Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

Open Access

R-convexity in R-vector spaces

Ali Ebrahimi Meymand[1] and Azadeh Alijani[1*]

*Correspondence: alijani@vru.ac.ir
[1]Department of Mathematics, Vali-e-Asr University of Rafsanjan, PO Box 518, Rafsanjan, Iran

Abstract

In this paper, for every relation R on a vector space V, we consider the R-vector space (V,R) and define the notions of R-convexity, R-convex hull, and R-extreme point in this space. Some examples are provided to compare them with the reference cases. The effects of some operations on R-convex sets are investigated. In particular, it is shown that the R-interior of an R-convex set is also an R-convex set under some restrictions on R. Also, we give some equivalent conditions for R-extremeness. Moreover, the notions of R-convex and R-affine maps on R-vector spaces are defined, and some results that assert the relation between an R-convex map f and its R-epigraph under some limitations on R are considered. Several propositions, such as R-continuous maps preserve R-compact sets and R-affine maps preserve R-convex sets, are presented, and some results on the composition of R-convex and R-affine maps are considered. Finally, some applications of R-convexity are investigated in optimization. More precisely, we show that the extrema values of R-affine R-continuous maps are reached on R-extreme points. Moreover, local and global minimum points of an R-convex map f on R-convex set K are considered.

MSC: Primary 54H25; secondary 26D15; 47H10

Keywords: R-convex set; R-convex function; R-extreme point; R-vector space

1 Introduction and preliminaries

Various generalizations of the classical concept of a convex function have been introduced, especially during the second half of the twentieth century. These generalizations have been explored in various fields, such as economics, engineering, statistics, and applied sciences, and they have provided interesting results in several branches related to mathematics such as convex analysis, nonlinear optimization, linear programming, geometric functional analysis, control theory, and dynamical systems; see for example [2, 13, 21], and the references therein. Recently, the extensions of convexity have been considered by many researchers. For example, Nikoufar et al. studied convexity in various branches of pure and applied mathematical areas [3, 18, 25]. Also, we refer the readers to η-convexity and coordinate convexity [9, 27, 37]; GA-convexity and GG-convexity [15, 20, 39]; s-convexity [1]; preinvexity [35]; strong convexity [29, 30, 38]; quasi-convexity [32]; Schur convexity [28, 34]; and pseudo-convexity [24]. Also, see the following recent related references: [12, 19, 31], and [36].

Over the last forty years, another type of extension of convexity, in which the convex coefficients need not commute with each other, has been considered. Examples include

C^*-convexity [22, 23], matrix convexity and operator convexity [8, 33], and the extension of C^*-convexity to $*$-rings [4–6], and [7]. The basic concepts of convex analysis can be seen in [26] and [14].

Recently, the notions of orthogonal metric spaces and metric spaces with relation have been considered by many researchers [10, 16, 17], and [11]. In [16], the authors introduced R-metric spaces and studied some of the properties of these spaces. We recall some notions and some notations as follows.

Suppose that (M,d) is a metric space and R is a relation on M. Then the triple (M,d,R) or in brief M is called an R-metric space. An R-sequence $\{x_n\}_{n\in\mathbb{N}}$ in an R-metric space M is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that x_nRx_{n+k} for each $n,k\in\mathbb{N}$, and R-sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to converge to x if, for every $\varepsilon>0$, there is an integer N such that $d(x_n,x)<\varepsilon$ for every $n\geq N$. In this case, we write $x_n \xrightarrow{R} x$, and the R-sequence $\{x_n\}_{n\in\mathbb{N}}$ in M is said to be an R-Cauchy sequence if, for every $\varepsilon>0$, there exists an integer N such that $d(x_n,x_m)<\varepsilon$ for $n\geq N$ and $m\geq N$. It is clear that x_nRx_m or x_mRx_n.

Also, the concepts of open and closed sets are defined in these spaces. For $E\subseteq M$, the element $x\in M$ is called an R-limit point of E if there exists an R-sequence $\{x_n\}_{n\in\mathbb{N}}$ in E such that $x_n\neq x$ for all $n\in\mathbb{N}$ and $x_n \xrightarrow{R} x$. The set of all R-limit points of E is denoted by E'^R, and the set E is R-closed if $E'^R\subseteq E$. Precisely, the R-closure of E is the set $\overline{E}^R=E\cup E'^R$. For $E\subseteq M$, if E^c is R-closed, then E is said R-open and E is called R-compact if every R-sequence $\{x_n\}_{n\in\mathbb{N}}$ in E has a convergent subsequence in E. The element $x\in E$ is an R-interior point for E if, for every R-sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_n \xrightarrow{R} x$, there exists $N\in\mathbb{N}$ such that $x_n\in E$ for every $n\geq N$. The set of all of R-interior points of E is denoted by $R-\operatorname{int}(E)$.

The map $f:M\to M$ is said to be R-continuous at $x\in M$ if, for every R-sequence $\{x_n\}_{n\in\mathbb{N}}$ in M that $x_n \xrightarrow{R} x$, we have $f(x_n) \xrightarrow{R} f(x)$. Also, f is said to be R-continuous on M if f is R-continuous at each $x\in M$.

The paper is organized as follows. We continue this introductory section with a review of the basic definitions and notations of relative metric spaces, i.e., metric spaces equipped with relations, that are needed for the next sections.

In Sect. 2, we first define the notions of R-vector space, vector space equipped with relation, and R-convexity in these spaces. After giving some examples that distinct the notions of convexity and R-convexity in general, the effect of some operations on R-convex sets is investigated. More precisely, we show that the R-interior of an R-convex set is R-convex under certain constraints on R.

Section 3 is devoted to studying R-extreme points, which are the relative extreme points of R-convex sets. After defining this notion and giving some examples, we prove that every extreme point is an R-extreme point, but the reverse is not necessarily true. Next, we define the R-convex hull of the sets and set some conditions that for R-convex set W, $R-\operatorname{co}(W)=W$. In the main theorem of this section, we give several equivalent conditions for R-extremeness, and in the last example of this section, we show that generally, the Krein–Milman type theorem does not hold. It seems that one can deduce a Krein–Milman type result for R-convex R-compact sets by putting additional restrictions on the relation R.

In Sect. 4, we introduce the notions of R-convex maps and R-affine maps on R-vector spaces. In classical convexity, f is a convex function if and only if the epigraph of f is a convex set. In this section, we prove such a result for R-convex maps, and then some corollaries of this theorem will be given. In continuation, by putting additional conditions on the relation R, we prove several propositions which assert that R-continuous maps take R-compact sets to R-compact sets, and R-affine maps preserve R-convexity. Also, the composition of an R-affine map and a preserving R-affine map is R-affine, and the composition of an increasing R-convex map and a preserving R-convex map is also an R-convex map.

The presented results in this manuscript make powerful tools for important applications in optimization theory. Finally, we concentrate on some applications of R-convexity in the optimization theory. More precisely, we show that the R-affine R-continuous maps take their extreme values on R-extreme points. Moreover, for an R-convex map f on R-convex set K, the set of all elements of K on which f takes its minimum is an R-convex set, and in R-vector metric space M, every local minimum x_0 of f is a global minimum of f on the set $[x_0]_R \cap K$, where $[x_0]_R = \{x \in M; x_0 R x\}$. Furthermore, if R is an equivalence relation with an additional condition, then for the global maximum x_0 of f, f is constant on $[x_0]_R \cap K$.

2 R-convex sets

In [10, 16], and [11], the authors considered some spaces with relations to them and obtained important and interesting results. It seems that these properties are independent of the relation and this fact was not considered. This section is devoted to preliminaries of R-vector spaces that are needed to study the R-convexity property for sets. Some examples are considered to clarify the contents.

Definition 2.1 Let R be a relation on a vector space V. Then V (or the pair (V,R)) is called to be an R-vector space.

In [16], the authors introduced R-convex sets for R-metric space $\mathbb{R}^k$. We recall this notion for an R-vector space as follows.

Definition 2.2 A subset W of an R-vector space V is said to be R-convex if $\lambda w_1 + (1-\lambda)w_2 \in W$ whenever $w_1, w_2 \in W$, $w_1 R w_2$, and $0 < \lambda < 1$. In this case, the combination $\lambda w_1 + (1-\lambda)w_2$ is called an R-convex combination of two elements w_1 and w_2.

The following remark and examples illustrate the relation between two notions "convexity" and "R-convexity".

Remark 2.3 Every convex set W in an R-vector space V is an R-convex set. However, the reverse of the result is not true.

Example 2.4 Suppose that $V = \mathbb{R}$ and R is the equality relation on V, and $W = \mathbb{N}$. Then $\mathbb{N}$ is an R-convex set, but it is not a convex set.

Example 2.5 Let $V = \mathbb{R}^2$ and

$$W = \{(x,y); x^2 + y^2 \leq 1\} \cup \{(x,y); (x-4)^2 + y^2 \leq 1\}$$

and

$$(x_1,y_1)R(x_2,y_2) \Longleftrightarrow (x_1,y_1),(x_2,y_2) \in \{(x,y); x^2 + y^2 \leq 1\}$$

or

$$(x_1, y_1), (x_2, y_2) \in \{(x, y); (x-4)^2 + y^2 \leq 1\}$$

Then W is an R-convex set but it is not convex.

Example 2.6 Let $V = \mathbb{R}$, $W = [-2, 0) \cup \{2\}$, and

$$xRy \Longleftrightarrow x, y \leq 0.$$

W is an R-convex set, but it is not convex.

Example 2.7 Let V be an R-vector space such that R is an equivalence relation. If there exists a $v_0 \in V$ such that $v_0 R v$ for all $v \in V$, then $R = V \times V$ and the notions of R-convexity and convexity are equivalent. Since for $v_1, v_2 \in V$ we have

$$v_0 R v_1 \text{ and } v_0 R v_2 \Longrightarrow v_1 R v_0 \text{ and } v_0 R v_2 \Longrightarrow v_1 R v_2.$$

The union and intersection of sets preserve R-convexity property. In the next proposition, we investigate these subjects.

Proposition 2.8 *Let V be an R-vector space. Then the following statements hold.*

i. *The intersection of every family of R-convex sets in V is also an R-convex set.*
ii. *For every chain of R-convex sets $\{E_i\}_{i \in I}$, the set $\bigcup_{i \in I} E_i$ is an R-convex set.*
iii. *If $\{E_i\}_{i \in I}$ is a sequence of R-convex sets in V, then $\limsup_{i \in I} E_i$ and $\liminf_{i \in I} E_i$ are also R-convex.*

Furthermore, not all properties of convex sets hold for R-convex sets, as is illustrated in the following two remarks.

Remark 2.9 The scalar multiplier of a convex set is convex. But it is not true for R-convex sets. Assume that E is an R-convex set and $\alpha \in \mathbb{C}$. Then the set αE is not necessarily R-convex. For example, set $E = (0, 1) \cup (2, 3)$ and for $x, y \in \mathbb{R}$,

$$xRy \Longleftrightarrow x, y \in (0, 1) \quad \text{or} \quad x, y \in (2, 3).$$

Hence E is an R-convex set, but for $\alpha = \frac{1}{3}$, the set $\alpha E = (0, \frac{1}{3}) \cup (\frac{2}{3}, 1)$ is not R-convex because

$$\frac{2}{9} R \frac{7}{9} \quad \text{and} \quad \frac{1}{2} \times \frac{2}{9} + \frac{1}{2} \times \frac{7}{9} = \frac{1}{2} \notin \alpha E.$$

Remark 2.10 For convex sets E_1 and E_2, the set $E_1 + E_2$ is also convex. But it is not valid for R-convex sets. To see this, let $E_1 = (0, 2) \cup (3, 5)$ and $E_2 = \{-1\}$, and for $x, y \in \mathbb{R}$,

$$xRy \Longleftrightarrow x, y \in \left(\frac{1}{2}, 3\right) \quad \text{or} \quad x, y \in (3, 5) \quad \text{or} \quad x = y = -1.$$

It can be verified that E_1 and E_2 are R-convex, but the set $E_1 + E_2 = (-1, 1) \cup (2, 4)$ is not R-convex because for the numbers $x = \frac{3}{4}$ and $y = 2.5$, xRy and some of their R-convex combinations are not in $E_1 + E_2$.

The closure of any convex set is convex. This will be investigated in the following example using an R-convex set and its R-closure.

Example 2.11 Let $V = \mathbb{R}$ and $E = (0,1) \cup (4,5)$, and

$$xRy \Longleftrightarrow (0 < x, y \leq 4 \text{ or } x, y > 4), \quad \forall x, y \in \mathbb{R}.$$

Then $\overline{E}^R = [0,1] \cup [4,5]$, which is not R-convex.

Remark 2.1 The set of all interior points of a convex set is convex, but this is not true for R-convex sets. In other words, the R-interior points of any R-convex set are not necessarily an R-convex set. See the following example as a counterexample.

Example 2.12 Suppose that $M := \mathbb{R}^2$, $E := \{(x,0) | -1 \leq x \leq 1\} \subseteq \mathbb{R}^2$, and the relation R is defined on M as follows:

$$(x,0)R(y,0) \Longleftrightarrow \bigl(x, y \in [-1,0] \text{ or } x, y \in [0,1] \text{ or } x, y \in \{-1,1\}\bigr)$$

and

$$(x,x)R(y,y), \quad \forall x, y \in \mathbb{R}.$$

Then E is an R-convex set. But $R - \operatorname{int}(E) = E - \{(0,0)\}$ is not an R-convex set since $(-1,0)R(1,0)$, but

$$\frac{1}{2}(-1,0) + \frac{1}{2}(1,0) = (0,0) \notin R - \operatorname{int}(E).$$

In the following theorem, we provide the conditions to preserve the R-convexity from E to $R - \operatorname{int}(E)$.

Theorem 2.13 *Let (M,d,R) be an R-metric vector space such that R is an equivalence relation on M, which has the following properties for every $x, y \in M$:*

i. $xRy \Longrightarrow xR(\lambda x + (1-\lambda)y)$, $\forall \lambda; 0 < \lambda < 1$.
ii. *If $x_n \xrightarrow{R} x$ and yRx, then yRx_n $(\forall n \geq N)$ for some $N \in \mathbb{N}$.*

Then the R-convexity of $E \subseteq M$ implies that $R - \operatorname{int}(E)$ is also an R-convex set.

Proof Suppose that E is an R-convex set, $x, y \in \operatorname{int}(E)$, xRy, and $\lambda \in (0,1)$. We must show that $z := \lambda x + (1-\lambda)y \in R - \operatorname{int}(E)$. Let $\{z_n\}$ be an arbitrary R-sequence in M such that $z_n \xrightarrow{R} z$. The set E is R-convex, so xRy implies that $z \in E$. Using condition i, xRy implies that xRz, and hence yRz (since R is an equivalence relation).

On the other hand, since $z_n \xrightarrow{R} z$ and xRz, by using condition ii, we conclude that xRz_n, $(\forall n \geq N_1)$, and hence yRz_n $(\forall n \geq N_1)$ for some $N_1 \in \mathbb{N}$. For each $m \in \mathbb{N}$, put

$$x_{n,m} := \frac{1}{m} z_n + \left(1 - \frac{1}{m}\right) x, \qquad y_{n,m} := \frac{1}{m} z_n + \left(1 - \frac{1}{m}\right) y.$$

Then, for each $n \geq N_1$, in view of condition i, we have $xRx_{n,m}$ and $yRy_{n,m}$ for all $m \in \mathbb{N}$. Since R is an equivalence relation on M, we can conclude from $xRx_{n,m}$ $(\forall m \in \mathbb{N})$ that

$\{x_{n,m}\}_{m=1}^{\infty}$ is an R-sequence in M, and $x_{n,m} \xrightarrow{R} x$ as $m \to \infty$. Similarly, $yRy_{n,m}$, $(\forall m \in \mathbb{N})$ implies that $\{y_{n,m}\}_{m=1}^{\infty}$ is an R-sequence in X, and $y_{n,m} \xrightarrow{R} y$ as $m \to \infty$. Thus, there are positive integers M_1 and M_2 such that $x_{n,m} \in E$, $\forall m \geq M_1$, and $y_{n,m} \in E$, $\forall m \geq M_2$. By taking $M_0 = \max\{M_1, M_2\}$, for each $m \geq M_0$, we have

$$\alpha x_{n,m} + (1-\alpha)y_{n,m} \in E, \quad (0 < \alpha < 1),$$

and furthermore, we have

$$\lambda x_{n,m} + (1-\lambda)y_{n,m} \in E, \quad \forall m \geq M_0.$$

Therefore, for all $m \geq M_0$, we conclude that

$$\begin{aligned} w &:= \frac{1}{m}z_n + \left(1-\frac{1}{m}\right)z \\ &= \frac{1}{m}z_n + \left(1-\frac{1}{m}\right)\bigl(\lambda x + (1-\lambda)y\bigr) \\ &= \lambda\left(\frac{1}{m}z_n + \left(1-\frac{1}{m}\right)x\right) + (1-\lambda)\left(\frac{1}{m}z_n + \left(1-\frac{1}{m}\right)y\right) \\ &= \lambda x_{n,m} + (1-\lambda)y_{n,m} \in E. \end{aligned}$$

Finally, if $d(z_j, z) < d(w, z)$ then $z_j \in E$, which implies that $z \in R - \operatorname{int}(E)$, and the proof is completed. □

3 *R*-extreme points

In this section, we define R-extreme point concept for an R-convex subset in R-vector spaces. Also, we define R-convex hull of the sets in R-vector spaces. The main results of this section are presented in Proposition 3.9 and Theorem 3.10, and some equivalent conditions for R-extremeness in the special R-vector spaces are obtained.

Definition 3.1 In an R-vector space V, an R-open line segment is a set of the following form:

$$(v_1, v_2) := \bigl\{\lambda v_1 + (1-\lambda)v_2; 0 < \lambda < 1\bigr\}$$

for $v_1, v_2 \in V$ such that $v_1 R v_2$. We say that this R-open line segment is proper if $v_1 \neq v_2$.

Definition 3.2 Let W be an R-convex set in an R-vector space V. Then a point $w \in W$ is called an R-extreme point for W if there is no proper R-open line segment that contains w and lies entirely in W. The set of all of R-extreme points of W is denoted by $R - \operatorname{ext}(W)$.

Remark 3.3 The following statements are valid:

i. For vector space V and $R = V \times V$, the extremeness and R-extremeness are equivalent.
ii. Every extreme point of an R-convex set is an R-extreme point of this set in R-vector spaces.

In the following, some examples are given to illustrate the concept of R-extreme points and the differences between the extreme points and R-extreme points.

Example 3.4 Consider the R-vector space $(\mathbb{R}, \leq)$.

i. If $W = [a,b]$, then $\operatorname{ext}(W) = R - \operatorname{ext}(W) = \{a,b\}$.
ii. If $W = (a,b)$, then $\operatorname{ext}(W) = R - \operatorname{ext}(W) = \varnothing$.
iii. In general, in R-vector space $(\mathbb{R}, \leq)$, a set W is convex if and only if W is R-convex and $\operatorname{ext}(W) = R - \operatorname{ext}(W)$.

Example 3.5 Suppose that $V = \mathbb{R}^2$ and $R := A \times A \subseteq \mathbb{R}^2$ such that $A = \{(x,y); x \leq 0\}$. If $W = \{(x,y); y \geq 0\}$, then W is R-convex and $\operatorname{ext}(W) = \varnothing$ but $R - \operatorname{ext}(W) = \{(x,y) \in W; x > 0\} \cup \{(0,0)\}$. To see this, assume that $(x,y) \in \mathbb{R}^2$ such that $x > 0$ and $y \geq 0$, then (x,y) cannot be written as an R-convex combination of two points of W. Also, if $y > 0$, then $(0,y) \notin$ R-ext(W) because

$$(0,y) = \frac{1}{2}\left(0, \frac{y}{2}\right) + \frac{1}{2}\left(0, \frac{3y}{2}\right).$$

Example 3.6 Let $V = \mathbb{R}^2$ and $W = \{(x,y) \in \mathbb{R}^2; y \geq 0\}$, and let

$$(x_1,y_1)R(x_2,y_2) \Longleftrightarrow x_1 \leq x_2 \text{ and } y_1 < y_2.$$

It is well known that W is a convex set and also an R-convex set, and $\operatorname{ext}(W) = \varnothing$ but $R - \operatorname{ext}(W) = \{(x,0); x \in \mathbb{R}\}$ because for every $x \in \mathbb{R}$ and $0 < \lambda < 1$, if

$$(x,0) = \lambda(z,w) + (1-\lambda)(t,v); \quad (z,w),(t,v) \in W, (z,w)R(t,v),$$

then $0 = \lambda w + (1-\lambda)v$ and $0 \leq w < v$, which is a contradiction, and hence $(x,0)$ cannot be written as an R-convex combination of elements of W.

Note that if we replace '$<$' with '$\leq$' in the relation R, then

$$R - \operatorname{ext}(W) = \operatorname{ext}(W) = \varnothing.$$

Example 3.7 Assume $V = \mathbb{R}^2$, $W = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 \leq 1\}$, and $A = \{(x,y) \in \mathbb{R}^2; x < 0\}$, and set $R = A \times A$. Clearly, W is a convex set, and so is R-convex. We know that $\operatorname{ext}(W) = \{(x,y) \in W; x^2 + y^2 = 1\}$, but $R - \operatorname{ext}(W)$ is different because

$$R - \operatorname{ext}(W) = \left\{(x,y) \in W; x^2 + y^2 = 1\right\} \cup \left\{(x,y) \in W; x \geq 0\right\}.$$

If the relation is replaced with the following relation:

$$(x_1,y_1)R(x_2,y_2) \Longleftrightarrow (x_1 \leq x_2, y_1 < y_2),$$

then $R - \operatorname{ext}(W) = \operatorname{ext}(W) = \{(x,y) \in W; x^2 + y^2 = 1\}$.

Now, we define the concept of R-convex hull of a set, and then we appoint some limitations on the relation R, to obtain a necessary and sufficient condition for a set to be an R-convex set.

Definition 3.8 Let W be a subset of an R-vector space V. The R-convex hull of W is denoted by $R-\mathrm{co}(W)$ and is defined as follows:

$$\left\{\sum_{i=1}^{n}\alpha_i w_i; 0<\alpha_i\leq 1, \sum_{i=1}^{n}\alpha_i=1, w_i\in W, w_i R\, w_{i+k}, 0\leq k\leq n-i, n\in\mathbb{N}\right\}.$$

Moreover, every element of $R-\mathrm{co}(W)$ is said to be an R-convex combination of elements of W.

Proposition 3.9 *Let V be an R-vector space such that the relation R has the following properties*:

i. $vR\,v$ *for all* $v\in V$.

ii. *If* $vR\,v_1$ *and* vRv_2, *then* $vR\,\lambda v_1+(1-\lambda)v_2$ *for* $v, v_1, v_2\in V$ *and every* $0<\lambda<1$.

Then every subset W of V is R-convex if and only if $W=R-\mathrm{co}(W)$.

Proof Firstly, assume that $W=R-\mathrm{co}(W)$, and $v_1, v_2\in W$ such that $v_1R\,v_2$. Then,

$$\lambda v_1+(1-\lambda)v_2\in R-\mathrm{co}(W)=W, \quad \forall\lambda; 0<\lambda<1.$$

Therefore, W is R-convex. Now, suppose that W is R-convex and $v\in W$. By the properties of R, for $v\in W$, by i, $vR\,v$, and for $\lambda=\frac{1}{2}$,

$$v=\frac{1}{2}v+\frac{1}{2}v\in R-\mathrm{co}(W).$$

This shows that $W\subseteq R-\mathrm{co}(W)$. Now, $R-\mathrm{co}(W)\subseteq W$ is obtained by induction.

For $n=2$, let $\{v_1, v_2\}\subseteq W$ such that $v_1R\,v_2$. Then $R-\mathrm{co}(\{v_1, v_2\})\subseteq W$ by the R-convexity of W.

Now, for $n=3$, let $\{v_1, v_2, v_3\}\subseteq W$ such that $v_1R\,v_2$, $v_1R\,v_3$, and $v_2R\,v_3$, and $\alpha_i\in(0,1)$ such that $\sum_{i=1}^{3}\alpha_i=1$. Then we can write

$$\sum_{i=1}^{3}\alpha_i v_i=\alpha_1v_1+(1-\alpha_1)\left(\frac{\alpha_2}{1-\alpha_1}v_2+\frac{\alpha_3}{1-\alpha_1}v_3\right)$$

such that $\frac{\alpha_2}{1-\alpha_1}v_2+\frac{\alpha_3}{1-\alpha_1}v_3\in W$, because $\frac{\alpha_2}{1-\alpha_1}+\frac{\alpha_3}{1-\alpha_1}=1$, and $v_2R\,v_3$ and by using R-convexity of W. Similarly, for every $n\in\mathbb{N}$, we obtain $R-\mathrm{co}\{v_1,\dots,v_n\}\subseteq W$. Thus $R-\mathrm{co}(W)\subseteq W$ and the proof is complete. □

Note that in Proposition 3.9 the given condition for R is necessary, and if this condition is omitted, then the result is not true. To see this, let $V=\mathbb{R}$ and $R:=$ '$<$'. It is clear that $(x,x)\notin R$ for every $x\in\mathbb{R}$. The set $K=[1,2]$ is R-convex but $R-\mathrm{co}(K)=(1,2)$.

In the last theorem of this section, some equivalent conditions for an element to be an R-extreme point are given.

Theorem 3.10 *Let V be an R-vector space such that R is reflexive and for $v, v_1, v_2\in V$, if $vR\,v_1$ and $vR\,v_2$, then $vR\,\lambda v_1+(1-\lambda)v_2$ for all $0<\lambda<1$. Then the following statements are equivalent for every R-convex subset W of V*:

i. $v \in R - \text{ext}(W)$.
ii. *If* $v = \frac{1}{2}v_1 + \frac{1}{2}v_2$, *where* $v_1, v_2 \in W$ *and* $v_1 R v_2$, *then* $v = v_1 = v_2$.
iii. *If* $v = \lambda v_1 + (1-\lambda)v_2$, *where* $0 < \lambda < 1$, $v_1, v_2 \in W$, *and* $v_1 R v_2$, *then* $v = v_1 = v_2$.
iv. $v \in R - \text{co}(\{v_1, \dots v_n\})$, *where* $v_i \in W$ *for* $i = 1, \dots, n$ *and* $n \in \mathbb{N}$, *then there exists* $j \in \{1, \dots n\}$ *such that* $v = v_j$.
v. $W \setminus \{v\}$ *is an R-convex set.*

Proof i $\longrightarrow$ ii. By definition of R-extreme point, it is clear.

ii $\longrightarrow$ iii. If $\lambda \neq \frac{1}{2}$, without loss of generality, we assume that $\frac{1}{2} < \lambda < 1$. Then the following equality is obtained:

$$v = \lambda v_1 + (1-\lambda)v_2 = \frac{1}{2}v_1 + \frac{1}{2}y; \qquad y = (2\lambda - 1)v_1 + (2 - 2\lambda)v_2 \in W,$$

and $v_1 R y$ by the assumption. Therefore, the part iii is valid.

iii $\longrightarrow$ iv. If $v = \sum_{i=1}^{n} \alpha_i v_i$ is an R-convex combination of $v_i \in W$, then we can write $v = \alpha_1 v_1 + (1-\alpha_1)\sum_{i=2}^{n} \frac{\alpha_i}{1-\alpha_1} v_i$. By induction, the properties of R, and Proposition 3.9, we have $\sum_{i=2}^{n} \frac{\alpha_i}{1-\alpha_1} v_i \in W$. The statement iii concludes that $v = v_1 = \sum_{i=2}^{n} \frac{\alpha_i}{1-\alpha_1} v_i$. Then $v = v_1$ and also $v = v_k$ for $k = 2, \dots, n$, similarly.

Now, we consider iv $\longrightarrow$ v. Let $v_1, v_2 \in W \setminus \{v\}$ such that $v_1 R v_2$, and $0 < \lambda < 1$. We must show that $\lambda v_1 + (1-\lambda)v_2 \in W \setminus \{v\}$. Since W is R-convex, the combination $\lambda v_1 + (1-\lambda)v_2 \in W$. Now, if $v = \lambda v_1 + (1-\lambda)v_2$, then $v \in R - \text{co}(\{v_1, v_2\})$ and $v = v_1$ or $v = v_2$ by part *iv*. This is a contradiction, and so $\lambda v_1 + (1-\lambda)v_2 \in W \setminus \{v\}$. Therefore, $W \setminus \{v\}$ is R-convex.

v $\longrightarrow$ i. Let $T = \{\lambda v_1 + (1-\lambda)v_2; 0 < \lambda < 1\} \subseteq W$ be a proper R-open line segment containing v. Then $v = \lambda v_1 + (1-\lambda)v_2$ for some $0 < \lambda < 1$. It is known that $v_1 \neq v_2$, then $v \neq v_1$ and $v \neq v_2$. Also, $W \setminus \{v\}$ is R-convex, and $v_1, v_2 \in W \setminus \{v\}$, and so $v \in W \setminus \{v\}$. But it is a contradiction, and hence v is an R-extreme point for W. □

One of the most important subjects is considering Krein–Milman theorem for R-vector spaces. In the following example, we see that this theorem is not valid for an R-compact R-convex set in R-vector spaces generally.

Example 3.1 Let $M = \mathbb{R}$ with the standard topology, $R =' \leq'$ and $K = (0, 1]$. Clearly, K is an R-compact and R-convex set, and $R - \text{ext}(K) = \{1\}$ and so $K \neq \overline{\text{co}}^R(R - \text{ext}(K))$.

At the end of this section, the question that remains is, "under what conditions dose the Krein-Milman Theorem for R-vector spaces hold?".

4 R-convex functions

An important part of subjections in mathematics is studying the properties of a type of map between two spaces. One type of the map is a convex map. This section introduces R-convex maps and relative concepts and considers their properties with respect to relation R.

Definition 4.1 Let V be an R-vector space, and $f : V \to V$ be a map.

i. Assume that R_1 is another relation on V. The map f is called to be R-convex with respect to R_1 if, for each $0 < \lambda < 1$ and $v_1, v_2 \in V$ such that $v_1 R v_2$, the following relation holds:

$$f\big(\lambda v_1 + (1-\lambda)v_2\big)R_1\big(\lambda f(v_1) + (1-\lambda)f(v_2)\big).$$

ii. The map $f : V \to \mathbb{R}$ is called to be R-convex if, for each $0 < \lambda < 1$ and $v_1, v_2 \in V$ such that $v_1 R v_2$, the following relation holds:

$$f\big(\lambda v_1 + (1-\lambda)v_2\big) \le \lambda f(v_1) + (1-\lambda)f(v_2).$$

iii. The map $f : V \to V$ (also the function $f : V \to \mathbb{R}$) is called to be R-affine if, for each $0 < \lambda < 1$ and $v_1, v_2 \in V$ such that $v_1 R v_2$, the following equation holds:

$$f\big(\lambda v_1 + (1-\lambda)v_2\big) = \lambda f(v_1) + (1-\lambda)f(v_2).$$

iv. The R-epigraph of a map $f : V \to V$, which is denoted by $R - \mathrm{epi}(f)$, is the set $\{(v, w); f(v)R w\}$.

Remark 4.2 Assume that V is a vector space.

i. Every convex map f on an R-vector space V is R-convex, however, the reverse of the result is not true.
ii. Let $R = V \times V$ be a relation on V. Then the notions of R-convexity and convexity for every map $f : V \longrightarrow \mathbb{R}$ are equivalent.

The following example illustrates that every R-convex map is not necessarily a convex map.

Example 4.3 Let $V = \mathbb{R}$ and

$$xRy \Longleftrightarrow (x, y \le 0 \text{ or } x, y > 0).$$

Then the map $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 & x \le 0, \\ x^2 - 3 & x > 0, \end{cases}$$

is an R-convex map on $\mathbb{R}$, but it is not a convex map. Because for $\alpha = \frac{1}{2}$, $v_1 = -1$, and $v_2 = 1$,

$$f\left(\frac{1}{2} \times (-1) + \frac{1}{2} \times 1\right) = f(0) = 0 \not\le \frac{1}{2}f(-1) + \frac{1}{2}f(1) = -\frac{1}{2}.$$

In the classical convexity, there is a straight relation between the convex functions and their epigraphs. In the following theorem and its corollaries, by giving some conditions, we obtain similar results for R-convex maps.

Theorem 4.4 *Let R_1 and R_2 be two relations on a vector space V, and let $f : V \longrightarrow V$ be a map. Also, assume that R_2 is transitive and reflexive with the following property: For*

$0 < \lambda < 1$ *and* $v_1, v_2, w_1, w_2 \in V$,

$$\big(f(v_1)R_2w_1,\ \textit{and}\, f(v_2)R_2w_2\big) \implies \big(\lambda f(v_1) + (1-\lambda)f(v_2)\big)R_2\big(\lambda w_1 + (1-\lambda)w_2\big).$$

Moreover, suppose that S *is a relation on* $V \times V$ *with two properties as follows*:

i. *If* $v_1R_1v_2$, *then* $(v_1,f(v_1))S(v_2,f(v_2))$.
ii. *For* $v_1, v_2, w_1, w_2 \in V$ *such that* $f(v_1)R_2w_1$ *and* $f(v_2)R_2w_2$, *if* $(v_1, w_1)S(v_2, w_2)$, *then* $v_1R_1v_2$.

Then f *is an* R_1*-convex map on* V *with respect to* R_2 *if and only if* $R_2 - \mathrm{epi}(f)$ *is an* S*-convex set.*

Proof Suppose that f is an R_1-convex map on V with respect to R_2. For (v_1, w_1) and (v_2, w_2) of $R_2 - \mathrm{epi}(f)$, where $(v_1, w_1)S(v_2, w_2)$, the definition of $R_2 - \mathrm{epi}(f)$ implies that

$$f(v_1)R_2w_1 \quad \text{and} \quad f(v_2)R_2w_2.$$

Hence, by the property of R_2, for each $0 < \lambda < 1$, we conclude that

$$\big(\lambda f(v_1) + (1-\lambda)f(v_2)\big)R_2\big(\lambda w_1 + (1-\lambda)w_2\big).$$

By ii, we have $v_1R_1v_2$, and by the R_1-convexity of f,

$$f\big(\lambda v_1 + (1-\lambda)v_2\big)R_2\big(\lambda f(v_1) + (1-\lambda)f(v_2)\big).$$

Now, since R_2 is transitive, we deduce $f(\lambda v_1 + (1-\lambda)v_2)R_2(\lambda w_1 + (1-\lambda)w_2)$. Therefore,

$$\lambda(v_1, w_1) + (1-\lambda)(v_2, w_2) = \big(\lambda v_1 + (1-\lambda)v_2, \lambda w_1 + (1-\lambda)w_2\big) \in R_2 - \mathrm{epi}(f),$$

which shows that $R_2 - \mathrm{epi}(f)$ is S-convex.

Conversely, let $R_2 - \mathrm{epi}(f)$ be an S-convex set. Suppose that $0 < \lambda < 1$ and v_1 and v_2 in V such that $v_1R_1v_2$. By the reflexivity of R_2 and the property i, the following statements hold:

$$\big(v_1,f(v_1)\big), \big(v_2,f(v_2)\big) \in R_2 - \mathrm{epi}(f) \quad \text{and} \quad \big(v_1,f(v_1)\big)S\big(v_2,f(v_2)\big).$$

Then the S-convexity of $R_2 - \mathrm{epi}(f)$ concludes

$$\lambda\big(v_1,f(v_1)\big) + (1-\lambda)\big(v_2,f(v_2)\big) \in R_2 - \mathrm{epi}(f),$$

and hence,

$$\big(\lambda v_1 + (1-\lambda)v_2, \lambda f(v_1) + (1-\lambda)f(v_2)\big) \in R_2 - \mathrm{epi}(f).$$

So,

$$f\big(\lambda v_1 + (1-\lambda)v_2\big)R_2\big(\lambda f(v_1) + (1-\lambda)f(v_2)\big),$$

and f is R_1-convex on V with respect to R_2. $\square$

The special cases of the above theorem for a real vector space with different relations are concluded in the following corollaries. It is well known that for $f:\mathbb{R}\longrightarrow\mathbb{R}$, the epigraph of f is $\{(x,y);f(x)\leq y\}$.

Corollary 4.5 *Let R be a relation on vector space $\mathbb{R}$, f be a map on $\mathbb{R}$, and S be a relation on $\mathbb{R}\times\mathbb{R}$ such that*

i. *If x_1Rx_2, then $(x_1,f(x_1))S(x_2,f(x_2))$.*
ii. *For $x_1,x_2,y_1,y_2\in\mathbb{R}$ such that $f(x_1)\leq y_1$ and $f(x_2)\leq y_2$, if $(x_1,y_1)S(x_2,y_2)$, then x_1Rx_2.*

Then f is R-convex with respect to the relation '$\leq$' if and only if epi(f) *is S-convex.*

Proof Since the relation '$\leq$' is reflexive and transitive on $\mathbb{R}$, so it is a straightforward conclusion of Theorem 4.4. □

Corollary 4.6 *Assume that R is a relation on the vector space $\mathbb{R}$, and f is a map on $\mathbb{R}$. Let S be the induced relation of R on $\mathbb{R}\times\mathbb{R}$ as follows:*

$$(x_1,y_1)S(x_2,y_2)\Longleftrightarrow x_1Rx_2,\quad \textit{for } x_1,x_2,y_1,y_2\in\mathbb{R}.$$

Then the function $f:\mathbb{R}\longrightarrow\mathbb{R}$ is R-convex with respect to '$\leq$' on $\mathbb{R}$ if and only if epi(f) *is S-convex.*

Proof It is concluded by Corollary 4.5. □

Corollary 4.7 *Let V be an R-vector space and $f:V\longrightarrow\mathbb{R}$ be a function. Also, S is a relation on $V\times V$ with the following properties:*

i. *If v_1Rv_2, then $(v_1,f(v_1))S(v_2,f(v_2))$.*
ii. *For $v_1,v_2,w_1,w_2\in V$ such that $f(v_1)\leq w_1$ and $f(v_2)\leq w_2$, if $(v_1,w_1)S(v_2,w_2)$, then v_1Rv_2.*

Then f is R-convex if and only if epi(f) *is an S-convex set.*

Proof It is a consequence of Theorem 4.4, since the relation '$\leq$' is reflexive and transitive. □

In the classical convexity, every convex function is a continuous function. But there exist some R-convex functions which are not R-continuous.

Example 4.8 Let $V=\mathbb{R}$, $a>1$, and

$$xRy\Longleftrightarrow (x,y\leq 0 \text{ or } x,y>0),\quad \forall x,y\in\mathbb{R}.$$

Then the function

$$f(x)=\begin{cases}x^2 & x\leq 0,\\ x^2-a & x>0,\end{cases}$$

is an R-convex function on the R-convex set $\mathbb{R}$, and it is not R-continuous. This is because by setting $x_n=\frac{1}{n}$ for all $n\in\mathbb{N}$, $\{x_n\}_{n\in\mathbb{N}}$ is an R-sequence converging to zero, and

$$f(x_n)=x_n^2-a\nrightarrow^R f(0)=0.$$

It is known that every continuous map preserves compact sets. In the following proposition, we show that every R-continuous map, by an additional condition, preserves R-compact sets.

Proposition 4.1 *Let M be an R-metric space, K be an R-compact subset of M, and $f: M \to M$ be an R-continuous function such that, for $x, y \in M$, $f(x)Rf(y)$ implies that xRy, then $f(K)$ is an R-compact set. That is, every R-continuous map preserves R-compact sets if $f(x)Rf(y)$ implies that xRy.*

Proof Let $\{f(x_n)\}_{n\in\mathbb{N}}$ be an R-sequence in $f(K)$. By the properties of f, $\{x_n\}_{n\in\mathbb{N}}$ is an R-sequence in R-compact set K, so there is a convergent R-subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of it. Suppose that

$$x_{n_k} \xrightarrow{R} x_0 \quad \text{for some } x_0 \in K.$$

f is R-continuous, so $f(x_{n_k}) \xrightarrow{R} f(x_0)$, and $f(K)$ is R-compact. □

The goal of the following proposition is to show the preservation of R-convex sets under the special R-affine maps.

Proposition 4.2 *Every R-affine function preserves R-convex sets if the following condition holds*:

$$f(x)Rf(y) \Longrightarrow xRy.$$

Proof Let K be an R-convex subset of an R-vector space V, and $f: V \to V$ be an R-affine function. Now assume that $f(\nu_1), f(\nu_2) \in f(K)$ for some $\nu_1, \nu_2 \in K$ such that $f(\nu_1)Rf(\nu_2)$ and $0 < \alpha < 1$. By the assumption, $f(\nu_1)Rf(\nu_2)$ implies that $\nu_1 R \nu_2$, and by the R-convexity of K, we have $(\alpha\nu_1 + (1-\alpha)\nu_2) \in K$. Also, f is an R-affine function, thus

$$\alpha f(\nu_1) + (1-\alpha)f(\nu_2) = f\big(\alpha\nu_1 + (1-\alpha)\nu_2\big) \in f(K).$$

So, $f(K)$ is an R-convex set. □

Proposition 4.3 *In an R-metric vector space, the following statements are valid*:

i. *Summation, subtraction, and scalar multiplication of R-affine maps are also R-affine.*
ii. *If f and g are R-affine maps and g is an R-preserving map, then fog is also R-affine.*

Proposition 4.4 *Let f be an increasing R-convex function on the R-metric vector space M, and let g be an R-preserving R-convex map on M. Then fog is also an R-convex map.*

Proof Let $x, y \in M$ such that xRy. Then $g(x)Rg(y)$. For $0 < \alpha < 1$,

$$\begin{aligned} fog\big(\alpha x + (1-\alpha)y\big) &= f\big(g\big(\alpha x + (1-\alpha)y\big)\big) \\ &\le f\big(\alpha g(x) + (1-\alpha)g(y)\big) \\ &\le \alpha f\big(g(x)\big) + (1-\alpha)f\big(g(y)\big). \end{aligned}$$

This shows that fog is R-convex. □

5 Some applications in optimization

An optimization problem considers minimizing or maximizing a given real function on a subset of its domain. In other words, in an optimization problem, one obtains the best available values for some functions that have different types corresponding to objective functions and types of their domains. The optimization theory and its techniques are useful and very important in a large area of applied mathematics. In this section, we study some results in optimization theory. More precisely, we study important results about the extreme values of some R-convex maps on R-convex sets. In the first theorem, we show that every R-continuous R-affine function attains its extrema at R-extreme points.

Theorem 5.1 *Suppose that* (M,R) *is an* R*-metric vector space,* K *is a subset of* M *where* $R-\operatorname{ext}(K)$ *is* R*-closed and* $R-\operatorname{ext}(K)\times R-\operatorname{ext}(K)\subset R$*, and* B *is an* R*-compact subset of* $\overline{\operatorname{co}}^R(R-\operatorname{ext}(K))$ *such that* $R-\operatorname{ext}(K)\subset B$*. Then every* R*-affine and* R*-continuous map* $f: M\longrightarrow\mathbb{R}$ *attains its maximum and minimum on* B *at* R*-extreme points of* K*. Moreover, the maximum and minimum of* f *on* B *is equal with its maximum and minimum on* $R-\operatorname{ext}(K)$*, respectively.*

Proof Let f take its maximum on B at $x_0\in B$. Then there exists an R-sequence $\{x_n\}_{n\in\mathbb{N}}\subset R-\operatorname{co}(R-\operatorname{ext}(K))$ such that $x_n\xrightarrow{R}x_0$. Notice that $x_n=\sum_{i=1}^{N_n}\lambda_{n,i}y_{n,i}$ where $N_n\in\mathbb{N}$ and $y_{n,i}\in R-\operatorname{ext}(K)$, $(1\leq i\leq N_n)$ and $\lambda_{n,i}\in(0,1]$ such that $\sum_{i=1}^{N_n}\lambda_{n,i}=1$. Thus,

$$\begin{aligned}
f(x_n)&=f\left(\sum_{i=1}^{N_n}\lambda_{n,i}y_{n,i}\right)\\
&=\sum_{i=1}^{N_n}\lambda_{n,i}f(y_{n,i})\\
&\leq\max_{1\leq i\leq N_n}f(y_{n,i})\sum_{i=1}^{N_n}\lambda_{n,i}\\
&=\max_{1\leq i\leq N_n}f(y_{n,i}),
\end{aligned}$$

so

$$f(x_n)\leq\max_{1\leq i\leq N_n}f(y_{n,i})=f(y_{n,i_n}).\tag{5.1}$$

Now, $\{y_{n,i_n}\}_{n\in\mathbb{N}}$ is an R-sequence in $R-\operatorname{ext}(K)\subset B$. Note that B is R-compact, which ensures the existence of a convergent R-subsequence $\{y_{n_k,i_{n_k}}\}_{k\in\mathbb{N}}$. Suppose that $y_{n_k,i_{n_k}}\xrightarrow{R}y_0$ as $k\to\infty$. Assuming $R-\operatorname{ext}(K)$ is R-closed, then $y_0\in R-\operatorname{ext}(K)$. Using relation (5.1) and the R-continuity of f, we conclude that

$$\begin{aligned}
f(x_0)&=f\left(R-\lim_{n\to\infty}x_n\right)\\
&=f\left(R-\lim_{k\to\infty}x_{n_k}\right)\\
&=R-\lim_{k\to\infty}f(x_{n_k})\\
&\leq R-\lim_{k\to\infty}f(y_{n_k,i_{n_k}})
\end{aligned}$$

$$= f\Big(R - \lim_{k\to\infty} y_{n_k, i_{n_k}}\Big) = f(y_0).$$

On the other hand, $f(x_0)$ is maximum of f on B, so $f(x_0) = f(y_0)$, and f attains its maximum on B at y_0. Similarly, we can prove the theorem for the minimum case. □

In the succeeding proposition, we show that the set of all elements on which an R-convex function takes its minimum is an R-convex set.

Proposition 5.2 *Let V be an R-vector space, K be an R-convex subset of V, and $f : K \longrightarrow \mathbb{R}$ be an R-convex function on K. Then, the set $B = \{x \in V; f(x) = \min_{y\in K} f(y)\}$ is R-convex.*

Proof Suppose that $x_1, x_2 \in B$, and f attains its minimum m on x_1 and x_2, and x_1Rx_2. Then, for each λ, $(0 < \lambda < 1)$, we have

$$m \leq f\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \lambda f(x_1) + (1-\lambda)f(x_2) = \lambda m + (1-\lambda)m = m.$$

So, $f(\lambda x_1 + (1-\lambda)x_2) = m$, and $\lambda x_1 + (1-\lambda)x_2 \in B$. Thus, B is an R-convex set. □

The following theorem asserts that every local minimum is a global minimum for R-convex functions.

Theorem 5.3 *Let (M,R) be an R-vector metric space, $K \subset M$ be an R-convex set, and $f : K \longrightarrow \mathbb{R}$ be an R-convex function which has a local minimum at x_0, then x_0 is also a global minimum of f on $[x_0]_R \cap K$, where $[x_0]_R = \{x \in M; x_0Rx\}$. Specially, x_0 is a global minimum on K if x_0Rx for all $x \in K$.*

Proof Suppose that f takes its minimum at x_0 on the neighborhood N of x_0, and $x \in [x_0]_R \cap K$. Then, for sufficiently small $\lambda > 0$, we have

$$f(x_0) \leq f\big((1-\lambda)x_0 + \lambda x\big) \leq (1-\lambda)f(x_0) + \lambda f(x),$$

and hence $\lambda(f(x) - f(x_0)) \geq 0$, which implies that $f(x_0) \leq f(x)$, and the proof is completed. In addition, if x_0Rx for all $x \in K$, then x_0 is a global minimum of f on K since $f(x_0) \leq f(x)$ for all $x \in K$. □

Corollary 5.4 *Let (M,R) be an R-vector metric space, $K \subset M$ be an R-convex set, and $f : K \longrightarrow \mathbb{R}$ be a strictly R-convex function which has a local minimum at x_0. Then the minimum point x_0 is unique on $[x_0]_R \cap K$.*

Proof Since f is strictly R-convex on K, as the proof of the previous theorem, we obtain $f(x_0) < f(x)$ for all $x \in [x_0]_R \cap K$ where $x \neq x_0$. □

Theorem 5.5 *Let (V,R) be an R-vector space such that R is an equivalence relation on V with the following property*:

$$aRb \Longrightarrow aR\big(\lambda a + (1-\lambda)b\big), \quad \forall \lambda \in (0,1).$$

If K is an R-convex subset of V and $f : K \longrightarrow \mathbb{R}$ is an R-convex function which has a global maximum at x_0, then f is constant on $[x_0]_R \cap K$.

Proof Let $y \in [x_0]_R \cap K$ such that $f(y) < f(x_0)$. Then x_0Ry, and so for $\alpha \in (0,1)$, $z = \alpha x_0 + (1-\alpha)y \in K$, and by the properties of R, we have yRz. On the other hand, $x_0 = \frac{1}{\alpha}z + \frac{\alpha-1}{\alpha}y$, and hence the R-convexity of f implies that

$$\begin{aligned} f(x_0) &= f\left(\left(1-\frac{1}{\alpha}\right)y + \frac{1}{\alpha}z\right) \\ &\le \left(1-\frac{1}{\alpha}\right)f(y) + \frac{1}{\alpha}f(z) \\ &< \left(1-\frac{1}{\alpha}\right)f(x_0) + \frac{1}{\alpha}f(x_0) \\ &= f(x_0). \end{aligned}$$

This suggests $f(x_0) < f(x_0)$, which is a contradiction. Therefore, $f(y) = f(x_0)$ for all $y \in [x_0]_R \cap K$. □

6 Conclusions

Convex functions and extreme points are important objects in convex analysis and especially in optimization theory. In this paper, we tried to extend these notions to R-vector spaces, where R is a relation on a vector space V, and then some results on this subject have been obtained, and we compared these results with the reference cases by giving some examples. Finally, some applications of R-convexity have been investigated in optimization theory. More precisely, we have shown that the extrema of R-affine R-continuous maps are reached on R-extreme points. Moreover, local and global minimum points of an R-convex map f on R-convex set K have been considered. In the forthcoming works, we will attempt to prove the Krein–Milman type theorems for R-compact R-convex subsets of R-vector metric spaces and some other applications of this theory.

Acknowledgements
The authors would like to express their sincere gratitude to anonymous referees for their helpful comments and recommendations which improved the quality of the paper.

Funding
The authors received no financial support for the research, authorship, or publication of this article.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors approved the final manuscript. AA did preliminary studies on the subject and obtained some results. AE performed an in-depth study of this subject, proved more theorems, and gave more examples. Then authors considered all results again, wrote, edited, and prepared the final version of the manuscript for submission.

Authors' information
Not applicable.

Publisher's Note

Received: 6 November 2021 Accepted: 9 May 2022 Published online: 04 June 2022

References

1. Adil Khan, M., Hanif, M., Khan, Z.A., Ahmad, K., Chu, Y.M.: Association of Jensen's inequality for *s*-convex function with Csiszr divergence. J. Inequal. Appl. **2019**, Article ID 162 (2019)
2. Cambini, A., Martein, L.: Generalized Convexity and Optimization Theory and Applications. Springer, Berlin (2009)
3. Ebadian, A., Nikoufar, I., Gordji, M.E.: Perspectives of matrix convex functions. Proc. Natl. Acad. Sci. **108**(18), 7313–7314 (2011)
4. Ebrahimi Meymand, A.: C^*-extreme points and C^*-faces of the epigraph of C^*-affine maps in $*$-rings. Wavelets Linear Algebra **5**(2), 21–28 (2019)
5. Ebrahimi Meymand, A.: The structure of the set of all C^*-convex maps in $*$-rings. Wavelets Linear Algebra **7**(2), 43–51 (2020)
6. Ebrahimi Meymand, A.: Noncommutative convexity in matricial $*$-rings. Khayyam J. Math. **8**(1), 7–16 (2022)
7. Ebrahimi Meymand, A., Esslamzadeh, G.H.: C^*-convexity and C^*-faces in $*$-rings. Turk. J. Math. **36**, 131–145 (2012)
8. Effros, E., Winkler, S.: Matrix convexity: operator analogues of bipolar and Hahn–Banach theorems. J. Funct. Anal. **144**, 117–152 (1997)
9. Eshaghi Gordji, M., Dragomir, S.S., Rostamian Delavar, M.: An inequality related to η-convex functions (II). Int. J. Nonlinear Anal. Appl. **6**(2), 27–33 (2015)
10. Eshaghi Gordji, M., Habibi, H.: Fixed point theory in generalized orthogonal metric space. J. Linear Topol. Algebra **6**, 251–260 (2017)
11. Eshaghi Gordji, M., Ramezani, M., De La Sen, M., Cho, Y.J.: On orthogonal sets and Banach fixed point theorem. Fixed Point Theory **18**, 569–578 (2017)
12. Goberna, M.A., Rodríguez, M.M.L., Vicente-Pérez, J.: Evenly convex sets, and evenly quasiconvex functions, revisited. J. Nonlinear Var. Anal. **4**, 189–206 (2020)
13. Hadjisavvas, N., Komlosi, S., Schaible, S.: Handbook of Generalised Convexity and Generalized Monotonicity. Springer, Boston (2005)
14. Hiriart-Urruty, J.B., Lemarechal, C.: Fundamentals of Convex Analysis. Springer, Berlin (2001)
15. Iscan, I.: Jensen-Mercer inequality for *GA*-convex functions and some related inequalities. J. Inequal. Appl. **2020**, 212 (2020)
16. Khalehoghli, S., Rahimi, H.R., Eshaghi Gordji, M.: Fixed point theorems in R-metric spaces with applications. AIMS Math. **5**(4), 3125–3137 (2020)
17. Khalehoghli, S., Rahimi, H.R., Eshaghi Gordji, M.: R-topological spaces and SR-topological spaces with their applications. Math. Sci. **14**, 249–255 (2020)
18. Khani, R., Najafzadeh, S., Ebadian, A., Nikoufar, I.: The n-valent convexity of Frasin integral operators. Ukr. Mat. Zh. **73**(2), 278–282 (2021)
19. Khazayel, B., Farajzadeh, A., Gunther, C., Tammer, C.: On the intrinsic core of convex cones in real linear spaces. SIAM J. Optim. **31**(2), 1276–1298 (2021)
20. Khurshid, Y., Adil Khan, M., Chu, Y.M.: Conformable integral inequalities of the Hermite–Hadamard type in terms of *GG*- and *GA*-convexities. J. Funct. Spaces **1**, 1–8 (2019)
21. Komlosi, S., Rapcsak, T., Schaible, S.: Generalised Convexity: Proceedings of the IVth International Workshop on Generalized Convexity Held at Janus Pannonius University Pecs, Hungary. Lecture Notes in Economics and Mathematical Systems, vol. 405. Springer, Berlin (1992)
22. Loebl, R., Paulsen, V.I.: Some remarks on C^*-convexity. Linear Algebra Appl. **35**, 63–78 (1981)
23. Magajna, B.: Operator systems and C^*-extreme points. Stud. Math. **247**, 45–62 (2019)
24. Mangasarian, O.: Pseudo-convex functions. SIAM J. Control **3**(2), 281–290 (1965)
25. Nikoufar, I.: Convex functions on compact C^*-convex sets. Wavelet Linear Algebra **7**(1), 57–62 (2020)
26. Roberts, A.W., Varberg, D.E.: Convex Functions. Academic Press, New York (1973)
27. Rostamian Delavar, M., Dragomir, S.S.: On η-convexity. Math. Inequal. Appl. **20**(1), 203–216 (2017)
28. Shi, H.N., Wang, P., Zhang, J.: Schur-convexity for compositions of complete symmetric function dual. J. Inequal. Appl. **2020**, 65 (2020)
29. Snchez, R.V., Sanabria, J.E.: Strongly convexity on fractal sets and some inequalities. Proyecciones **39**(1), 1–13 (2020)
30. Song, Y.Q., Adil Khan, M., Zaheer Ullah, S., Chu, Y.M.: Integral inequalities involving strongly convex functions. J. Funct. Spaces **2018**, Article ID 6595921 (2018)
31. Tung, L.T.: Optimality conditions and duality for E-differentiable semi-infinite programming with multiple interval-valued objective functions under generalized E-convexity. J. Nonlinear Funct. Anal. **2020**, Article ID 21 (2020)
32. Ullah, S., Farid, G., Khan, K.A., Waheed, A., Mehmood, S.: Generalized fractional inequalities for quasi-convex functions. Adv. Differ. Equ. **2019**, 15 (2019)
33. Webster, C., Winkler, S.: The Krein–Milman theorem in operator convexity. Trans. Am. Math. Soc. **351**, 307–322 (1999)
34. Wu, S.H., Chu, Y.M.: Schur *m*-power convexity of generalized geometric Bonferroni mean involving three parameters. J. Inequal. Appl. **2019**, 57 (2019)
35. Yang, X.: Generalized Preinvexity and Second Order Duality in Multiobjective Programming. Springer, Berlin (2018)
36. Yen, L.H., Muu, L.D.: A normal-subgradient algorithm for fixed point problems and quasiconvex equilibrium problems. Appl. Set-Valued Anal. Optim. **2**, 329–337 (2020)
37. Zaheer Ullah, S., Adil Khan, M., Chu, Y.M.: A note on generalized convex functions. J. Inequal. Appl. **2019**, 291 (2019)
38. Zaheer Ullah, S., Adil Khan, M., Chu, Y.M.: Majorization theorems for strongly convex functions. J. Inequal. Appl. **2019**, 58 (2019)
39. Zhang, X.M., Chu, Y.M., Zhang, X.H.: The Hermite–Hadamard type inequality of *GA*-convex functions and its applications. J. Inequal. Appl. **4**, Article ID 507560 (2010)