i}z^{t_i+k-s_i}\right]
\end{aligned}$$

and

$$\begin{aligned}
T^*_\varphi C_{\mu,\lambda}z^k &= T_{\overline{\varphi}}\mu\overline{\lambda}^k z^k \\
&= \mu\overline{\lambda}^k P\left(\sum_{i=0}^{\infty}\left[\overline{a_i z}^{m_i}z^{n_i+k} + \overline{b_i}\overline{z}^{t_i}z^{s_i+k}\right]\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=0}^{\infty}\left[\mu\overline{\lambda}^{k}\frac{(n_i+k+\alpha)(n_i+k+\beta)}{(n_i+k-m_i+\alpha)(n_i+k-m_i+\beta)}\overline{a_i}z^{n_i+k-m_i}\right.\\
&\quad\left.+\mu\overline{\lambda}^{k}\frac{(s_i+k+\alpha)(s_i+k+\beta)}{(s_i+k-t_i+\alpha)(s_i+k-t_i+\beta)}\overline{b_i}z^{s_i+k-t_i}\right].
\end{aligned}$$

Since T_φ is a complex symmetric operator with the conjugation $C_{\mu,\lambda}$, we have that

$$n_i = m_i \quad \text{and} \quad s_i = t_i$$

for any $i \geq 0$, i.e., φ is of the form $\varphi(z) = \sum_{i=0}^{\infty}(a_i|z|^{2m_i} + b_i|z|^{2t_i})$ or

$$\frac{(m_i+k+\alpha)(m_i+k+\beta)}{(m_i+k-n_i+\alpha)(m_i+k-n_i+\beta)}\overline{a_i}\overline{\lambda}^{m_i-n_i} = \frac{(s_i+k+\alpha)(s_i+k+\beta)}{(s_i+k-t_i+\alpha)(s_i+k-t_i+\beta)}\overline{b_i} \tag{8}$$

and

$$\frac{(t_i+k+\alpha)(t_i+k+\beta)}{(t_i+k-s_i+\alpha)(t_i+k-s_i+\beta)}\overline{b_i}\overline{\lambda}^{t_i-s_i} = \frac{(n_i+k+\alpha)(n_i+k+\beta)}{(n_i+k-m_i+\alpha)(n_i+k-m_i+\beta)}\overline{a_i} \tag{9}$$

for all $i \in \mathbb{N}$. By equations (8) and (9), we obtain $s_i = m_i$, $t_i = n_i$, and $a_i = b_i\lambda^{n_i-m_i}$ for all $i \geq 0$, and so φ is of the form

$$\varphi(z) = \sum_{i=0}^{\infty} a_i\left(\overline{z}^{n_i}z^{m_i} + \lambda^{m_i-n_i}\overline{z}^{m_i}z^{n_i}\right).$$

On the one hand, suppose that φ is of the form

$$\varphi(z) = \sum_{i=0}^{\infty} a_i\left(\overline{z}^{n_i}z^{m_i} + \lambda^{m_i-n_i}\overline{z}^{m_i}z^{n_i}\right).$$

Then, by similar calculations, we have that

$$\begin{aligned}
&C_{\mu,\lambda}T_\varphi\sum_{k=0}^{\infty}c_k z^k\\
&= C_{\mu,\lambda}P\left(\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\left(a_ic_k\overline{z}^{n_i}z^{m_i+k} + a_ic_k\lambda^{m_i-n_i}\overline{z}^{m_i}z^{n_i+k}\right)\right)\\
&= C_{\mu,\lambda}\left(\sum_{i=0}^{\infty}\left[\sum_{k=n_i-m_i}^{\infty}\frac{(m_i+k+\alpha)(m_i+k+\beta)}{(m_i+k-n_i+\alpha)(m_i+k-n_i+\beta)}a_ic_kz^{m_i+k-n_i}\right.\right.\\
&\quad\left.\left.+\sum_{k=0}^{\infty}\frac{(n_i+k+\alpha)(n_i+k+\beta)}{(n_i+k-m_i+\alpha)(n_i+k-m_i+\beta)}a_i\lambda^{m_i-n_i}c_kz^{n_i+k-m_i}\right]\right)\\
&= \sum_{i=0}^{\infty}\left[\sum_{k=n_i-m_i}^{\infty}\mu\frac{(m_i+k+\alpha)(m_i+k+\beta)}{(m_i+k-n_i+\alpha)(m_i+k-n_i+\beta)}\overline{a_ic_k}\overline{\lambda}^{m_i+k-n_i}z^{m_i+k-n_i}\right.\\
&\quad\left.+\sum_{k=0}^{\infty}\mu\frac{(n_i+k+\alpha)(n_i+k+\beta)}{(n_i+k-m_i+\alpha)(n_i+k-m_i+\beta)}\overline{a_ic_k}\overline{\lambda}^{k}z^{n_i+k-m_i}\right]
\end{aligned}$$

and

$$\begin{aligned}
&T_\varphi^* C_{\mu,\lambda} \sum_{k=0}^{\infty} c_k z^k \\
&= T_{\overline{\varphi}} \sum_{k=0}^{\infty} \overline{c_k} \mu \overline{\lambda}^k z^k \\
&= P\left(\sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \overline{c_k} \mu \overline{\lambda}^k \left[\overline{a_i} \overline{z}^{m_i} z^{n_i+k} + \overline{a_i} \overline{\lambda}^{m_i-n_i} \overline{z}^{n_i} z^{m_i+k} \right] \right) \\
&= \sum_{i=0}^{\infty} \left[\sum_{k=0}^{\infty} \overline{c_k} \mu \overline{\lambda}^k \frac{(n_i+k+\alpha)(n_i+k+\beta)}{(n_i+k-m_i+\alpha)(n_i+k-m_i+\beta)} \overline{a_i} z^{n_i+k-m_i} \right. \\
&\quad \left. + \sum_{k=n_i-m_i}^{\infty} \overline{c_k} \mu \overline{\lambda}^k \frac{(m_i+k+\alpha)(m_i+k+\beta)}{(m_i+k-n_i+\alpha)(m_i+k-n_i+\beta)} \overline{a_i} \overline{\lambda}^{m_i-n_i} z^{m_i+k-n_i} \right].
\end{aligned}$$

Therefore, we know that

$$C_{\mu,\lambda} T_\varphi \sum_{k=0}^{\infty} c_k z^k = T_\varphi^* C_{\mu,\lambda} \sum_{k=0}^{\infty} c_k z^k,$$

and hence T_φ is complex symmetric with the conjugation $C_{\mu,\lambda}$.

Similarly, if φ is of the form $\varphi(z) = \sum_{i=0}^{\infty}(a_i|z|^{2m_i} + b_i|z|^{2t_i})$, then we can show that T_φ is complex symmetric with the conjugation $C_{\mu,\lambda}$. This completes the proof. □

Remark 3.2 Let

$$\varphi(z) = \sum_{i=0}^{\infty} \left(a_i \overline{z}^{n_i} z^{m_i} + b_i \overline{z}^{s_i} z^{t_i} \right)$$

for $a_i, b_i \in \mathbb{C}$ and let $n_i - m_i = t_i - s_i$ hold. By Theorem 2.6, T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is complex symmetric with the conjugation $C_{\mu,\lambda}$ if and only if $\varphi(\lambda\overline{z}) = \varphi(z)$. Indeed, since

$$\begin{aligned}
\varphi(\lambda\overline{z}) - \varphi(z) &= \sum_{i=0}^{\infty} \left(a_i \overline{(\lambda\overline{z})}^{n_i} (\lambda\overline{z})^{m_i} + b_i \overline{(\lambda\overline{z})}^{s_i} (\lambda\overline{z})^{t_i} \right) - \sum_{i=0}^{\infty} \left(a_i \overline{z}^{n_i} z^{m_i} + b_i \overline{z}^{s_i} z^{t_i} \right) \\
&= \sum_{i=0}^{\infty} \left(a_i \overline{\lambda}^{n_i} z^{n_i} (\lambda\overline{z})^{m_i} + b_i \overline{\lambda}^{s_i} z^{s_i} \left(\lambda^{t_i} \overline{z}^{t_i} \right) \right) - \sum_{i=0}^{\infty} \left(a_i \overline{z}^{n_i} z^{m_i} + b_i \overline{z}^{s_i} z^{t_i} \right) \\
&= \sum_{i=0}^{\infty} \left[a_i \lambda^{m_i-n_i} z^{n_i} \overline{z}^{m_i} + b_i \lambda^{t_i-s_i} z^{s_i} \overline{z}^{t_i} - a_i \overline{z}^{n_i} z^{m_i} - b_i \overline{z}^{s_i} z^{t_i} \right] = 0
\end{aligned}$$

if and only if $s_i = m_i$, $t_i = n_i$, and $a_i = b_i \lambda^{n_i-m_i}$.

Corollary 3.3 (i) *Let* $\varphi(z) = \sum_{i=0}^{\infty}(a_i \overline{z}^{n_i} z^{m_i} + b_i \overline{z}^{m_i} z^{n_i})$ *for some* $a_i, b_i \in \mathbb{C}$. *Then* T_φ *on* $S^2_{\alpha,\beta}(\mathbb{D})$ *is complex symmetric with the conjugation* $C_{\mu,\lambda}$ *if and only if* $a_i = b_i \lambda^{n_i-m_i}$.

(ii) *If* $\varphi(z) = \sum_{i=0}^{\infty}(a_i \overline{z}^{m_i+1} z^{m_i} + b_i \overline{z}^{m_i} z^{m_i+1})$ *for some* $a_i, b_i \in \mathbb{C}$, *then* T_φ *on* $S^2_{\alpha,\beta}(\mathbb{D})$ *is complex symmetric with the conjugation* $C_{\mu,\lambda}$ *if and only if* $a_i = b_i\lambda$ *for all* i.

(iii) *If $\varphi(z) = \sum_{i=0}^{\infty} 2a_i|z|^{2i}$ for $a_i \in \mathbb{C}$, then T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is complex symmetric with the conjugation $C_{\mu,\lambda}$.*

Proof (i) If $s_i = m_i$ and $t_i = n_i$ in Theorem 3.1, then we obtain statement (i).

(ii) If we put $n_i = m_i + 1$ in (i), then we get statement (ii).

(iii) If $s_i = m_i$, $t_i = n_i$, and $a_i = b_i$, then we have this result. □

Example 3.4 Let $\varphi(z) = \sum_{j=0}^{\infty}(a_j\overline{z}^{n_j}z^{m_j} + a_je^{i\theta}\overline{z}^{m_j}z^{n_j})$ for some $a_i \in \mathbb{C}$ and for some real θ. Then T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is complex symmetric with the conjugation $C_{\mu,\lambda}$.

Corollary 3.5 *Let $\varphi(z) = \sum_{i=0}^{\infty}(a_i\overline{z}^{n_i}z^{m_i} + b_i\overline{z}^{s_i}z^{t_i})$ for $a_i, b_i \in \mathbb{C}$ and let $n_i - m_i = t_i - s_i$ hold. If one of $s_i = m_i$, $t_i = n_i$, and $a_i = b_i\lambda^{n_i-m_i}$ does not hold, then T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is not complex symmetric with the conjugation $C_{\mu,\lambda}$.*

Proof The proof follows from Theorem 3.1. □

Remark 3.6 In Theorem 3.1, the condition "$n_i - m_i = t_i - s_i$" is a necessary condition. If not, we may not consider complex symmetric Toeplitz operators with such non-harmonic symbols. For example, let $\varphi(z) = a\overline{z}z^3 + b\overline{z}^2z$ for $a, b \in \mathbb{C}$. Then, for $k \geq 2$, we have

$$C_{\mu,\lambda}T_\varphi z^k = \mu\frac{(k+\alpha+3)(k+\beta+3)}{(k+\alpha+2)(k+\beta+2)}\overline{a}\overline{\lambda}^{k+2}z^{k+2} + \mu\frac{(k+\alpha+1)(k+\beta+1)}{(k+\alpha-1)(k+\beta-1)}\overline{b}\overline{\lambda}^{k-1}z^{k-1}$$

and

$$T^*_\varphi C_{\mu,\lambda}z^k = \mu\overline{\lambda}^k\frac{(k+\alpha+1)(k+\beta+1)}{(k+\alpha-2)(k+\beta-2)}\overline{a}z^{k-2} + \mu\overline{\lambda}^k\frac{(k+\alpha+2)(k+\beta+2)}{(k+\alpha+1)(k+\beta+1)}\overline{b}z^{k+1}.$$

Thus $C_{\mu,\lambda}T_\varphi z^k \neq T^*_\varphi C_{\mu,\lambda}z^k$ for any $k \geq 2$, and so T_φ is not complex symmetric Toeplitz operators.

Corollary 3.7 *Let $\varphi(z) = a\overline{z}^nz^m + bz^k$, where $n, m, k \in \mathbb{N}$ with $n > m$ and $a, b \in \mathbb{C}$ with $|a| \neq |b|$. Then T_φ on $S^2_{\alpha,\beta}(\mathbb{D})$ is never complex symmetric with the conjugation $C_{\mu,\lambda}$.*

Remark 3.8 If $\varphi(z) = a\overline{z}^nz^m$ for $a \in \mathbb{C}$ and $m, n \in \mathbb{N}$ with $m \neq n$ or $\varphi(z) = \overline{z}^2z + bz$ for $b \in \mathbb{C}$ with $b \neq 1$. By Theorem 3.1 and Corollary 3.7, T_φ is never complex symmetric with the conjugation $C_{\mu,\lambda}$ in $S^2_{\alpha,\beta}(\mathbb{D})$ and the Hardy space $H^2(\mathbb{T})$.

4 Conclusion

In this paper, we make characterizations for Toeplitz operators to be complex symmetric on the generalized derivative Hardy space $S^2_{\alpha,\beta}(\mathbb{D})$ with respect to the conjugations C_ξ, $C_{\mu,\lambda}$ as in Theorem 2.6. Moreover, in Theorem 2.9, we deduce the necessary and sufficient conditions for complex symmetric Toeplitz operators with any conjugation C. Next, for the conjugation $C_{\mu,\lambda}$, we also obtain complex symmetric Toeplitz operators with non-harmonic symbols of the form $\varphi(z) = \sum_{i=0}^{\infty}(a_i\overline{z}^{n_i}z^{m_i} + b_i\overline{z}^{s_i}z^{t_i})$ in Theorem 3.1. The results of this paper provide an answer in the generalized derivative Hardy space $S^2_{\alpha,\beta}(\mathbb{D})$ as in the question raised in [8].

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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
JL wrote the initial draft after calculation of results, EK originated the idea of this research and supervised the results, the methodology was given by JEL. All authors read and approved the final manuscript. These authors contributed equally to this work.

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References
1. Brown, A., Halmos, P.R.: Algebraic properties of Toeplitz operators. J. Reine Angew. Math. **213**, 89–102 (1964)
2. Bu, Q., Chen, Y., Zhu, S.: Complex symmetric Toeplitz operators. Integral Equ. Oper. Theory **93**, 15, 19pp (2021)
3. Douglas, R.G.: Banach Algebra Techniques in Operator Theory. Academic Press, New York (1972)
4. Garcia, S.R., Prodan, E., Putinar, M.: Mathematical and physical aspects of complex symmetric operators. J. Phys. A, Math. Theor. **47**, 353001, 57pp (2014)
5. Garcia, S.R., Putinar, M.: Complex symmetric operators and applications. Trans. Am. Math. Soc. **358**, 1285–1315 (2006)
6. Gu, C., Luo, S.: Composition and multiplication operators on the derivative Hardy space. Complex Var. Elliptic Equ. **63**, 599–624 (2018)
7. Kang, D., Ko, E., Lee, J.E.: Remarks on complex symmetric Toeplitz operators. Linear Multilinear Algebra https://doi.org/10.1080/03081087.2020.1842847
8. Ko, E., Lee, J.E.: On complex symmetric Toeplitz operators. J. Math. Anal. Appl. **434**, 20–34 (2016)
9. Ko, E., Lee, J.E., Lee, J.: Multiplication and Toeplitz operators on the generalized derivative Hardy space. Complex Anal. Oper. Theory **13**(4), 4143–4164 (2019)
10. Li, R., Yang, Y., Lu, Y.: A class of complex symmetric Toeplitz operator on Hardy and Bergman spaces. J. Math. Anal. Appl. **489**, 124173, 12pp (2020)
11. Zhu, K.: Operator Theory in Function Spaces, vol. 138. Am. Math. Soc., Providence (2007)