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A new approach on generalized quasimetric spaces induced by partial metric spaces

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Abstract

In this paper, we introduce the concept of generalized quasimetric spaces by a new approach, and present some examples in the partial metric spaces. Furthermore, we obtain some results on (strong) complete partial metric spaces.

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Keywords: Partial metric; Quasimetric; Fixed-point theorem; Strong complete partial metric spaces

1 Introduction

In 1931, Wilson [1] initiated the notion of quasimetric spaces, which was defined without the symmetric condition comparing to the axioms of the standard metric. Later, Matthews [2] defined the concept of partial metric space in 1994, in which the distance of each object to itself is not necessarily zero. Additionally, he constructed quasimetric q and weighted metric p^m by partial metric p, where $q(x,y) = p(x,y) - p(x,x)$ and $p^m(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, respectively. Over the past few decades, these methods of construction have appeared in many papers on partial metric spaces, and the fixed-pointed theory has been one of the most important topics in topology ([3–12]).

The object of this paper tries to give a generalized quasimetric $\hat{p}$, i.e., d_p [6] is its special case, $\tilde{p}$ [13] and p^s [9] are equivalent. Furthermore, we obtain some results on (strong) complete partial metric spaces.

2 Preliminaries

Throughout this paper, X is always a nonempty set, the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}^+$ always denote the set of real numbers, of all positive real numbers and of all positive integers, respectively.

Definition 2.1 ([1]) A *quasimetric* is a function $d : X \times X \to [0,+\infty)$ satisfying the following conditions: $\forall x,y,z \in X$,

(M1) $x = y \Leftrightarrow d(x,y) = d(y,x) = 0$;

(M2) $d(x,z) \leq d(x,y) + d(y,z)$.

A quasimetric d is called a *metric* if it also satisfies

(M3) $d(x,y) = d(y,x)$.

A *(quasi)metric space* is a pair (X,d) such that d is a (quasi)metric on X.

Definition 2.2 ([2]) A *partial metric* is a function $p : X \times X \to [0,+\infty)$ satisfying the following conditions: $\forall x,y,z \in X$,

(P1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;

(P2) $p(x,x) \le p(x,y)$;

(P3) $p(x,y) = p(y,x)$;

(P4) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

A *partial metric space* is a pair (X,p) such that p is a partial metric on X.

Apparently, each metric is precisely a partial metric on X, and a partial metric p is a metric if and only if $p(x,x) = 0$ for all $x \in X$. Similar to the definition of open balls in metric spaces, that is $B^d_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$, Matthews used $B^p_\varepsilon(x) = \{y \in X : p(x,y) < \varepsilon\}$ to denote open p-balls for all $x \in X$ and $\varepsilon > 0$, we can see that some open p-balls may be empty (see more details in [2]).

Lemma 2.3 *For each partial metric p: $X \times X \to [0,+\infty)$, set $\hat{p}(x,y) = p(x,y) - [\alpha p(x,x) + \beta p(y,y)]$, where $0 \le \alpha, \beta \le 1$, $\alpha + \beta = 1$. Then, the following statements hold*:

(1) *$\hat{p}$ is a quasimetric.*

(2) *$\tilde{p}$ is a metric if and only if $\alpha = \beta = \frac{1}{2}$, where we denote*

$$\tilde{p}(x,y) = p(x,y) - \frac{p(x,x) + p(y,y)}{2}.$$

(3) *$\hat{q}$ is a metric, where $\hat{q}(x,y) = \max\{\hat{p}(x,y), \hat{p}(y,x)\}$.*

Proof (1) We verify the conditions (M1) and (M2) one by one.

(M1): $(\Rightarrow)$ Suppose that $x = y$. It is clear that $\hat{p}(x,y) = \hat{p}(y,x) = 0$.

$(\Leftarrow)$ Suppose that $\hat{p}(x,y) = \hat{p}(y,x) = 0$. Then, $p(x,y) = \alpha p(x,x) + \beta p(y,y)$, and $p(y,x) = \alpha p(y,y) + \beta p(x,x)$, which implies $p(x,y) + p(y,x) = p(x,x) + p(y,y)$. Since $p(x,x) \le p(x,y)$ by (P2), we have $p(x,x) + p(y,x) \le p(x,x) + p(y,y)$, namely $p(y,x) \le p(y,y)$. By (P2) and (P3), we have $p(y,x) = p(y,y)$. Analogously, we can deduce $p(x,y) = p(x,x)$. Hence, $p(x,y) = p(x,x) = p(y,y)$, which implies $x = y$ by (P1).

(M2): By (P4), we have

$$\begin{aligned} \hat{p}(x,y) + \hat{p}(y,z) &= p(x,y) - \big[\alpha p(x,x) + \beta p(y,y)\big] + p(y,z) - \big[\alpha p(y,y) + \beta p(z,z)\big] \\ &= p(x,y) + p(y,z) - p(y,y) - \big[\alpha p(x,x) + \beta p(z,z)\big] \\ &\ge p(x,z) - \big[\alpha p(x,x) + \beta p(z,z)\big] = \hat{p}(x,z), \end{aligned}$$

for all $x,y,z \in X$. Therefore, $\hat{p}$ is a quasimetric.

(2) and (3) are trivial in that $\tilde{p}$ and $\hat{q}$ satisfy (M1)–(M3). □

Remark 2.4

(1) If $\alpha = 1$ and $\beta = 0$, then $\hat{p}$ is q (see [2]).

(2) If $\alpha = \beta = \frac{1}{2}$, then $\tilde{p}$ and p^s are equivalent (see [4]).

Proposition 2.5 *Let X be a nonempty set, p be a partial metric, and $(X,\hat{p})$ be the corresponding quasimetric space defined in Lemma 2.3, i.e., $\hat{p}(x,y) = p(x,y) - [\alpha p(x,x) + \beta p(y,y)]$ for any $x,y \in X$. Then, the following statements hold.*

(1) *The set of all open p-balls $B_\varepsilon^p(x)$ is the basis of a topology $\mathcal{T}(p)$ on X, where $B_\varepsilon^p(x) = \{y \in X : p(x,y) < \varepsilon\}$ for any $\varepsilon > 0$. We call $\mathcal{T}(p)$ the topology generated by the partial metric p on X.*

(2) *The set of all open $\hat{p}$-balls $B_\varepsilon^{\hat{p}}(x)$ is the basis of a topology $\mathcal{T}(\hat{p})$ on X, where $B_\varepsilon^{\hat{p}}(x) = \{y \in X : \hat{p}(x,y) < \varepsilon\}$ for any $\varepsilon > 0$. We call $\mathcal{T}(\hat{p})$ the topology generated by the quasimetric $\hat{p}$ on X.*

Proof (1) It is trivial by Theorem 3.1 in [2].

(2) It is not difficult to prove that $X = \bigcup_{x\in X} B_\varepsilon^{\hat{p}}(x)$, where $\varepsilon > 0$.

Moreover, we have $B_\varepsilon^{\hat{p}}(x) \cap B_\delta^{\hat{p}}(y) = \bigcup\{B_\eta^{\hat{p}}(z) : z \in B_\varepsilon^{\hat{p}}(x) \cap B_\delta^{\hat{p}}(y)\}$, where $\eta = \beta p(z,z) + \min\{\varepsilon - p(x,z) + \alpha p(x,x), \delta - p(y,z) + \alpha p(y,y)\}$. □

Theorem 2.6 *Let X be a nonempty set, p be a partial metric and $\hat{p}(x,y) = p(x,y) - [\alpha p(x,x) + \beta p(y,y)]$, where $0 \le \alpha, \beta \le 1$, $\alpha + \beta = 1$ and $\alpha \ne 1/2$, $\beta \ne 1/2$, for any $x,y \in X$. The following statements hold*:

(1) *Each partial metric p on X generates a T_0 topology $\mathcal{T}(p)$ on X.*

(2) *Each quasimetric $\hat{p}$ on X generates a T_0 topology $\mathcal{T}(\hat{p})$ on X.*

(3) *$\mathcal{T}(p) = \mathcal{T}(\hat{p})$.*

(4) *$(X, \mathcal{T}(p))$ and $(X, \mathcal{T}(\hat{p}))$ are first countable.*

Proof (1) It is trivial by Theorem 3.3 in [2].

(2) By Lemma 2.3(1), we know that $\hat{p}$ is a quasimetric. Suppose that $x \ne y$. By (P2) and (P3), we have $\alpha p(x,x) + \beta p(y,y) \le \alpha p(x,y) + \beta p(x,y) = p(x,y)$. Set $\varepsilon = \frac{p(x,y)-[\alpha p(x,x)+\beta p(y,y)]}{2}$. Then, $x \in B_\varepsilon^{\hat{p}}(x)$ and $y \notin B_\varepsilon^{\hat{p}}(x)$. Therefore, $(X, \mathcal{T}(\hat{p}))$ is a T_0 topology space.

(3) For any $x \in X$ and $\varepsilon > 0$, suppose $y \in B_\varepsilon^p(x)$, namely, $p(x,y) < \varepsilon$. Since $\alpha p(x,x) + \beta p(y,y) \le p(x,y)$, we have $\alpha p(x,x) + \beta p(y,y) < \varepsilon$. Set $\delta = \varepsilon - [\alpha p(x,x) + \beta p(y,y)]$. We can deduce $p(x,y) < \delta + [\alpha p(x,x) + \beta p(y,y)]$, which implies $y \in B_\delta^{\hat{p}}(x)$. Therefore, $B_\varepsilon^p(x) \subseteq B_\delta^{\hat{p}}(x)$.

On the other hand, for any $x \in X$ and $\varepsilon > 0$, suppose $y \in B_\varepsilon^{\hat{p}}(x)$. We have $p(x,y) - [\alpha p(x,x) + \beta p(y,y)] < \varepsilon$. Set $\eta = \varepsilon + [\alpha p(x,x) + \beta p(y,y)]$. Then, we can deduce $p(x,y) < \eta$, which implies $y \in B_\eta^p(x)$, thus $B_\varepsilon^{\hat{p}}(x) \subseteq B_\eta^p(x)$. Hence, $\mathcal{T}(p) = \mathcal{T}(\hat{p})$.

(4) Set $\varepsilon \in \mathbb{Q}^+$, where $\mathbb{Q}^+$ denotes the set of all positive rational numbers. For any $x \in X$, $B_\varepsilon^p(x)$ and $B_\varepsilon^{\hat{p}}(x)$ are countable neighborhoods at x in $(X, \mathcal{T}(p))$ and $(X, \mathcal{T}(\hat{p}))$, respectively. □

3 Some results on (strong) complete partial metric spaces

Definition 3.1 Let (X,p) be a partial metric space and $\{x_n\}$ be a sequence in X.

(1) A sequence $\{x_n\}$ *converges to a point* $x \in X$ if $p(x,x) = \lim_{n\to+\infty} p(x,x_n)$;

(2) A sequence $\{x_n\}$ is called a *Cauchy sequence* if $\lim_{n,m\to+\infty} p(x_n,x_m)$ exists and is finite;

(3) (X,p) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges, with respect to $\mathcal{T}(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m) = \lim_{n\to+\infty} p(x_n,x)$.

Lemma 3.2 *Let (X,p) be a partial metric space and $(X,\tilde{p})$ be the corresponding metric space defined in Lemma 2.3(2), i.e., $\tilde{p}(x,y) = p(x,y) - \frac{p(x,x)+p(y,y)}{2}$, for all $x,y \in X$. Let $(X,\hat{q})$ be the corresponding metric space, where $\hat{q}(x,y) = \max\{\hat{p}(x,y),\hat{p}(y,x)\}$, and $\hat{p}(x,y) = p(x,y) - [\alpha p(x,x) + \beta p(y,y)]$, $0 \le \alpha,\beta \le 1$, $\alpha + \beta = 1$ and $\alpha \ne \frac{1}{2}$, $\beta \ne \frac{1}{2}$, for all $x,y \in X$. The following statements hold*:

(1) *A sequence is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in $(X,\hat{q})$.*

(2) *(X,p) is complete if and only if $(X,\hat{q})$ is complete.*

(3) *(X,p) is complete if and only if $(X,\tilde{p})$ is complete.*

(4) *$\lim_{n\to+\infty} \tilde{p}(x_n,x) = 0$ if and only if*

$$p(x,x) = \lim_{n\to+\infty} p(x_n,x) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Proof (1) ($\Rightarrow$) Let $\{x_n\}$ be a Cauchy sequence in (X,p). There exists $\eta \in [0,+\infty)$ such that $\lim_{n,m\to+\infty} p(x_n,x_m) = \eta$. Then, for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}^+$ such that

$$\left|p(x_n,x_m) - \eta\right| < \frac{\varepsilon}{2}, \quad \forall n,m > n_\varepsilon.$$

Then, we have that

$$\begin{aligned}
\left|\hat{p}(x_n,x_m)\right| &= \left|p(x_n,x_m) - \left[\alpha p(x_n,x_n) + \beta p(x_m,x_m)\right]\right| \\
&\le \left|p(x_n,x_m) - \eta\right| + \alpha\left|p(x_n,x_n) - \eta\right| + \beta\left|p(x_m,x_m) - \eta\right| \\
&< \frac{\varepsilon}{2} + \alpha \cdot \frac{\varepsilon}{2} + \beta \cdot \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in $(X,\hat{q})$.

($\Leftarrow$) Suppose $\{x_n\}$ is a Cauchy sequence in $(X,\hat{q})$ and let $\varepsilon > 0$. Then, there exists $n_\varepsilon \in \mathbb{N}^+$, such that

$$\hat{q}(x_n,x_m) < \frac{|\alpha - \beta|\varepsilon}{2}, \quad \forall n,m > n_\varepsilon.$$

Set $\varepsilon = 1$. Then, there exists $n_0 \in \mathbb{N}^+$ such that

$$\hat{q}(x_n,x_m) < \frac{|\alpha - \beta|}{2}, \quad \forall n,m > n_0.$$

We prove that $\{x_n\}$ is a Cauchy sequence in (X,p) in the following steps.

Step 1: Since $p(x_n,x_{n_0}) = p(x_{n_0},x_n)$ by (P3) for all $n \ge n_0$, we have

$$\hat{p}(x_n,x_{n_0}) + \left[\alpha p(x_n,x_n) + \beta p(x_{n_0},x_{n_0})\right] = \hat{p}(x_{n_0},x_n) + \left[\alpha p(x_{n_0},x_{n_0}) + \beta p(x_n,x_n)\right].$$

Thus, we have $(\alpha - \beta)p(x_n,x_n) = \hat{p}(x_{n_0},x_n) + (\alpha - \beta)p(x_{n_0},x_{n_0}) - \hat{p}(x_n,x_{n_0})$, which implies that

$$p(x_n,x_n) = \frac{1}{\alpha - \beta}\left[\hat{p}(x_{n_0},x_n) - \hat{p}(x_n,x_{n_0})\right] + p(x_{n_0},x_{n_0}).$$

Then, we have

$$\begin{aligned}\left|p(x_n,x_n)\right| &\le \frac{1}{|\alpha-\beta|}\left[\left|\hat{p}(x_{n_0},x_n)\right| + \left|\hat{p}(x_n,x_{n_0})\right|\right] + p(x_{n_0},x_{n_0})\\ &\le \frac{2}{|\alpha-\beta|}\left|\hat{q}(x_n,x_{n_0})\right| + p(x_{n_0},x_{n_0})\\ &< 1 + p(x_{n_0},x_{n_0}),\end{aligned}$$

for all $n \ge n_0$, which implies that the sequence $\{p(x_n,x_n)\}$ is bounded in $\mathbb{R}$. Hence, the sequence $\{p(x_n,x_n)\}$ exists with a subsequence $\{p(x_{n_k},x_{n_k})\}$ that is convergent and we denote $\lim_{n_k\to+\infty} p(x_{n_k},x_{n_k}) = a$.

Step 2: By Step 1, we have

$$\begin{aligned}\left|p(x_n,x_n) - p(x_m,x_m)\right| &= \frac{1}{|\alpha-\beta|}\left|\hat{p}(x_m,x_n) - \hat{p}(x_n,x_m)\right|\\ &\le \frac{1}{|\alpha-\beta|}\left[\left|\hat{p}(x_m,x_n)\right| + \left|\hat{p}(x_n,x_m)\right|\right]\\ &< \frac{2}{|\alpha-\beta|}\left|\hat{q}(x_m,x_n)\right| < \varepsilon,\end{aligned}$$

for all $n,m > n_\varepsilon$. In addition, since

$$p(x_n,x_n) = \frac{1}{\alpha-\beta}\left[\hat{p}(x_m,x_n) - \hat{p}(x_n,x_m)\right] + p(x_m,x_m),$$

we have

$$\lim_{n\to+\infty} p(x_n,x_n) = \lim_{m\to+\infty} p(x_m,x_m) = a,$$

for all $n,m > n_1$, where $n_1 = \max\{n_\varepsilon,n_0\}$.

Furthermore,

$$\begin{aligned}&\left|p(x_n,x_m) - a\right|\\ &\quad= \left|p(x_n,x_m) - \left[\alpha p(x_n,x_n) + \beta p(x_m,x_m)\right] + \left[\alpha p(x_n,x_n) + \beta p(x_m,x_m)\right] - a\right|\\ &\quad\le \left|p(x_n,x_m) - \left[\alpha p(x_n,x_n) + \beta p(x_m,x_m)\right]\right| + \left|\alpha p(x_n,x_n) + \beta p(x_m,x_m) - a\right|\\ &\quad= \hat{p}(x_n,x_m) + \alpha\left|p(x_n,x_n) - a\right| + \beta\left|p(x_m,x_m) - a\right|\\ &\quad< \frac{|\alpha-\beta|}{2}\cdot\varepsilon + \alpha\cdot\varepsilon + \beta\cdot\varepsilon = \frac{|\alpha-\beta|+2}{2}\cdot\varepsilon,\end{aligned}$$

for all $n,m > n_1$. This implies that $\{x_n\}$ is a Cauchy sequence in (X,p).

(2) ($\Leftarrow$) First, without loss of generality, we claim that $0 \le \beta < \frac{1}{2}$ (in fact, by $0 \le \alpha,\beta \le 1$, $\alpha+\beta = 1$ and $\alpha \ne \frac{1}{2}$, $\beta \ne \frac{1}{2}$, then, we have $\alpha < \frac{1}{2}$ or $\beta < \frac{1}{2}$).

Step 1: Let $\{x_n\}$ be a Cauchy sequence in (X,p). It is clear that $\{x_n\}$ is a Cauchy sequence in $(X,\hat{q})$ by Lemma 3.2(1). Since $(X,\hat{q})$ is complete, there exists $x \in X$ such that $\lim_{n\to+\infty} \hat{q}(x,x_n) = 0$, i.e., for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}^+$ such that

$$\left|\hat{q}(x,x_n)\right| < \frac{1-2\beta}{2}\cdot\varepsilon$$

for all $n > n_0$. Since $\hat{q}(x,y) = \max\{\hat{p}(x,y), \hat{p}(y,x)\}$, we have $\lim_{n\to+\infty} \hat{p}(x,x_n) = 0$. This shows that $\{x_n\}$ is a convergent sequence in $(X,\hat{p})$.

On the other hand, we have

$$\begin{aligned}
&\bigl|p(x_n,x_n) - p(x,x)\bigr| \\
&\quad = \bigl|(\alpha+\beta)p(x,x) - (\alpha+\beta)p(x_n,x_n)\bigr| \\
&\quad = \bigl|\bigl[\alpha p(x,x) + \beta p(x_n,x_n) - p(x,x_n)\bigr] + \bigl[p(x,x_n) - \alpha p(x_n,x_n) - \beta p(x,x)\bigr] \\
&\qquad - 2\beta\bigl[p(x_n,x_n) - p(x,x)\bigr]\bigr| \\
&\quad \le \bigl|p(x,x_n) - \alpha p(x,x) - \beta p(x_n,x_n)\bigr| + \bigl|p(x,x_n) - \alpha p(x_n,x_n) - \beta p(x,x)\bigr| \\
&\qquad + 2\beta\bigl|p(x_n,x_n) - p(x,x)\bigr|.
\end{aligned}$$

Then,

$$\begin{aligned}
&(1-2\beta)\bigl|p(x_n,x_n) - p(x,x)\bigr| \\
&\quad \le \bigl|p(x,x_n) - \bigl[\alpha p(x,x) + \beta p(x_n,x_n)\bigr]\bigr| + \bigl|p(x_n,x) - \bigl[\alpha p(x_n,x_n) + \beta p(x,x)\bigr]\bigr| \\
&\quad = \hat{p}(x,x_n) + \hat{p}(x_n,x) \\
&\quad < 2\hat{q}(x,x_n) < (1-2\beta)\cdot\varepsilon,
\end{aligned}$$

for all $n > n_0$. Therefore, we can deduce $|p(x_n,x_n) - p(x,x)| < \varepsilon$, which implies $\lim_{n\to+\infty} p(x_n, x_n) = p(x,x)$.

Step 2: Since $\hat{p}(x,y) = p(x,y) - [\alpha p(x,x) + \beta p(y,y)]$, by Step 1, we have $\lim_{n\to+\infty} \hat{p}(x,x_n) = \lim_{n\to+\infty} p(x,x_n) - \alpha p(x,x) - \beta \lim_{n\to+\infty} p(x_n,x_n)$. Then, we can deduce $\lim_{n\to+\infty} p(x,x_n) = \lim_{n\to+\infty} p(x_n,x) = p(x,x)$.

In addition, by (P4) we have $p(x_n,x_m) \le p(x_n,x) + p(x,x_m) - p(x,x)$. Hence, $\lim_{n,m\to+\infty} p(x_n,x_m) \le p(x,x)$. Moreover, by (P2), we have $p(x_n,x_m) \ge p(x_n,x_n)$, which implies $\lim_{n,m\to+\infty} p(x_n,x_m) \ge p(x,x)$. Then, we have

$$\lim_{n,m\to+\infty} p(x_n,x_m) = p(x,x).$$

Therefore, (X,p) is complete.

($\Rightarrow$) Let $\{x_n\}$ be a Cauchy sequence in $(X,\hat{q})$. Then, $\{x_n\}$ is a Cauchy sequence in (X,p) by Lemma 3.2(2). There exists a point $x \in X$, such that $\lim_{n,m\to+\infty} p(x_n,x_m) = \lim_{n\to+\infty} p(x,x_n) = p(x,x)$. Therefore, for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}^+$ such that

$$\bigl|p(x,x_n) - p(x,x)\bigr| < \varepsilon$$

and

$$\bigl|p(x_n,x_n) - p(x,x)\bigr| < \varepsilon,$$

for all $n \ge n_0$. Then, we have

$$\bigl|\hat{p}(x,x_n)\bigr| = \bigl|p(x,x_n) - \bigl[\alpha p(x,x) + \beta p(x_n,x_n)\bigr]\bigr|$$

$$
\begin{aligned}
&= \alpha\left|\left[p(x,x_n) - p(x,x)\right]\right| + \beta\left|\left[p(x,x_n) - p(x_n,x_n)\right]\right| \\
&< \alpha\cdot\varepsilon + \beta\left|p(x,x_n) - p(x,x)\right| + \beta\left|p(x_n,x_n) - p(x,x)\right| \\
&< \alpha\cdot\varepsilon + 2\beta\cdot\varepsilon = \beta\cdot\varepsilon.
\end{aligned}
$$

Therefore, we have $\lim_{n\to+\infty}\hat{p}(x,x_n) = 0$.

Analogously, we have

$$
\begin{aligned}
\left|\hat{p}(x_n,x)\right| &= \left|p(x_n,x) - \left[\alpha p(x_n,x_n) + \beta p(x,x)\right]\right| \\
&< \beta\cdot\varepsilon + \alpha\left|p(x,x_n) - p(x,x)\right| + \alpha\left|p(x_n,x_n) - p(x,x)\right| \\
&< \beta\cdot\varepsilon + 2\alpha\cdot\varepsilon = \alpha\cdot\varepsilon.
\end{aligned}
$$

This implies $\lim_{n\to+\infty}\hat{p}(x_n,x) = 0$.

Furthermore, by (M2), we have $\hat{p}(x_n,x_m) \le \hat{p}(x_n,x) + \hat{p}(x,x_m)$. Therefore, $\lim_{n,m\to+\infty}\hat{p}(x_n, x_m) = 0$, which implies $(X,\hat{p})$ is complete. It is not difficult to show $(X,\hat{q})$ is complete.

(3) It is trivial by Lemma 3.2 in [14].

(4) It is trivial by Lemma 2.1 in [4]. □

Corollary 3.3 *Let (X,p) be a partial metric space. Then, $\lim_{n\to+\infty}\hat{q}(x,x_n) = 0$ if and only if $\lim_{n,m\to+\infty}p(x_n,x_m) = \lim_{n\to+\infty}p(x,x_n) = p(x,x)$.*

Lemma 3.4 ([4]) *Let (X,d) be a complete metric space, $\varphi: X \to [0,+\infty)$ be a lower semi-continuous function, and $T: X \to X$ be a given mapping. The following statements hold:*

(1) *Suppose that for any $0 < a < b < +\infty$, there exists $0 < \gamma(a,b) < 1$ such that for all $x,y \in X$, $a \le d(x,y) + \frac{\varphi(x)+\varphi(y)}{2} \le b$ implies $d(Tx,Ty) + \frac{\varphi(Tx)+\varphi(Ty)}{2} \le \gamma(a,b)[d(x,y) + \frac{\varphi(x)+\varphi(y)}{2}]$. Then, T has a unique fixed point $x^* \in X$. Moreover, we have $\varphi(x^*) = 0$.*

(2) *Suppose that for all $x,y \in X$, there exist $a,b,c \in [0,+\infty)$ with $a+b+c < 1$ such that $d(Tx,Ty) + \varphi Tx + \varphi Ty \le a[d(x,y) + \varphi(x) + \varphi(y)] + b[d(x,Tx) + \varphi(x) + \varphi(Tx)] + c[d(y,Ty) + \varphi(y) + \varphi(Ty)]$. Then, T has a unique fixed point $x^* \in X$. Moreover, we have $\varphi(x^*) = 0$.*

Theorem 3.5 *Let (X,p) be a complete partial metric space and $T: X \to X$ be a given mapping. The following statements hold:*

(1) *Suppose for any $a,b \in (0,+\infty)$, there exists $0 < \gamma(a,b) < 1$ such that for all $x,y \in X$, $a \le p(x,y) \le b$ implies $p(Tx,Ty) \le \gamma(a,b)p(x,y)$. Then, T has a unique fixed point $x^* \in X$. Moreover, we have $p(x^*,x^*) = 0$.*

(2) *Suppose for all $x,y \in X$, there exist $a,b,c \in (0,+\infty)$ and $a+b+c < 1$ such that $p(Tx,Ty) \le ap(x,y) + bp(x,Tx) + cp(y,Ty)$. Then, T has a unique fixed point $x^* \in X$. Moreover, we have $p(x^*,x^*) = 0$.*

Proof (1) We have $\tilde{p}(x,y) = p(x,y) - \frac{p(x,x)+p(y,y)}{2}$ by Lemma 2.3(2). Then,

$$p(x,y) = \tilde{p}(x,y) + \frac{p(x,x)+p(y,y)}{2},$$

for all $x,y \in X$. Since (X,p) is complete, we have that $(X,\tilde{p})$ is complete by Lemma 3.2(3). Define a function $\varphi: X \to [0,+\infty)$. Set $\varphi(x) = p(x,x)$ for all $x \in X$. Since $p(x,y) = \tilde{p}(x,y) +$

$\frac{p(x,x)+p(y,y)}{2}$, there exists $0 < \gamma(a,b) < 1$ for any $a,b,c \in (0,+\infty)$. From Lemma 3.4, we can deduce that $a \leq \tilde{p}(x,y) + \frac{p(x,x)+p(y,y)}{2} \leq b$ implies $\tilde{p}(Tx,Ty) + \frac{\varphi(Tx)+\varphi(Ty)}{2} \leq \gamma(a,b)[\tilde{p}(x,y) + \frac{\varphi(x)+\varphi(y)}{2}]$.

On the other hand, let $\{x_n\}$ be a sequence in X such that $\lim_{n\to+\infty}\tilde{p}(x_n,x) = 0$, where $x \in X$. Then, we have $\lim_{n\to+\infty} p(x_n,x) = p(x,x)$ by Lemma 3.2(4), i.e., $\lim_{n\to+\infty}\varphi(x_n) = \varphi(x)$, so φ is continuous. By Lemma 3.4(1), the result follows.

(2) It is not difficult to show that

$$\begin{aligned}
&2\tilde{p}(Tx,Ty) + p(Tx,Tx) + p(Ty,Ty) \\
&\quad\leq a\bigl[2\tilde{p}(x,y) + p(x,x) + p(y,y)\bigr] + b\bigl[2\tilde{p}(x,Tx) + p(x,x) + p(Tx,Tx)\bigr] \\
&\qquad + c\bigl[2\tilde{p}(y,Ty) + p(y,y) + p(Ty,Ty)\bigr].
\end{aligned}$$

Set $d = 2\tilde{p}$ and $\varphi(x) = p(x,x)$. By Lemma 3.4(2), then this statement holds. □

Example 3.6 Let $X = [0,+\infty)$. Define $p : X \times X\times \to [0,+\infty)$ as follows: $p(x,y) = \max\{x,y\}$ for all $x,y \in X$. It is clear that (X,p) is a partial metric space. Define a mapping $T: X \to X$ by $Tx = \frac{x}{1+x}$ for all $x \in X$, and taking $\gamma(a,b) = \frac{a+b}{1+a+b}$ for all $a,b \in (0,+\infty)$. Thus, all the conditions of Theorem 3.5(1) are satisfied and obviously $x = 0$ is a fixed point of T.

Definition 3.7 Let p be a partial metric and $(X,\hat{p})$ be the corresponding quasimetric space defined in Theorem 2.6, i.e., $\hat{p}(x,y) = p(x,y) - [\alpha p(x,x) + \beta p(y,y)]$, $0 \leq \alpha,\beta \leq 1, \alpha + \beta = 1$ and $\alpha \neq \frac{1}{2}$, $\beta \neq \frac{1}{2}$, for all $x,y \in X$. (X,p) is said to be a *strong complete partial metric space* if $\lim_{m>n\to+\infty}\hat{p}(x_n,x_m) = 0$ can imply $\lim_{n\to+\infty} x_n = x$ for some $x \in X$.

Remark 3.8 A strong complete partial metric space is a complete partial metric space, but the converse may not be true.

In fact, by (P4), we have

$$\begin{aligned}
&p(x_n,x_m) - p(x,x) \\
&\quad\leq p(x_n,x) + p(x,x_m) - 2p(x,x) \\
&\quad= \hat{p}(x_n,x) + \hat{p}(x,x_m) + \alpha p(x_n,x_n) + \beta p(x_m,x_m) - p(x,x),
\end{aligned}$$

for all $n,m \in \mathbb{N}^+$. Since (X,p) is a strong complete partial metric space, we have $\lim_{m>n\to+\infty}\hat{p}(x_n,x_m) = 0$ and $\lim_{n\to+\infty} x_n = x$, which implies that $\lim_{m,n\to+\infty}[p(x_n,x_m) - p(x,x)] = 0$, namely, (X,p) is complete.

The following example shows that a complete partial metric space may not be a strong complete partial metric space.

Example 3.9 Let $A = \{a_i : a_i = 2i, i \in \mathbb{N}^+\}$ and $B = \{b_i : b_i = 2i+1, i \in \mathbb{N}^+\}$ be two disjoint infinitely countable sets, and $X = A \cup B$. Define a function $p : X \times X \to [0,+\infty)$ by

$$P(x,y) = \begin{cases} 1, & x = y \in A \text{ or } x = y \in B; \\ 1 + \frac{1}{i} + \frac{1}{j}, & x \neq y \text{ and } \{x,y\} \in \{\{a_i,a_j\},\{a_i,b_j\},\{b_i,b_j\}\}. \end{cases}$$

It is not difficult to prove that (X,p) is a complete partial metric space. Set $x_n = 2n$, $x_m = 2m+1$ for all $m > n$, where $n,m \in \mathbb{N}^+$. Then, we have $\hat{p}(x_n,x_m) = p(x_n,x_m) - [\alpha p(x_n,x_n) + \beta p(x_m,x_m)] = \frac{1}{n} + \frac{1}{m}$, and we can deduce $\lim_{m>n\to+\infty} \hat{p}(x_n,x_m) = 0$. However, $\lim_{n\to+\infty} x_n$ does not exist.

Theorem 3.10 *Let p be a partial metric and $(X,\hat{p})$ be the corresponding metric space defined in Theorem 2.6, i.e., $\hat{p}(x,y) = p(x,y) - [\alpha p(x,x) + \beta p(y,y)]$, $0 \le \alpha,\beta \le 1$, $\alpha + \beta = 1$ and $\alpha \ne \frac{1}{2}$, $\beta \ne \frac{1}{2}$, for all $x,y \in X$, and satisfies the following conditions*:

(1) *$(X,\hat{p})$ is a strong complete partial metric space.*

(2) *$f : X \to \mathbb{R}$ is a lower semicontinuous function bounded from below.*

(3) *Let $\varepsilon > 0$, there exists $x_0 \in X$ such that $f(x_0) \le \inf_{x\in X} f(x) + \varepsilon$ for all $x \in X$.*

Then, there exists a point x^* such that $\hat{p}(x_0,x^*) < 1$.

Proof For any $\varepsilon > 0$, we denote $T(x,y) = \varepsilon p(x,y)$. Suppose

$$S_n = \bigl\{t : f(t) + T(t,x_n) \le f(x_n) + \alpha T(t,t) + \beta T(x_n,x_n)\bigr\}.$$

It is easy to see that $x_n \in S_n$, namely, $S_n \ne \varnothing$. Take $x_{n+1} \in S_n$, such that

$$f(x_{n+1}) - \inf_{S_n} f \le \frac{f(x_n) - \inf_{S_n} f}{2}.$$

We have that the sequence $\{f(x_n)\}$ is nonincreasing and bounded from below. Hence, $\{f(x_n)\}$ is a Cauchy sequence. We prove in the following steps:

Step 1: By (P4), we can deduce that

$$\begin{aligned} f(t) + T(t,x_n) &= f(t) + \varepsilon p(t,x_n) \\ &\le f(t) + \varepsilon\bigl[p(t,x_{n+1}) + p(x_{n+1},x_n) - p(x_{n+1},x_{n+1})\bigr] \\ &= f(t) + T(t,x_{n+1}) + T(x_{n+1},x_n) - T(x_{n+1},x_{n+1}). \end{aligned}$$

For any $t \in S_{n+1}$, we have

$$f(t) + T(t,x_{n+1}) \le f(x_{n+1}) + \alpha T(t,t) + \beta T(x_{n+1},x_{n+1}).$$

Then,

$$\begin{aligned} &f(t) + T(t,x_n) \\ &\quad \le f(x_{n+1}) + \alpha T(t,t) + \beta T(x_{n+1},x_{n+1}) + T(x_{n+1},x_n) - T(x_{n+1},x_{n+1}). \end{aligned}$$

Since $x_{n+1} \in S_n$, we have that

$$f(x_{n+1}) + T(x_{n+1},x_n) \le f(x_n) + \alpha T(x_{n+1},x_{n+1}) + \beta T(x_n,x_n).$$

Then,

$$\begin{aligned}
&f(t) + T(t,x_n) \\
&\quad \le f(x_n) + \alpha T(x_{n+1},x_{n+1}) + \beta T(x_n,x_n) + \alpha T(t,t) + \beta T(x_{n+1},x_{n+1}) \\
&\qquad - T(x_{n+1},x_{n+1}) \\
&\quad = f(x_n) + \alpha T(t,t) + \beta T(x_n,x_n),
\end{aligned}$$

which implies that $t \in S_n$, then $S_{n+1} \subset S_n$.

Hence, for any $x_m \in S_m$, we have

$$f(x_m) + T(x_m,x_n) \le f(x_n) + \alpha T(x_m,x_m) + \beta T(x_n,x_n),$$

for all $m \ge n$.

Step 2: By step 1, we have

$$T(x_m,x_n) - \alpha T(x_m,x_m) - \beta T(x_n,x_n) \le f(x_n) - f(x_m),$$

for all $m \ge n$. Since $\{f(x_n)\}$ is a Cauchy sequence, we have

$$\lim_{m>n\to+\infty}\big[T(x_m,x_n) - \alpha T(x_m,x_m) - \beta T(x_n,x_n)\big] = 0,$$

which implies that $\lim_{m>n\to+\infty}[p(x_m,x_n) - \alpha p(x_m,x_m) - \beta p(x_n,x_n)] = 0$, namely, $\lim_{m>n\to+\infty}\hat{p}(x_m,x_n) = 0$. Since $(X,\hat{p})$ is strong complete, there exists some point x^*, such that $\lim_{m\to+\infty} x_m = x^*$, and $\lim_{m\to+\infty} f(x_m) = f(x^*)$.

Furthermore, for any $m \ge n$, we have

$$f(x_m) + T(x_m,x_n) \le f(x_n) + \alpha T(x_m,x_m) + \beta T(x_n,x_n),$$

which implies that

$$f\big(x^*\big) + T\big(x^*,x_n\big) \le f(x_n) + \alpha T\big(x^*,x^*\big) + \beta T(x_n,x_n),$$

so $x^* \in S_n$.

Step 3: From step 1 and step 2, we have

$$\begin{aligned}
&T\big(x^*,x_0\big) - \alpha T\big(x^*,x^*\big) - \beta T(x_0,x_0) \\
&\quad = \lim_{n\to+\infty}\big[T(x_n,x_0) - \alpha T(x_n,x_n) - \beta T(x_0,x_0)\big] \\
&\quad = f(x_0) - \alpha f\big(x^*\big) - \beta f\big(x^*\big) \\
&\quad = f(x_0) - f\big(x^*\big) \\
&\quad \le f(x_0) - \inf_{x\in X} f\big(x^*\big) < \varepsilon,
\end{aligned}$$

which implies $p(x^*,x_0) - \alpha p(x^*,x^*) - \beta p(x_0,x_0) < 1$, namely, $\hat{p}(x_0,x^*) < 1$. □

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