Bombardelli and Varošanec *Journal of Inequalities and Applications* (2022) 2022:65
https://doi.org/10.1186/s13660-022-02797-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

$M_\varphi M_\psi$-convexity and separation theorems

Mea Bombardelli[1*] and Sanja Varošanec[1]

*Correspondence:
mea.bombardelli@math.hr
[1]Department of Mathematics, Faculty of Science, University of Zagreb, Zagreb, Croatia

Abstract

A characterization of pairs of functions that can be separated by an $M_\varphi M_\psi$-convex function and related results are obtained. Also, a Hyers–Ulam stability result for $M_\varphi M_\psi$-convex functions is given.

MSC: 26E60; 26A51; 26D07; 39B72

Keywords: $M_\varphi M_\psi$-convex function; Separation theorem; Hyers–Ulam stability; Quasi-arithmetic mean

1 Preliminaries

The concept of classical convexity has been generalized in various ways. Among numerous generalizations, we pay attention to the $M_\varphi M_\psi$-convexity described in [11].

Let φ and ψ be two continuous, strictly monotone functions defined on intervals I and J respectively. By M_φ we denote a quasi-arithmetic mean:

$$M_\varphi(x,y;t) := \varphi^{-1}\big(t\varphi(x) + (1-t)\varphi(y)\big), \quad x,y \in I, t \in [0,1].$$

It is obvious that the power mean M_p corresponds to $\varphi(x) = x^p$ if $p \neq 0$ and to $\varphi(x) = \log x$ if $p = 0$. If it is clear from the text that the weight next to $\varphi(x)$ equals t, then we omit parameter t and simply write $M_\varphi(x,y)$.

We say that a function $f : I \to J$ is $M_\varphi M_\psi$-convex if

$$f\big(M_\varphi(x,y)\big) \le M_\psi\big(f(x),f(y)\big)$$

for all $x,y \in I$ and $t \in [0,1]$. The $M_\varphi M_\psi$-concavity and $M_\varphi M_\psi$-affinity are defined in a natural way. If ψ is strictly increasing (strictly decreasing), then f is $M_\varphi M_\psi$-convex if and only if $\psi \circ f \circ \varphi^{-1}$ is convex (concave) in the usual sense [11, p. 68].

The most known examples are classes of $M_\varphi M_\psi$-convex functions where M_φ and M_ψ belong to $\{A,G,H\}$, where A, G, and H are weighted arithmetic, geometric, and harmonic mean, respectively. Some of them are known under specific names. For example, AG-convex function is usually known as log-convex function, GG-convex function is called multiplicatively convex function, HA-convex function is named harmonically convex function. Of course, AA-convex function is the usual convex function.

A lot of examples of AG-convex or log-convex functions connected with various functionals, which have appeared in the investigation of n-convexity, are given in [5] and [6, pp. 105, 155-160, 177]. Every polynomial with nonnegative coefficients is GG-convex or multiplicatively convex function, every real analytic function $f(x) = \sum a_n x^n$ with $a_n \geq 0$ is GG-convex on $[0,R\rangle$ where R is the radius of convergence [11, Chap. 2]. Particularly, functions exp, sinh, cosh on $\langle 0,\infty\rangle$, arcsin on $\langle 0,1]$ are GG-convex. Examples of special functions which are GG-convex are the following: the gamma function, the Lobacevski function, and the integral sine. In [3], an example of HG-convex function is given. Namely, the function $V_n^{-1}(p) = 2^{-n}\frac{\Gamma(1+n/p)}{\Gamma(1+1/p)^n}$ which is connected with the volume of the ellipsoid $\{x \in \mathbb{R}^n : \|x\|_{L^p} \leq 1\}$ is HG-convex on $(0,\infty)$. Also, it is AG-convex.

The aim of this paper is to give a separation (sandwich) theorem in this settings. A characterization of pairs of functions that can be separated by a convex function is given in [2], and it is stated as follows.

Theorem 1.1 *Let $f, g : I \to \mathbb{R}$ be two functions. The following statements are equivalent*:

(i) *For all $x, y \in I$ and $t \in [0,1]$,*

$$f\big(tx + (1-t)y\big) \leq tg(x) + (1-t)g(y).$$

(ii) *There exists a convex function $h : I \to \mathbb{R}$ such that*

$$f \leq h \leq g.$$

As a consequence of the above-mentioned theorem, the Hyers–Ulam stability result for convex functions is obtained also in [2]. Namely, if $\varepsilon > 0$ and $f : I \to \mathbb{R}$ is a function such that

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) + \varepsilon, \quad x, y \in I, t \in [0,1],$$

then there exists a convex function $h : I \to \mathbb{R}$ such that

$$\big|f(x) - h(x)\big| \leq \frac{\varepsilon}{2}, \quad x \in I.$$

Finally, we mention a sandwich theorem involving affine functions which are considered in [12].

Theorem 1.2 *Let $I \subseteq \mathbb{R}$ be an interval and f and g be real functions defined on I. The following conditions are equivalent*:

(i) *There exists an affine function $h : I \to \mathbb{R}$ such that $f \leq h \leq g$ on I.*

(ii) *There exist a convex function $h_1 : I \to \mathbb{R}$ and a concave function $h_2 : I \to \mathbb{R}$ such that $f \leq h_1 \leq g$ and $f \leq h_2 \leq g$ on I.*

(iii) *The following inequalities hold*:

$$f\big(tx + (1-t)y\big) \leq tg(x) + (1-t)g(y),$$
$$g\big(tx + (1-t)y\big) \geq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

In this paper we show that the above-mentioned theorems have their counterparts in the setting of $M_\varphi M_\psi$-convex functions. We prove that two functions f, g can be separated by an $M_\varphi M_\psi$-convex function h if and only if

$$f\big(M_\varphi(x,y)\big) \le M_\psi\big(g(x),g(y)\big)$$

for all $x, y \in I$ and $t \in [0,1]$. In the same section we give a result for an $M_\varphi M_\psi$-affine function which is a generalization of Theorem 1.2. The last section is devoted to the counterpart of the Hyers–Ulam stability theorem.

2 Separation theorems

Theorem 2.1 *Let φ and ψ be two continuous, strictly monotone functions defined on intervals I and J respectively. Let $f, g : I \to J$ be real functions.*

The following statements are equivalent:

(i) *There exists an $M_\varphi M_\psi$-convex function $h : I \to J$ such that*

$$f \le h \le g.$$

(ii) *The following inequality holds:*

$$f\big(M_\varphi(x,y;t)\big) \le M_\psi\big(g(x),g(y);t\big) \tag{1}$$

for all $x, y \in I$, $t \in [0,1]$.

Proof Assume that ψ is an increasing function. Then ψ^{-1} is also increasing.

First we prove that (i) implies (ii).

Since $h \le g$,

$$t\psi\big(h(x)\big) + (1-t)\psi\big(h(y)\big) \le t\psi\big(g(x)\big) + (1-t)\psi\big(g(y)\big)$$

and then

$$\psi^{-1}\big(t\psi\big(h(x)\big) + (1-t)\psi\big(h(y)\big)\big) \le \psi^{-1}\big(t\psi\big(g(x)\big) + (1-t)\psi\big(g(y)\big)\big),$$

i.e.,

$$M_\psi\big(h(x),h(y)\big) \le M_\psi\big(g(x),g(y)\big). \tag{2}$$

Using the fact that $f \le h$, h is $M_\varphi M_\psi$-convex and inequality (2)

$$\begin{aligned} f\big(M_\varphi(x,y)\big) &\le h\big(M_\varphi(x,y)\big) \\ &\le M_\psi\big(h(x),h(y)\big) \le M_\psi\big(g(x),g(y)\big). \end{aligned}$$

Now assume that (ii) holds. For any $u, v \in \operatorname{Im}\varphi$, there exist $x, y \in I$ such that $u = \varphi(x)$, $v = \varphi(y)$. From (1) it follows

$$\psi\big(f\big(\varphi^{-1}\big(tu + (1-t)v\big)\big)\big) \le t\psi\big(g\big(\varphi^{-1}(u)\big)\big) + (1-t)\psi\big(g\big(\varphi^{-1}(v)\big)\big).$$

This can be written as

$$F\big(tu + (1-t)v\big) \le tG(u) + (1-t)G(v), \tag{3}$$

where $F = \psi \circ f \circ \varphi^{-1}$ and $G = \psi \circ g \circ \varphi^{-1}$, $F, G : \operatorname{Im}\varphi \to \mathbb{R}$. Inequality (3) holds for all $u, v \in \operatorname{Im}\varphi$ and for all $t \in [0,1]$.

Now we may apply Theorem 1.1 to conclude that there exists a convex function $H : \operatorname{Im}\varphi \to \mathbb{R}$ such that

$$F \le H \le G. \tag{4}$$

Then $H \circ \varphi$ is well defined.

Since $F(u) \le H(u) \le G(u)$, i.e., $\psi(f(x)) \le (H \circ \varphi)(x) \le \psi(g(x))$, and since ψ is a continuous, strictly increasing function defined on the interval J, the value $(H \circ \varphi)(x)$ is in the domain of ψ. This allows us to define $h = \psi^{-1} \circ H \circ \varphi$, $h : I \to J$. As H is convex, it follows that h is $M_\varphi M_\psi$-convex, and from (4) it follows that $f \le h \le g$, i.e., (i) holds.

If ψ is decreasing, the proof is analogous. □

Theorem 2.2 *Let φ and ψ be two continuous, strictly monotone functions defined on intervals I and J respectively. Let $f, g : I \to J$ be real functions.*

The following statements are equivalent:

(i) *There exists an $M_\varphi M_\psi$-affine function h such that*

$$f \le h \le g.$$

(ii) *The following inequalities*:

$$\begin{aligned} f\big(M_\varphi(x,y;t)\big) &\le M_\psi\big(g(x),g(y);t\big), \\ g\big(M_\varphi(x,y;t)\big) &\ge M_\psi\big(f(x),f(y);t\big) \end{aligned} \tag{5}$$

hold for all $x, y \in I$ and $t \in [0,1]$.

Proof Let h be an $M_\varphi M_\psi$-affine function such that $f \le h \le g$. This means that

$$h\big(M_\varphi(x,y)\big) = M_\psi\big(h(x),h(y)\big), \quad \forall x, y \in I.$$

Let $F = \psi \circ f \circ \varphi^{-1}$, $G = \psi \circ g \circ \varphi^{-1}$, $H = \psi \circ h \circ \varphi^{-1}$. It is easy to show that H is an affine function.

Let ψ be an increasing function. Then $F \le H \le G$ on $\operatorname{Im}\varphi$. (If ψ is decreasing, then $G \le H \le F$, and the proof is similar.)

Applying Theorem 1.2 ((i) implies (iii)), we obtain

$$F\big(tu + (1-t)v\big) \le tG(u) + (1-t)G(v), \tag{6}$$

$$G\big(tu + (1-t)v\big) \ge tF(u) + (1-t)F(v) \tag{7}$$

for all $u, v \in \operatorname{Im}\varphi$ and $t \in [0,1]$.

From (6), for all $x, y \in I$ and $t \in [0,1]$, it follows

$$\begin{aligned}&\left(\psi \circ f \circ \varphi^{-1}\right)\left(t\varphi(x) + (1-t)\varphi(y)\right) \le t(\psi \circ g)(x) + (1-t)(\psi \circ g)(y),\\ &f\left(\varphi^{-1}\left(t\varphi(x) + (1-t)\varphi(y)\right)\right) \le \psi^{-1}\left(t\psi\left(g(x)\right) + (1-t)\psi\left(g(y)\right)\right),\end{aligned}$$

i.e., $f(M_\varphi(x,y;t)) \le M_\psi(g(x),g(y);t)$.

In the same way, $g(M_\varphi(x,y;t)) \ge M_\psi(f(x),f(y);t)$.

Now assume (ii).

From (5) it follows

$$\begin{aligned}&F\left(tu + (1-t)v\right) \le tG(u) + (1-t)G(v),\\ &G\left(tu + (1-t)v\right) \ge tF(u) + (1-t)F(v), \quad \forall u, v \in \operatorname{Im}\varphi, \forall t \in [0,1],\end{aligned}$$

where $F = \psi \circ f \circ \varphi^{-1}$ and $G = \psi \circ g \circ \varphi^{-1}$, $F, G : \operatorname{Im}\varphi \to \mathbb{R}$.

From Theorem 1.2 ((iii) implies (i)) we conclude that there exists an affine function $H : \operatorname{Im}\varphi \to \mathbb{R}$ such that $F(w) \le H(w) \le G(w)$ for all $w \in \operatorname{Im}\varphi$.

Then, as in the proof of the previous theorem, $h = \psi^{-1} \circ H \circ \varphi$, $h : I \to \mathbb{R}$ is well defined, and $f \le h \le g$. It is easy to verify that h is an $M_\varphi M_\psi$-affine function. □

3 Hyers–Ulam stability

Theorem 3.1 *Let φ be a continuous strictly monotone function on an interval I. Let $\varepsilon > 0$ be a fixed number. A function $f : I \to \mathbb{R}$ satisfies*

$$f\left(M_\varphi(x,y)\right) \le tf(x) + (1-t)f(y) + \varepsilon \tag{8}$$

for all $x, y \in I$, $t \in [0,1]$, if and only if there exists an $M_\varphi A$-convex function $h : I \to \mathbb{R}$ such that

$$\left|f(x) - h(x)\right| \le \frac{1}{2}\varepsilon, \quad \forall x \in I. \tag{9}$$

Proof Assume that f satisfies (8). For $g = f + \varepsilon$, we have

$$A\left(f(x), f(y)\right) + \varepsilon = A\left(g(x), g(y)\right).$$

Therefore, from (8) it follows

$$f\left(M_\varphi(x,y)\right) \le A\left(g(x), g(y)\right),$$

which is a form of condition (ii) from Theorem 2.1.

We conclude that there exists an $M_\varphi A$-convex function $h_1 : I \to \mathbb{R}$ such that $f \le h_1 \le g$, i.e., $f \le h_1 \le f + \varepsilon$.

Let $h = h_1 - \frac{1}{2}\varepsilon$. Then $-\frac{1}{2}\varepsilon \le f(x) - h(x) \le \frac{1}{2}\varepsilon$ for all $x \in I$, so (9) holds.

Since

$$h(M_\varphi(x,y) = h_1(M_\varphi(x,y) - \frac{1}{2}\varepsilon \le A\bigl(h_1(x), h_1(y)\bigr) - \frac{1}{2}\varepsilon = A\bigl(h(x), h(y)\bigr),$$

h is also $M_\varphi A$-convex, which completes the proof.

Now let $h : I \to \mathbb{R}$ be an $M_\varphi A$-convex function such that (9) holds. This condition can be written in the form

$$f(x) - \frac{1}{2}\varepsilon \le h(x) \le f(x) + \frac{1}{2}\varepsilon.$$

Using Theorem 2.1 we can conclude that functions $f_1 = f - \frac{1}{2}\varepsilon$ and $f_2 = f + \frac{1}{2}\varepsilon$ satisfy

$$f_1\bigl(M_\varphi(x,y)\bigr) \le A\bigl(f_2(x), f_2(y)\bigr).$$

This is equivalent to

$$f\bigl(M_\varphi(x,y)\bigr) - \frac{1}{2}\varepsilon \le A\bigl(f(x), f(y)\bigr) + \frac{1}{2}\varepsilon,$$

which proves (8). □

As we mentioned in the first section, the corresponding results for convex functions, i.e., for AA-convex functions, are given in [2] and [12]. A special case of Theorem 2.2, where $\psi = \varphi$, is given in [8]. Particular cases of Theorem 2.1 and Theorem 3.1 for HA-convex functions are given in [4].

Results about the separation problem for some other classes of functions which are not particular cases of the class of $M_\varphi M_\psi$-convex functions, i.e., for strongly convex functions, m-convex and h-convex functions, set-valued functions, and convex functions with control function, are given in [7, 9, 10, 13], and [1] respectively.

Acknowledgements
Not applicable.

Funding
The publication charges for this manuscript are supported by the University of Zagreb, Faculty of Science, Department of Mathematics.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SV: conceptualization, writing the original draft, computation. MB: computation, analyzing the results, writing and editing. Both authors read and approved the final manuscript.

Publisher's Note

Received: 17 November 2021 Accepted: 4 May 2022 Published online: 23 May 2022

References

1. Adamek, M.: On a generalization of sandwich type theorems. Aequationes Math. **92**, 641–647 (2018)
2. Baron, K., Matkowski, J., Nikodem, K.: A sandwich with convexity. Math. Pannon. **5**(1), 139–144 (1994)
3. Borwein, D., Borwein, J., Fee, G., Girgensohn, R.: Refined convexity and special cases of the Blaschke-Santalo inequality. Math. Inequal. Appl. **4**, 631–638 (2001)
4. Bracamonte, M., Giménez, J., Medina, J., Vivas-Cortez, M.: A sandwich theorem and stability result of Hyers-Ulam type for harmonically convex functions. Lect. Mat. **38**, 5–18 (2017)
5. Khan, A.R., Pečarić, J., Varošanec, S.: Positivity of sums and integrals for convex functions of higher order of n variables. Math. Inequal. Appl. **19**(1), 221–247 (2016)
6. Khan, A.R., Praljak, M., Pečarić, J., Varošanec, S.: General Linear Inequalities. Element, Zagreb (2017)
7. Lara, T., Matkowski, J., Merentes, N., Quintero, R., Wróbel, M.: A generalization of m-convexity and a sandwich theorem. Ann. Math. Sil. **31**, 107–126 (2017)
8. Matkowski, J., Zgraja, T.: A separation theorem for $M\varphi$-convex functions. Math. Pannon. **9**(1), 103–109 (1998)
9. Merentes, N., Nikodem, K.: Remarks on strongly convex functions. Aequationes Math. **80**, 193–199 (2010)
10. Mitroi-Symeonidis, F.C.: Convexity and sandwich theorems. Eur. J. Res. Appl. Sci. **1**, 9–11 (2015)
11. Niculescu, C., Persson, L.: Convex Functions and Their Applications, a Contemporary Approach. CMS Books in Mathematics, vol. 23. Springer, New York (2006)
12. Nikodem, K., Wasowicz, S.: A sandwich theorem and Hyers–Ulam stability of affine functions. Aequationes Math. **49**, 160–164 (1995)
13. Olbryś, A.: On separation by h-convex functions. Tatra Mt. Math. Publ. **62**, 105–111 (2015)