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Some inequalities and numerical results estimating error of approximation for tensor product kind bivariate quantum beta-type operators and pertaining to GBS variant

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Abstract

In this study, we investigate some inequalities estimating the error of approximation for new defined tensor product kind quantum beta-type operators on rectangular regions, and we give an inequality in weighted mean. Moreover, we introduce generalized Boolean sum (GBS) operators pertaining to tensor product kind bivariate quantum beta-type operators, and we also give inequalities estimating the error of approximation for GBS variant concerning the mixed modulus of continuity for Bögel continuous and Bögel differentiable functions, respectively. Finally, we present some applications illustrating numerical results of approximation errors with the help of Maple software.

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1 Introduction

Stancu [1] introduced for each $n \in \mathbb{N}$ the following sequence of beta-type operators including improper integral:

$$L_n(g,x) = \frac{1}{B(nx,n+1)} \int_0^\infty \frac{u^{nx-1}}{1+u^{nx+n+1}} g(u)\,du.$$

Here, g is a real-valued bounded function defined on $[0,\infty)$, which is measurable on $[a,b] \subset [0,\infty)$, and L_n is linear and positive for each $n \in \mathbb{N}$. Abel and Gupta estimated the rate of convergence of the sequence of these beta-type operators for the functions of bounded variations [2]. Besides, Gupta et al. obtained some results for the functions with derivatives of bounded variation in [3]. The quantum generalization of the sequence of the operators L_n was defined by Aral and Gupta in [4]. They gave some direct results and an asymptotic formula.

We initially commence by recalling certain notations of quantum calculus. For nonnegative integers $k = 0, 1, 2, \dots$, the quantum integer denoted by $[k]_q$ is defined by

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1. \end{cases}$$

It is obvious that the quantum integer $[k]_q$ is reduced to the classical nonnegative integer k as $q \to 1^-$. The quantum factorial of nonnegative integer k denoted by $[k]_q!$ is defined as follows:

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k = 1, 2, \dots, \\ 1, & k = 0. \end{cases}$$

Let n, k be nonnegative integers such that $0 \leq k \leq n$. The quantum binomial coefficients denoted by $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

For $A > 0$, the quantum improper integral depending on A is defined by

$$\int_0^{\frac{\infty}{A}} f(u)\, d_q u = \sum_{n=-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}(1-q),$$

provided that the series to the right-hand side of the equality is convergent. The quantum beta function including quantum improper integral depending on A is defined as follows:

$$B_q(a,b) = K(A,a) \int_0^{\frac{\infty}{A}} \frac{u^{a-1}}{(1+u)_q^{a+b}}\, d_q u,$$

where the quantum Pochhammer symbol is denoted by

$$(a+b)_q^n = \prod_{j=0}^{n-1} \left(a + q^j b\right).$$

Furthermore, $K(y,u)$ (see [5]) denotes the following notation:

$$K(y,u) = \frac{1}{y+1} y^u \left(1 + \frac{1}{y}\right)_q^u (1+y)_q^{1-u},$$

which satisfies the following recurrence formula:

$$K(A, u+1) = q^u K(A,u).$$

For $u, v > 0$, another definition of quantum beta function is given as follows:

$$B_q(u,v) = \int_0^1 y^{u-1}(1-qy)_q^{v-1}\, d_q y.$$

The relation between quantum beta function and quantum gamma function is known with the following equality:

$$B_q(u,v) = \frac{\Gamma_q(u)\Gamma_q(v)}{\Gamma_q(u+v)}.$$

Extensive details about the quantum calculus can be found in references [5, 6].

Aral ve Gupta [4] defined the following sequence of quantum beta-type operators:

$$S_n^q\bigl(g(\tau),x\bigr) = \frac{K(A,[n]_q x)}{B_q([n]_q x,[n]_q+1)}\int_0^{\infty/A}\frac{u^{[n]_q x-1}}{(1+u)_q^{[n]_q x+[n]_q+1}}g\bigl(q^{[n]_q x}u\bigr)\,d_q u. \tag{1.1}$$

Here g is a quantum improper integrable function defined on $[0,\infty)$, $q\in(0,1)$. Aral and Gupta calculated the following equalities:

$$S_n^q(1,x) = 1, \tag{1.2}$$

$$S_n^q(\tau,x) = x, \tag{1.3}$$

$$S_n^q\bigl(\tau^2,x\bigr) = \frac{([n]_q x+1)x}{q([n]_q-1)}, \tag{1.4}$$

and they also obtained the mth-order moment as follows:

$$S_n^q\bigl(\tau^m,x\bigr) = \frac{\Gamma_q([n]_q x+m)\Gamma_q([n]_q-m+1)}{\Gamma_q([n]_q x)\Gamma_q([n]_q+1)q^{m(m-1)/2}}. \tag{1.5}$$

Recently, it has been studied on approximation properties of varied bivariate operators. For instance, readers can see references [7–10].

In this study, we define tensor product kind bivariate quantum beta-type operators. We investigate inequalities estimating the error of approximations and present numerical results. In Sect. 2, we construct tensor product kind bivariate quantum beta-type operators, and we give some auxiliary results. In Sect. 3, we give inequalities estimating the error of approximation on rectangular regions. In Sect. 4, we give an inequality in weighted mean. In Sect. 5, we construct generalized Boolean sum operators of tensor product kind bivariate quantum beta-type operators and give inequalities estimating the error of approximation by means of the mixed modulus of continuity for Bögel continuous and Bögel differentiable functions. In Sect. 6, we present numerical results.

2 Tensor product kind bivariate operators and auxiliary results

In this section, we introduce tensor product kind bivariate quantum beta-type operators and give some auxiliary results.

Definition 1 Let $\mathbb{R}_+ = [0,\infty)$, $A_1,A_2>0$, $q_1,q_2\in(0,1)$ and h be a bivariate continuous and bounded function defined on $\mathbb{R}_+\times\mathbb{R}_+$. For all $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ and n_1, $n_2\in\mathbb{N}$, we define tensor product kind bivariate quantum beta-type operators by

$$\begin{aligned} S_{n_1,n_2}^{q_1,q_2}\big(h(\tau,\sigma);x,y\big) &= \frac{K(A_1,[n_1]_{q_1}x)}{B_{q_1}([n_1]_{q_1}x,[n_1]_{q_1}+1)}\frac{K(A_2,[n_2]_{q_2}y)}{B_{q_2}([n_2]_{q_2}y,[n_2]_{q_2}+1)} \\ &\quad\times\int_0^{\frac{\infty}{A_1}}\int_0^{\frac{\infty}{A_2}}\frac{u^{[n_1]_{q_1}x-1}}{(1+u)_{q_1}^{[n_1]_{q_1}x+[n_1]_{q_1}+1}}\frac{v^{[n_2]_{q_2}y-1}}{(1+v)_{q_2}^{[n_2]_{q_2}y+[n_2]_{q_2}+1}} \\ &\quad\times h\big({q_1}^{[n_1]_{q_1}x}u,{q_2}^{[n_2]_{q_2}y}v\big)\,d_{q_2}v\,d_{q_1}u. \end{aligned}$$

It is obvious that the tensor product kind bivariate quantum beta-type operators are linear positive operators.

Lemma 1 *The following equalities are valid for the tensor product kind quantum beta-type operators*:

(i) $S_{n_1,n_2}^{q_1,q_2}\big(h(\tau,\sigma);x,y\big)=S_{n_1}^{x}\big(S_{n_2}^{y}\big(h(\tau,\sigma);q_2\big);q_1\big)$,

(ii) $S_{n_1,n_2}^{q_1,q_2}\big(h(\tau,\sigma);x,y\big)=S_{n_2}^{y}\big(S_{n_1}^{x}\big(h(\tau,\sigma);q_1\big);q_2\big)$,

where

$$\begin{aligned} S_{n_1}^{x}\big(h(\tau,\sigma);q_1\big) &= \frac{K(A_1,[n_1]_{q_1}x)}{B_{q_1}([n_1]_{q_1}x,[n_1]_{q_1}+1)}\int_0^{\frac{\infty}{A_1}}\frac{u^{[n_1]_{q_1}x-1}}{(1+u)_{q_1}^{[n_1]_{q_1}x+[n_1]_{q_1}+1}} \\ &\quad\times h\big({q_1}^{[n_1]_{q_1}x}u,\sigma\big)\,d_{q_1}u \end{aligned}$$

and

$$\begin{aligned} S_{n_2}^{y}\big(h(\tau,\sigma);q_2\big) &= \frac{K(A_2,[n_2]_{q_2}y)}{B_{q_2}([n_2]_{q_2}y,[n_2]_{q_2}+1)}\int_0^{\frac{\infty}{A_2}}\frac{v^{[n_2]_{q_2}y-1}}{(1+v)_{q_2}^{[n_2]_{q_2}y+[n_2]_{q_2}+1}} \\ &\quad\times h\big(\tau,{q_2}^{[n_2]_{q_2}y}v\big)\,d_{q_2}v. \end{aligned}$$

Proof The proof of lemma is obvious by considering the definition of $S_{n_1}^{x}$ and $S_{n_2}^{y}$; therefore, we omit the proof. □

Lemma 2 *We have the following equalities for the tensor product kind bivariate quantum beta-type operators*:

(i) $S_{n_1,n_2}^{q_1,q_2}(1;x,y)=1$,

(ii) $S_{n_1,n_2}^{q_1,q_2}(\tau;x,y)=x$,

(iii) $S_{n_1,n_2}^{q_1,q_2}(\sigma;x,y)=y$,

(iv) $S_{n_1,n_2}^{q_1,q_2}\big(\tau^2;x,y\big)=\dfrac{([n_1]_{q_1}x+1)x}{q_1([n_1]_{q_1}-1)}$,

(v) $S_{n_1,n_2}^{q_1,q_2}\big(\sigma^2;x,y\big)=\dfrac{([n_2]_{q_2}y+1)y}{q_2([n_2]_{q_2}-1)}$,

(vi) $S_{n_1,n_2}^{q_1,q_2}\big(\tau^3;x,y\big)=\dfrac{([n_1]_{q_1}x+2)([n_1]_{q_1}x+1)x}{{q_1}^3([n_1]_{q_1}-1)([n_1]_{q_1}-2)}$,

$$\text{(vii)}\quad S_{n_1,n_2}^{q_1,q_2}\left(\sigma^3;x,y\right)=\frac{([n_2]_{q_2}y+2)([n_2]_{q_2}y+1)y}{q_2{}^3([n_2]_{q_2}-1)([n_2]_{q_2}-2)},$$

$$\text{(viii)}\quad S_{n_1,n_2}^{q_1,q_2}\left(\tau^4;x,y\right)=\frac{([n_1]_{q_1}x+3)([n_1]_{q_1}x+2)([n_1]_{q_1}x+1)x}{q_1{}^6([n_1]_{q_1}-1)([n_1]_{q_1}-2)([n_1]_{q_1}-3)},$$

$$\text{(ix)}\quad S_{n_1,n_2}^{q_1,q_2}\left(\sigma^4;x,y\right)=\frac{([n_2]_{q_2}y+3)([n_2]_{q_2}y+2)([n_2]_{q_2}y+1)y}{q_2{}^6([n_2]_{q_2}-1)([n_2]_{q_2}-2)([n_2]_{q_2}-3)}.$$

Proof Using Lemma 1, with a tiny calculation, (i)–(iii) can be obtained. Along with Lemma 1, by using the formula for the mth order moment given in (1.5), (iv)–(ix) can be simply proved; therefore, we omit the proof. □

Lemma 3 *Let $n_1, n_2 \in \mathbb{N}$ be sufficiently large, for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$, we have*

$$S_{n_1,n_2}^{q_1,q_2}\left((\tau-x)^2;x,y\right)=\begin{cases}O(\frac{1}{[n_1]_{q_1}}), & 0\le x\le 1,\\ x^2O(\frac{1}{[n_1]_{q_1}}), & 1<x,\end{cases}$$

$$S_{n_1,n_2}^{q_1,q_2}\left((\sigma-y)^2;x,y\right)=\begin{cases}O(\frac{1}{[n_2]_{q_2}}), & 0\le y\le 1,\\ y^2O(\frac{1}{[n_2]_{q_2}}), & 1<y,\end{cases}$$

$$S_{n_1,n_2}^{q_1,q_2}\left((\tau-x)^4;x,y\right)=\begin{cases}O(\frac{1}{[n_1]_{q_1}}), & 0\le x\le 1,\\ x^4O(\frac{1}{[n_1]_{q_1}}), & 1<x,\end{cases}$$

$$S_{n_1,n_2}^{q_1,q_2}\left((\sigma-y)^4;x,y\right)=\begin{cases}O(\frac{1}{[n_2]_{q_2}}), & 0\le y\le 1,\\ y^4O(\frac{1}{[n_2]_{q_2}}), & 1<y.\end{cases}$$

Proof By considering Lemma 1, since $1-q_1\le\frac{1}{[n_1]_{q_1}}$, we can write

$$\begin{aligned}S_{n_1,n_2}^{q_1,q_2}\left((\tau-x)^2;x,y\right)&=\frac{(1-q_1+\frac{q_1}{[n_1]_{q_1}})x^2+\frac{1}{[n_1]_{q_1}}x}{q_1(1-\frac{1}{[n_1]_{q_1}})}\\&\le\frac{1+q_1}{[n_1]_{q_1}}x^2+\frac{1}{[n_1]_{q_1}}x\\&\le\frac{1}{[n_1]_{q_1}}\left(2x^2+x\right),\end{aligned}$$

which implies

$$S_{n_1,n_2}^{q_1,q_2}\left((\tau-x)^2;x,y\right)=\begin{cases}O(\frac{1}{[n_1]_{q_1}}), & 0\le x\le 1,\\ x^2O(\frac{1}{[n_1]_{q_1}}), & 1<x.\end{cases}$$

Similarly,

$$S_{n_1,n_2}^{q_1,q_2}\left((\sigma-y)^2;x,y\right)=\begin{cases}O(\frac{1}{[n_2]_{q_2}}), & 0\le y\le 1,\\ y^2O(\frac{1}{[n_2]_{q_2}}), & 1<y.\end{cases}$$

By considering Lemma 1, we get

$$
\begin{aligned}
&S_{n_1,n_2}^{q_1,q_2}\bigl((\tau-x)^4;x,y\bigr)\\
&\quad=\frac{1-4{q_1}^3+6{q_1}^5-3{q_1}^6+\frac{12{q_1}^3-30{q_1}^5+18{q_1}^6}{[n_1]_{q_1}}+\frac{36{q_1}^5-33{q_1}^6}{{[n_1]_{q_1}}^2}+\frac{18{q_1}^6}{{[n_1]_{q_1}}^3}}{(1-\frac{1}{[n_1]_{q_1}})(1-\frac{2}{[n_1]_{q_1}})(1-\frac{3}{[n_1]_{q_1}}){q_1}^6}x^4\\
&\qquad+\frac{\frac{6-12{q_1}^3+6{q_1}^5}{[n_1]_{q_1}}+\frac{36{q_1}^3-30{q_1}^5}{{[n_1]_{q_1}}^2}+\frac{36{q_1}^5}{{[n_1]_{q_1}}^3}}{(1-\frac{1}{[n_1]_{q_1}})(1-\frac{2}{[n_1]_{q_1}})(1-\frac{3}{[n_1]_{q_1}}){q_1}^6}x^3\\
&\qquad+\frac{\frac{11-8{q_1}^3}{{[n_1]_{q_1}}^2}+\frac{24}{{[n_1]_{q_1}}^3}}{(1-\frac{1}{[n_1]_{q_1}})(1-\frac{2}{[n_1]_{q_1}})(1-\frac{3}{[n_1]_{q_1}}){q_1}^6}x^2.
\end{aligned}
$$

Since $1-4{q_1}^3+6{q_1}^5-3{q_1}^6=(1-q_1)(1+q_1+{q_1}^2)(1-3{q_1}^3)\le 1-q_1\le\frac{1}{[n_1]_{q_1}}$ and $\frac{1}{[n_1]_{q_1}}\le\frac{1}{{[n_1]_{q_1}}^2}\le\frac{1}{{[n_1]_{q_1}}^3}$, we obtain

$$S_{n_1,n_2}^{q_1,q_2}\bigl((\tau-x)^4;x,y\bigr)=O\biggl(\frac{1}{[n_1]_{q_1}}\biggr)\bigl(x^4+x^3+x^2\bigr),\quad x\ge 0.$$

That implies

$$S_{n_1,n_2}^{q_1,q_2}\bigl((\tau-x)^4;x,y\bigr)=\begin{cases}O(\frac{1}{[n_1]_{q_1}}), & 0\le x\le 1,\\ x^4O(\frac{1}{[n_1]_{q_1}}), & 1<x.\end{cases}$$

Similarly, we get

$$S_{n_1,n_2}^{q_1,q_2}\bigl((\sigma-y)^4;x,y\bigr)=\begin{cases}O(\frac{1}{[n_2]_{q_2}}), & 0\le y\le 1,\\ y^4O(\frac{1}{[n_2]_{q_2}}), & 1<y.\end{cases}$$ □

3 Inequalities on rectangular regions

In this part, we give some inequalities estimating the error of approximation in view of the complete modulus of continuity, the partial modulus of continuity, and the Lipschitz class for the bivariate continuous functions on rectangular regions. Firstly, we give an auxiliary result possessing an important role in the proofs.

Lemma 4 *Let $p,r>1$ such that $\frac{1}{p}+\frac{1}{r}=1$. Let f and g be any q-improper integrable functions defined on $[0,\infty)$. Then we have the following inequality*:

$$\int_0^{\infty/A}\bigl|f(u)\bigr|g(u)\,d_qu\le\biggl(\int_0^{\infty/A}\bigl|f(u)\bigr|^pg(u)\,d_qu\biggr)^{1/p}\biggl(\int_0^{\infty/A}g(u)\,d_qu\biggr)^{1/r}.$$

Proof Let $p,r > 1$ satisfying $\frac{1}{p} + \frac{1}{r} = 1$ and f, g be any q-improper integrable functions defined on $[0,\infty)$. By the definition of the q-improper integral, we can write

$$\begin{aligned}\int_0^{\infty/A} \left|f(u)\right| g(u)\, d_q u &= \sum_{n=-\infty}^{\infty} \left|f\left(\frac{q^n}{A}\right)\right| g\left(\frac{q^n}{A}\right)\frac{q^n}{A}(1-q) \\ &= \sum_{n=0}^{\infty} \left|f\left(\frac{q^n}{A}\right)\right| g\left(\frac{q^n}{A}\right)\frac{q^n}{A}(1-q) \\ &\quad + \sum_{n=-\infty}^{1} \left|f\left(\frac{q^n}{A}\right)\right| g\left(\frac{q^n}{A}\right)\frac{q^n}{A}(1-q).\end{aligned} \tag{3.1}$$

By rearranging the second series in (3.1) for $n = -k$, we obtain

$$\begin{aligned}\int_0^{\infty/A} \left|f(u)\right| g(u)\, d_q u &= \sum_{n=0}^{\infty} \left|f\left(\frac{q^n}{A}\right)\right| g\left(\frac{q^n}{A}\right)\frac{q^n}{A}(1-q) \\ &\quad + \sum_{k=1}^{\infty} \left|f\left(\frac{q^{-k}}{A}\right)\right| g\left(\frac{q^{-k}}{A}\right)\frac{q^{-k}}{A}(1-q).\end{aligned} \tag{3.2}$$

By applying the Hölder inequality to (3.2), we get

$$\begin{aligned}\int_0^{\infty/A} \left|f(u)\right| g(u)\, d_q u &= \left\{\sum_{n=0}^{\infty} \left|f\left(\frac{q^n}{A}\right)\right|^p g\left(\frac{q^n}{A}\right)\frac{q^n}{A}(1-q)\right\}^{1/p} \\ &\quad \times \left\{\sum_{n=0}^{\infty} g\left(\frac{q^n}{A}\right)\frac{q^n}{A}(1-q)\right\}^{1/r} \\ &\quad + \left\{\sum_{k=1}^{\infty} \left|f\left(\frac{q^{-k}}{A}\right)\right|^p g\left(\frac{q^{-k}}{A}\right)\frac{q^{-k}}{A}(1-q)\right\}^{1/p} \\ &\quad \times \left\{\sum_{k=1}^{\infty} g\left(\frac{q^{-k}}{A}\right)\frac{q^{-k}}{A}(1-q)\right\}^{1/r}.\end{aligned} \tag{3.3}$$

By regulating (3.3) for $k = -n$, we have

$$\begin{aligned}\int_0^{\infty/A} \left|f(u)\right| g(u)\, d_q u &\leq \left\{\sum_{n=-\infty}^{\infty} \left|f\left(\frac{q^n}{A}\right)\right|^p g\left(\frac{q^n}{A}\right)\frac{q^n}{A}(1-q)\right\}^{1/p} \\ &\quad \times \left\{\sum_{n=-\infty}^{\infty} g\left(\frac{q^n}{A}\right)\frac{q^n}{A}(1-q)\right\}^{1/r}.\end{aligned}$$

Lastly, by considering the definition of q-improper integral, we reach that

$$\int_0^{\infty/A} \left|f(u)\right| g(u)\, d_q u \leq \left(\int_0^{\infty/A} \left|f(u)\right|^p g(u)\, d_q u\right)^{1/p} \left(\int_0^{\infty/A} g(u)\, d_q u\right)^{1/r}. \qquad \square$$

Let A be a compact subset of $\mathbb{R}^2$. For $h \in C(A)$, the complete modulus of continuity for the bivariate function h is defined as follows:

$$\omega(h;\delta_1,\delta_2) = \sup\{\left|h(x_1,y_1) - h(x_2,y_2)\right| : |x_1 - x_2| \leq \delta_1, |y_1 - y_2| \leq \delta_2\},$$

where $\delta_1 > 0$, $\delta_2 > 0$ and $(x_1,y_1),(x_2,y_2) \in A$, which possesses the following property:

$$\left|h(x_1,y_1) - h(x_2,y_2)\right| \leq \omega(h;\delta_1,\delta_2)\left(1 + \frac{|x_1 - x_2|}{\delta_1}\right)\left(1 + \frac{|y_1 - y_2|}{\delta_2}\right). \tag{3.4}$$

It is clear that

$$\lim_{\delta_1,\delta_2\to 0^+} \omega(h;\delta_1,\delta_2) = 0.$$

Let rectangular regions be denoted by I_R such that $I_R := I_1 \times I_{2,}$ where $I_i = [0,r_i]$ for $r_1,r_2 > 0$ and $i = 1,2$.

Let $\{q_{1,n_1}\}$ and $\{q_{2,n_2}\}$ be any sequences such that $q_{1,n_1}, q_{2,n_2} \in (0,1)$ satisfying the following condition:

$$\lim_{n_1\to\infty} q_{1,n_1} = \lim_{n_2\to\infty} q_{2,n_2} = 1. \tag{3.5}$$

Theorem 1 *Let $\{q_{1,n_1}\}$ and $\{q_{2,n_2}\}$ be any sequences such that $q_{1,n_1}, q_{2,n_2} \in (0,1)$ satisfying the condition given in (3.5). If any $h \in C(I_R)$, then the following inequality holds*:

$$\left|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(h(\tau,\sigma);x,y\bigr) - h(x,y)\right| \leq 4\omega\Bigl(h;\sqrt{\mu_{n_1}^{x,q_{1,n_1}}},\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\Bigr).$$

Here

$$\mu_{n_1}^{x,q_{1,n_1}} = \frac{((1-q_{1,n_1})[n_1]_{q_{1,n_1}} + q_{1,n_1})x^2 + x}{q_{1,n_1}([n_1]_{q_{1,n_1}} - 1)}, \tag{3.6}$$

$$\mu_{n_2}^{y,q_{2,n_2}} = \frac{((1-q_{2,n_2})[n_2]_{q_{2,n_2}} + q_{2,n_2})y^2 + y}{q_{2,n_2}([n_2]_{q_{2,n_2}} - 1)}. \tag{3.7}$$

Proof By applying the operators $S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}$ to (3.4) and by considering the linearity and positivity of the operators $S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}$, we can write

$$\begin{aligned}
&\left|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(h(\tau,\sigma);x,y\bigr) - h(x,y)\right| \\
&\quad\leq S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(\left|h(\tau,\sigma) - h(x,y)\right|;x,y\bigr) \\
&\quad\leq \omega(h;\delta_1,\delta_2)\biggl\{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y) + \frac{1}{\delta_1}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(|\tau - x|;x,y\bigr) \\
&\qquad + \frac{1}{\delta_2}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(|\sigma - y|;x,y\bigr) + \frac{1}{\delta_1\delta_2}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(|\tau - x||\sigma - y|;x,y\bigr)\biggr\}.
\end{aligned}$$

By applying Lemma 4 for $p = r = 2$, we obtain

$$\begin{aligned}
&\left|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(h(\tau,\sigma);x,y\bigr) - h(x,y)\right| \\
&\quad\leq \omega(h;\delta_1,\delta_2)\biggl\{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\delta_1}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2;x,y\big)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)} \\
&+ \frac{1}{\delta_2}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\sigma-y)^2;x,y\big)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)} \\
&+ \frac{1}{\delta_1\delta_2}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2(\sigma-y)^2;x,y\big)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)}\bigg\}.
\end{aligned}
$$

In view of Lemmas 1 and 2, after simple calculation, by selecting $\delta_1 = \sqrt{\mu_{n_1}^{x,q_{1,n_1}}}$ and $\delta_2 = \sqrt{\mu_{n_2}^{y,q_{2,n_2}}}$, we get the desired result. □

For $h \in C(A)$, the partial modulus of continuity with respect to x and y is defined as follows:

$$
\begin{aligned}
\omega_1(h;\delta_1) &= \sup\big\{\big|h(x_1,y)-h(x_2,y)\big| : |x_1-x_2| \le \delta_1\big\}, \\
\omega_2(h;\delta_2) &= \sup\big\{\big|h(x,y_1)-h(x,y_2)\big| : |y_1-y_2| \le \delta_2\big\},
\end{aligned}
$$

where $\delta_1 > 0$, $\delta_2 > 0$, (x_1,y), (x_2,y), (x,y_1) and $(x,y_2) \in A$.

It is clear that $\omega_1(h;\delta_1)$, $\omega_2(h;\delta_2)$ possess similar properties as to the usual modulus of continuity; therefore, we have the following inequalities:

$$
\begin{aligned}
\big|h(x_1,y)-h(x_2,y)\big| &\le \omega_1(h;\delta_1)\left(1+\frac{(x_1-x_2)^2}{\delta_1}\right), \quad \delta_1 > 0, \\
\big|h(x,y_1)-h(x,y_2)\big| &\le \omega_2(h;\delta_2)\left(1+\frac{(y_1-y_2)^2}{\delta_2}\right), \quad \delta_2 > 0.
\end{aligned}
$$

Theorem 2 *Let $\{q_{1,n_1}\}$ and $\{q_{2,n_2}\}$ be any sequences such that $q_{1,n_1}, q_{2,n_2} \in (0,1)$ satisfying the condition given in (3.5). If any $h \in C(I_R)$, then the following inequality holds*:

$$
\big|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big)-h(x,y)\big| \le 2\omega_1\big(h;\mu_{n_1}^{x,q_{1,n_1}}\big)+2\omega_2\big(h;\mu_{n_2}^{y,q_{2,n_2}}\big),
$$

where $\mu_{n_1}^{x,q_{1,n_1}}$ and $\mu_{n_2}^{y,q_{2,n_2}}$ are as in (3.6) and (3.7), respectively.

Proof By the definition of partial modulus of continuity, by considering the linearity and positivity of the operators $S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}$, we can write

$$
\begin{aligned}
\big|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big)-h(x,y)\big| &\le S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(\big|h(\tau,\sigma)-h(x,y)\big|;x,y\big) \\
&\le S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(\big|h(\tau,\sigma)-h(x,\sigma)\big|;x,y\big) \\
&\quad + S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(\big|h(x,\sigma)-h(x,y)\big|;x,y\big) \\
&\le \omega_1(h;\delta_1)\left(1+\frac{1}{\delta_1}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2;x,y\big)\right) \\
&\quad + \omega_2(h;\delta_1)\left(1+\frac{1}{\delta_2}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\sigma-y)^2;x,y\big)\right).
\end{aligned}
$$

In view of Lemmas 1 and 2, after simple calculation, by selecting $\delta_1 = \mu_{n_1}^{x,q_{1,n_1}}$ and $\delta_2 = \mu_{n_2}^{y,q_{2,n_2}}$, we complete the proof of the theorem. □

Now, we consider the following functions of Lipschitz class of $h \in C(A)$ denoted by $\mathrm{Lip}_M(h;\theta,A)$ satisfying

$$\left|h(x_1,y_1),-h(x_2,y_2)\right| \leq M_h|x_1-x_2|^\theta|y_1-y_2|^\theta, \quad M_h>0,$$

where (x_1,y_1), $(x_2,y_2) \in A$ and $0<\theta\leq 1$. Here is the aim of selecting this Lipschitz class to being convenient that Lemma 4 is applicable.

Theorem 3 *Let $\{q_{1,n_1}\}$ and $\{q_{2,n_2}\}$ be any sequences such that $q_{1,n_1}, q_{2,n_2} \in (0,1)$ satisfy the condition given in (3.5). If any $h \in \mathrm{Lip}_M(h;\theta,I_R)$, then the following inequality holds*:

$$\left|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big)-h(x,y)\right| \leq M_h\Big\{\sqrt{\mu_{n_1}^{x,q_{1,n_1}}\mu_{n_2}^{y,q_{2,n_2}}}\Big\}^\theta, \quad M_h>0.$$

Here, $\mu_{n_1}^{x,q_{1,n_1}}$ and $\mu_{n_2}^{y,q_{2,n_2}}$ are as in (3.6) and (3.7), respectively.

Proof Since $h \in \mathrm{Lip}_M(h;\theta,I_R)$, by Lemma 1 and considering the linearity and positivity of the operators $S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}$, we can write

$$\begin{aligned}\left|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big)-h(x,y)\right| &\leq S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(\left|h(\tau,\sigma)-h(x,y)\right|;x,y\big)\\ &\leq M_h S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(|\tau-x|^\theta|\sigma-y|^\theta;x,y\big).\end{aligned}$$

In the last inequality, by applying Lemma 4 for $p=\frac{2}{\theta}$, $r=\frac{2}{2-\theta}$, we obtain

$$\begin{aligned}&\left|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big)-h(x,y)\right|\\ &\quad\leq M_h\big\{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2(\sigma-y)^2;x,y\big)\big\}^{\theta/2}\big\{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)\big\}^{2-\theta/2}\\ &\quad=M_h\Big\{\sqrt{\mu_{n_1}^{x,q_{1,n_1}}\mu_{n_2}^{y,q_{2,n_2}}}\Big\}^\theta,\end{aligned}$$

which completes the proof of the theorem. □

4 An inequality in weighted mean

Let $\rho(x,y)=1+x^2+y^2$ be a weight function and $\mathbb{R}_+=[0,\infty)$. Let us denote by B_ρ the space of all functions h defined on $\mathbb{R}_+\times\mathbb{R}_+$ satisfying

$$\left|h(x,y)\right| \leq M_h\rho(x,y), \quad M_h>0.$$

Let C_ρ denote the subspace of all continuous functions h of B_ρ with the norm

$$\|h\|_\rho=\sup_{(x,y)\in\mathbb{R}_+\times\mathbb{R}_+}\frac{|h(x,y)|}{\rho(x,y)}.$$

Let C_ρ^0 denote the subspace of all functions in C_ρ such that

$$\lim_{\sqrt{x^2+y^2}\to\infty}\frac{h(x,y)}{\rho(x,y)}$$

exists finitely.

İspir and Atakut [11] introduced the following weighted modulus of continuity for each $h \in C_\rho^0$ and $\delta_1, \delta_2 > 0$ by

$$\Omega_\rho(h; \delta_1, \delta_2) = \sup_{(x,y)\in\mathbb{R}_+\times\mathbb{R}_+} \sup_{\substack{0\leq k\leq\delta_1\\ 0\leq l\leq\delta_2}} \frac{|h(x+k, y+l)|}{\rho(x,y)\rho(k,l)},$$

which has the following property:

$$\Omega_\rho(h; \lambda_1\delta_1, \lambda_2\delta_2) \leq 4(1+\lambda_1)(1+\lambda_2)\bigl(1+{\delta_1}^2\bigr)\bigl(1+{\delta_2}^2\bigr)\Omega_\rho(h; \delta_1, \delta_2) \tag{4.1}$$

for $\lambda_1, \lambda_2 > 0$. From the definition of Ω_ρ, we have

$$\begin{aligned} \bigl|h(\tau,\sigma) - h(x,y)\bigr| &\leq \bigl(1+x^2+y^2\bigr)\bigl(1+(\tau-x)^2\bigr)\bigl(1+(\sigma-y)^2\bigr) \\ &\quad \times \Omega_\rho\bigl(h; |\tau-x|, |\sigma-y|\bigr). \end{aligned} \tag{4.2}$$

Now, we estimate the error of approximation by means of the weighted modulus of continuity.

Theorem 4 *Let $\{q_{1,n_1}\}$ and $\{q_{2,n_2}\}$ be any sequences such that $q_{1,n_1}, q_{2,n_2} \in (0,1)$ satisfying the condition given in (3.5) and $n_1, n_2 \in \mathbb{N}$ be sufficiently large. If any $h \in C_\rho^0$, then we have the following inequality*:

$$\sup_{(x,y)\in\mathbb{R}_+\times\mathbb{R}_+} \frac{|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h(\tau,\sigma);x,y) - h(x,y)|}{\rho^\varphi(x,y)} \leq K\Omega_\rho\Bigl(h; \sqrt{\mu_{n_1}^{x,q_{1,n_1}}}, \sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\Bigr),$$

where K is a certain constant, $\varphi \geq 2$, $\mu_{n_1}^{x,q_{1,n_1}}$, and $\mu_{n_2}^{y,q_{2,n_2}}$ are as in (3.6) and (3.7), respectively.

Proof By considering inequalities (4.1) and (4.2), we can write

$$\begin{aligned} &\bigl|h(\tau,\sigma) - h(x,y)\bigr| \\ &\quad \leq 4\bigl(1+x^2+y^2\bigr)\bigl(1+(\tau-x)^2\bigr)\bigl(1+(\sigma-y)^2\bigr)\biggl(1+\frac{|\tau-x|}{\delta_1}\biggr)\biggl(1+\frac{|\sigma-y|}{\delta_2}\biggr) \\ &\quad\quad \times \bigl(1+{\delta_1}^2\bigr)\bigl(1+{\delta_2}^2\bigr)\Omega_\rho(h;\delta_1,\delta_2) \\ &\quad = 4\bigl(1+x^2+y^2\bigr)\bigl(1+{\delta_1}^2\bigr)\bigl(1+{\delta_2}^2\bigr)\biggl(1+\frac{|\tau-x|}{\delta_1}+(\tau-x)^2+\frac{|\tau-x|}{\delta_1}(\tau-x)^2\biggr) \\ &\quad\quad \times \biggl(1+\frac{|\sigma-y|}{\delta_2}+(\sigma-y)^2+\frac{|\sigma-y|}{\delta_2}(\sigma-y)^2\biggr)\Omega_\rho(h;\delta_1,\delta_2). \end{aligned}$$

By applying the operators $S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}$ to the above inequality, taking the linearity and positivity of $S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}$ into account, we obtain

$$\begin{aligned} &\bigl|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(h(\tau,\sigma);x,y\bigr) - h(x,y)\bigr| \\ &\quad \leq S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(\bigl|h(\tau,\sigma) - h(x,y)\bigr|;x,y\bigr) \end{aligned}$$

$$\begin{aligned}
&\leq 4\left(1+x^2+y^2\right)\left(1+{\delta_1}^2\right)\left(1+{\delta_2}^2\right)\\
&\quad\times S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(1+\frac{|\tau-x|}{\delta_1}+(\tau-x)^2+\frac{|\tau-x|}{\delta_1}(\tau-x)^2;x,y\right)\\
&\quad\times S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(1+\frac{|\sigma-y|}{\delta_2}+(\sigma-y)^2+\frac{|\sigma-y|}{\delta_2}(\sigma-y)^2;x,y\right)\\
&\quad\times \Omega_\rho(h;\delta_1,\delta_2)\\
&=4\left(1+x^2+y^2\right)\left(1+{\delta_1}^2\right)\left(1+{\delta_2}^2\right)\\
&\quad\times\left\{1+\frac{1}{\delta_1}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(|\tau-x|;x,y\right)+S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\tau-x)^2;x,y\right)\right.\\
&\quad\left.+\frac{1}{\delta_1}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(|\tau-x|(\tau-x)^2;x,y\right)\right\}\\
&\quad\times\left\{1+\frac{1}{\delta_2}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(|\sigma-y|;x,y\right)+S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\sigma-y)^2;x,y\right)\right.\\
&\quad\left.+\frac{1}{\delta_2}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(|\sigma-y|(\sigma-y)^2;x,y\right)\right\}\Omega_\rho(h;\delta_1,\delta_2).
\end{aligned}$$

By using Lemma 4 for $p=q=2$, we calculate that

$$\begin{aligned}
&\left|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(h(\tau,\sigma);x,y\right)-h(x,y)\right|\\
&\leq 4\left(1+x^2+y^2\right)\left\{1+\frac{1}{\delta_1}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\tau-x)^2;x,y\right)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)}\right.\\
&\quad+S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\tau-x)^2;x,y\right)\\
&\quad\left.+\frac{1}{\delta_1}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\tau-x)^4;x,y\right)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\tau-x)^2;x,y\right)}\right\}\\
&\quad\times\left\{1+\frac{1}{\delta_2}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\sigma-y)^2;x,y\right)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)}\right.\\
&\quad+S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\sigma-y)^2;x,y\right)\\
&\quad\left.+\frac{1}{\delta_2}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\sigma-y)^4;x,y\right)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left((\sigma-y)^2;x,y\right)}\right\}\\
&\quad\times\left(1+{\delta_1}^2\right)\left(1+{\delta_2}^2\right)\Omega_\rho(h;\delta_1,\delta_2).
\end{aligned}$$

By considering Lemmas 1, 2, and 3 and choosing $\delta_1=\sqrt{\mu_{n_1}^{x,q_{1,n_1}}}$ and $\delta_2=\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}$, we have four cases:

For $0\leq x\leq 1$, $0\leq y\leq 1$, there exists $K_1>0$ such that

$$\begin{aligned}
\left|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\left(h(\tau,\sigma);x,y\right)-h(x,y)\right| &\leq K_1\left(1+x^2+y^2\right)\\
&\quad\times\Omega_\rho\left(h;\sqrt{\mu_{n_1}^{x,q_{1,n_1}}},\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\right).
\end{aligned}\tag{4.3}$$

For $0 \le x \le 1$, $y > 1$, there exists $K_2 > 0$ such that

$$\begin{aligned} \left|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big) - h(x,y)\right| &\le K_2\big(1+x^2+y^2\big)\big(1+y^2\big) \\ &\quad \times \Omega_\rho\Big(h;\sqrt{\mu_{n_1}^{x,q_{1,n_1}}},\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\Big). \end{aligned} \tag{4.4}$$

For $x > 1$, $0 \le y \le 1$, there exists $K_3 > 0$ such that

$$\begin{aligned} \left|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big) - h(x,y)\right| &\le K_3\big(1+x^2+y^2\big)\big(1+x^2\big) \\ &\quad \times \Omega_\rho\Big(h;\sqrt{\mu_{n_1}^{x,q_{1,n_1}}},\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\Big). \end{aligned} \tag{4.5}$$

For $x > 1$, $y > 1$, there exists $K_4 > 0$ such that

$$\begin{aligned} \left|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big) - h(x,y)\right| &\le K_4\big(1+x^2+y^2\big)\big(1+x^2\big)\big(1+y^2\big) \\ &\quad \times \Omega_\rho\Big(h;\sqrt{\mu_{n_1}^{x,q_{1,n_1}}},\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\Big). \end{aligned} \tag{4.6}$$

By (4.3)–(4.6), for all $(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $\varphi \ge 2$, we have

$$\begin{aligned} \frac{|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h(\tau,\sigma);x,y) - h(x,y)|}{\rho^\varphi(x,y)} &\le K_5 \frac{1+x^2+1+y^2+(1+x^2)(1+y^2)}{(1+x^2+y^2)^{\varphi-1}} \\ &\quad \times \Omega_\rho\Big(h;\sqrt{\mu_{n_1}^{x,q_{1,n_1}}},\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\Big), \end{aligned}$$

which implies

$$\sup_{(x,y)\in\mathbb{R}_+\times\mathbb{R}_+} \frac{|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h(\tau,\sigma);x,y) - h(x,y)|}{\rho^\varphi(x,y)} \le K\Omega_\rho\Big(h;\sqrt{\mu_{n_1}^{x,q_{1,n_1}}},\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\Big). \qquad \square$$

5 GBS variant and inequalities

Bögel introduced Bögel continuous and Bögel bounded functions. We recall the basic notations given by Bögel. Details can be found in [12–14].

Let A be a compact subset of $\mathbb{R}^2$. Any function $h: A \to \mathbb{R}$ is called Bögel continuous function at $(a,b) \in A$ if

$$\lim_{(\tau,\sigma)\to(a,b)} \Delta_{(\tau,\sigma)} h(a,b) = 0,$$

where $\Delta_{(\tau,\sigma)} h(x,y)$ denotes the mixed difference defined by

$$\Delta_{(\tau,\sigma)} h(x,y) = h(x,y) - h(x,\sigma) - h(\tau,y) + h(\tau,\sigma).$$

Let A be a subset of $\mathbb{R}^2$. A function $h: A \to \mathbb{R}$ is Bögel bounded function on A if there exists $M > 0$ such that

$$\left|\Delta_{(\tau,\sigma)} h(x,y)\right| \le M$$

for each $(\tau,\sigma), (x,y) \in A$. If A is a compact subset of $\mathbb{R}^2$, then each Bögel continuous function is a Bögel bounded function.

Let $C_b(A)$ denote the space of all real-valued Bögel continuous functions defined on A endowed with the norm

$$\|h\|_B = \sup\bigl\{\bigl|\Delta_{(\tau,\sigma)}h(x,y)\bigr| : (x,y),(\tau,\sigma)\in A\bigr\}.$$

Let us denote by $C(A)$ and $B(A)$ the space of all real valued, continuous and bounded functions defined on $A\subset\mathbb{R}^2$, respectively. $C(A)$ and $B(A)$ are Banach spaces endowed with the norm

$$\|h\| = \sup\bigl\{\bigl|h(x,y)\bigr| : (x,y)\in A\bigr\}.$$

It is obvious that $C(A)\subset C_b(A)$.

The mixed modulus of continuity of $h\in C_b(A)$ is defined by

$$\omega_{\text{mixed}}(h;\delta_1,\delta_2) = \sup_{(x,y)\in A}\ \sup_{\substack{0\le|k|\le\delta_1\\ 0\le|l|\le\delta_2}}\bigl|\Delta_{(x+k,y+l)}h(x,y)\bigr|,$$

where $(k,l)\in\mathbb{R}_+\times\mathbb{R}_+$ such that $(x+k,y+l)\in A$ and $\delta_1>0$, $\delta_2>0$. For $(\tau,\sigma),(x,y)\in A$ and $\delta_1>0$, $\delta_2>0$, the mixed difference $\Delta_{(\tau,\sigma)}h(x,y)$ produces the following inequality:

$$\bigl|\Delta_{(\tau,\sigma)}h(x,y)\bigr| \le \left(1+\frac{|\tau-x|}{\delta_1}\right)\left(1+\frac{|\sigma-y|}{\delta_2}\right)\omega_{\text{mixed}}(h;\delta_1,\delta_2). \tag{5.1}$$

Now, we define generalized Boolean sum (GBS) operators of the tensor product kind bivariate quantum beta-type operators as follows:

$$B_{n_1,n_2}^{q_1,q_2}\bigl(h(\tau,\sigma);x,y\bigr) = S_{n_1,n_2}^{q_1,q_2}\bigl(h(\tau,y)+h(x,\sigma)-h(\tau,\sigma);x,y\bigr)$$

for all $(\tau,\sigma),(x,y)\in I_R$ and $h\in C(I_R)$, where $I_R := I_1\times I_2$ is rectangular regions such that $I_i=[0,r_i]$ for $i=1,2$ and $r_1,r_2>0$.

We present the following inequalities estimating the rate of convergence of GBS variant.

Theorem 5 *Let $\{q_{1,n_1}\}$ and $\{q_{2,n_2}\}$ be any sequences such that $q_{1,n_1},q_{2,n_2}\in(0,1)$ satisfying the condition given in (3.5). If any $h\in C_b(I_R)$, then for all $(x,y)\in I_R$, the following inequality holds*:

$$\bigl|B_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(h(\tau,\sigma);x,y\bigr)-h(x,y)\bigr| \le 4\omega_{\text{mixed}}\Bigl(h;\sqrt{\mu_{n_1}^{x,q_{1,n_1}}},\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\Bigr),$$

where $\mu_{n_1}^{x,q_{1,n_1}}$ and $\mu_{n_2}^{y,q_{2,n_2}}$ are as in (3.6) and (3.7).

Proof By considering the definition of $\Delta_{(\tau,\sigma)}h(x,y)$ and (5.1), by applying Lemma 4 for $p=r=2$, we can write

$$\begin{aligned}
&\bigl|B_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(h(\tau,\sigma);x,y\bigr)-h(x,y)\bigr| \\
&\quad\le S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(\bigl|\Delta_{(\tau,\sigma)}h(x,y)\bigr|;x,y\bigr) \\
&\quad\le \omega_{\text{mixed}}(h;\delta_1,\delta_2)\biggl\{1+\frac{1}{\delta_1}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(|\tau-x|;x,y\bigr)+\frac{1}{\delta_2}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\bigl(|\sigma-y|;x,y\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\delta_1\delta_2}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(|\tau-x||\sigma-y|;x,y\big)\bigg\} \\
&\leq \omega_{\text{mixed}}(h;\delta_1,\delta_2)\bigg\{1+\frac{1}{\delta_1}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2;x,y\big)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)} \\
&+\frac{1}{\delta_2}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\sigma-y)^2;x,y\big)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)} \\
&+\frac{1}{\delta_1\delta_2}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2(\sigma-y)^2;x,y\big)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)}\bigg\} \\
&\times \omega_{\text{mixed}}(h;\delta_1,\delta_2).
\end{aligned}
$$

By considering Lemmas 1 and 2 and by choosing $\delta_1=\sqrt{\mu_{n_1}^{x,q_{1,n_1}}}$ and $\delta_2=\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}$, we complete the proof. □

A function h is Bögel differentiable at the point $(a,b)\in A\subset\mathbb{R}^2$ if the limit

$$
\lim_{(\tau,\sigma)\to(a,b)}\frac{\Delta_{(\tau,\sigma)}h(a,b)}{(\tau-a)(\sigma-b)}
$$

exists, which is denoted by $D_Bh(a,b)$.

Let us denote by $D_b(A)$ the set of all Bögel differentiable functions defined on $A\subset\mathbb{R}^2$.

Theorem 6 *Let $\{q_{1,n_1}\}$ and $\{q_{2,n_2}\}$ be any sequences such that $q_{1,n_1},q_{2,n_2}\in(0,1)$ satisfying the condition given in (3.5). If any $h\in D_b(I_R)$ such that $D_Bh\in B(I_R)$, then for all $(x,y)\in I_R$, the following inequality holds*:

$$
\begin{aligned}
&\big|B_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big)-h(x,y)\big| \\
&\leq \tilde{K}\sqrt{\frac{1}{[n_1]_{q_{1,n_1}}[n_2]_{q_{2,n_2}}}}\Big(\omega_{\text{mixed}}\Big(h;\sqrt{\mu_{n_1}^{x,q_{1,n_1}}},\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\Big)+\|D_Bh\|\Big),
\end{aligned}
$$

where $\mu_{n_1}^{x,q_{1,n_1}}$ and $\mu_{n_2}^{y,q_{2,n_2}}$ are as in (3.6) and (3.7), $\tilde{K}$ is certain constant.

Proof Let $h\in D_b(I_R)$. From the definition of Bögel differentiability, we have

$$
\Delta_{(\tau,\sigma)}h(x,y)=(\tau-x)(\sigma-y)D_Bh(\eta,\xi) \tag{5.2}
$$

for all $x<\eta<\tau$, $y<\xi<\sigma$. Therefore, we can write

$$
D_Bh(\eta,\xi)=\Delta_{(\tau,\sigma)}D_Bh(\eta,\xi)+D_Bh(\eta,y)+D_Bh(x,\xi)-D_Bh(x,y). \tag{5.3}
$$

Since $D_Bh\in B(I_R)$, we have $|D_Bh(x,y)|\leq\|D_Bh\|$ for $(x,y)\in I_R$. By considering (5.2) and (5.3), we obtain

$$
\begin{aligned}
&\big|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(\Delta_{(\tau,\sigma)}h(x,y);x,y\big)\big| \\
&=\big|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)(\sigma-y)D_Bh(\eta,\xi);x,y\big)\big| \\
&\leq S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(|\tau-x||\sigma-y|\big|D_Bh(\eta,\xi)\big|;x,y\big)
\end{aligned}
$$

$$
\begin{aligned}
&\leq S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(|\tau-x||\sigma-y|\big|\Delta_{(\tau,\sigma)}D_Bh(\eta,\xi)\big|;x,y\big)\\
&\quad+S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(|\tau-x||\sigma-y|\big\{\big|D_Bh(\eta,y)\big|+\big|D_Bh(x,\xi)\big|+\big|D_Bh(x,y)\big|\big\};x,y\big)\\
&\leq S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(|\tau-x||\sigma-y|\omega_{\text{mixed}}\big(D_Bh;|\eta-x|,|\xi-y|\big);x,y\big)\\
&\quad+3\|D_Bh\|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(|\tau-x||\sigma-y|;x,y\big).
\end{aligned}
\tag{5.4}
$$

Additionally, we have

$$
\begin{aligned}
\omega_{\text{mixed}}\big(D_Bh;|\tau-\eta|,|\sigma-\xi|\big)&\leq\omega_{\text{mixed}}\big(D_Bh;|\tau-x|,|\sigma-y|\big)\\
&\leq\left(1+\frac{|\tau-x|}{\delta_1}\right)\left(1+\frac{|\sigma-y|}{\delta_2}\right)\\
&\quad\times\omega_{\text{mixed}}(D_Bh;\delta_1,\delta_2).
\end{aligned}
\tag{5.5}
$$

From (5.4) and (5.5), we can write

$$
\begin{aligned}
&\big|B_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big)-h(x,y)\big)\big|\\
&\quad=\big|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(\Delta_{(\tau,\sigma)}h(x,y);x,y\big)\big|\\
&\quad\leq\Bigg\{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(|\tau-x||\sigma-y|;x,y\big)+\frac{1}{\delta_1}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2|\sigma-y|;x,y\big)\\
&\quad\quad+\frac{1}{\delta_2}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(|\tau-x|(\sigma-y)^2;x,y\big)+\frac{1}{\delta_1\delta_2}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2(\sigma-y)^2;x,y\big)\Bigg\}\\
&\quad\quad\times\omega_{\text{mixed}}(D_Bh;\delta_1,\delta_2)\\
&\quad\quad+3\|D_Bh\|S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(|\tau-x||\sigma-y|;x,y\big).
\end{aligned}
$$

By applying Lemma 4 for $p=r=2$, we get

$$
\begin{aligned}
&\big|B_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big)-h(x,y)\big|\\
&\quad\leq\Bigg\{\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2(\sigma-y)^2;x,y\big)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)}\\
&\quad\quad+\frac{1}{\delta_1}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2(\sigma-y)^2;x,y\big)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2;x,y\big)}\\
&\quad\quad+\frac{1}{\delta_2}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2(\sigma-y)^2;x,y\big)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\sigma-y)^2;x,y\big)}\\
&\quad\quad+\frac{1}{\delta_1\delta_2}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2(\sigma-y)^2;x,y\big)\Bigg\}\\
&\quad\quad\times\omega_{\text{mixed}}(D_Bh;\delta_1,\delta_2)\\
&\quad\quad+3\|D_Bh\|\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2(\sigma-y)^2;x,y\big)}\sqrt{S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(1;x,y)}.
\end{aligned}
$$

By Lemma 3, we have

$$
S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2;x,y\big)=O\left(\frac{1}{[n_1]_{q_{1,n_1}}}\right),\quad x\in I_1,
$$

$$S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\sigma-y)^2;x,y\big)=O\bigg(\frac{1}{[n_2]_{q_{2,n_2}}}\bigg),\quad y\in I_2.$$

Also, we have

$$\begin{aligned}S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2(\sigma-y)^2;x,y\big)&=S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\tau-x)^2;x,y\big)\\&\quad\times S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big((\sigma-y)^2;x,y\big).\end{aligned}$$

Therefore, by Lemmas 1 and 2, and by choosing $\delta_1=\sqrt{\mu_{n_1}^{x,q_{1,n_1}}}$ and $\delta_2=\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}$, we estimate that

$$\begin{aligned}&\big|B_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big)-h(x,y)\big|\\&\quad\le4\sqrt{\frac{c_1c_2}{[n_1]_{q_{1,n_1}}[n_2]_{q_{2,n_2}}}}\,\omega_{\mathrm{mixed}}\Big(D_Bh;\sqrt{\mu_{n_1}^{x,q_{1,n_1}}},\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\Big)+3\|D_Bh\|\sqrt{\frac{c_1c_2}{[n_1]_{q_{1,n_1}}[n_2]_{q_{2,n_2}}}},\end{aligned}$$

which implies

$$\begin{aligned}&\big|B_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}\big(h(\tau,\sigma);x,y\big)-h(x,y)\big|\\&\quad\le\tilde{K}\sqrt{\frac{1}{[n_1]_{q_{1,n_1}}[n_2]_{q_{2,n_2}}}}\Big(\omega_{\mathrm{mixed}}\Big(h;\sqrt{\mu_{n_1}^{x,q_{1,n_1}}},\sqrt{\mu_{n_2}^{y,q_{2,n_2}}}\Big)+\|D_Bh\|\Big),\end{aligned}$$

where $\tilde{K}=4\sqrt{c_1c_2}$. Thus, the proof of theorem is completed. □

6 Some applications and numerical results

In this part, we present the approximation errors of the tensor product kind bivariate quantum beta-type operators and the GBS variant for certain functions on the certain rectangular regions.

Let us denote by $E(h,g)$ the approximation error of the function h to the function g.

Example 1 Let $I_1=I_2=[0,4]$, then $I_R=I_1\times I_2=[0,4]\times[0,4]$ and $h_0(x,y)=xy$ for each $(x,y)\in I_R$. Let $n_1=n_2=n\in\mathbb{N}$ and let us choose $q_{1,n_1}=q_n$, $q_{2,n_2}=q_n$ such that $q_n=1-\frac{1}{n}$, then $\delta_1:=\delta_{1,n_1}=\delta_{2,n_2}:=\delta_2$. h_0 is continuous on I_R. Since $C(I_R)\subset C_b(I_R)$, h_0 is also Bögel continuous on I_R. Then the following numerical results of the approximation error of $S_{n,n}^{q_n,q_n}(h_0)$ to h_0 are presented in Tables 1–4, while those of $B_{n,n}^{q_n,q_n}(h_0)$ to h_0 in Table 5.

Table 1 The approximation error of $S_{n,n}^{q_n,q_n}(h_0)$ to h_0 by means of the complete modulus of continuity

	$n=1\times10^6$	$n=1\times10^5$	$n=1\times10^4$
$\delta_1=\delta_2$	$0.6902146233\times10^{-2}$	$0.2182668528\times10^{-1}$	$0.6902780895\times10^{-1}$
$\omega(h_0;\delta_1,\delta_2)$	$0.5516953024\times10^{-1}$	0.1741370781	0.5474576332
$E_1(S_{n,n}^{q_n,q_n}(h_0),h_0)$	0.2206781210	0.6965483124	2.189830533

Table 2 The approximation error of $S_{n,n}^{q_n,q_n}(h_0)$ to h_0 by means of the partial modulus of continuities

	$n=1\times10^6$	$n=1\times10^5$	$n=1\times10^4$
$\delta_1=\delta_2$	$0.4763962262\times10^{-4}$	$0.4764041901\times10^{-3}$	$0.4764838409\times10^{-2}$
$\omega_1(h_0;\delta_1)=\omega_2(h_0;\delta_2)$	$0.1905584905\times10^{-3}$	$0.1905616760\times10^{-2}$	$0.1905935364\times10^{-1}$
$E_2(S_{n,n}^{q_n,q_n}(h_0),h_0)$	$0.7622339620\times10^{-3}$	$0.7622467040\times10^{-2}$	$0.7623741456\times10^{-1}$

Table 3 The approximation error of $S_{n,n}^{q_n,q_n}(h_0)$ to h_0 by means of the Lipschitz functions on I_R for $\theta = 0.1 \times 10^{-5}$

	$n = 1 \times 10^6$	$n = 1 \times 10^5$	$n = 1 \times 10^4$
$\delta_1 = \delta_2$	$0.6902146233 \times 10^{-2}$	$0.2182668528 \times 10^{-1}$	$0.6902780895 \times 10^{-1}$
M_{h_0}	$0.5517007928 \times 10^{-1}$	0.1741384101	0.5474605603
$E_3(S_{n,n}^{q_n,q_n}(h_0), h_0)$	$0.5516953024 \times 10^{-1}$	0.1741370781	0.5474576333

Table 4 The approximation error of $S_{n,n}^{q_n,q_n}(h_0)$ to h_0 by means of the Lipschitz functions on I_R for $n = 1 \times 10^6$

	$\theta = 0.1 \times 10^{-5}$	$\theta = 0.1 \times 10^{-4}$	$\theta = 0.1 \times 10^{-3}$
$\delta_1 = \delta_2$	$0.6902146233 \times 10^{-2}$	$0.6902146233 \times 10^{-2}$	$0.6902146233 \times 10^{-2}$
M_{h_0}	$0.5517007928 \times 10^{-1}$	$0.5517502090 \times 10^{-1}$	$0.5522446144 \times 10^{-1}$
$E_4(S_{n,n}^{q_n,q_n}(h_0), h_0)$	$0.5516953024 \times 10^{-1}$	$0.5516953024 \times 10^{-1}$	$0.5516953025 \times 10^{-1}$

Table 5 The approximation error of $B_{n,n}^{q_n,q_n}(h_0)$ to h_0 by means of the mixed modulus of continuity

	$n = 1 \times 10^6$	$n = 1 \times 10^5$	$n = 1 \times 10^4$
$\delta_1 = \delta_2$	$0.6902146233 \times 10^{-2}$	$0.2182668528 \times 10^{-1}$	$0.6902780895 \times 10^{-1}$
$\omega_{\text{mixed}}(h_0; \delta_1, \delta_2)$	$0.4763962262 \times 10^{-4}$	$0.4764041903 \times 10^{-3}$	$0.4764838408 \times 10^{-2}$
$E_5(B_{n,n}^{q_n,q_n}(h_0), h_0)$	$0.1905584905 \times 10^{-3}$	$0.1905616761 \times 10^{-2}$	$0.1905935363 \times 10^{-1}$

Table 6 The approximation error of $S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h_1)$ to h_1 by means of the complete modulus of continuity

	$n = 1 \times 10^5$	$n = 1 \times 10^4$	$n = 1 \times 10^3$
δ_1	$0.1182874529 \times 10^{-1}$	$0.3740651406 \times 10^{-1}$	0.1183130922
δ_2	$0.2182679440 \times 10^{-1}$	$0.6903126124 \times 10^{-1}$	0.2185772687
$\omega(h_1; \delta_1, \delta_2)$	$0.8826171210 \times 10^{-2}$	$0.2765639169 \times 10^{-1}$	$0.8463458139 \times 10^{-1}$
$E_6(S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h_1), h_1)$	$0.3530468484 \times 10^{-1}$	0.1106255668	0.3385383256

Table 7 The approximation error of $S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h_1)$ to h_1 by means of the partial modulus of continuities

	$n = 1 \times 10^5$	$n = 1 \times 10^4$	$n = 1 \times 10^3$
δ_1	$0.1399192151 \times 10^{-3}$	$0.1399247294 \times 10^{-2}$	$0.1399798779 \times 10^{-1}$
δ_2	$0.4764089543 \times 10^{-3}$	$0.4765315029 \times 10^{-2}$	$0.4777602239 \times 10^{-1}$
$\omega_1(h_1; \delta_1)$	$0.3787201321 \times 10^{-4}$	$0.3787350577 \times 10^{-3}$	$0.3788843285 \times 10^{-2}$
$\omega_2(h_1; \delta_2)$	$0.1934252196 \times 10^{-3}$	$0.1934738320 \times 10^{-2}$	$0.1939013872 \times 10^{-1}$
$E_7(S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h_1), h_1)$	$0.4625944656 \times 10^{-3}$	$0.4626946755 \times 10^{-2}$	$0.4635796401 \times 10^{-1}$

Example 2 Let $I_1 = [2,3]$ and $I_2 = [2,4]$, then $I_R = I_1 \times I_2 = [2,3] \times [2,4]$ and $h_1(x,y) = xye^{-y}$ for each $(x,y) \in I$. Let us choose $n_1 = 2n+1$, $n_2 = n-1$ for each $n \in \mathbb{N}$, $q_{1,n_1} = 1 - \frac{1}{2n+1}$, $q_{2,n_2} = 1 - \frac{1}{n-1}$, $\delta_1 := \delta_{1,n_1}$, and $\delta_{2,n_2} := \delta_2$. h_1 is continuous on I_R. Since $C(I_R) \subset C_b(I_R)$, h_1 is also Bögel continuous on I_R. Then we have the numerical results of the error of $S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h_1)$ to h_1 in Tables 6 and 7, while those of $B_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h_1)$ to h_1 in Table 8. Since there does not exist $M_{h_1} > 0$ such that $|h_1(x_1,y_1), -h_1(x_2,y_2)| \leq M_{h_1}|x_1 - x_2|^{\theta}|y_1 - y_2|^{\theta}$, that is, $h_1 \notin \mathrm{Lip}_M(h; \theta, I_R)$; therefore, the approximation error of $S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h_1)$ to h_1 can not be obtained by means of Lipschitz functions.

Table 8 The approximation error of $B_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h_1)$ to h_1 by means of the mixed modulus of continuity

	$n = 1 \times 10^5$	$n = 1 \times 10^4$	$n = 1 \times 10^3$
δ_1	$0.1182874529 \times 10^{-1}$	$0.3740651406 \times 10^{-1}$	0.1183130922
δ_2	$0.2182679440 \times 10^{-1}$	$0.6903126124 \times 10^{-1}$	0.2185772687
$\omega_{\text{mixed}}(h_1;\delta_1,\delta_2)$	$0.3493860331 \times 10^{-4}$	$0.3491971504 \times 10^{-3}$	$0.3474832397 \times 10^{-2}$
$E_8(S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}(h_1),h_1)$	$0.1397544132 \times 10^{-3}$	$0.1396788602 \times 10^{-2}$	$0.1389932959 \times 10^{-1}$

7 Conclusions and discussion

Let (q_{1,n_1}) and (q_{2,n_2}) be sequences satisfying condition (3.5). Under condition (3.5), we have

$$\lim_{n_1\to\infty} \mu_{n_1}^{x,q_{1,n_1}} = 0, \quad x \in I_1,$$

$$\lim_{n_2\to\infty} \mu_{n_2}^{y,q_{2,n_2}} = 0, \quad y \in I_2,$$

and

$$\lim_{n_1\to\infty} \frac{1}{[n_1]_{q_{1,n_1}}} = 0,$$

$$\lim_{n_2\to\infty} \frac{1}{[n_2]_{q_{2,n_2}}} = 0.$$

Consequently, all the results in this study demonstrate the error of approximation for the new defined tensor product kind bivariate quantum beta-type operators and the associated GBS variant in different respects. Numerical results of Examples 1 and 2 concretely illustrate that the approximation of $S_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}$ and $B_{n_1,n_2}^{q_{1,n_1},q_{2,n_2}}$ becomes better for increasing value of n under condition (3.5) and show that the associated GBS variant possesses at least better numerical results than the tensor product kind bivariate quantum beta-type operator.

Lastly, this study can be extended to the following future problems. By considering the reference [15], A-statistical convergence of the bivariate beta-type operators can be investigated; by using the definition of post-quantum beta function and post-quantum gamma function in [16], a post-quantum analogue of the bivariate quantum beta-type operators can be defined and its approximation properties can be investigated; and by considering the reference [17], a bivariate exponential beta-type operator can be defined and its approximation properties can be investigated.

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