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Convergence analysis of the shrinking approximants for fixed point problem and generalized split common null point problem

Yasir Arfat[1], Olaniyi S. Iyiola[2], Muhammad Aqeel Ahmad Khan[3], Poom Kumam[1,4,5*], Wiyada Kumam[6] and Kanokwan Sitthithakerngkiet[7]

*Correspondence:
poom.kum@kmutt.ac.th
[1]KMUTT Fixed Point Research Laboratory, KMUTT-Fixed Point Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
[4]Center of Excellence in Theoretical and Computational Science (TaCS-CoE) and KMUTT Fixed Point Research Laboratory, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Department of Mathematics, Fachulity of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
Full list of author information is available at the end of the article

Abstract

In this paper, we compute a common solution of the fixed point problem (FPP) and the generalized split common null point problem (GSCNPP) via the inertial hybrid shrinking approximants in Hilbert spaces. We show that the approximants can be easily adapted to various extensively analyzed theoretical problems in this framework. Finally, we furnish a numerical experiment to analyze the viability of the approximants in comparison with the results presented in (Reich and Tuyen in Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. 114:180, 2020).

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1 Introduction

The triplet $(\Xi, \langle\cdot,\cdot\rangle, \|\cdot\|)$ represents a real Hilbert space, the inner product, and the induced norm, respectively. For an operator $U : K \to K$, $\mathrm{Fix}(U)$ denotes the set of all fixed points of the operator U, where K is a nonempty closed convex subset of Ξ. Recall that the operator U is called η-demimetric [46], where $\eta \in (-\infty, 1)$, if $\mathrm{Fix}(U) \neq \emptyset$ and

$$\langle p - q, (Id - U)p\rangle \geq \frac{1}{2}(1 - \eta)\|(Id - U)p\|^2 \quad \text{for all } p \in K \text{ and } q \in \mathrm{Fix}(U),$$

where Id denotes the identity operator.

The η-demimetric operator is equivalently defined by

$$\|Up - q\|^2 \leq \|p - q\|^2 + \eta\|p - Up\|^2 \quad \text{for all } p \in K \text{ and } q \in \mathrm{Fix}(U).$$

The class of η-demimetric operators plays a prominent role in metric fixed point theory and has been analyzed in various instances of fixed point problems [47, 48, 50]. We remark that various nonlinear operators have been analyzed in connections with variational inequality problems, fixed point problems, equilibrium problems, convex feasibility problems, signal processing, and image reconstruction [3–6, 11, 12, 14–19, 23, 24, 26, 28–30, 32, 34, 35, 37, 42, 43, 53–57]. In 2007, Aoyama et al. [2] suggested a Halpern [33] type approximants for an infinite family of nonexpansive operators satisfying the *AKTT-Condition* $\sum_{k=1}^{\infty} \sup_{p\in X} \|U_{k+1}p - U_k p\| < \infty$ for any bounded subset X of Ξ. The following construction of operator S_k for a countably infinite family of η-demimetric operators does not require the *AKTT-Condition* and hence improves the performance of the approximants:

$$
\begin{aligned}
&Q_{k,k+1} = Id,\\
&Q_{k,k} = \lambda_k U'_k Q_{k,k+1} + (1-\lambda_k)Id,\\
&Q_{k,k-1} = \lambda_{k-1} U'_{k-1} Q_{k,k} + (1-\lambda_{k-1})Id,\\
&\vdots\\
&Q_{k,m} = \lambda_m U'_m Q_{k,m+1} + (1-\lambda_m)Id,\\
&\vdots\\
&Q_{k,2} = \lambda_2 U'_2 Q_{k,3} + (1-\lambda_2)Id,\\
&S_k = Q_{k,1} = \lambda_1 U'_1 Q_{k,2} + (1-\lambda_1)Id,
\end{aligned}
$$

where $0 \le \lambda_m \le 1$ and $U'_m = \rho p + (1-\rho)((1-\gamma)Id + \gamma U_m)p$ for all $p \in K$ with U_m being η-demimetric operator, $\rho \in (0,1)$, and $0 < \gamma < 1-\eta$. It is well known in the context of operator S_k that each U'_m is nonexpansive and the limit $\lim_{k\to\infty} Q_{k,m}$ exists. Moreover,

$$
Sp = \lim_{k\to\infty} S_k p = \lim_{k\to\infty} Q_{k,1}p \quad \text{for all } p \in K.
$$

This implies that $\operatorname{Fix}(S) = \bigcap_{k=1}^{\infty} \operatorname{Fix}(S_k)$ [36, 49].

The following concept of a split convex feasibility problem (SCFP) is presented in [20]:

Let H and W be nonempty closed convex subsets of real Hilbert spaces Ξ_1 and Ξ_2, respectively. In SCFP, we compute

$$
p \in H \quad \text{such that } Vp \in W, \tag{1.1}
$$

where $V : \Xi_1 \to \Xi_2$ is a bounded linear operator. The SCFP is a particular case of the following split common null point problem (SCNPP) of maximal monotone operators:

Let $A_1 \subseteq \Xi_1 \times \Xi_1$ and $A_2 \subseteq \Xi_2 \times \Xi_2$ be two monotone operators such that $\Gamma = A_1^{-1}(0) \cap V^{-1}(A_2^{-1}(0)) \neq \emptyset$. In SCNPP, we compute $p \in \Gamma$. Some interesting results on the SCNPP via iterative approximants can be found in [13, 21, 22, 44]. It is worth mentioning that the concept of SCNPP has been extended to the concept of a generalized split common null point problem (GSCNPP) in Hilbert spaces [39, 40]. In GSCNPP, we compute

$$p \in \Gamma := A_1^{-1}(0) \cap V_1^{-1}\big(A_2^{-1}(0)\big) \cap \cdots \cap V_1^{-1}\big(V_2^{-1} \cdots \big(V_{N-1}^{-1}\big(A_N^{-1}(0)\big)\big)\big), \tag{1.2}$$

where $A_j : \Xi_j \to 2^{\Xi_j}$, $j \in \{1, 2, \ldots, N\}$, is a finite family of maximal monotone operators, and $V_j : \Xi_j \to \Xi_{j+1}$, $j \in \{1, 2, \ldots, N-1\}$, is a finite family of bounded linear operators such that $V_j \neq 0$.

From the perspective of optimization, problem (1.2) has been analyzed via different iterative approximants. A variant of the classical CQ-algorithm, essentially due to Byrne [13], is employed in [39], whereas shrinking projection approximants are analyzed in [40] to obtain the strong convergence results in Hilbert spaces. It is therefore natural to ask whether we can device strongly convergent approximants to compute a solution of GSCNPP and fixed point point problem of an infinite family of operators without employing the *AKTT-Condition*.

To answer the above question, we consider the following GSCNPP and FPP:

$$p \in \Omega = \Gamma \cap \operatorname{Fix}(S) \neq \emptyset. \tag{1.3}$$

For the computation of a solution of problem (1.3), we employ hybrid shrinking approximants embedded with the inertial extrapolation technique, essentially due to Polyak [38] (see also [1, 7–10]), in Hilbert spaces.

The rest of the paper is organized as follows: In Sect. 2, we present mathematical preliminaries. We establish strong convergence results of the approximants and their variant, namely the Halpern-type approximants in Sect. 3. In Sect. 4, we elaborate the adaptability of the approximants for various extensively analyzed theoretical problems in this framework. Section 6 provides a numerical experiment to analyze the viability of the approximants in comparison with the existing results.

2 Preliminaries

We start this section with mathematical preliminary notions. We always assume that K is a nonempty closed convex subset of a real Hilbert space Ξ_1.

Recall the nearest point projector $\Pi_K^{\Xi_1}$ of Ξ_1 onto $K \subset \Xi_1$ is defined so that for every $p \in \Xi_1$, we have a unique $\Pi_K^{\Xi_1} p$ in K such that

$$\big\| p - \Pi_K^{\Xi_1} p \big\| \leq \| p - q \| \quad \text{for all } q \in K.$$

Note here that the nearest point projector has the following properties:

(i) $\| \Pi_K^{\Xi_1} p - \Pi_K^{\Xi_1} q \|^2 \leq \langle p - q, \Pi_K^{\Xi_1} p - \Pi_K^{\Xi_1} q \rangle$ for all $p, q \in K$ (firmly nonexpansive);

(ii) $\langle p - \Pi_K^{\Xi_1} p, \Pi_K^{\Xi_1} p - q \rangle \geq 0$ for all $p \in \Xi_1$ and $q \in K$ (characterization property).

Let the sets $\mathcal{D}(A_1) = \{p \in \Xi_1 \mid A_1 p \neq \emptyset\}$, $\mathcal{R}(A_1) = \{u \in \Xi_1 \mid (\exists p \in \Xi_1) u \in A_1 p\}$, $\mathcal{G}(A_1) = \{(p,u) \in \Xi_1 \times \Xi_1 \mid u \in A_1 p\}$, and $\mathcal{Z}(A_1) = \{p \in \Xi_1 \mid 0 \in A_1 p\}$ denote the domain, range, graph, and zeros of a set-valued operator $A_1 \subseteq \Xi_1 \times \Xi_1$, respectively. If a set-valued operator A_1 satisfies $\langle p - q, t - w\rangle \geq 0$ for all $(p,t),(q,w) \in \operatorname{gra}(A_1)$, then A_1 is called monotone. Recall also that a monotone operator A_1 is called maximal monotone if its graph is not strictly contained in the graph of any other monotone operator on Ξ_1. The well-defined single-valued operator $\mathcal{J}_\theta^{A_1} := (Id + \theta A_1)^{-1} : \mathcal{R}(Id + \theta A_1) \to \mathcal{D}(A_1)$ is known as the resolvent of A_1, where $\theta > 0$. The resolvent operator $\mathcal{J}_\theta^{A_1}$ is closely related to A_1 as $q \in A_1^{-1}(0)$ if and only if $q = \mathcal{J}_\theta^{A_1}(q)$.

Lemma 2.1 ([46]) *Let $K \subset \Xi_1$, and let $U : K \to \Xi_1$ be an η-demimetric operator with $\eta \in (-\infty, 1)$. Then* $\operatorname{Fix}(U)$ *is closed and convex.*

Lemma 2.2 ([50]) *Let $K \subset \Xi_1$, and let $U : K \to \Xi_1$ be an η-demimetric operator with $\eta \in (-\infty, 1)$ and* $\operatorname{Fix}(U) \neq \emptyset$. *Let γ be a real number such that $0 < \gamma < 1 - \eta$ and set $M = (1-\gamma)Id + \gamma U$. Then M is a quasinonexpansive operator of K into Ξ_1.*

Lemma 2.3 ([45]) *Let Ξ_1 be a real Hilbert space, and let (d_k) be a sequence in Ξ_1. Then*:

(i) *If $d_k \rightharpoonup d$ and $\|d_k\| \to \|d\|$ as $k \to \infty$, then $d_k \to d$ as $k \to \infty$ (the Kadec–Klee property)*;

(ii) *If $d_k \rightharpoonup d$ as $k \to \infty$, then $\|d\| \leq \liminf_{k\to\infty} \|d_k\|$.*

Lemma 2.4 ([27]) *Let $A_1 \subseteq \Xi_1 \times \Xi_1$ be a maximal monotone operator. Then for $\theta \geq \tilde{\theta} > 0$, we have*

$$\left\|p - \mathcal{J}_{\tilde{\theta}}^{A_1} p\right\| \leq 2\left\|p - \mathcal{J}_\theta^{A_1} p\right\| \quad \text{for all } p \in \Xi_1.$$

Lemma 2.5 ([31]) *Let $K \subset \Xi_1$. Then the operator $Id - U$ satisfies the demiclosedness principle with respect to the origin, that is, $(Id - U)(d) = 0$, provided that there exists a sequence (d_k) in K that converges weakly to some d and $((Id - U)d_k)$ converges strongly to* 0.

Lemma 2.6 ([52]) *Let $K \subset \Xi_1$, and let (U'_m) be a sequence of nonexpansive operators such that $\bigcap_{k=1}^\infty \operatorname{Fix}(U'_k) \neq \emptyset$ and $0 \leq \beta_m \leq b < 1$. Then for a bounded subset D of K, we have*

$$\lim_{k\to\infty} \sup_{p\in D} \|Sp - S_k p\| = 0.$$

3 Convergence analysis of the approximants

For the computation of a solution of (1.3), we propose the following approximants:

Algorithm 1 Hybrid Shrinking Approximants (Algo.1)

Initialization: Suppose $H_0 = W_0 = \Xi_1$. Assume nonincreasing sequences $\rho_k, \tau_k \subset (0,1)$, $\mu_k \in [0,1)$, $\gamma_k \in (0,\infty)$, and let V_0 be the identity operator on Ξ_1 and $\bar{V}_{j-1} = \{V_{j-1}V_{j-2}\cdots V_0\}$ for all $j \in \{1,2,\dots,N\}$. Choose the inertial parameter

$$\mu_k = \begin{cases} \min\{\frac{\tau_k}{\|d_k - d_{k-1}\|}, \mu\} & \text{if } d_k \neq d_{k-1}; \\ \mu & \text{otherwise,} \end{cases}$$

where $\mu \in [0,1)$, and (τ_k) is a sequence of positives satisfying $\sum_{k=1}^{\infty} \tau_k < \infty$.

Step 0. Choose arbitrarily $d_0, d_1 \in \Xi_1$ and set $k \geq 1$;

Iterative Steps: Given $d_k \in \Xi_k$, calculate a_k, b_k, and c_k as follows:

Step 1. Compute

$$\begin{cases} a_k = (1+\mu_k)d_k - \mu_k d_{k-1}; \\ b_k = \rho_k a_k + (1-\rho_k)S_k a_k; \\ c_k = \tau_k \bar{V}_{j-1} b_k + (1-\tau_k)(\mathcal{J}_{\theta_{j,k}}^{A_j}(\bar{V}_{j-1} b_k)). \end{cases}$$

The approximants abort if $a_k = b_k = c_k = d_k$, and then d_k is the required approximation. Otherwise,

Step 2. Compute

$$\begin{aligned} H_{k+1} &= \{p \in \Xi_1 : \|b_k - p\|^2 \leq \|d_k - p\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 + 2\mu_k \langle d_k - p, d_k - d_{k-1}\rangle\}, \\ W_{k+1} &= \{p \in \Xi_1 : \|c_k - \bar{V}_{j-1}p\| \leq \|\bar{V}_{j-1} b_k - \bar{V}_{j-1} p\|\}, \\ d_{k+1} &= \Pi_{H_{k+1}\cap W_{k+1}}^{\Xi_1} d_1 \quad \forall k \geq 1. \end{aligned}$$

Fix $k =: k+1$ and reiterate **Step 1.**

We assume the following control conditions on the approximants:

(C1) $\sum_{k=1}^{\infty} \mu_k \|d_{k-1} - d_k\| < \infty$;

(C2) $0 < a \leq \rho_k \leq b < 1$;

(C3) $\liminf_{k\to\infty} \tau_k > 0$;

(C4) $\min_j\{\inf_k\{\theta_{j,k}\}\} \geq m > 0$.

Theorem 3.1 *Any approximants defined via Algorithm* 1, *under control conditions* (C1)–(C4), *converge strongly to an element in* Ω.

Proof We divide the proof into different steps for understanding.

Step 1. We show that the approximants (d_k) defined in Algorithm 1 are stable.

Claim: H_k and W_k are closed and convex subsets of Ξ_1 for all $k \geq 0$.

Consider, for each $k \geq 0$, the following representation of the subsets H_k and W_k:

$$H_k = \Big\{ p \in \Xi_1 : \\ \langle b_k - d_k, p\rangle \leq \frac{1}{2}\big(\|b_k\|^2 - \|d_k\|^2 + \mu_k^2\|d_k - d_{k-1}\|^2 + 2\mu_k\langle d_k - p, d_k - d_{k-1}\rangle\big)\Big\},$$

$$
\begin{aligned}
W_k &= \left\{ p \in \Xi_1 : \langle \bar{V}_{j-1} b_k - c_k, \bar{V}_{j-1} p \rangle \le \frac{1}{2} \bigl(\| \bar{V}_{j-1} b_k \|^2 - \| c_k \|^2 \bigr) \right\}, \\
&= \left\{ p \in \Xi_1 : \bigl\langle \bar{V}_{j-1}^* (\bar{V}_{j-1} b_k - c_k), p \bigr\rangle \le \frac{1}{2} \bigl(\| \bar{V}_{j-1} b_k \|^2 - \| c_k \|^2 \bigr) \right\}.
\end{aligned}
$$

The claim follows from the above representations of closed and convex subsets H_k and W_k of Ξ_1 for all $k \ge 1$. Further, the sets Γ and $\mathrm{Fix}(S)$ (from Lemma 2.1) are closed and convex. Hence we have that Ω is nonempty, closed, and convex. Let $q \in \Omega$, $\Omega \subset H_0 = \Xi_1$. Now it follows from Algorithm 1 that

$$
\begin{aligned}
\|a_k - q\|^2 &= \bigl\| (1 + \mu_k) d_k - \mu_k (d_{k-1} - q) \bigr\|^2 \\
&\le \|d_k - q\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 + 2\mu_k \langle d_k - q, d_k - d_{k-1} \rangle.
\end{aligned} \tag{3.1}
$$

From (3.1) and Lemma 2.2 we obtain

$$
\begin{aligned}
\|b_k - q\|^2 &= \bigl\| \rho_k a_k + (1 - \rho_k) S_k a_k - q \bigr\|^2 \\
&\le \rho_k \|a_k - q\|^2 + (1 - \rho_k) \|S_k a_k - q\|^2 - \rho_k (1 - \rho_k) \bigl\| (Id - S_k) a_k \bigr\|^2 \\
&\le \rho_k \|a_k - q\|^2 + (1 - \rho_k) \|a_k - q\|^2 - \rho_k (1 - \rho_k) \bigl\| (Id - S_k) a_k \bigr\|^2 \\
&\le \|a_k - q\|^2 - \rho_k (1 - \rho_k) \bigl\| (Id - S_k) a_k \bigr\|^2 \\
&\le \|d_k - q\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 + 2\mu_k \langle d_k - q, d_k - d_{k-1} \rangle.
\end{aligned} \tag{3.2}
$$

This shows that Ω is contained in H_k for all $k \ge 1$. Now assume that $\Omega \subset W_k$ for some $k \ge 1$. Using the nonexpansiveness of $\mathcal{J}_{\theta_{j,k}}^{A_j}$, (3.1), and (3.2), we get

$$
\begin{aligned}
\|c_k - \bar{V}_{j-1} q\|^2 &= \bigl\| \tau_k (\bar{V}_{j-1} b_k - \bar{V}_{j-1} q) + (1 - \tau_k) \bigl(\mathcal{J}_{\theta_{j,k}}^{A_j} (\bar{V}_{j-1} b_k) - \bar{V}_{j-1} q \bigr) \bigr\|^2 \\
&\le \tau_k \| \bar{V}_{j-1} b_k - \bar{V}_{j-1} q \|^2 + (1 - \tau_k) \bigl\| \mathcal{J}_{\theta_{j,k}}^{A_j} (\bar{V}_{j-1} b_k) - \mathcal{J}_{\theta_{j,k}}^{A_j} (\bar{V}_{j-1} q) \bigr\|^2 \\
&\le \tau_k \| \bar{V}_{j-1} b_k - \bar{V}_{j-1} q \|^2 + (1 - \tau_k) \| \bar{V}_{j-1} b_k - \bar{V}_{j-1} q \|^2 \\
&\le \| \bar{V}_{j-1} b_k - \bar{V}_{j-1} q \|^2.
\end{aligned} \tag{3.3}
$$

It follows from estimate (3.3) that $\Omega \subset W_{k+1}$, and hence $\Omega \subset H_{k+1} \cap W_{k+1}$. Consequently, by Step 1 the approximants (d_k) defined in Algorithm 1 are stable.

Step 2. We next show that $\lim_{k \to \infty} \|d_k - d_1\|$ exists.

Observe that

$$
\|d_{k+1} - d_1\| \le \|q - d_1\| \quad \text{for all } q \in \Omega \subset H_{k+1},
$$

since $d_{k+1} = \Pi_{H_{k+1} \cap W_{k+1}}^{\Xi_1} d_1$. In particular,

$$
\|d_{k+1} - d_1\| \le \bigl\| \Pi_{\Omega}^{\Xi_1} d_1 - d_1 \bigr\|.
$$

These estimates establish the boundedness of the approximants $(\|d_k - d_1\|)$. Since $d_k = \Pi_{H_k \cap W_k}^{\Xi_1} d_1$ and $d_{k+1} = \Pi_{H_{k+1} \cap W_{k+1}}^{\Xi_1} d_1 \in H_{k+1}$, we have

$$
\|d_k - d_1\| \le \|d_{k+1} - d_1\|.
$$

This yields that the approximants ($\|d_k - d_1\|$) are nondecreasing, and hence

$$\lim_{k\to\infty} \|d_k - d_1\| \text{ exists.} \tag{3.4}$$

Step 3. We now show that $\tilde{q} \in \Omega$.

We first compute

$$\begin{aligned}
\|d_{k+1} - d_k\|^2 &= \|d_{k+1} - d_1\|^2 + \|d_k - d_1\|^2 - 2\langle d_k - d_1, d_{k+1} - d_1\rangle \\
&= \|d_{k+1} - d_1\|^2 + \|d_k - d_1\|^2 - 2\langle d_k - d_1, d_{k+1} - d_k + d_k - d_1\rangle \\
&= \|d_{k+1} - d_1\|^2 - \|d_k - d_1\|^2 - 2\langle d_k - d_1, d_{k+1} - d_k\rangle \\
&\le \|d_{k+1} - d_1\|^2 - \|d_k - d_1\|^2.
\end{aligned}$$

By (3.3) the above computation yields

$$\lim_{k\to\infty} \|d_{k+1} - d_k\| = 0. \tag{3.5}$$

In view of the control condition (C1), we get

$$\lim_{k\to\infty} \|a_k - d_k\| = \lim_{k\to\infty} \mu_k \|d_k - d_{k-1}\| = 0. \tag{3.6}$$

As a consequence of estimates (3.5) and (3.6), we also obtain that

$$\lim_{k\to\infty} \|a_k - d_{k+1}\| = 0. \tag{3.7}$$

Since $d_{k+1} \in H_{k+1}$, we have

$$\|b_k - d_{k+1}\| \le \|d_k - d_{k+1}\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 + 2\mu_k \langle d_k - d_{k+1}, d_k - d_{k-1}\rangle.$$

This estimate, in the light of estimate (3.5) and the control condition (C1), yields that

$$\lim_{k\to\infty} \|b_k - d_{k+1}\| = 0. \tag{3.8}$$

Similarly, we infer from estimates (3.5) and (3.8) that

$$\lim_{k\to\infty} \|b_k - d_k\| = 0, \tag{3.9}$$

and from the estimates (3.6) and (3.9) that

$$\lim_{k\to\infty} \|b_k - a_k\| = 0. \tag{3.10}$$

In view of the control condition (C2), consider the variant of estimate (3.2)

$$\begin{aligned}
a(1-b)\big\|(Id - S_k)a_k\big\|^2 &\le \|d_k - q\|^2 - \|b_k - q\|^2 \\
&\le \big(\|d_k - q\| + \|b_k - q\|\big)\|d_k - b_k\|.
\end{aligned}$$

Letting $k \to \infty$ and using (3.9) and (C2), we have

$$\lim_{k\to\infty} \|a_k - S_k a_k\| = 0. \tag{3.11}$$

Observe that

$$\begin{aligned}\|b_k - S_k b_k\| &\le \|b_k - a_k\| + \|a_k - S_k a_k\| + \|S_k a_k - S_k b_k\| \\ &\le 2\|b_k - a_k\| + \|a_k - S_k a_k\|.\end{aligned}$$

The above computation, in view of estimates (3.10) and (3.11), yields

$$\lim_{k\to\infty} \|b_k - S_k b_k\| = 0. \tag{3.12}$$

Note that $d_{k+1} = \Pi^{\Xi_1}_{H_k \cap W_k}(d_1) \in W_k$. Therefore we have

$$\|c_k - \bar{V}_{j-1} d_{k+1}\| \le \|\bar{V}_{j-1} b_k - \bar{V}_{j-1} d_{k+1}\| \le \|\bar{V}_{j-1}\| \|b_k - d_{k+1}\|.$$

Employing estimate (3.8), the above computation yields

$$\lim_{k\to\infty} \|c_k - \bar{V}_{j-1} d_{k+1}\| = 0. \tag{3.13}$$

Reasoning as above, we infer from estimates (3.8) and (3.13) that

$$\lim_{k\to\infty} \|c_k - \bar{V}_{j-1} b_k\| = 0 \tag{3.14}$$

and from estimates (3.9) and (3.14) that

$$\lim_{k\to\infty} \|c_k - \bar{V}_{j-1} d_k\| = 0. \tag{3.15}$$

Since (d_k) is bounded, there exists a subsequence (d_{k_t}) of (d_k) such that $d_{k_t} \rightharpoonup \tilde{q} \in \Xi_1$ as $t \to \infty$. Therefore $b_{k_t} \rightharpoonup \tilde{q}$ and $c_{k_t} \rightharpoonup \bar{V}_{j-1}\tilde{q}$ as $t \to \infty$. From the definition of $\bar{V}_{j-1}$ we have $\bar{V}_{j-1} d_{k_t} \rightharpoonup \bar{V}_{j-1}\tilde{q}$ as $t \to \infty$ for all $j \in \{1, 2, \dots, N\}$.

Using (3.14), we estimate that

$$\lim_{t\to\infty} \left\| \bar{V}_{j-1} c_{k_t} - \mathcal{J}^{A_j}_{\theta_{j,k}}(\bar{V}_{j-1} c_{k_t}) \right\| = 0$$

for all $j \in \{1, 2, \dots, N\}$. Then from Lemma 2.4 and (C4) we obtain the inequality

$$\left\| \bar{V}_{j-1} c_{k_t} - \mathcal{J}^{A_j}_{\theta}(\bar{V}_{j-1} c_{k_t}) \right\| \le 2 \left\| \bar{V}_{j-1} c_{k_t} - \mathcal{J}^{A_j}_{\theta_{j,k}}(\bar{V}_{j-1} c_{k_t}) \right\|.$$

This estimate implies that

$$\lim_{t\to\infty} \left\| \bar{V}_{j-1} c_{k_t} - \mathcal{J}^{A_j}_{\theta}(\bar{V}_{j-1} c_{k_t}) \right\| = 0 \tag{3.16}$$

for all $j \in \{1,2,\dots,N\}$. By Lemma 2.5 we have $\bar{V}_{j-1}\tilde{q} \in \mathrm{Fix}(\mathcal{J}_{\theta}^{A_j})$ for all $j \in \{1,2,\dots,N\}$, that is, $\tilde{q} \in \Gamma$. It remains to show that $\tilde{q} \in \mathrm{Fix}(S)$. Observe that

$$\begin{aligned}\|b_k - Sb_k\| &\le \|b_k - S_k b_k\| + \|S_k b_k - Sb_k\| \\ &\le \|b_k - S_k b_k\| + \sup_{p\in D}\|S_k p - Sp\|.\end{aligned}$$

Using (3.12) and Lemma 2.6, this estimate implies that $\lim_{k\to\infty}\|b_k - Sb_k\| = 0$. This, together with the fact that $b_{k_t} \rightharpoonup \tilde{q}$, implies by Lemma 2.5 that $\tilde{q} \in \mathrm{Fix}(S) = \bigcap_{k=1}^{\infty}\mathrm{Fix}(S_k)$. Hence $\tilde{q} \in \Omega$.

Step 4. The final part is showing that $d_k \to q = \Pi_{\Omega}^{\Xi_1} d_1$.

Since $q = \Pi_{\Omega}^{\Xi_1} d_1$ and $\tilde{q} \in \Omega$, Lemma 2.3 implies that

$$\begin{aligned}\|q - d_1\| &\le \|\tilde{q} - d_1\| \le \liminf_{t\to\infty}\|d_{k_t} - d_1\| \\ &\le \limsup_{t\to\infty}\|d_{k_t} - d_1\| \le \|q - d_1\|.\end{aligned}$$

Using the uniqueness of q yields the equality $\tilde{q} = q$. From Step 2 it follows that $\|d_{k_t} - d_1\| \le \|q - d_1\|$, and from Lemma 2.3 we obtain $\lim_{k\to\infty} d_k = \tilde{q} = q = \Pi_{\Omega}^{\Xi_1} d_1$. □

We first apply Theorem 3.1 to the following problem:

$$\Omega := \bigcap_{j=1}^{s} A_j^{-1}(0) \cap V^{-1}\left(\bigcap_{\ell=s+1}^{N} A_{\ell}^{-1}(0)\right) \cap \mathrm{Fix}(S),$$

where $A_j \subseteq \Xi_1 \times \Xi_1$, $j \in \{1,2,\dots,s\}$, and $A_\ell \subseteq \Xi_1 \times \Xi_1$, $\ell = \{s+1, s+2,\dots,N\}$, are finite families of maximal monotone operators, and $S_k : \Xi_1 \to \Xi_1$ is an infinite family of η-demimetric operators.

Corollary 3.2 *Assume that $\Omega \neq \emptyset$. Then the approximants initialized by arbitrary $d_0, d_1 \in \Xi_1$ and $H_0 = W_0 = \Xi_1$ with the nonincreasing sequences $\rho_k, \tau_k \subset (0,1)$, $\mu_k \in [0,1)$, and $\gamma_k \in (0,\infty)$ for $k \ge 1$ defined as*

$$\begin{cases} a_k = (1+\mu_k)d_k - \mu_k d_{k-1}, \\ b_k = \rho_k a_k + (1-\rho_k)S_k a_k, \\ c_k = \tau_k \bar{V}_{j-1}b_k + (1-\tau_k)(\mathcal{J}_{\theta_{j,k}}^{A_j}(\bar{V}_{j-1}b_k)); \\ H_{k+1} = \{p \in \Xi_1 : \\ \qquad \|b_k - p\|^2 \le \|d_k - p\|^2 + \mu_k^2\|d_k - d_{k-1}\|^2 + 2\mu_k\langle d_k - p, d_k - d_{k-1}\rangle\}, \\ W_{k+1} = \{p \in \Xi_1 : \|c_k - \bar{V}_{j-1}p\| \le \|\bar{V}_{j-1}b_k - \bar{V}_{j-1}p\|\}, \\ d_{k+1} = \Pi_{H_{k+1}\cap W_{k+1}}^{\Xi_1} d_1 \quad \forall k \ge 1, \end{cases} \tag{3.17}$$

under the control conditions (C1)–(C4), *converge strongly to an element in* Ω.

We now consider the following Halpern-type variant of Algorithm 1:

Algorithm 2 Hybrid Shrinking Halpern Approximants (Algo.2)

Initialization: Choose arbitrarily $t, d_0, d_1 \in \Xi_1$, and $H_0 = W_0 = \Xi_1$, set $k \geq 1$ and non-increasing sequences $\rho_k, \tau_k \subset (0,1)$, $\mu_k \in [0,1)$, and $\gamma_k \in (0,\infty)$, let V_0 be the identity operator on Ξ_1, and let $\bar{V}_{j-1} = \{V_{j-1}V_{j-2}\cdots V_0\}$ for all $j \in \{1,2,\dots,N\}$. Choose the inertial parameter

$$\mu_k = \begin{cases} \min\{\frac{\tau_k}{\|d_k - d_{k-1}\|}, \mu\} & \text{if } d_k \neq d_{k-1}, \\ \mu & \text{otherwise,} \end{cases}$$

where $\mu \in [0,1)$, and (τ_k) is a sequence of positives satisfying $\sum_{k=1}^{\infty} \tau_k < \infty$.

Iterative Steps: Given $d_k \in \Xi_k$, calculate b_k and c_k as follows:

Step 1. Compute

$$\begin{cases} a_k = (1+\mu_k)d_k - \mu_k d_{k-1}, \\ b_k = \rho_k t + (1-\rho_k)S_k a_k, \\ c_k = \tau_k \bar{V}_{j-1} b_k + (1-\tau_k)(\mathcal{J}^{A_j}_{\theta_{j,k}}(\bar{V}_{j-1} b_k)). \end{cases}$$

The approximants abort if $k > k_{\max}$ for some chosen sufficiently large number $k_{\max}$, and then d_k is the required approximation. Otherwise,

Step 2. Compute

$$\begin{aligned} H_{k+1} &= \bigl\{p \in \Xi_1 : \|b_k - p\|^2 \leq \rho_k \|t-p\|^2 + (1-\rho_k)\bigl(\|d_k - p\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 \\ &\qquad + 2\mu_k \langle d_k - p, d_k - d_{k-1}\rangle\bigr)\bigr\}, \\ W_{k+1} &= \bigl\{p \in \Xi_1 : \|c_k - \bar{V}_{j-1}p\| \leq \|\bar{V}_{j-1}b_k - \bar{V}_{j-1}p\|\bigr\}, \\ d_{k+1} &= \Pi^{\Xi_1}_{H_{k+1}\cap W_{k+1}} t \quad \forall k \geq 1. \end{aligned}$$

Fix $k =: k+1$ and reiterate **Step 1.**

Theorem 3.3 *Any approximants defined via Algorithm 2, under the control conditions* (C1)–(C4), *converge strongly to an element in* Ω.

Proof Observe that for each $k \geq 1$, the subsets H_k have the following form:

$$\begin{aligned} H_k &= \bigl\{p \in \Xi_1 : \|b_k - p\|^2 \leq \rho_k \|t-p\|^2 + (1-\rho_k)\bigl(\|d_k - p\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 \\ &\qquad + 2\mu_k \langle d_k - z, d_k - d_{k-1}\rangle\bigr)\bigr\}. \end{aligned}$$

Arguing similarly as in the proof of Theorem 3.1 (Steps 1–2), we deduce that Ω, H_k, and W_k are closed and convex. Moreover, $\Omega \subset H_{k+1} \cap W_{k+1}$ for all $k \geq 1$. Furthermore, the sequence (d_k) is bounded, and

$$\lim_{k\to\infty} \|d_{k+1} - d_k\| = 0. \tag{3.18}$$

Since $d_{k+1} = \Pi^{\Xi_1}_{H_k \cap W_k}(t) \in H_k$, we have

$$\begin{aligned}&\|b_k - d_{k+1}\|^2\\&\quad\leq \rho_k\|t - d_{k+1}\|^2 + (1-\rho_k)\big(\|d_k - d_{k+1}\|^2 + \mu_k^2\|d_k - d_{k-1}\|^2 + 2\mu_k\langle d_k - z, d_k - d_{k-1}\rangle\big).\end{aligned}$$

Letting $k \to \infty$, using (3.18) along (C1)–(C2), and the boundedness of (d_k), we obtain

$$\lim_{k\to\infty}\|b_k - d_{k+1}\| = 0.$$

Similarly, we get

$$\lim_{k\to\infty}\|b_k - d_k\| = 0.$$

Let $b_k = \rho_k t + (1-\rho_k)S_k a_k$. An easy calculation along (C1)–C2) implies that

$$\|S_k a_k - a_k\| \leq \frac{1}{(1-\rho_k)}\|b_k - a_k\| + \frac{\rho_k}{(1-\rho_k)}\|t - a_k\|.$$

This estimate implies that

$$\lim_{k\to\infty}\|S_k a_k - a_k\| = 0.$$

The rest of the proof of Theorem 3.3 follows immediately from the proof of Theorem 3.1 and is therefore omitted. □

4 Applications

Our main result in the previous section has various interesting applications of great importance in the field. We present some of these applications.

4.1 Generalized split feasibility problems

In the context of generalized split feasibility problems [20], we recall that the indicator function j_K is a proper lower semicontinuous convex function (PCLS), where $K \subset \Xi_1$. Therefore ∂j_K, the subdifferential of j_K, satisfies the maximal monotonicity such that $\partial j_K(p) = N_p^K$, where N_p^K denotes the normal cone of K at u. From this we can deduce that ∂j_K coincides with $\Pi_K^{\Xi_1}$. Assume that

$$\Theta := K_1 \cap V_1^{-1}(K_2) \cap \cdots \cap V_1^{-1}\big(V_2^{-1}\cdots\big(V_{N-1}^{-1}(K_N)\big)\big) \neq \emptyset,$$

where $K_j \subset \Xi_j$, $j \in \{1, 2, \ldots, N\}$.

Theorem 4.1 *Assume that $\Omega = \Theta \cap \mathrm{Fix}(S) \neq \emptyset$. Then the approximants initialized by arbitrary $d_0, d_1 \in \Xi_1$ and $H_0 = W_0 = \Xi_1$ with the nonincreasing sequences $\rho_k, \tau_k \subset (0,1)$,*

$\mu_k \in [0,1)$, *and* $\gamma_k \in (0,\infty)$ *for* $k \geq 1$ *defined as*

$$\begin{cases} a_k = (1+\mu_k)d_k - \mu_k d_{k-1}, \\ b_k = \rho_k a_k + (1-\rho_k)S_k a_k, \\ c_k = \tau_k \bar{V}_{j-1} b_k + (1-\tau_k)(\Pi_{K_j}^{\Xi_j}(\bar{V}_{j-1} b_k)), \\ H_{k+1} = \{z \in \Xi_1 : \\ \qquad \|b_k - z\|^2 \leq \|d_k - z\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 + 2\mu_k \langle d_k - z, d_k - d_{k-1}\rangle\}, \\ W_{k+1} = \{z \in \Xi_1 : \|c_k - \bar{V}_{j-1} z\| \leq \|\bar{V}_{j-1} b_k - \bar{V}_{j-1} z\|\}, \\ d_{k+1} = \Pi_{H_{k+1}\cap W_{k+1}}^{\Xi_1} d_1 \quad \forall k \geq 1, \end{cases} \tag{4.1}$$

under the control conditions (C1)–(C4), *converge strongly to an element in* Ω.

4.2 Generalized split variational inequality problems

The well-known variational inequality problem deals with computation of a point $p \in K$ such that

$$\langle \mathcal{A}p, q - p\rangle \geq 0 \quad \forall q \in K,$$

where $\mathcal{A} : K \to \Xi_1$ is a nonlinear monotone operator defined with respect to $K \subset \Xi_1$. By $\mathrm{Sol}(K,\mathcal{A})$ we denote the set of all solutions associated with the variational inequality problem. We consider the following problem:

$$\Theta := \mathrm{Sol}(K_1,\mathcal{A}_1) \cap V_1^{-1}\big(\mathrm{Sol}(K_2,\mathcal{A}_2)\big) \cap \cdots \cap V_1^{-1}\big(V_2^{-1} \cdots \big(V_{N-1}^{-1}\big(\mathrm{Sol}(K_N,\mathcal{A}_N)\big)\big)\big) \neq \emptyset.$$

Theorem 4.2 *Assume that* $\Omega = \Theta \cap \mathrm{Fix}(S) \neq \emptyset$. *Then the approximants initialized by arbitrary* $d_0, d_1 \in \Xi_1$ *and* $H_0 = W_0 = \Xi_1$ *with the nonincreasing sequences* $\rho_k, \tau_k \subset (0,1)$, $\mu_k \in [0,1)$, *and* $\gamma_k \in (0,\infty)$ *for* $k \geq 1$ *defined as*

$$\begin{cases} a_k = (1+\mu_k)d_k - \mu_k d_{k-1}, \\ b_k = \rho_k a_k + (1-\rho_k)S_k a_k; \\ c_k = \tau_k \bar{V}_{j-1} b_k + (1-\tau_k)\Pi_{K_j}(Id - \theta_{j,k}\mathcal{A}_j)\bar{V}_{j-1} b_k, \\ H_{k+1} = \{z \in \Xi_1 : \\ \qquad \|b_k - z\|^2 \leq \|d_k - z\|^2 + \mu_k^2 \|d_k - d_{k-1}\|^2 + 2\mu_k \langle d_k - z, d_k - d_{k-1}\rangle\}, \\ W_{k+1} = \{z \in \Xi_1 : \|c_k - \bar{V}_{j-1} z\| \leq \|\bar{V}_{j-1} b_k - \bar{V}_{j-1} z\|\}, \\ d_{k+1} = \Pi_{H_{k+1}\cap W_{k+1}}^{\Xi_1} d_1 \quad \forall k \geq 1, \end{cases} \tag{4.2}$$

under the control conditions (C1)–(C4), *converge strongly to an element in* Ω.

Proof Let $h_{\mathcal{A}_j} \subset \Xi_j \times \Xi_j$ be defined by

$$h_{\mathcal{A}_j} p = \begin{cases} \mathcal{A}_j p + N_{K_j}(p) & \text{if } p \in K_j, \\ \emptyset & \text{if } p \notin K_j, \end{cases}$$

where $N_{K_j}(p) := \{q \in \Xi_j : \langle t - p, q\rangle \leq 0 \text{ for all } t \in K_j\}$, $j = \{1,2,\ldots,N\}$.

Note that $h_{\mathcal{A}_j}$ is maximal monotone [41] such that

$$0 \in h_{\mathcal{A}_j}(p) \iff p \in \mathrm{Sol}(K_j, \mathcal{A}_j) \iff p = \Pi_{K_j}\big(p - \theta_{j,k}\mathcal{A}_j(p)\big).$$

The rest of the proof now follows from Theorem 3.1. □

4.3 Generalized split minimization problems

Let the set of minimizers associated with the function $\phi : \Xi_1 \to (-\infty, \infty]$ be denoted as

$$\arg\min(\phi) := \big\{p \in \Xi_1 : \phi(p) \le \phi(q) \text{ for all } q \in \Xi_1\big\}.$$

If ϕ is a proper convex lower semicontinuous (PCLS) function, then $\partial\phi$ is a maximal monotone operator. Moreover, $q \in (\partial\phi)^{-1}0 \Leftrightarrow 0 \in \partial\phi(q)$ (see [25]). Now observe that

$$\begin{aligned}\Theta := {} & \arg\min_{x\in\Xi_1} \phi_1(x) \cap V_1^{-1}\big(\arg\min_{x\in\Xi_2} \phi_2(x)\big) \cap \cdots \\ & \cap V_1^{-1}\big(h_2^{-1} \cdots \big(V_{N-1}^{-1}\big(\arg\min_{x\in\Xi_N} \phi_N(x)\big)\big)\big) \neq \emptyset,\end{aligned}$$

where $\phi_j : \Xi_j \to (-\infty, \infty]$ is as defined above.

Theorem 4.3 *Assume that* $\Omega = \Theta \cap \mathrm{Fix}(S) \neq \emptyset$. *Then the approximants initialized by arbitrary* $d_0, d_1 \in \Xi_1$ *and* $H_0 = W_0 = \Xi_1$ *with the nonincreasing sequences* $\rho_k, \tau_k \subset (0,1)$, $\mu_k \in [0,1)$, *and* $\gamma_k \in (0,\infty)$ *for* $k \ge 1$ *defined as*

$$\begin{cases} a_k = (1+\mu_k)d_k - \mu_k d_{k-1}, \\ b_k = \rho_k a_k + (1-\rho_k)S_k a_k, \\ c_k = \tau_k \bar{V}_{j-1} b_k + (1-\tau_k)\mathcal{J}_{\theta_{j,k}}^{\partial\phi_j} \bar{V}_{j-1} b_k, \\ H_{k+1} = \{p \in \Xi_1 : \\ \qquad \|b_k - p\|^2 \le \|d_k - p\|^2 + \mu_k^2\|d_k - d_{k-1}\|^2 + 2\mu_k\langle d_k - p, d_k - d_{k-1}\rangle\}, \\ W_{k+1} = \{p \in \Xi_1 : \|c_k - \bar{V}_{j-1}p\| \le \|\bar{V}_{j-1}b_k - \bar{V}_{j-1}p\|\}, \\ d_{k+1} = \Pi_{H_{k+1}\cap W_{k+1}}^{\Xi_1} d_1 \quad \forall k \ge 1, \end{cases} \tag{4.3}$$

under the control conditions (C1)–(C4), *converge strongly to an element in* Ω.

4.4 Signal processing

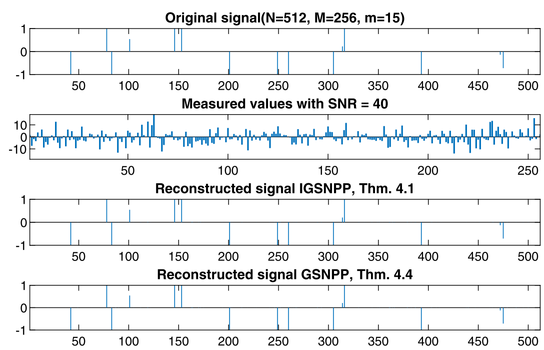
This subsection deals with the case of signal recovery problem, which we aim to solve by applying Theorem 4.1. The following underdetermined formalism denotes the signal recovery problem:

$$Vd = \kappa - \vartheta, \tag{4.4}$$

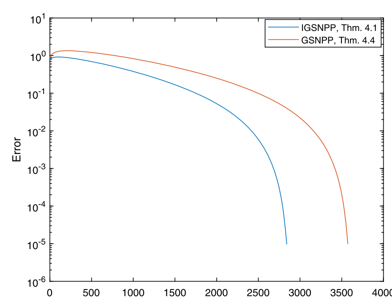
where $\kappa \in \mathbb{R}^M$ is the measured noise data with noise ϑ, $d \in \mathbb{R}^N$ is the sparse original data for recovery, and $V : \mathbb{R}^N \to \mathbb{R}^M$ $(M < N)$ is the bounded linear observation matrix. Formalism (4.4) is equivalent to the well-known least absolute shrinkage and selection operator

Table 1 Comparison of Theorems 4.1 and 4.4 (Reich et al. [40])

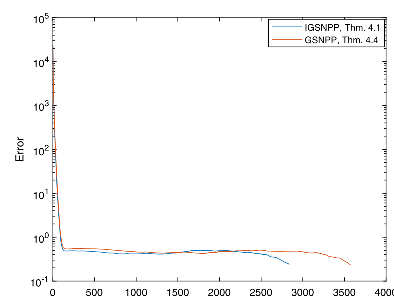
	No. of Iterations		CPU Time	
	Test I	Test II	Test I	Test II
IGSNPP, Theorem 4.1	2841	6069	0.4969	6.4199
GSNPP, Theorem 4.4 [40]	3573	7903	0.7023	9.8853



(A) Numerical Test 1



(B) Comparison of Error and No. of Iterations



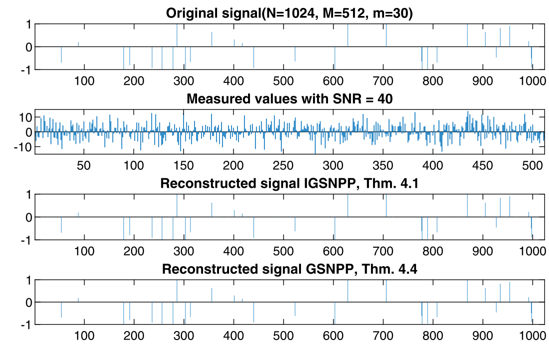
(C) Comparison of objective function values and No. of Iterations

Figure 1 Comparison of two algorithms for Numerical Test 1

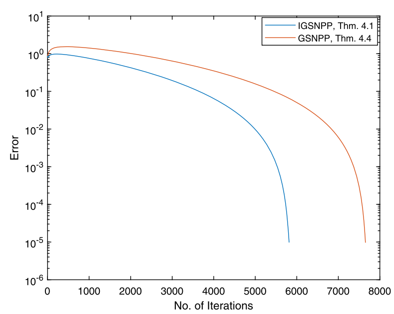
(LASSO) problem [51] in the following convex constrained optimization formalism:

$$\min_{d\in\mathbb{R}^N}\left\{\frac{1}{2}\|Vd-\kappa\|^2\right\} \quad \text{subject to } t \geq \|d\|_1. \tag{4.5}$$

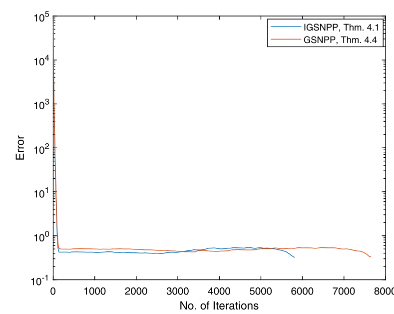
If we set $\Theta = K_1 \cap V^{-1}(K_2) \neq \emptyset$ with $K_1 = \{d \mid t \geq \|d\|_1\}$ and $K_2 = \{\kappa\}$, then the LASSO problem can be easily solved via Theorem 4.1. To conduct the numerical experiment, we generate (i) the matrix $V^{N\times M}$ from the standard normal distributions with zero mean and unit variance, (ii) d having $m \ll N$ nonzero elements via a uniform distribution in $[-2, 2]$, and (iii) κ from a Gaussian noise with signal-to-noise ratio SNR = 40. The approximants are initiated with randomly chosen d_0, d_1 and abort when the following mean square error



(A) Numerical Test 2



(B) Comparison of Error and No. of Iterations



(C) Comparison of objective function values and No. of Iterations

Figure 2 Comparison of two algorithms for Numerical Test 2

is satisfied:

$$E_k = \frac{1}{N}\left\| d_k - d^* \right\| < 10^{-4}.$$

Here d^* is called the estimated signal of d.

For Theorem 4.1, we choose $\mu_k = \frac{1}{(100\times k+1)^{1.04}}$, $\rho_k = \frac{1}{k^{1.02}}$, $t = m - 0.001$, and $\vartheta = 0$.

We recover the signals for the following two tests:

Numerical Test 1 Choose $N = 512$, $M = 256$, and $m = 15$.

Numerical Test 2 Choose $N = 1024$, $M = 512$, and $m = 30$.

From Table 1 and Figs. 1 and 2 we conclude that IGSNPP as in Theorem 4.1 reconstruct the original signal (A) faster than the algorithm for GSNPP as in Theorem 4.4 [40] in the compressed sensing. Moreover, the graph of error function values (B) and objective function values (C) generated by IGSNPP as in Theorem 4.1 converge faster as compared to the algorithm for GSNPP as in Theorem 4.4 [40].

5 Numerical experiment and results

In this section, we focus on numerical implementation of our proposed algorithm. Comparison with Reich et al. [40]) shows the effectiveness and efficiency of our proposed algorithm. All codes were written in MATLAB R2020a and performed on a laptop Intel(R) Core(TM) i3-3217U @ 1.80 GHz, RAM 4.00 GB.

Example 5.1 Let $\Xi_1 = \mathbb{R}^2$ and $\Xi_2 = \mathbb{R}^4$ with the inner product defined by $\langle x, y\rangle = xy$, for all $x, y \in \mathbb{R}^2, \mathbb{R}^4$ and the induced usual norm $|\cdot|$.

Consider the following problem: find an element $q \in \mathbb{R}^2$ such that

$$q \in \Omega = \Theta_1 \cap V^{-1}(\Theta_2) \cap \operatorname{Fix}(S) \neq \emptyset,$$

where

$$\Theta_1 = \{x \in \mathbb{R}^2 \mid \|x - a_1\|^2 \leq R_1^2\}, \qquad \Theta_2 = \{x \in \mathbb{R}^4 \mid \|x - a_2\|^2 \leq R_2^2\},$$

and $V : \mathbb{R}^2 \to \mathbb{R}^4$ is a bounded linear matrix randomly generated in the closed interval $[-5, 5]$. Let the operators $S_k : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S_k(x) = (-(k+1)x_1, -(k+1)x_2)$ for $k = 1, 2$. Then S_k is a η-demimetric operator with $\eta_1 = \frac{1}{3}$ and $\eta_2 = \frac{1}{2}$, respectively. It is easy to observe that $\bigcap_{k=1}^2 \operatorname{Fix}(S_k) = \{0\}$ and $\Theta := \{\Theta_1 \cap V^{-1}(\Theta_2)\} = 0$. Hence $\Omega = \Theta \cap \operatorname{Fix}(S) = 0$. Furthermore, the coordinate of the center a is randomly generated in the closed interval $[-1, 1]$, and the radii R_1 and R_2 are randomly generated in the closed intervals $[5, 9]$ and $[9, 17]$, respectively. The coordinates of the initial point d_0, d_1 are randomly generated in the closed interval $[-5, 5]$. Choose $\mu = 0.9$, $m = 0.01$, $\rho_k = \frac{1}{100k+1}$, and $\beta_1 = \frac{1}{100k+1}$. We provide a numerical test of the hybrid shrinking approximants defined in Theorem 4.1 (i.e., Theorem 4.1 with $\mu_k \neq 0$) with the noninertial variant of Theorem 4.4 (i.e., Theorem 4.4, Reich et al. [40]). It is remarked that the function E_k is defined by

$$E_k = \frac{1}{2}\left[\left\|d_k - \Pi_{\Theta_1}^{\mathbb{R}^2}(d_k)\right\|^2 + \left\|Vd_k - \Pi_{\Theta_2}^{\mathbb{R}^4}(Vd_k)\right\|^2\right] \quad \text{for } k \geq 1.$$

Note that at the kth step, $E_k = 0$, and then $d_k \in \Theta$, which implies that d_k is a solution of this problem. The stopping criterion is defined as $E_k < 10^{-5}$. The different choices of d_0, d_1 are given as follows:

Case I: $d_0 = [6, 8]^T$, $d_1 = [3, 7]^T$.

Case II: $d_0 = [6.5, 7.2]^T$, $d_1 = [-1.4, -9.7]^T$.

Case III: $d_0 = [3, -4.7]^T$, $d_1 = [1.2, 4]^T$.

Table 2 Comparison of Theorems 4.1 and 4.4 (Reich et al. [40]) with different values of μ_k

	No. of Iterations			CPU Time		
	Case I	Case II	Case III	Case I	Case II	Case III
Theorem 4.1 ($\mu = 0.15$)	13	18	15	0.03426	0.04521	0.03218
Theorem 4.1 ($\mu = 0.35$)	12	17	14	0.04036	0.08146	0.05919
Theorem 4.1 ($\mu = 0.75$)	11	15	12	0.05731	0.09032	0.07984
Theorem 4.1 ($\mu = 0.95$)	10	14	11	0.09104	1.09821	0.09978
Theorem 4.4 (Reich et al. [40])	57	67	49	2.93910	3.12310	2.11351

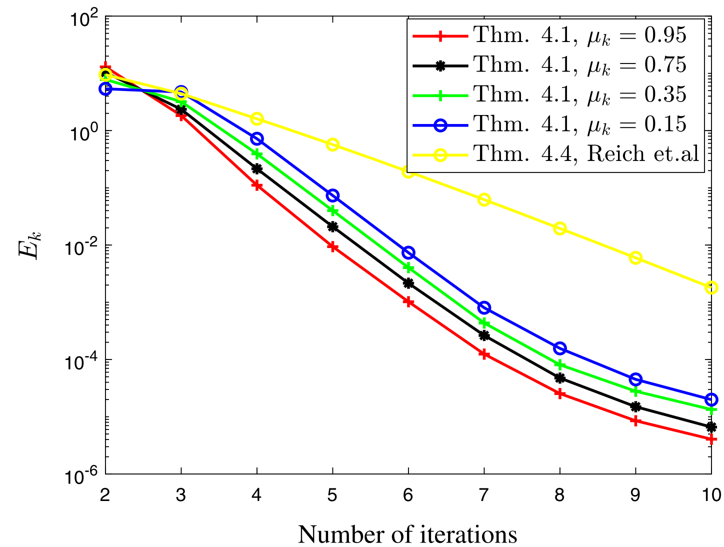


Figure 3 Example 5.1: Case I

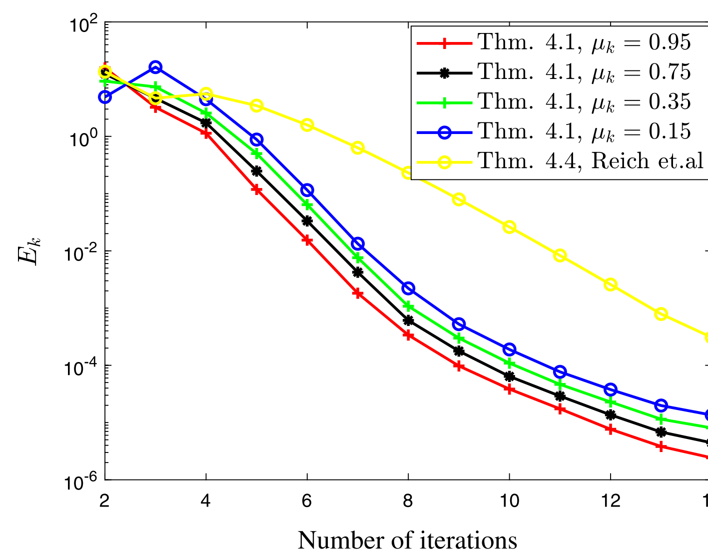


Figure 4 Example 5.1: Case II

Remark 5.2

(i) The example presented above serves for two purposes:
- impact of different values of μ_k on our proposed algorithm
- comparison with the noninertial ($\mu_k = 0$) type algorithm proposed by Reich et al. [40] given in Theorem 4.4.

(ii) The numerical results presented in Table 2 and Figs. 3–5 indicate that our proposed approximants are efficient, easy to implement, and do well for any values of $\mu_k \neq 0$ in both number of iterations and CPU time required.

(iii) We observe that the CPU time of Theorem 4.1 increases, but the number of iterations decreases as the parameter μ approaches 1.

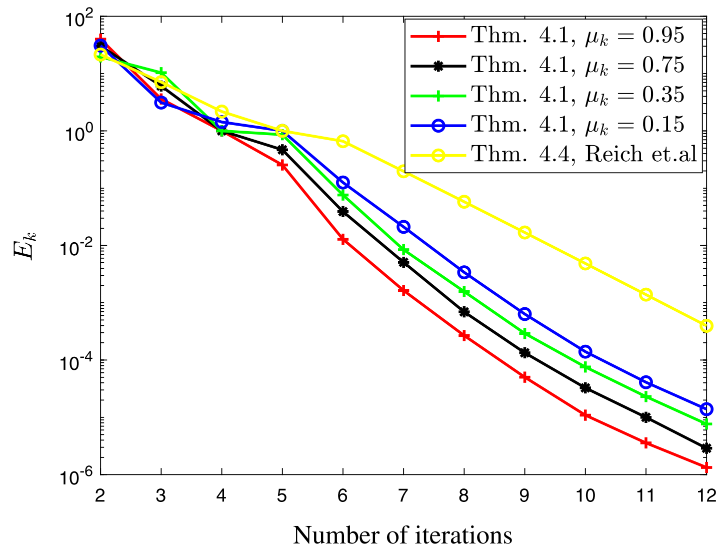


Figure 5 Example 5.1: Case III

(iv) We observe from the numerical implementation above and our proposed algorithm outperformed the noninertial version proposed by Reich et al. [40] given in Theorem 4.4 both in the number of iterations and CPU time required to reach the stopping criterion.

6 Conclusions

The problem for computing a common solution via unifying approximants, of a finite family of GSCNPP and the FPP for a countably infinite family of nonlinear operators has its own importance in the fields of monotone operator theory and fixed point theory. We proved that the approximants perform in an effective and efficient way when compared with the existing approximants, in particular, those studied in Hilbert spaces. The theoretical framework of the algorithm has been strengthened with an appropriate numerical example. Moreover, this framework has also been implemented to various instances of the split inverse problems. We would like to emphasize that the above mentioned problems occur naturally in many applications. Therefore iterative algorithms are inevitable in this field of investigation. As a consequence, our theoretical framework constitutes an important topic of future research.

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Availability of data and materials

Data sharing not applicable to this paper as no datasets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Conceptualization of the paper was carried out by YA, MA, and PK. Methodology by YA and MA. Formal analysis, investigation, and writing the original draft preparation by YA, MA and OS. Software and validation by OS, WK and KS. Writing, reviewing, and editing by YA, MA, and PK. Project administration by PK, WK, and KS. All authors read and approved the final manuscript.

Author details
[1]KMUTT Fixed Point Research Laboratory, KMUTT-Fixed Point Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand. [2]Department of Mathematics, Clarkson University, Potsdam, NY, USA. [3]Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore 54000, Pakistan. [4]Center of Excellence in Theoretical and Computational Science (TaCS-CoE) and KMUTT Fixed Point Research Laboratory, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Department of Mathematics, Fachulity of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand. [5]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan. [6]Applied Mathematics for Science and Engineering Research Unit (AMSERU), Program in Applied Statistics, Department of Mathematics and Computer Science, Faculty of Science and Technology, Rajamangala University of Technology Thanyaburi (RMUTT), Pathum Thani 12110, Thailand. [7]Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok (KMUTNB), 1518, Wongsawang, Bangsue, Bangkok 10800, Thailand.

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