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Fixed point results via extended $\mathcal{FZ}$-simulation functions in fuzzy metric spaces

Abdelhamid Moussaoui[1*], Nawab Hussain[2], Said Melliani[1], Hayel Nasr[3] and Mohammad Imdad[3]

*Correspondence:
a.moussaoui@usms.ma
[1]Laboratory of Applied Mathematics & Scientific Computing, Sultan Moulay Slimane University, F.S.T Beni Mellal, PO Box 523, 23000, Morocco
Full list of author information is available at the end of the article

Abstract
In this paper, we introduce a new class of control functions, namely extended $\mathcal{FZ}$-simulation functions, and employ it to define a new contractive condition. We also prove some new fixed and best proximity point results in the context of an M-complete fuzzy metric space. The presented theorems unify, generalize, and improve several existing results in the literature.

Keywords: Fixed point theory; Fuzzy metric; Fuzzy contraction; $\mathcal{FZ}$-simulation function

1 Introduction

Fixed point theory is one of the central parts of research in functional analysis that provides several mathematical concepts and fruitful tools for the resolution of many problems arising from different fields of engineering and sciences. Due to its potential applicability, the Banach contraction principle is one of the most crucial results, and it asserts that every self-contraction $\mathcal{G}$ defined on a complete metric space X admits a unique fixed point. This influential result has been generalized and extended in different approaches and several abstract spaces (see [1–19]). In particular, Khojasteh *et al.* [2] proposed a new approach to the study of fixed point theory based on the notion of simulation functions which exhibit a significant unifying power over several known results. Roldán *et al.* [20] slightly revised the previous notion of simulation function by reformulating the definition given in [2]. In sequential study, Demma *et al.* [13] extended and generalized the concept of simulation functions on a b-metric framework by providing a new concept of b-simulation functions, and then in connection with existing fixed point results of [2], the authors addressed several new ones. Roldán and Samet [11] developed the family of extended simulation functions with respect to a lower semi-continuous mapping. The usefulness and applicability of these control functions have inspirited many authors to diversify it further in different metric spaces (see e.g. [4, 8, 9, 11, 13, 15, 20–23]).

On the other hand, the non-self-mapping $\mathcal{G} : U \to V$ with $U \cap V = \emptyset$ does not have a fixed point. In this case, it is of interest to find an element x in U such that $d(x, \mathcal{G}x)$ is minimum. Since $d(U, V) \leq d(x, \mathcal{G}x)$, for all $x \in U$, the point x in U which satisfies the condition

$d(x,\mathcal{G}x) = d(\mathcal{U},\mathcal{V})$ is called best proximity point. A best proximity theorem enunciates sufficient conditions for the existence of a best proximity point of the mapping $\mathcal{G}$. In fact, best proximity theorems are natural generalizations of fixed point theorems.

The concept of fuzzy metric space was introduced by Kramosil and Michalek [24] and further modified by George and Veeramani [25] with the purpose of obtaining a Hausdorff topology. Later on, Gregori and Sapena [26] introduced the concept of fuzzy contractive mappings and proved a fixed point result in the setting of fuzzy metric space. In [27], Mihet proposed the class of ψ-contractive mappings, which is larger than the class of fuzzy contractive mappings given in [26]. Following this direction, Wardowski [28] presented and studied the concept of $\mathcal{H}$-contractive mappings. Very recently, inspired by the approach in [2], Melliani and Moussaoui [3] (see also [4]) initiated the study of $\mathcal{FZ}$-contractions involving a new class of simulation functions which provides a unique and common point of view for several previously known concepts in the context of fuzzy metric spaces such as fuzzy contractive, fuzzy ψ-contractive, and $\mathcal{H}$-contractive mappings.

In the present paper, we introduce a new class of control functions, namely extended $\mathcal{FZ}$-simulation functions, we prove some fixed points results in the context of an M-complete fuzzy metric space by defining a new contractive condition via the same class. The presented theorems unify, generalize, and improve several existing results in the literature.

2 Preliminaries

Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ will stand for the set of all positive integer numbers and the set of all real numbers, respectively.

Definition 1 ([29]) A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous triangular norm if it satisfies the following conditions:

(T_1) $*$ is continuous;
(T_2) $*$ is commutative and associative;
(T_3) $a * 1 = a$ for all $a \in [0,1]$;
(T_4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Example 1 The following instances are classical examples of continuous t-norm:

1. $a * b = \min\{a,b\}$ for all $a,b \in [0,1]$;
2. $a * b = \max\{0, a+b-1\}$ for all $a,b \in [0,1]$;
3. $a * b = a \cdot b$ for all $a,b \in [0,1]$.

Definition 2 ([25]) The 3-tuple $(X,M,*)$ is said to be a fuzzy metric space (in the sense of George and Veeramani) if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions:

$(\mathcal{FM}_1)$ $M(x,y,t) > 0$;
$(\mathcal{FM}_2)$ $M(x,y,t) = 1$ if and only if $x = y$;
$(\mathcal{FM}_3)$ $M(x,y,t) = M(y,x,t)$;
$(\mathcal{FM}_4)$ $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$;
$(\mathcal{FM}_5)$ $M(x,y,.) : (0,\infty) \to (0,1]$ is continuous;

for all $x,y,z \in X$ and $s,t > 0$.

An ordered triple $(X, M, *)$ is said to be a strong fuzzy metric space if the triangular inequality $(\mathcal{FM}_4)$ of Definition 2 is replaced by the following one:

$(\mathcal{FM}_4)'$: $M(x,y,t) * M(y,z,t) \leq M(x,z,t)$ for all $x,y,z \in X$ and $t > 0$.

For further details and topological results, the reader is refereed to [24–26, 30].

Remark 1 In view of $(\mathcal{FM}_1)$ and $(\mathcal{FM}_2)$ we have $0 < M(x,y,t) < 1$ for all $x \neq y$ and $t > 0$ (see [31]).

Example 2 ([25]) Let (X,d) be a metric space. Define $a * b = \min\{a,b\}$ for all $a,b \in [0,1]$ and the function $M_d : X \times X \times (0,\infty) \to [0,1]$ by

$$M_d(x,y,t) = \frac{t}{t + d(x,y)} \quad \text{for all } x,y \in X, t > 0.$$

Then $(X, M_d, *)$ is a fuzzy metric space, M_d is called standard fuzzy metric induced by d.

Lemma 1 ([32]) *$M(x,y,\cdot)$ is nondecreasing for all x, y in X.*

Definition 3 ([25, 32]) Let $(X,M,*)$ be a fuzzy metric space.

1. A sequence $\{x_n\} \subseteq X$ is said to be convergent and converges to $x \in X$ if and only if $\lim_{n\to\infty} M(x_n,x,t) = 1$ for all $t > 0$.
2. A sequence $\{x_n\} \subseteq X$ is said to be an M-Cauchy sequence if and only if, for each $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1 - \varepsilon$ for all $n,m \geq n_0$.
3. A sequence $\{x_n\} \subseteq X$ is said to be a G-Cauchy sequence if $M(x_n,x_{n+p},t) = 1$ for all $p \in \mathbb{N}$ and $t > 0$.
4. A fuzzy metric space in which every M-Cauchy (G-Cauchy) sequence is convergent is called an M-complete (G-complete) fuzzy metric space.

Definition 4 ([26]) Let $(X,M,*)$ be a fuzzy metric space. A mapping $\mathcal{G} : X \to X$ is said to be fuzzy contractive mapping if there exists $\lambda \in (0,1)$ such that

$$\frac{1}{M(\mathcal{G}x,\mathcal{G}y,t)} - 1 \leq \lambda\left(\frac{1}{M(x,y,t)} - 1\right),$$

for each $x,y \in X$ and $t > 0$.

Let Ψ be the class of all functions $\psi : (0,1] \to (0,1]$ such that ψ is continuous, nondecreasing and $\psi(\ell) > \ell$ for all $\ell \in (0,1)$.

Definition 5 ([27]) Let $(X,M,*)$ be a fuzzy metric space. A mapping $\mathcal{G} : X \to X$ is said to be a fuzzy ψ-contractive mapping if

$$M(\mathcal{G}x,\mathcal{G}y,t) \geq \psi\bigl(M(x,y,t)\bigr) \quad \text{for all } x,y \in X, t > 0.$$

Let $\mathcal{H}$ be a family of the mappings $\eta : (0,1] \longrightarrow [0,\infty)$ satisfying the following conditions:

$\mathcal{C}_1$) η transforms $(0,1]$ onto $[0,\infty)$;

$\mathcal{C}_2$) η is strictly decreasing.

Definition 6 ([28]) Let $(X,M,*)$ be a fuzzy metric space. A mapping $\mathcal{G}:X\to X$ is said to be fuzzy $\mathcal{H}$-contractive with respect to $\eta\in\mathcal{H}$ if there exists $\lambda(0,1)$ such that

$$\eta\big(M(\mathcal{G}x,\mathcal{G}y,t)\big)\leq\lambda\eta\big(\big(M(x,y,t)\big)\quad\text{for all } x,y\in X, t>0.$$

Definition 7 ([3, 4]) The function $\zeta:(0,1]\times(0,1]\to\mathbb{R}$ is said to be an $\mathcal{FZ}$-simulation function if the following properties hold:

(ζ1) $\zeta(1,1)=0$;

(ζ2) $\zeta(t,s)<\frac{1}{s}-\frac{1}{t}$ for all $t,s\in(0,1)$;

(ζ3) If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,1]$ such that $\lim_{n\to\infty}t_n=\lim_{n\to\infty}s_n<1$, then $\limsup_{n\to\infty}\zeta(t_n,s_n)<0$.

The collection of all $\mathcal{FZ}$-simulation functions is denoted by $\mathcal{FZ}$.

Definition 8 ([3, 4]) Let $(X,M,*)$ be a fuzzy metric space, $\mathcal{G}:X\longrightarrow X$ be a mapping, and $\zeta\in\mathcal{FZ}$. Then $\mathcal{G}$ is called an $\mathcal{FZ}$-contraction with respect to ζ if the following condition is satisfied:

$$\zeta\big(M(\mathcal{G}x,\mathcal{G}y,t),M(x,y,t)\big)\geq 0\quad\text{for all } x,y\in X, t>0.$$

Example 3 ([3, 4]) Each fuzzy contractive mapping is an $\mathcal{FZ}$-contraction with respect to the $\mathcal{FZ}$-simulation function given by

$$\zeta(t,s)=\lambda\left(\frac{1}{s}-1\right)-\frac{1}{t}+1\quad\text{for all } s,t\in(0,1],$$

where $\lambda\in(0,1)$.

Example 4 ([3, 4]) Each ψ-contractive mapping is an $\mathcal{FZ}$-contraction with respect to the $\mathcal{FZ}$-simulation function given by

$$\zeta(t,s)=\frac{1}{\psi(s)}-\frac{1}{t}\quad\text{for all } s,t\in(0,1] \text{ and } \psi\in\Psi.$$

The authors in [3] proved the following result.

Theorem 1 *Let $(X,M,*)$ be an M-complete strong fuzzy metric space and $\mathcal{G}:X\longrightarrow X$ be an $\mathcal{FZ}$-contraction with respect to $\zeta\in\mathcal{FZ}$. Then $\mathcal{G}$ has a unique fixed point.*

Let $(X,M,*)$ be a fuzzy metric space, $\varphi:X\to(0,1]$ be a given function, and $\mathcal{G}:X\to X$ be a mapping. The set of all fixed points of T will be denoted by

$$\mathrm{Fix}(\mathcal{G})=\{x\in X:\mathcal{G}x=x\}.$$

The set of all ones of the function φ will be denoted by

$$\mathcal{O}_\varphi=\big\{x\in X:\varphi(x)=1\big\}.$$

Sezen *et al.* [33] presented the notion of fuzzy φ-fixed point as follows.

Definition 9 ([33]) Let X be a nonempty set, $\varphi : X \to (0,1]$ be a given function, and $\mathcal{G} : X \to X$. An element $z \in X$ is said to be a fuzzy φ-fixed point of the mapping $\mathcal{G}$ if and only if $z \in \mathrm{Fix}(\mathcal{G}) \cap \mathcal{O}_\varphi$.

Let $F : (0,1]^3 \to (0,1]$ be a given function, and consider the following axioms:
(F_1) $F(u,v,w) \le \min\{u,v\}$ for all $u,v,w \in (0,1]$;
(F_2) $F(1,1,1) = 1$;
$(F_2)'$ $F(u,1,1) = u$ for all $u \in (0,1]$;
(F_3) F is continuous.
We consider the following classes of functions:

$$\mathcal{F}_M = \{F : (0,1]^3 \to (0,1] : F \textit{ satisfies } (F_1), (F_2), \textit{ and } (F_3)\}$$

and

$$\mathcal{F} = \{F : (0,1]^3 \to (0,1] : F \textit{ satisfies } (F_1), (F_2)', \textit{ and } (F_3)\}.$$

Example 5 ([15, 33]) The following functions $F : (0,1]^3 \to (0,1]$ belong to $\mathcal{F}$ and $\mathcal{F}_M$:
(1) $F(u,v,w) = u \cdot v \cdot w$ for all $u,v,w \in (0,1]$;
(2) $F(u,v,w) = \min\{u,v\} \cdot w$ for all $u,v,w \in (0,1]$.

The main result of [33] is the following.

Theorem 2 *Let $(X,M,*)$ be a G-complete fuzzy metric space, $\mathcal{G} : X \to X$, and $\varphi : X \to (0,1]$ be a lower semi-continuous function. Suppose that there exist two functions $\psi \in \Psi$ and $F \in \mathcal{F}_M$ such that, for all $x,y \in X$, $t > 0$,*

$$F\bigl(M(\mathcal{G}x,\mathcal{G}y,t),\varphi(\mathcal{G}x),\varphi(\mathcal{G}y)\bigr) \ge \psi\bigl(F\bigl(M(x,y,t),\varphi(x),\varphi(y)\bigr)\bigr). \tag{1}$$

Then $\mathcal{G}$ has a unique φ-fixed point.

3 A new class of control functions

In this section, we enlarge the class of $\mathcal{FZ}$-simulation functions by introducing the class of extended $\mathcal{FZ}$-simulation functions.

Definition 10 The function $e : (0,1] \times (0,1] \to \mathbb{R}$ is said to be an extended $\mathcal{FZ}$-simulation function if the following properties hold:
($\mathcal{E}1$) $e(t,s) < \frac{1}{s} - \frac{1}{t}$ for all $t,s \in (0,1)$;
($\mathcal{E}2$) If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a < 1$ and $s_n < a$, then $\limsup_{n\to\infty} e(t_n,s_n) < 0$;
($\mathcal{E}3$) For any sequence $\{t_n\}$ in $(0,1)$, we have

$$\lim_{n\to\infty} t_n = a \in (0,1], \quad e(t_n,a) \ge 0 \quad \Longrightarrow \quad a = 1.$$

We denote the collection of all extended $\mathcal{FZ}$-simulation functions by $\mathcal{FZ}_e$.

Proposition 3 *Every $\mathcal{FZ}$-simulation function is an extended $\mathcal{FZ}$-simulation function.*

Proof Let $\zeta : (0,1] \times (0,1] \to \mathbb{R}$ be an $\mathcal{FZ}$-simulation function. It is easy to show that ζ satisfies $(\mathcal{E}1)$ and $(\mathcal{E}2)$, we shall prove $(\mathcal{E}3)$. Reasoning by contradiction, let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = a \leq 1$ and $\zeta(t_n, a) \geq 0$. Assume that $a < 1$, and applying $(\zeta 3)$ with $s_n = a \in (0,1)$, we get

$$\limsup_{n\to\infty} \zeta(t_n, a) = \limsup_{n\to\infty} \zeta(t_n, s_n) < 0,$$

which yields a contradiction, hence $a = 1$. □

The converse inclusion is not true, we confirm this by the following example.

Example 6 Let $e : (0,1] \times (0,1] \to \mathbb{R}$ be the function defined by

$$e(t,s) = \begin{cases} 1, & \text{if } t = s = 1, \\ \frac{1}{\psi(s)} - \frac{1}{t}, & \text{otherwise,} \end{cases}$$

where $\psi \in \Psi$. Clearly, e is not $\mathcal{FZ}$-simulation function, since $e(1,1) \neq 0$ and $(\zeta 1)$ is not satisfied. Now, we show that e is an extended $\mathcal{FZ}$-simulation function. For all $t, s \in (0,1)$, we have $e(t,s) = \frac{1}{\psi(s)} - \frac{1}{t} < \frac{1}{s} - \frac{1}{t}$, which proves $(\mathcal{E}1)$. If $\{t_n\}$, $\{s_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a < 1$ and $s_n < a$, using the fact that $\psi(u) > u$ for all $u \in (0,1)$, we have

$$\limsup_{n\to\infty} e(t_n, s_n) = \frac{1}{\psi(a)} - \frac{1}{a} < 0.$$

Therefore, e satisfies $(\mathcal{E}2)$.

Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = a \in (0,1]$, $e(t_n, a) \geq 0$, we shall prove that $a = 1$. Suppose that $a < 1$, we have

$$e(t_n, a) = \frac{1}{\psi(a)} - \frac{1}{t_n} \geq 0.$$

Taking the limit as $n \to \infty$, we get

$$\frac{1}{\psi(a)} - \frac{1}{a} \geq 0.$$

Hence, $\psi(a) \leq a$, which contradicts the fact that $\psi(u) > u$ for all $u \in (0,1)$. Therefore $a = 1$ and e is an extended $\mathcal{FZ}$-simulation function.

4 Main results

First we introduce the following concept of $(\mathcal{FZ}_e^{\varphi}, F)$-contraction.

Definition 11 Let $(X, M, *)$ be a fuzzy metric space, $\varphi : X \to (0,1]$ be a given function, and $F \in \mathcal{F}$. A mapping $\mathcal{G} : X \to X$ is said to be an $(\mathcal{FZ}_e^{\varphi}, F)$-contraction, if there exists $e \in \mathcal{FZ}_e$ such that

$$e\big(F\big(M(\mathcal{G}x, \mathcal{G}y, t), \varphi(\mathcal{G}x), \varphi(\mathcal{G}y)\big), \mathcal{N}_F^{\varphi}(x, y, t)\big) \geq 0 \tag{2}$$

for all $x, y \in X$ and all $t > 0$, where

$$\begin{aligned} \mathcal{N}_F^{\varphi}(x,y,t) = \min\bigl\{&F\bigl(M(x,y,t),\varphi(x),\varphi(y)\bigr), F\bigl(M(x,\mathcal{G}x,t),\varphi(x),\varphi(\mathcal{G}x)\bigr),\\ &F\bigl(M(y,\mathcal{G}y,t),\varphi(y),\varphi(\mathcal{G}y)\bigr)\bigr\}. \end{aligned}$$

Our first main result is the following theorem.

Theorem 4 *Let $(X, M, *)$ be an M-complete fuzzy metric space, $\varphi : X \to (0,1]$ be a given function, and $F \in \mathcal{F}$. Suppose that the following conditions hold:*

(i) *$\mathcal{G} : X \to X$ is an $(\mathcal{F}\mathcal{Z}_e^{\varphi}, F)$-contraction with respect to $e \in \mathcal{F}\mathcal{Z}_e$;*

(ii) *φ is lower semi-continuous.*

Then $\mathrm{Fix}(\mathcal{G}) \subseteq \mathcal{O}_\varphi$. *Moreover, the mapping $\mathcal{G}$ has a unique φ-fixed point.*

Proof First, we show that $\mathrm{Fix}(\mathcal{G}) \subseteq \mathcal{O}_\varphi$. Assume that $u \in X$ is a fixed point of $\mathcal{G}$. Applying (2) with $x = y = u$, we obtain

$$\begin{aligned} 0 &\le e\bigl(F\bigl(M(\mathcal{G}u,\mathcal{G}u,t),\varphi(\mathcal{G}u),\varphi(\mathcal{G}u)\bigr), \mathcal{N}_F^{\varphi}(u,u,t)\bigr)\\ &= e\bigl(F\bigl(1,\varphi(u),\varphi(u)\bigr), \mathcal{N}_F^{\varphi}(u,u,t)\bigr), \end{aligned} \tag{3}$$

where

$$\begin{aligned} \mathcal{N}_F^{\varphi}(u,u,t) = \min\bigl\{&F\bigl(M(u,u,t),\varphi(u),\varphi(u)\bigr), F\bigl(M(u,\mathcal{G}u,t),\varphi(u),\varphi(\mathcal{G}u)\bigr),\\ &F\bigl(M(u,\mathcal{G}u,t),\varphi(u),\varphi(\mathcal{G}u)\bigr)\bigr\}\\ = \min\bigl\{&F\bigl(1,\varphi(u),\varphi(u)\bigr), F\bigl(1,\varphi(u),\varphi(u)\bigr), F\bigl(1,\varphi(u),\varphi(u)\bigr)\bigr\}\\ = F\bigl(&1,\varphi(u),\varphi(u)\bigr). \end{aligned}$$

We claim that $F(1,\varphi(u),\varphi(u)) = 1$. Reasoning by contradiction, suppose that $F(1,\varphi(u),\varphi(u)) < 1$. Regarding $(\mathcal{E}1)$, inequality (3) yields that

$$\begin{aligned} 0 &\le e\bigl(F\bigl(M(\mathcal{G}u,\mathcal{G}u,t),\varphi(\mathcal{G}u),\varphi(\mathcal{G}u)\bigr), \mathcal{N}_F^{\varphi}(u,u,t)\bigr)\\ &= e\bigl(F\bigl(1,\varphi(u),\varphi(u)\bigr), \mathcal{N}_F^{\varphi}(u,u,t)\bigr)\\ &< \frac{1}{\mathcal{N}_F^{\varphi}(u,u,t))} - \frac{1}{F(1,\varphi(u),\varphi(u))}\\ &= \frac{1}{F(1,\varphi(u),\varphi(u))} - \frac{1}{F(1,\varphi(u),\varphi(u))}\\ &= 0, \end{aligned}$$

which is a contradiction. Then

$$F\bigl(1,\varphi(u),\varphi(u)\bigr) = 1.$$

From (F_1), we deduce that

$$1 = F\bigl(1,\varphi(u),\varphi(u)\bigr) \le \min\bigl\{1,\varphi(u)\bigr\} \le \varphi(u),$$

which means that $\varphi(u) = 1$, and hence $u \in \mathcal{O}_\varphi$, and so $\mathrm{Fix}(\mathcal{G}) \subseteq \mathcal{O}_\varphi$.

Next, let $x_0 \in X$ be an arbitrary point and $\{x_n\}$ be the Picard sequence defined by $x_n = \mathcal{G}^n x_0$, $n \in \mathbb{N}$. If there exists some $m \in \mathbb{N}$ such that $x_m = x_{m+1}$, then x_m is a fixed point of $\mathcal{G}$ and hence a fuzzy φ-fixed point of $\mathcal{G}$ (as $\mathrm{Fix}(\mathcal{G}) \subseteq \mathcal{O}_\varphi$), which completes the proof. For this reason, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, which means that $M(x_n, x_{n+1}, t) < 1$ for all $t > 0$.

If there exists some $k_0 \in \mathbb{N}$ such that $F(M(x_{k_0}, x_{k_0+1}, t), \varphi(x_{k_0}), \varphi(x_{k_0+1})) = 1$, then we could deduce from condition (F_1) that

$$\begin{aligned} F\bigl(M(x_{k_0}, x_{k_0+1}, t), \varphi(x_{k_0}), \varphi(x_{k_0+1})\bigr) &\leq \min\bigl\{M(x_{k_0}, x_{k_0+1}, t), \varphi(x_{k_0})\bigr\} \\ &\leq M(x_{k_0}, x_{k_0+1}, t) < 1, \end{aligned}$$

which is a contradiction. As consequence,

$$F\bigl(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\bigr) < 1 \quad \text{for all } n \in \mathbb{N}.$$

Since $\mathcal{G}$ is an $(\mathcal{FZ}_e^\varphi, F)$-contraction with respect to $e \in \mathcal{FZ}_e$, we have

$$0 \leq e\bigl(F\bigl(M(\mathcal{G}x_n, \mathcal{G}x_{n+1}, t), \varphi(\mathcal{G}x_n), \varphi(\mathcal{G}x_{n+1})\bigr), \mathcal{N}_F^\varphi(x_n, x_{n+1}, t)\bigr). \tag{4}$$

Now, we define $\vartheta_n(t) = F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) < 1$, $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathcal{N}_F^\varphi(x_n, x_{n+1}, t) = {} & \min\bigl\{F\bigl(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\bigr), \\ & F\bigl(M(x_n, \mathcal{G}x_n, t), \varphi(x_n), \varphi(\mathcal{G}x_n)\bigr), \\ & F\bigl(M(x_{n+1}, \mathcal{G}x_{n+1}, t), \varphi(x_{n+1}), \varphi(\mathcal{G}x_{n+1})\bigr)\bigr\} \\ = {} & \min\{F\bigl(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\bigr), \\ & F\bigl(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\bigr), \\ & F\bigl(M(x_{n+1}, x_{n+2}, t), \varphi(x_{n+1}), \varphi(x_{n+2})\bigr) \\ = {} & \min\bigl\{\vartheta_n(t), \vartheta_n(t), \vartheta_{n+1}(t)\bigr\} \\ = {} & \min\bigl\{\vartheta_n(t), \vartheta_{n+1}(t)\bigr\} < 1. \end{aligned}$$

Regarding $(\mathcal{E}1)$, inequality (4) yields that

$$\begin{aligned} 0 &\leq e\bigl(F\bigl(M(\mathcal{G}x_n, \mathcal{G}x_{n+1}, t), \varphi(\mathcal{G}x_n), \varphi(\mathcal{G}x_{n+1})\bigr), \mathcal{N}_F^\varphi(x_n, x_{n+1}, t)\bigr) \\ &= e\bigl(F\bigl(M(x_{n+1}, x_{n+2}, t), \varphi(x_{n+1}), \varphi(x_{n+2})\bigr), \min\bigl\{\vartheta_n(t), \vartheta_{n+1}(t)\bigr\}\bigr) \\ &= e\bigl(\vartheta_{n+1}(t), \min\bigl\{\vartheta_n(t), \vartheta_{n+1}(t)\bigr\}\bigr) \\ &< \frac{1}{\min\{\vartheta_n(t), \vartheta_{n+1}(t)\})} - \frac{1}{\vartheta_{n+1}(t)}, \end{aligned}$$

which means that

$$\min\bigl\{\vartheta_n(t), \vartheta_{n+1}(t)\bigr\} < \vartheta_{n+1}(t).$$

Therefore $\vartheta_n(t) < \vartheta_{n+1}(t)$. Then, it follows that $\{\vartheta_n(t)\}$ is an increasing sequence of positive real numbers in $(0, 1]$. Consequently, there exists $l(t) \leq 1$ such that $\lim_{n\to\infty} \vartheta_n(t) =$

$l(t) \leq 1$ for all $t > 0$. We shall prove that $l(t) = 1$. On the contrary, we assume that $l(t) < 1$ for some $t > 0$. Denote $\tau_n(t) = \vartheta_{n+1}(t)$ and $\delta_n(t) = \min\{\vartheta_n(t), \vartheta_{n+1}(t)\} = \vartheta_n(t)$, we have

$$\lim_{n\to\infty} \tau_n(t) = \lim_{n\to\infty} \delta_n(t) = l(t).$$

Since $\{\delta_n(t)\}$ is strictly increasing, we have $\delta_n(t) < l(t)$. Regarding $(\mathcal{E}2)$, we get

$$\limsup_{n\to\infty} e\bigl(\tau_n(t), \delta_n(t)\bigr) < 0,$$

which is in contradiction with $e(\tau_n(t), \delta_n(t)) \geq 0$ for all $n \in \mathbb{N}$. Accordingly, we deduce that

$$\lim_{n\to\infty} \vartheta_n(t) = \lim_{n\to\infty} F\bigl(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\bigr) = 1. \tag{5}$$

Due to (F_1), we have

$$F\bigl(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\bigr) \leq \min\bigl\{M(x_n, x_{n+1}, t), \varphi(x_n)\bigr\} \leq \varphi(x_n)$$

and

$$F\bigl(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\bigr) \leq \min\bigl\{M(x_n, x_{n+1}, t), \varphi(x_n)\bigr\} \leq M(x_n, x_{n+1}, t).$$

Taking $n \to \infty$ and keeping (5) in mind, we obtain

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1 \quad \text{for all } t > 0 \quad \text{and} \tag{6}$$

$$\lim_{n\to\infty} \varphi(x_n) = 1. \tag{7}$$

Next, we show that $\{x_n\}$ is an M-Cauchy sequence in X. Arguing by contradiction, we assume that $\{x_n\}$ is not an M-Cauchy sequence. Then there exist $\epsilon \in (0,1)$, $t_0 > 0$ and two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ such that

$$M(x_{n_k}, x_{m_k}, t_0) < 1 - \epsilon. \tag{8}$$

Taking into account Lemma 1, we have

$$M\left(x_{n_k}, x_{m_k}, \frac{t_0}{2}\right) < 1 - \epsilon. \tag{9}$$

By choosing m_k as the smallest index satisfying (9), we get

$$M\left(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}\right) \geq 1 - \epsilon. \tag{10}$$

On account of (8) and (10), the triangular inequality yields

$$\begin{aligned} 1 - \epsilon &> M(x_{n_k}, x_{m_k}, t_0) \\ &\geq M\left(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}\right) * M\left(x_{m_k-1}, x_{m_k}, \frac{t_0}{2}\right) \\ &\geq (1 - \epsilon) * M\left(x_{m_k-1}, x_{m_k}, \frac{t_0}{2}\right). \end{aligned}$$

Taking the limit of both sides as $k \to \infty$, using (6) and (T_3), we derive that

$$\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, t_0) = 1 - \epsilon. \tag{11}$$

Since $\mathcal{G}$ is an $(\mathcal{FZ}_e^{\varphi}, F)$-contraction, we have that, for all $k \in \mathbb{N}$,

$$\begin{aligned}
0 &\le e\big(F\big(M(\mathcal{G}x_{n_k-1}, \mathcal{G}x_{m_k-1}, t_0), \varphi(\mathcal{G}x_{n_k-1}), \varphi(\mathcal{G}x_{m_k-1})\big), \mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0)\big) \\
&= e\big(F\big(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})\big), \mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0)\big) \\
&< \frac{1}{\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0)} - \frac{1}{F(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k}))},
\end{aligned}$$

which implies that

$$\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0) < F\big(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})\big), \tag{12}$$

where

$$\begin{aligned}
\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0) = {} & \min\big\{F\big(M(x_{n_k-1}, x_{m_k-1}, t_0), \varphi(x_{n_k-1}), \varphi(x_{m_k-1})\big), \\
& F\big(M(x_{n_k-1}, \mathcal{G}x_{n_k-1}, t_0), \varphi(x_{n_k-1}), \varphi(\mathcal{G}x_{n_k-1})\big), \\
& F\big(M(x_{m_k-1}, \mathcal{G}x_{m_k-1}, t_0), \varphi(x_{m_k-1}), \varphi(\mathcal{G}x_{m_k-1})\big)\big\} \\
= {} & \min\big\{F(M(x_{n_k-1}, x_{m_k-1}, t_0), \varphi(x_{n_k-1}), \varphi(x_{m_k-1}), \\
& F(M(x_{n_k-1}, x_{n_k}, t_0), \varphi(x_{n_k-1}), \varphi(x_{n_k}), \\
& F\big(M(x_{m_k-1}, x_{m_k}, t_0), \varphi(x_{m_k-1}), \varphi(x_{m_k})\big)\big\}.
\end{aligned}$$

Passing to the limit as $k \to \infty$ in the above equality, using (6), (7), $(F_2)'$ and taking into account the continuity of F, we obtain

$$\begin{aligned}
\lim_{k\to\infty} \mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0) &= \min\Big\{F\Big(\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0), 1, 1\Big), \\
&\qquad F(1,1,1), F(1,1,1)\Big\} \\
&= \lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0).
\end{aligned}$$

Therefore, (12) gives rise to

$$\begin{aligned}
\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0) &\le F\Big(\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, t_0), 1, 1\Big) \\
&= \lim_{k\to\infty} M(x_{n_k}, x_{m_k}, t_0) \\
&= 1 - \epsilon.
\end{aligned} \tag{13}$$

By the triangular inequality, we have

$$M(x_{n_k-1}, x_{m_k-1}, t_0) \ge M\left(x_{n_k-1}, x_{n_k}, \frac{t_0}{2}\right) * M\left(x_{n_k}, x_{m_k-1}, \frac{t_0}{2}\right).$$

Letting $k \to \infty$ in the last inequality and using (6) and (10), we get

$$\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0) \geq 1 * (1-\epsilon) = 1-\epsilon. \tag{14}$$

From (13) and (14), we derive that

$$\lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0) = 1-\epsilon. \tag{15}$$

On the other hand, by (6), (10) and regarding $(F_2)'$, we have

$$\lim_{k\to\infty} F\bigl(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})\bigr) = F(1-\epsilon, 1, 1) = 1-\epsilon.$$

In particular, it follows from (12), (F_1), and (8) that

$$\begin{aligned}\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0) &< F\bigl(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})\bigr),\\ &\leq \min\bigl\{M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k})\bigr\}\\ &\leq M(x_{n_k}, x_{m_k}, t_0).\\ &< 1-\epsilon.\end{aligned}$$

Take the sequences $\alpha_k = F(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k}))$, and $\beta_k = \mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0))$ for all $k \in \mathbb{N}$. From the above observations, (11) and (15), we conclude that $\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = 1-\epsilon$ and $\beta_k < 1-\epsilon$. Thus, we can apply axiom $(\mathcal{E}2)$ to these sequences; as a consequence

$$0 \leq \lim_{k\to\infty} \sup e(\alpha_k, \beta_k) < 0,$$

which is a contradiction. Thus, we deduce that $\{x_n\}$ is an M-Cauchy sequence. Since $(X, M, *)$ is an M-complete fuzzy metric space, there exists $u \in X$ such that

$$\lim_{n\to\infty} M(x_n, u, t) = 1, \quad \forall t > 0. \tag{16}$$

Due to the lower semi-continuity of φ, (7) and (16), we derive that

$$\varphi(u) = 1. \tag{17}$$

Therefore, $u \in \mathcal{O}_{\varphi}$. Next, we shall show that u is a fixed point of $\mathcal{G}$ arguing by contradiction. Suppose that $M(u, \mathcal{G}u, t) < 1$ for some $t > 0$. Let us define

$$\mu(t) = F\bigl(M(u, \mathcal{G}u, t), 1, \varphi(\mathcal{G}u)\bigr), \qquad \acute{\alpha}_n(t) = F\bigl(M(x_{n+1}, \mathcal{G}u, t), \varphi(x_{n+1}), \varphi(\mathcal{G}u)\bigr) \quad \text{and}$$
$$\acute{\beta}_n(t) = \mathcal{N}_F^{\varphi}(x_n, u, t) \quad \text{for all } n \in \mathbb{N}.$$

Using (F_1), we obtain

$$\begin{aligned}\mu(t) = F\bigl(M(u, \mathcal{G}u, t), 1, \varphi(\mathcal{G}u)\bigr) &\leq \min\bigl\{M(u, \mathcal{G}u, t), 1\bigr\}\\ &= M(u, \mathcal{G}u, t) < 1.\end{aligned} \tag{18}$$

Taking the limit as $n \to \infty$ and using the continuity of F

$$\begin{aligned}\lim_{n\to\infty} \acute{\alpha}_n(t) &= \lim_{n\to\infty} F\big(M(x_{n+1}, \mathcal{G}u, t), \varphi(x_{n+1}), \varphi(\mathcal{G}u)\big) \\ &= F\big(M(u, \mathcal{G}u, t), 1, \varphi(\mathcal{G}u)\big) \\ &= \mu(t).\end{aligned}$$

On the other hand,

$$\begin{aligned}\acute{\beta}_n(t) &= \mathcal{N}_F^{\varphi}(x_n, u, t) \\ &= \min\big\{F(M(x_n, u, t), \varphi(x_n), \varphi(u), F\big(M(x_n, \mathcal{G}x_n, t), \varphi(x_n), \varphi(\mathcal{G}x_n)\big), \\ &\quad F\big(M(u, \mathcal{G}u, t), \varphi(u), \varphi(\mathcal{G}u)\big)\big\} \\ &= \min\big\{F\big(M(x_n, u, t), \varphi(x_n), 1\big), F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1}), \\ &\quad F\big(M(u, \mathcal{G}u, t), \varphi(u), \varphi(\mathcal{G}u)\big)\big\}.\end{aligned}$$

As F is continuous, we have

$$\begin{aligned}&\lim_{n\to\infty} F\big(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\big) = F(1,1,1) = 1 \quad \textit{and} \\ &\lim_{n\to\infty} F\big(M(x_n, u, t), \varphi(x_n), 1\big) = F(1,1,1) = 1.\end{aligned}$$

Particulary, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$\acute{\beta}_n(t) = F(\big(M(u, \mathcal{G}u, t), 1, \varphi(\mathcal{G}u)\big) = \mu(t),$$

and $\{\acute{\alpha}_n(t)\}_{n\geq n_0} \subset (0,1]$ is a sequence converging to $\mu(t) < 1$ such that, for all $n \geq n_0$,

$$\begin{aligned}e\big(\acute{\alpha}_n(t), \mu(t)\big) &= e\big(\acute{\alpha}_n(t), \acute{\beta}_n(t)\big) \\ &= e\big(F\big(M(x_{n+1}, \mathcal{G}u, t), \varphi(x_{n+1}), \varphi(\mathcal{G}u)\big), \mathcal{N}_F^{\varphi}(x_n, u, t)\big)\}) \\ &= e\big(F\big(M(\mathcal{G}x_n, \mathcal{G}u, t), \varphi(\mathcal{G}x_n), \varphi(\mathcal{G}u)\big), \mathcal{N}_F^{\varphi}(x_n, u, t)\big)\}) \\ &\geq 0.\end{aligned}$$

Regarding ($\mathcal{E}3$), the last inequality yields that $\mu(t) = 1$, which contradicts (18). As a consequence, $M(u, \mathcal{G}u, t) = 1$, which together with (17) means that u is a fuzzy φ-fixed point of $\mathcal{G}$.

As a final step, we shall show the uniqueness of a fuzzy φ-fixed point of $\mathcal{G}$. We argue by contradiction. Suppose that there are two distinct φ-fixed points $u, v \in X$ of the mapping $\mathcal{G}$. Then $M(u, v, t) < 1$ for all $t > 0$. Since we have $\mathrm{Fix}(\mathcal{G}) \subseteq \mathcal{O}_{\varphi}$, it follows that $\varphi(u) = \varphi(v) = 1$. Now, using (2), we have

$$0 \leq e\big(F\big(M(\mathcal{G}u, \mathcal{G}v, t), \varphi(\mathcal{G}u), \varphi(\mathcal{G}v)\big), \mathcal{N}_F^{\varphi}(u, v, t)\big), \tag{19}$$

where

$$\begin{aligned}\mathcal{N}_F^{\varphi}(u,v,t)) &= \min\bigl\{F\bigl(M(u,v,t),\varphi(u),\varphi(v)\bigr),F\bigl(M(u,\mathcal{G}u,t),\varphi(u),\varphi(\mathcal{G}u)\bigr),\\ &\quad F\bigl(M(v,\mathcal{G}v,t),\varphi(v),\varphi(\mathcal{G}v)\bigr)\bigr\}\\ &= \min\bigl\{F\bigl(M(u,v,t),1,1\bigr),F(1,1,1),F(1,1,1)\bigr\}\\ &= F\bigl(M(u,v,t),1,1\bigr)\\ &= M(u,v,t).\end{aligned}$$

Regarding $(\mathcal{E}1)$, inequality (19) yields that

$$\begin{aligned}0 &\le e\bigl(F\bigl(M(\mathcal{G}u,\mathcal{G}v,t),\varphi(\mathcal{G}u),\varphi(\mathcal{G}v)\bigr),\mathcal{N}_F^{\varphi}(u,v,t)\bigr)\\ &= e\bigl(F\bigl(M(u,v,t),1,1\bigr),M(u,v,t)\bigr)\\ &= e\bigl(M(u,v,t),M(u,v,t)\bigr)\\ &< \frac{1}{M(u,v,t)}-\frac{1}{M(u,v,t)}=0,\end{aligned}$$

a contradiction, thus $u=v$. Therefore, the fuzzy φ-fixed point of $\mathcal{G}$ is unique. This completes the proof. □

To support our result, we provide an illustrative example. Precisely, we show that our result (Theorem 4) can be used to cover this example, while Theorem 2 is not applicable.

Example 7 Let $X=[-2,2]$ endowed with the standard fuzzy metric $M(x,y,t)=\frac{t}{t+d(x,y)}$, where $d(x,y)$ is the usual metric $d(x,y)=|x-y|$ for all $x,y\in[-2,2]$. It is clear that $(X,M,*)$ is an M-complete fuzzy metric space. Consider the mapping $\mathcal{G}:X\to X$ defined by

$$\mathcal{G}x=\begin{cases}-1, & x=\frac{1}{2},\\ \frac{-x}{20}, & \textit{otherwise}.\end{cases}$$

Now, we define two auxiliary functions $F:(0,1]^3\to(0,1]$ and $\varphi:X\to(0,1]$ by $F(a,b,c)=a\cdot b\cdot c$ for all $a,b,c\in(0,1]$ and $\varphi(x)=1$ for all $x\in X$. It is obvious that $F\in\mathcal{F}$ and φ is a lower semi-continuous function. Now, consider the function $e:(0,1]\times(0,1]\to\mathbb{R}$ defined by

$$e(t,s)=\frac{3}{4}\left(\frac{1}{s}-1\right)-\left(\frac{1}{t}-1\right).$$

In order to show that $\mathcal{G}$ is an $(\mathcal{F}\mathcal{Z}_e^{\varphi},F)$-contraction mapping, we distinguish the following cases:

Case I: Let $x=y=\frac{1}{2}$

$$\begin{aligned}&e\left(F\left(M\left(\mathcal{G}\frac{1}{2},\mathcal{G}\frac{1}{2},t\right),\varphi\left(\mathcal{G}\frac{1}{2}\right),\varphi\left(\mathcal{G}\frac{1}{2}\right)\right),\mathcal{N}_F^{\varphi}\left(\frac{1}{2},\frac{1}{2},t\right)\right)\\ &\quad= e\left(M(-1,-1,t),\min\left\{M\left(\frac{1}{2},\frac{1}{2},t\right),M\left(\frac{1}{2},\mathcal{G}\frac{1}{2},t\right),M\left(\frac{1}{2},\mathcal{G}\frac{1}{2},t\right)\right\}\right)\end{aligned}$$

$$
\begin{aligned}
&= e\left(1, \min\left\{1, M\left(\frac{1}{2}, -1, t\right), M\left(\frac{1}{2}, -1, t\right)\right\}\right) \\
&= e\left(1, M\left(\frac{1}{2}, -1, t\right)\right) = \frac{3}{4}\left(\frac{1}{M(\frac{1}{2}, -1, t)} - 1\right) - \left(\frac{1}{1} - 1\right) \\
&= \frac{3}{4}\left(\frac{d(\frac{1}{2}, -1)}{t}\right) = \frac{9}{8t} \geq 0.
\end{aligned}
$$

Case II: Let $x, y \in X - \{\frac{1}{2}\}$

$$
\begin{aligned}
&e\bigl(F\bigl(M(\mathcal{G}x, \mathcal{G}y, t), \varphi(\mathcal{G}x), \varphi(\mathcal{G}y)\bigr), \mathcal{N}_F^{\varphi}(x, y, t)\bigr) \\
&\quad = e\left(M\left(\frac{-x}{20}, \frac{-y}{20}, t\right), \min\left\{M(x, y, t), M\left(x, \frac{-x}{20}, t\right), M\left(y, \frac{-y}{20}, t\right)\right\}\right) \\
&\quad = \frac{3}{4}\left(\frac{1}{\min\{M(x, y, t), M(x, \frac{-x}{20}, t), M(y, \frac{-y}{20}, t)\}} - 1\right) - \left(\frac{1}{M(\frac{-x}{20}, \frac{-y}{20}, t)} - 1\right) \\
&\quad \geq \frac{3}{4}\left(\frac{1}{M(x, y, t)} - 1\right) - \left(\frac{1}{M(\frac{-x}{20}, \frac{-y}{20}, t)} - 1\right) \\
&\quad = \frac{3}{4}\left(\frac{d(x, y)}{t}\right) - \left(\frac{d(\frac{-x}{20}, \frac{-y}{20})}{t}\right) = \frac{3}{4}\left(\frac{d(x, y)}{t}\right) - \frac{1}{20}\left(\frac{d(x, y)}{t}\right) \\
&\quad = \frac{7}{10}\left(\frac{d(x, y)}{t}\right) \geq 0.
\end{aligned}
$$

Case III: Let $x = \frac{1}{2}$ and $y \in X - \{\frac{1}{2}\}$

$$
\begin{aligned}
&e\left(F\left(M\left(\mathcal{G}\frac{1}{2}, \mathcal{G}y, t\right), \varphi\left(\mathcal{G}\frac{1}{2}\right), \varphi(\mathcal{G}y)\right), \mathcal{N}_F^{\varphi}\left(\frac{1}{2}, y, t\right)\right) \\
&\quad = e\left(M\left(-1, \frac{-y}{20}, t\right), \min\left\{M\left(\frac{1}{2}, y, t\right), M\left(\frac{1}{2}, -1, t\right), M\left(y, \frac{-y}{20}, t\right)\right\}\right) \\
&\quad = \frac{3}{4}\left(\frac{1}{\min\{M(\frac{1}{2}, y, t), M(\frac{1}{2}, -1, t), M(y, \frac{-y}{20}, t)\}} - 1\right) - \left(\frac{1}{M(-1, \frac{-y}{20}, t)} - 1\right) \\
&\quad \geq \frac{3}{4}\left(\frac{1}{M(\frac{1}{2}, -1, t)} - 1\right) - \left(\frac{1}{M(-1, \frac{-y}{20}, t)} - 1\right) \\
&\quad = \frac{3}{4}\left(\frac{d(\frac{1}{2}, -1)}{t}\right) - \left(\frac{d(-1, \frac{-y}{20})}{t}\right) \\
&\quad = \frac{9}{8t} - \frac{1}{t}\left(1 - \frac{y}{20}\right) = \frac{5 + 2y}{40t} \geq 0.
\end{aligned}
$$

Hence, $\mathcal{G}$ is an $(\mathcal{FZ}_e^{\varphi}, F)$-contraction mapping. Therefore, all the hypotheses of Theorem 4 are satisfied, and hence $\mathcal{G}$ has a φ-fixed point (namely $x = 0$).

Finally, we show that Theorem 2 is not applicable in this example. In fact, suppose that there is $\psi \in \Psi$ such that the contraction condition (1) of Theorem 2 holds, that is, for all $x, y \in X$, we have

$$
F\bigl(M(\mathcal{G}x, \mathcal{G}y, t), \varphi(\mathcal{G}x), \varphi(\mathcal{G}y)\bigr) \geq \psi\bigl(F\bigl(M(x, y, t), \varphi(x), \varphi(y)\bigr)\bigr).
$$

Choose $x = 0$ and $y = \frac{1}{2}$ and take into the account that $\psi(t) > t$ for all $t \in (0,1)$, we have

$$\begin{aligned}
F\big(M(\mathcal{G}x,\mathcal{G}y,t),\varphi(\mathcal{G}x),\varphi(\mathcal{G}y)\big) &= F\left(M\left(\mathcal{G}0,\mathcal{G}\frac{1}{2},t\right),\varphi(\mathcal{G}0),\varphi\left(\mathcal{G}\frac{1}{2}\right)\right) \\
&= M\left(\mathcal{G}0,\mathcal{G}\frac{1}{2},t\right) \\
&= \frac{t}{t+1} \\
&< \frac{t}{t+\frac{1}{2}} \\
&= M\left(0,\frac{1}{2},t\right) \\
&< \psi\left(M\left(0,\frac{1}{2},t\right)\right) = \psi\big(F\big(M(x,y,t),\varphi(x),\varphi(y)\big)\big),
\end{aligned}$$

which is a contradiction. This shows that it is impossible to find a function $\psi \in \Psi$ such that the contraction condition (1) holds. Therefore, Theorem 2 is not applicable.

Corollary 1 ([34]) *Let $(X,M,*)$ be an M-complete fuzzy metric space and $\mathcal{G}: X \to X$ be a given mapping such that, for all $x,y \in X$, $t > 0$ and for some $\lambda \in (0,1)$,*

$$\frac{1}{M(\mathcal{G}x,\mathcal{G}y,t)} - 1 \le \lambda\left(\frac{1}{\min\{M(x,y,t),M(x,\mathcal{G}x,t),M(y,\mathcal{G}y,t)\}} - 1\right).$$

Then $\mathcal{G}$ has a unique fixed point.

Proof The result follows by defining $F(a,b,c) = a \cdot b \cdot c$ for all $a,b,c \in (0,1]$, $\varphi(x) = 1$ for all $x \in X$, and $e(t,s) = \lambda(\frac{1}{s} - 1) - \frac{1}{t} + 1$ for all $s,t \in (0,1]$ in Theorem 4. □

Corollary 2 *Let $(X,M,*)$ be an M-complete fuzzy metric space and $\mathcal{G}: X \to X$. Suppose that there exists some $e \in \mathcal{FZ}_e$ such that, for all $x,y \in X$, $t > 0$,*

$$e\big(M(\mathcal{G}x,\mathcal{G}y,t),\min\big\{M(x,y,t),M(x,\mathcal{G}x,t),M(y,\mathcal{G}y,t)\big\}\big) \ge 0.$$

Then $\mathcal{G}$ has a unique fixed point.

Proof The result follows by defining $F(a,b,c) = a \cdot b \cdot c$ for all $a,b,c \in (0,1]$ and $\varphi(x) = 1$ for all $x \in X$ in Theorem 4. □

Corollary 3 *Let $(X,M,*)$ be an M-complete fuzzy metric space, and let $\mathcal{G}: X \to X$ be a given mapping. Suppose that there exists some $\psi \in \Psi$ such that, for all $x,y \in X$, $t > 0$,*

$$M(\mathcal{G}x,\mathcal{G}y,t) \ge \psi\big(\min\big\{M(x,y,t),M(x,\mathcal{G}x,t),M(y,\mathcal{G}y,t)\big\}\big).$$

Then $\mathcal{G}$ has a unique fixed point.

Proof Define $e: (0,1] \times (0,1] \to \mathbb{R}$ by $e(t,s) = \frac{1}{\psi(s)} - \frac{1}{t}$ for all $s,t \in (0,1]$, $F(a,b,c) = a \cdot b \cdot c$ for all $a,b,c \in (0,1]$, and $\varphi(x) = 1$ for all $x \in X$. By Proposition 3, one can see that e is an extended $\mathcal{FZ}$-simulation function. The result follows from Theorem 4. □

Corollary 4 *Let $(X, M, *)$ be an M-complete fuzzy metric space, and let $\mathcal{G} : X \to X$ be a given mapping and $\eta \in \mathcal{H}$ such that, for all $x, y \in X$, $t > 0$,*

$$\eta\big(F\big(M(\mathcal{G}x, \mathcal{G}y, t), \varphi(\mathcal{G}x), \varphi(\mathcal{G}y)\big)\big) \leq k\eta\big(\mathcal{N}_F^{\varphi}(x, y, t)\big),$$

where $k \in (0, 1)$. Then $\mathcal{G}$ has a unique φ-fixed point.

Proof The result follows by defining $e : (0, 1] \times (0, 1] \to \mathbb{R}$ by $e(t, s) = \frac{1}{\eta^{-1}(k\eta(s))} - \frac{1}{t}$ for all $s, t \in (0, 1]$ in Theorem 4. □

Corollary 5 *Let $(X, M, *)$ be an M-complete fuzzy metric space and $\mathcal{G} : X \to X$ be a given mapping. Assume that*

$$\frac{1}{F(M(\mathcal{G}x, \mathcal{G}y, t), \varphi(\mathcal{G}x), \varphi(\mathcal{G}y))} - 1 \leq \phi\left(\frac{1}{\mathcal{N}_F^{\varphi}(x, y, t)} - 1\right)$$

for all $x, y \in X$ and $t > 0$, where $\phi : [0, \infty) \longrightarrow [0, \infty)$ is a right-continuous function with $\phi(t) < t$ for all $t > 0$. Then $\mathcal{G}$ has a unique fuzzy φ-fixed point.

Proof In view of Theorem 4, where $e(t, s) = \phi(\frac{1}{s} - 1) - (\frac{1}{t} - 1)$ for all $t, s \in (0, 1]$, the result follows. □

5 Best proximity point results

In this section, we obtain a sufficient condition to ensure the existence of a φ-best proximity point in the setting of fuzzy metric spaces. Our results can be viewed as an extension of some related results in the existing literature.

Let U and V be two nonempty subsets of a fuzzy metric space $(X, M, *)$ and $\mathcal{G} : U \to V$ be a non-self-mapping. We will use the following notations:

$$U_0(t) = \big\{u \in U : M(u, v, t) = M(U, V, t) \text{ for some } v \in V\big\},$$
$$V_0(t) = \big\{v \in V : M(u, v, t) = M(U, V, t) \text{ for some } u \in U\big\},$$
$$\text{where } M(U, V, t) = \sup\big\{M(u, v, t) : u \in U, v \in V\big\}.$$

The set of all best proximity points of the non-self-mapping $\mathcal{G} : U \to V$ will be denoted by

$$B_{\text{est}}(\mathcal{G}) = \big\{u \in U : M(u, \mathcal{G}u, t) = M(U, V, t)\big\}.$$

Definition 12 ([33]) Let X be a nonempty set, $\varphi : X \to (0, 1]$ be a given function, and $\mathcal{G} : U \to V$ be a non-self-mapping. An element $u^* \in U$ is said to be a fuzzy φ-best proximity point of $\mathcal{G}$ if and only if u^* is a best proximity point of $\mathcal{G}$ and $\varphi(u^*) = 1$.

Definition 13 Let U and V be two nonempty closed subsets of a fuzzy metric space $(X, M, *)$. We say that the operator $\mathcal{G} : U \to V$ is an $(\mathcal{FZ}_e^{\varphi}, F)$-fuzzy proximal contraction with respect to $e \in \mathcal{FZ}_e$ if there exist a function $\varphi : X \to (0, 1]$ and $F \in \mathcal{F}$ such that

$$\begin{cases} M(u, \mathcal{G}x, t) = M(U, V, t), \\ M(v, \mathcal{G}y, t) = M(U, V, t) \end{cases} \Rightarrow \quad e\big(F\big(M(u, v, t), \varphi(u), \varphi(v)\big), \mathcal{N}_F^{\varphi}(x, y, t)\big) \geq 0, \tag{20}$$

for all $u, v, x, y \in U$ and $t > 0$, where

$$\begin{aligned}\mathcal{N}_F^{\varphi}(x,y,t) = \min\bigl\{&F\bigl(M(x,y,t),\varphi(x),\varphi(y)\bigr),F\bigl(M(x,u,t),\varphi(x),\varphi(u)\bigr),\\ &F\bigl(M(y,v,t),\varphi(y),\varphi(v)\bigr)\bigr\}.\end{aligned}$$

Theorem 5 *Let U and V be two nonempty subsets of an M-complete fuzzy metric space $(X, M, *)$ such that $U_0(t)$ is nonempty and $\varphi : X \to (0,1]$, $F \in \mathcal{F}$. Suppose that $\mathcal{G} : U \to V$ is an $(\mathcal{FZ}_e^{\varphi}, F)$-fuzzy proximal contraction with respect to $e \in \mathcal{FZ}_e$. Suppose also*

(i) *$U_0(t)$ is closed with respect to the topology induced by M;*
(ii) *$\mathcal{G}(U_0(t)) \subseteq V_0(t)$;*
(iii) *φ is lower semi-continuous.*

Then $\mathcal{G}$ has a unique fuzzy φ-best proximity point, that is, there exists $x^* \in U$ such that $B_{\text{est}}(\mathcal{G}) \cap \mathcal{O}_{\varphi} = \{x^*\}$.

Proof First, we show that $B_{\text{est}}(\mathcal{G}) \subseteq \mathcal{O}_{\varphi}$. Assume that $\sigma \in U$ is a best proximity point of $\mathcal{G}$, which means that $M(\sigma, \mathcal{G}\sigma, t) = M(U, V, t)$. Applying (20) with $\sigma = u = v = x = y$, we have $0 \leq e(F(1, \varphi(\sigma), \varphi(\sigma)), \mathcal{N}_F^{\varphi}(\sigma, \sigma, t))$, where

$$\begin{aligned}\mathcal{N}_F^{\varphi}(\sigma,\sigma,t) &= \min\bigl\{F\bigl(M(\sigma,\sigma,t),\varphi(\sigma),\varphi(\sigma)\bigr),F\bigl(M(\sigma,\sigma,t),\varphi(\sigma),\varphi(\sigma)\bigr),\\ &\qquad F\bigl(M(\sigma,\sigma,t),\varphi(\sigma),\varphi(\sigma)\bigr)\bigr\}\\ &= F\bigl(1,\varphi(\sigma),\varphi(\sigma)\bigr).\end{aligned}$$

We shall indicate that $F(1, \varphi(\sigma), \varphi(\sigma)) = 1$. Reasoning by contradiction, suppose that $F(1, \varphi(\sigma), \varphi(\sigma)) < 1$, and using $(\mathcal{E}1)$ we derive

$$\begin{aligned}0 &\leq e\bigl(F\bigl(M(\sigma,\sigma,t),\varphi(\sigma),\varphi(\sigma 11)\bigr),\mathcal{N}_F^{\varphi}(\sigma,\sigma,t)\bigr)\\ &= e\bigl(F\bigl(1,\varphi(\sigma),\varphi(\sigma)\bigr),\mathcal{N}_F^{\varphi}(\sigma,\sigma,t)\bigr)\\ &< \frac{1}{\mathcal{N}_F^{\varphi}(\sigma,\sigma,t)} - \frac{1}{F(1,\varphi(\sigma),\varphi(\sigma))}\\ &= \frac{1}{F(1,\varphi(\sigma),\varphi(\sigma))} - \frac{1}{F(1,\varphi(\sigma),\varphi(\sigma))}\\ &= 0,\end{aligned}$$

which is a contradiction. Therefore, $F(1, \varphi(\sigma), \varphi(\sigma)) = 1$.

By (F_1), we have

$$F\bigl(1,\varphi(\sigma),\varphi(\sigma)\bigr) = 1 \leq \min\bigl\{1,\varphi(\sigma)\bigr\} \leq \varphi(\sigma),$$

which yields $\varphi(\sigma) = 1$, and then $B_{\text{est}}(\mathcal{G}) \subseteq \mathcal{O}_{\varphi}$.

Next, let $x_0 \in X$ be an element in $U_0(t)$. Taking into account that $\mathcal{G}x_0 \in \mathcal{G}(U_0(t)) \subseteq V_0(t)$, we can find $x_1 \in U_0(t)$ such that $M(x_1, \mathcal{G}x_0, t) = M(U, V, t)$. Since $\mathcal{G}x_1 \in \mathcal{G}(U_0(t)) \subseteq V_0(t)$, so that there exists $x_2 \in U_0(t)$ such that $M(x_2, \mathcal{G}x_1, t) = M(U, V, t)$. Recursively, a sequence $\{x_n\} \subset U_0(t)$ can be constructed as follows:

$$M(x_{n+1}, \mathcal{G}x_n, t) = M(U, V, t) \quad \text{for all } n \in \mathbb{N}. \tag{21}$$

If $x_k = x_{k+1}$ for some $k \in \mathbb{N}$, then

$$M(x_k, \mathcal{G}x_k, t) = M(x_{k+1}, \mathcal{G}x_k, t) = M(\mathcal{U}, \mathcal{V}, t).$$

Therefore, x_k is the required best proximity point and hence a fuzzy φ-best proximity point of $\mathcal{G}$ (as $B_{\mathrm{est}}(\mathcal{G}) \subseteq \mathcal{O}_\varphi$), which completes the proof. Due to this reason, for the rest of the proof, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, that is, $M(x_n, x_{n+1}, t) < 1$ for all $t > 0$. Now, if there exists some $n_0 \in \mathbb{N}$ such that $F(M(x_{n_0}, x_{n_0+1}, t), \varphi(x_{n_0}), \varphi(x_{n_0+1})) = 1$, condition (F_1) yields that

$$\begin{aligned} 1 &= F\big(M(x_{n_0}, x_{n_0+1}, t), \varphi(x_{n_0}), \varphi(x_{n_0+1})\big) \leq \min\big\{M(x_{n_0}, x_{n_0+1}, t), \varphi(x_{n_0})\big\} \\ &\leq M(x_{n_0}, x_{n_0+1}, t) < 1, \end{aligned}$$

a contradiction. Accordingly, we deduce that

$$F\big(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\big) < 1 \quad \text{for all } n \in \mathbb{N}.$$

Next, we denote $\gamma_n(t) = F(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})) < 1$, $n \in \mathbb{N}$. By (20) and (21), we get

$$0 \leq e\big(F\big(M(x_{n+1}, x_{n+2}, t), \varphi(x_{n+1}), \varphi(x_{n+2})\big), \mathcal{N}_F^\varphi(x_n, x_{n+1}, t)\big), \tag{22}$$

where

$$\begin{aligned} \mathcal{N}_F^\varphi(x_n, x_{n+1}, t) = {} & \min\big\{F\big(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\big), \\ & F\big(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\big), \\ & F\big(M(x_{n+1}, x_{n+2}, t), \varphi(x_{n+1}), \varphi(x_{n+2})\big)\big\} \\ = {} & \min\big\{\gamma_n(t), \gamma_n(t), \gamma_{n+1}(t)\big\} \\ = {} & \min\big\{\gamma_n(t), \gamma_{n+1}(t)\big\} < 1. \end{aligned}$$

Using property $(\mathcal{E}1)$, we deduce that

$$\begin{aligned} 0 &\leq e\big(F\big(M(x_{n+1}, x_{n+2}, t), \varphi(x_{n+1}), \varphi(x_{n+2})\big), \mathcal{N}_F^\varphi(x_n, x_{n+1}, t)\big) \\ &= e\big(\gamma_{n+1}(t), \min\big\{\gamma_n(t), \gamma_{n+1}(t)\big\}\big) \\ &< \frac{1}{\min\{\gamma_n(t), \gamma_{n+1}(t)\}} - \frac{1}{\gamma_{n+1}(t)}, \end{aligned}$$

which yields $\gamma_n(t) < \gamma_{n+1}(t)$. Therefore, we deduce that $\{\gamma_n(t)\}$ is an increasing sequence of real numbers in $(0, 1]$. Thus, there exists $h(t) \leq 1$ such that $\lim_{n\to\infty} \gamma_n(t) = h(t) \leq 1$ for all $n \in \mathbb{N}$. In particular, as $\{\gamma_n(t)\}$ is strictly increasing, then $h(t) > \gamma_n(t)$. We shall prove that $h(t) = 1$ for all $t > 0$. Suppose, on the contrary, that $h(t) < 1$ for some $t > 0$. If we choose the sequences $\varpi_n(t) = \gamma_{n+1}(t)$ and $\theta_n(t) = \min\{\gamma_n(t), \gamma_{n+1}(t)\}$, we have $\lim_{n\to\infty} \varpi_n(t) = \lim_{n\to\infty} \theta_n(t) = h(t)$ and $\theta_n(t) < h(t)$. By condition $(\mathcal{E}2)$, we derive that

$$\limsup_{n\to\infty} e\big(\varpi_n(t), \theta_n(t)\big) < 0,$$

which contradicts equation (22). Accordingly, we deduce that

$$\lim_{n\to\infty}\gamma_n(t)=\lim_{n\to\infty}F\big(M(x_n,x_{n+1},t),\varphi(x_n),\varphi(x_{n+1})\big)=1\quad\text{for all } t>0.$$

Moreover, using (F_1) we get

$$F\big(M(x_n,x_{n+1},t),\varphi(x_n),\varphi(x_{n+1})\big)\le\min\big\{M(x_n,x_{n+1},t),\varphi(x_n)\big\}\le\varphi(x_n)$$

and

$$F\big(M(x_n,x_{n+1},t),\varphi(x_n),\varphi(x_{n+1})\big)\le\min\big\{M(x_n,x_{n+1},t),\varphi(x_n)\big\}\le M(x_n,x_{n+1},t),$$

which implies

$$\lim_{n\to\infty}\varphi(x_n)=1\quad\text{and}\quad\lim_{n\to\infty}M(x_n,x_{n+1},t)=1,\quad\forall t>0.\tag{23}$$

As a next step, we shall prove that the sequence $\{x_n\}$ is Cauchy. Reasoning by contradiction, assume that $\{x_n\}$ is not an M-Cauchy sequence. Then there exists $\epsilon\in(0,1)$, $t_0>0$ and subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ so that, for $m_k>n_k\ge k$, we have

$$M(x_{n_k},x_{m_k},t_0)<1-\epsilon.\tag{24}$$

By Lemma 1, we have

$$M\left(x_{n_k},x_{m_k},\frac{t_0}{2}\right)<1-\epsilon.\tag{25}$$

If we choose m_k as the least natural number satisfying (25), we have

$$M\left(x_{n_k},x_{m_k-1},\frac{t_0}{2}\right)\ge1-\epsilon.\tag{26}$$

Taking into account (24) and (26), we deduce that

$$\begin{aligned}1-\epsilon&>M(x_{n_k},x_{m_k},t_0)\\&\ge M\left(x_{n_k},x_{m_k-1},\frac{t_0}{2}\right)*M\left(x_{m_k-1},x_{m_k},\frac{t_0}{2}\right)\\&>(1-\epsilon)*M\left(x_{m_k-1},x_{m_k},\frac{t_0}{2}\right).\end{aligned}$$

Letting $k\to\infty$ and using (23), we get

$$\lim_{k\to\infty}M(x_{n_k},x_{m_k},t_0)=1-\epsilon.$$

Denote $r_k=F(M(x_{n_k},x_{m_k},t_0),\varphi(x_{n_k}),\varphi(x_{m_k}))$ and $s_k=\mathcal{N}_F^{\varphi}(x_{n_k-1},x_{m_k-1},t_0)$ for all $k\in\mathbb{N}$. Since $\mathcal{G}$ is an $(\mathcal{F}\mathcal{Z}_e^{\varphi},F)$-fuzzy proximal contraction and

$$M(x_{n_k},\mathcal{G}x_{n_k-1},t_0)=M(x_{m_k},\mathcal{G}x_{m_k-1},t_0)=M(U,V,t)$$

for all $k \in \mathbb{N}$. So, by (20), we have

$$\begin{aligned} 0 &\le e\big(F\big(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})\big), \mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0)\big) \\ &< \frac{1}{\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0)} - \frac{1}{F(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k}))}. \end{aligned}$$

Hence

$$\mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0) < F\big(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})\big), \tag{27}$$

where

$$\begin{aligned} \mathcal{N}_F^{\varphi}(x_{n_k-1}, x_{m_k-1}, t_0) = \min\big\{ &F\big(M(x_{n_k-1}, x_{m_k-1}, t_0), \varphi(x_{n_k-1}), \varphi(x_{m_k-1})\big), \\ &F\big(M(x_{n_k-1}, x_{n_k}, t_0), \varphi(x_{n_k-1}), \varphi(x_{n_k})\big), \\ &F\big(M(x_{m_k-1}, x_{m_k}, t_0), \varphi(x_{m_k-1}), \varphi(x_{m_k})\big)\big\}. \end{aligned}$$

By following a similar reasoning to that in the proof of Theorem 4, one can show that

$$\begin{aligned} &\lim_{k\to\infty} s_k = \lim_{k\to\infty} M(x_{n_k-1}, x_{m_k-1}, t_0) = 1-\epsilon \quad \text{and} \\ &\lim_{k\to\infty} r_k = \lim_{k\to\infty} F\big(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})\big) = 1-\epsilon. \end{aligned}$$

Particularly, it follows from (27), (F_1), and (24) that

$$\begin{aligned} s_k &< F\big(M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k}), \varphi(x_{m_k})\big) \\ &\le \min\big\{M(x_{n_k}, x_{m_k}, t_0), \varphi(x_{n_k})\big\} \\ &\le M(x_{n_k}, x_{m_k}, t_0) \\ &< 1-\epsilon. \end{aligned}$$

On account of the above observations, we deduce that $\lim_{k\to\infty} r_k = \lim_{k\to\infty} s_k = 1-\epsilon$ and $s_k < 1-\epsilon$. Regarding axiom ($\mathcal{E}2$), we obtain

$$0 \le \limsup_{k\to\infty} e(r_k, s_k) < 0,$$

which is a contradiction. This contradiction proves that $\{x_n\}$ is an M-Cauchy sequence. Since $U_0(t)$ is a closed subset of the M-complete fuzzy metric space $(X, M, *)$, there exists $x^* \in U_0(t)$ such that

$$\lim_{n\to\infty} M\big(x_n, x^*, t\big) = 1 \quad \forall t > 0. \tag{28}$$

By the lower semi-continuity of φ, (23) and (28), we have

$$\varphi\big(x^*\big) = 1. \tag{29}$$

As $\mathcal{G}(U_0(t)) \subseteq V_0(t)$ and $x^* \in U_0(t)$, there exists $\omega \in U_0(t)$ such that

$$M(w, \mathcal{G}x^*, t) = M(U, V, t). \tag{30}$$

Now, we shall prove that $x^* = w$, reasoning by contradiction. Suppose that $M(x^*, w, t) < 1$ for some $t > 0$. Define

$$\begin{aligned}
&a(t) = F\big(M\big(x^*, w, t\big), 1, \varphi(w)\big),\\
&\acute{r}_n(t) = F\big(M(x_{n+1}, w, t), \varphi(x_{n+1}), \varphi(w)\big) \quad \text{and}\\
&\acute{s}_n(t) = \mathcal{N}_F^{\varphi}\big(x_n, x^*, t\big) \quad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Using (F_1), we obtain

$$a(t) = F\big(M\big(x^*, w, t\big), 1, \varphi(w)\big) \leq \min\big\{M\big(x^*, w, t\big), 1\big\} = M\big(x^*, w, t\big) < 1. \tag{31}$$

Taking the limit as $n \to \infty$ and using the continuity of F, we have

$$\begin{aligned}
\lim_{n\to\infty} \acute{r}_n(t) &= \lim_{n\to\infty} F\big(M(x_{n+1}, w, t), \varphi(x_{n+1}), \varphi(w)\big)\\
&= F\big(M\big(x^*, w, t\big), 1, \varphi(w)\big)\\
&= a(t).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\acute{s}_n(t) &= \mathcal{N}_F^{\varphi}\big(x_n, x^*, t\big)\\
&= \min\big\{F\big(M\big(x_n, x^*, t\big), \varphi(x_n), \varphi\big(x^*\big)\big),\\
&\quad F\big(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\big),\\
&\quad F\big(M\big(x^*, w, t\big), \varphi\big(x^*\big), \varphi(w)\big)\big\}\\
&= \min\big\{F\big(M\big(x_n, x^*, t\big), \varphi(x_n), 1\big),\\
&\quad F\big(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\big),\\
&\quad F\big(M\big(x^*, w, t\big), \varphi\big(x^*\big), \varphi(w)\big)\big\}.
\end{aligned}$$

Due to the continuity of F, we have

$$\begin{aligned}
&\lim_{n\to\infty} F\big(M(x_n, x_{n+1}, t), \varphi(x_n), \varphi(x_{n+1})\big) = F(1,1,1) = 1 \quad \textit{and}\\
&\lim_{n\to\infty} F\big(M\big(x_n, x^*, t\big), \varphi(x_n), 1\big) = F(1,1,1) = 1.
\end{aligned}$$

As consequence, there exists $n_0 \in \mathbb{N}$ such that

$$\acute{s}_n(t) = F\big(M\big(x^*, w, t\big), 1, \varphi(w)\big) = a(t), \quad n \geq n_0.$$

In particular, $\{\acute{r_n}(t)\}_{n\geq n_0} \subset (0,1]$ is a sequence converging to $a(t) < 1$ such that, for all $n \geq n_0$,

$$e\big(\acute{r_n}(t), a(t)\big) = e\big(\acute{r_n}(t), \acute{s_n}(t)\big) = e\big(F\big(M(x_{n+1}, w, t), \varphi(x_{n+1}), \varphi(w)\big), \mathcal{N}_F^{\varphi}\big(x_n, x^*, t\big)\big) \geq 0.$$

By means of (20), the previous inequality with axiom $(\mathcal{E}3)$ ensures that $a(t) = 1$. This contradicts (31). Hence

$$M\big(x^*, w, t\big) = 1.$$

Thus, $x^* = w$, by considering (30), we derive that

$$M\big(x^*, \mathcal{G}x^*, t\big) = M(\mathcal{U}, \mathcal{V}, t).$$

By (29), we conclude that x^* is a fuzzy φ-best proximity point of $\mathcal{G}$.

Finally, we shall show the uniqueness of the fuzzy φ-best proximity point of $\mathcal{G}$, that is, $B_{\text{est}}(\mathcal{G}) \cap \mathcal{O}_\varphi$ is singleton. We argue by contradiction. Suppose that $x^*, w^* \in X$ are two distinct fuzzy φ-best proximity fixed points of the mapping $\mathcal{G}$. Then $M(x^*, w^*, t) < 1$ for all $t > 0$. Hence

$$M\big(x^*, \mathcal{G}x^*, t\big) = M(\mathcal{U}, \mathcal{V}, t) \quad \text{and} \quad M\big(w^*, \mathcal{G}w^*, t\big) = M(\mathcal{U}, \mathcal{V}, t).$$

As $\mathcal{G}$ is an $(\mathcal{FZ}_e^{\varphi}, F)$-fuzzy proximal contraction, we have

$$0 \leq e\big(F\big(M\big(x^*, w^*, t\big), \varphi\big(x^*\big), \varphi\big(w^*\big)\big), \mathcal{N}_F^{\varphi}\big(x^*, w^*, t\big)\big),$$

where

$$\begin{aligned}
\mathcal{N}_F^{\varphi}\big(x^*, w^*, t\big) = {} & \min\big\{F\big(M\big(x^*, w^*, t\big), \varphi\big(x^*\big), \varphi\big(w^*\big)\big), F\big(M\big(x^*, x^*, t\big), \varphi\big(x^*\big), \varphi\big(x^*\big)\big), \\
& F\big(M\big(w^*, w^*, t\big), \varphi\big(w^*\big), \varphi\big(w^*\big)\big)\big\} \\
= {} & \min\big\{F\big(M\big(x^*, w^*, t\big), 1\big), F(1,1,1), F(1,1,1)\big\} \\
= {} & F\big(M\big(x^*, w^*, t\big), 1, 1\big).
\end{aligned}$$

Then, using property $(\mathcal{E}1)$, we get

$$\begin{aligned}
0 & \leq e\big(F\big(M\big(x^*, w^*, t\big), \varphi\big(x^*\big), \varphi\big(w^*\big)\big), \mathcal{N}_F^{\varphi}\big(x^*, w^*, t\big)\big) \\
& < \frac{1}{F(M(x^*, w^*, t), \varphi(x^*), \varphi(w^*))} - \frac{1}{\mathcal{N}_F^{\varphi}(x^*, w^*, t)} \\
& = \frac{1}{F(M(x^*, w^*, t), 1, 1)} - \frac{1}{F(M(x^*, w^*, t), 1, 1)} \\
& = 0,
\end{aligned}$$

which leads to a contradiction. Hence $M(x^*, w^*, t) < 1$, which implies $x^* = w^*$. This completes the proof. □

Corollary 6 *Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty subsets of an M-complete fuzzy metric space $(X, M, *)$ such that $\mathcal{U}_0(t)$ is nonempty. Assume that the mappings $\mathcal{G} : X \to X$, $\varphi : X \to (0,1]$, $\psi \in \Psi$, and $F \in \mathcal{F}$ satisfy the following conditions*:

$$\begin{cases} M(u, \mathcal{G}x, t) = M(\mathcal{U}, \mathcal{V}, t), \\ M(v, \mathcal{G}y, t) = M(\mathcal{U}, \mathcal{V}, t) \end{cases} \quad \Rightarrow \quad F\big(M(u, v, t), \varphi(u), \varphi(v)\big) \geq \psi\left(\mathcal{N}_F^{\varphi}(x, y, t)\right);$$

(i) $\mathcal{U}_0(t)$ *is closed with respect to the topology induced by* M;
(ii) $\mathcal{G}(\mathcal{U}_0(t)) \subseteq \mathcal{V}_0(t)$;
(iii) φ *is continuous*.

Then $\mathcal{G}$ has a unique fuzzy φ-best proximity point, that is, there exists $x^* \in \mathcal{U}$ such that $B_{\text{est}}(\mathcal{G}) \cap \mathcal{O}_{\varphi} = \{x^*\}$.

6 Conclusion

In this study, we established the concept of extended $\mathcal{FZ}$-simulation functions with a view to consider a new class of fuzzy contractions, namely $(\mathcal{FZ}_e^{\varphi}, F)$-contractions. Such a family generalized, extended, and unified several results and enriched various classical types of fuzzy contractions in the literature. We must underline that by properly specifying the control function e, we can particularize and derive different consequences of our main results. Nevertheless, further research is needed in this regard, because it is plausible to explore the existence and uniqueness of a common fixed point or a coincidence point of two self-mappings in a more general setting, for example, partially ordered fuzzy metric spaces.

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AM: conceptualization, investigation, writing-original draft, review and editing. NH,SM, and MI: conceptualization, methodology, supervision. HN: conceptualization, writing-original draft, review, and editing. All authors read and approved the final manuscript.

Author details
[1]Laboratory of Applied Mathematics & Scientific Computing, Sultan Moulay Slimane University, F.S.T Beni Mellal, PO Box 523, 23000, Morocco. [2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia. [3]Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India.

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