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An accelerated common fixed point algorithm for a countable family of G-nonexpansive mappings with applications to image recovery

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Abstract

In this paper, we define a new concept of left and right coordinate affine of a directed graph and then employ it to introduce a new accelerated common fixed point algorithm for a countable family of G-nonexpansive mappings in a real Hilbert space with a graph. We prove, under certain conditions, weak convergence theorems for the proposed algorithm. As applications, we also apply our results to solve convex minimization and image restoration problems. Moreover, we show that our algorithm provides better convergence behavior than other methods in the literature.

Keywords: Convex minimization; Coordinate affine; G-nonexpansive; Image restoration problem; Inertial techniques; Weak convergence

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H with norm $\|\cdot\|$. A mapping T of C into itself is said to be

(i) *Lipschitzian* if there exists $\gamma \geq 0$ such that

$$\|Tx - Ty\| \leq \gamma\|x - y\|$$

for all $x, y \in C$, where γ is called the coefficient of T;

(ii) *nonexpansive* if T is Lipschitzian with $\gamma = 1$.

The element $x \in C$ is a fixed point of T if $Tx = x$, and $F(T) := \{x \in C : x = Tx\}$ denotes the set of all fixed points of T.

For the past seven decades, several iterative methods were proposed to find approximating fixed point theorems of nonexpansive mappings; see, for instance, [1, 2].

One of the famous and well-known iterative methods, the Picard iteration process, is defined by

$$x_{n+1} = Tx_n$$

for $n \geq 1$, and the initial point x_1 is chosen arbitrarily. Furthermore, the Picard iteration process was improved and studied extensively by many mathematicians such as follows.

The *Mann iteration process* [3] is defined by

$$x_{n+1} = (1 - \rho_n)x_n + \rho_n Tx_n \tag{1.1}$$

for $n \geq 1$, the initial point x_1 is chosen arbitrarily, and $\{\rho_n\}$ is a sequence in $[0, 1]$.

The *Ishikawa iteration process* [4] is defined by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1 - \rho_n)x_n + \rho_n Ty_n \end{cases} \tag{1.2}$$

for $n \geq 1$, the initial point x_1 is chosen arbitrarily, and $\{\beta_n\}$ and $\{\rho_n\}$ are sequences in $[0, 1]$.

The *S-iteration process* [5] is defined by

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ x_{n+1} = (1 - \rho_n)Tx_n + \rho_n Ty_n \end{cases} \tag{1.3}$$

for $n \geq 1$, the initial point x_1 is chosen arbitrarily, and $\{\beta_n\}$ and $\{\rho_n\}$ are sequences in $[0, 1]$.

In 2017, Agarwal, O'Regan, and Sahu [5] proved that the iteration process (1.3) is independent of the Mann and Ishikawa iteration processes and converges faster. In 2012, Aleomraninejad et al. [6] used the idea of combination of fixed point and graph theories in proving a convergence theorem for G-nonexpansive mappings in a Banach space. In 2015, Tiammee et al. [7] proved the Browder convergence theorem and a strong convergence theorem of the Halpern iterative scheme for G-nonexpansive mappings in a Hilbert space endowed with a graph. Later, Tripak [8], by using the Ishikawa iteration, proved weak and strong convergence theorems for finding a common fixed point for two G-nonexpansive mappings in a Banach space. In 2019, Sridarat et al. [9], using the SP-iteration, proved weak and strong convergence theorems for finding a common fixed point of three G-nonexpansive mappings in a uniformly convex Banach space with a graph. In 2020, Yambangwai et al. [10], using a modified three-step iteration method, proved weak and strong convergence theorems for three G-nonexpansive mappings in a uniformly convex Banach space with a graph. They also applied their results to find solutions of constrained minimization problems and split feasibility problems. Recently, Suantai et al. [11] modified the shrinking projection method with the parallel monotone hybrid method for approximating common fixed points of a finite family of G-nonexpansive mappings. They proved a strong convergence theorem under suitable conditions in Hilbert spaces endowed with graphs and applied it to signal recovery.

The main objectives of this paper are introducing an iterative method for finding a common fixed point of a countable family of G-nonexpansive mappings, analyzing the convergence behavior of the recommended algorithm in comparison with the others, and giving some applications to solve the image restoration problem.

2 Preliminaries

In what follows, X is a real normed space. Let C be a nonempty subset of X. Let $G = (V(G), E(G))$ be a directed graph with $V(G) = C$ and $E(G) \supseteq \triangle$, where $\triangle = \{(u, u) : u \in C\}$.

Assume that G has no parallel edges. We denote by G^{-1} the graph obtained from G by reversing the direction of edges. Then

$$E(G^{-1}) = \{(u,v) \in C \times C : (v,u) \in E(G)\}.$$

Recall that a graph G is said to be *connected* if there is a path between any two vertices of the graph G. For more detail on some basic notions of the graphs, we refer the readers to [12].

A mapping $T : C \to C$ is said to be

(i) *G-contraction* [13] if
  (a) T is edge-preserving, i.e., $(Tu, Tv) \in E(G)$ for all $(u,v) \in E(G)$, and
  (b) there exists $\rho \in [0,1)$ such that $\|Tu - Tv\| \le \rho \|u - v\|$ for all $(u,v) \in E(G)$, where ρ is called a contraction factor;

(ii) *G-nonexpansive* [7] if
  (a) T is edge-preserving, and
  (b) $\|Tu - Tv\| \le \|u - v\|$ for all $(u,v) \in E(G)$.

If $\{u_n\}$ is a sequence in X, then $u_n \rightharpoonup u$ denotes weak convergence of the sequence $\{u_n\}$ to u. For $v \in C$, if there is a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \rightharpoonup v$, then v is called a weak cluster point of $\{u_n\}$. By $\omega_w(u_n)$ we denote the set of all weak cluster points of $\{u_n\}$.

Let $\{T_n\}$ and ψ be families of nonexpansive mappings of C into itself such that $\emptyset \neq F(\psi) \subset \Lambda := \bigcap_{n=1}^{\infty} F(T_n)$, where $F(\psi)$ is the set of all common fixed points of all $T \in \psi$. A sequence $\{T_n\}$ satisfies the *NST*-condition (I) with ψ [14] if for any bounded sequence $\{u_n\}$ in C,

$$\lim_{n\to\infty} \|T_n u_n - u_n\| = 0 \quad \text{implies} \quad \lim_{n\to\infty} \|Tu_n - u_n\| = 0$$

for all $T \in \psi$. If $\psi = \{T\}$, then $\{T_n\}$ satisfies the *NST*-condition (I) with T. In 2009, Nakajo et al. [15] have given the definition of the *NST**-condition: a sequence $\{T_n\}$ satisfies the *NST**-condition if

$$\lim_{n\to\infty} \|T_n u_n - u_n\| = \lim_{n\to\infty} \|u_{n+1} - u_n\| = 0 \quad \text{implies} \quad \omega_w(u_n) \subset \Lambda$$

for every bounded sequence $\{u_n\}$ in C.

We recall the definition of forward–backward operator of lower semicontinuous and convex functions of $f, g : \mathbb{R}^n \to (-\infty, +\infty]$ as follows: A forward–backward operator T is defined by $T := prox_{\lambda g}(I - \lambda \nabla f)$ for $\lambda > 0$, where ∇f is the gradient operator of function f and $prox_{\lambda g} x := \arg\min_{y \in H}\{g(y) + \frac{1}{2\lambda}\|y - x\|^2\}$ (see [16, 17]). The operator $prox_{\lambda g}$ was defined by Moreau [18], who called it the proximity operator with respect to λ and function g. We know that T is a nonexpansive mapping whenever $\lambda \in (0, 2/L)$, where L is a Lipschitz constant of ∇f.

Remark 2.1 ([19]) Let $g : \mathbb{R}^n \to \mathbb{R}$ be given by $g(x) = \lambda \|x\|_1$. The proximity operator of g is defined by the formula

$$prox_{\lambda\|\cdot\|_1}(x) = \big(sign(x_i) \max(|x_i| - \lambda, 0)\big)_{i=1}^{n},$$

where $x = (x_1, x_2, \dots, x_n)$ and $\|x\|_1 = \sum_{i=1}^{n} |x_i|$.

Lemma 2.2 ([20]) *Let g be a lower semicontinuous and proper convex function from a Hilbert space H into $\mathbb{R} \cup \{\infty\}$, and let f be a convex differentiable function from H into $\mathbb{R}$ with L-Lipschitz gradient ∇f for some $L > 0$. Let T be the forward–backward operator of g and f. A sequence $\{T_n\}$ satisfies the NST-condition (I) with T if $\{T_n\}$ is the forward–backward operator of g and f such that $a_n \to a$ with $a, a_n \in (0, 2/L)$.*

Lemma 2.3 ([21]) *For a real Hilbert space H, we have:*

(i) *For all $u, v \in H$ and $\gamma \in [0,1]$,*

$$\left\|\gamma u + (1-\gamma)v\right\|^2 = \gamma\|u\|^2 + (1-\gamma)\|v\|^2 - \gamma(1-\gamma)\|u-v\|^2;$$

(ii) *For any $u, v \in H$,*

$$\|u \pm v\|^2 = \|u\|^2 \pm 2\langle u, v\rangle + \|v\|^2.$$

Lemma 2.4 ([22]) *Let $\{u_n\}$, $\{v_n\}$, and $\{\vartheta_n\}$ be sequences of nonnegative real numbers such that*

$$u_{n+1} \le (1+\vartheta_n)u_n + v_n$$

for $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \vartheta_n < \infty$ and $\sum_{n=1}^{\infty} v_n < \infty$, then $\lim_{n\to\infty} u_n$ exists.

Lemma 2.5 ([23]) *Let H be a real Hilbert space, and let $\{u_n\}$ be a sequence in H such that there exists a nonempty set $\Lambda \subset H$ satisfying the following conditions:*

(i) *For any $p \in \Lambda$, $\lim_{n\to\infty} \|u_n - p\|$ exists;*

(ii) *Any weak cluster point of $\{u_n\} \in \Lambda$.*

Then there exists $q^* \in \Lambda$ such that $u_n \rightharpoonup q^*$.

Lemma 2.6 ([24]) *Let $\{u_n\}$ and $\{\mu_n\}$ be sequences of nonnegative real numbers such that*

$$u_{n+1} \le (1+\mu_n)u_n + \mu_n u_{n-1}$$

for $n \in \mathbb{N}$. Then

$$u_{n+1} \le M \cdot \prod_{j=1}^{n}(1+2\mu_j),$$

where $M = \max\{u_1, u_2\}$. Moreover, if $\sum_{n=1}^{\infty} \mu_n < \infty$, then $\{u_n\}$ is bounded.

3 Main results

In this section, by using the inertial technique we prove a weak convergence theorem for a new accelerated algorithm for a countable family of G-nonexpansive mappings in a real Hilbert space with a directed graph.

Let C be a nonempty closed and convex subset of a real Hilbert space H with a directed graph $G = (V(G), E(G))$ such that $V(G) = C$. Let $\{T_n\}$ be a family of G-nonexpansive mappings of C into itself such that $\emptyset \ne \Lambda := \bigcap_{n=1}^{\infty} F(T_n)$.

Algorithm 3.1 An Inertial Mann Algorithm

1: **Initial.** Take arbitrary $x_0, x_1 \in C$ and $n = 1$, $\rho_n \in [a,b] \subset (0,1)$, and $\mu_n \geq 0$ such that $\sum_{n=1}^{\infty} \mu_n < \infty$.

2: **Step 1**. Compute y_n and x_{n+1} by using

$$\begin{cases} y_n = x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} = (1-\rho_n)y_n + \rho_n T_n y_n. \end{cases}$$

Then update $n := n + 1$ and go to Step 1.

The sequence μ_n is called an inertial step size. Before giving a weak convergence theorem for Algorithm 3.1 for a family of G-nonexpansive mappings, we need to introduce a concept of coordinate affine of the graph $G = (V(G), E(G))$.

Definition 3.1 Assume that $\Lambda := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\Lambda \times \Lambda \subseteq E(G)$. Then $E(G)$ is said to be

(i) left coordinate affine if $\alpha(x,y) + \beta(u,y) \in E(G)$ for all (x,y), $(u,y) \in E(G)$ and all α, $\beta \in \mathbb{R}$ such that $\alpha + \beta = 1$.

(ii) right coordinate affine if $\alpha(x,y) + \beta(x,z) \in E(G)$ for all (x,y), $(x,z) \in E(G)$ and all α, $\beta \in \mathbb{R}$ such that $\alpha + \beta = 1$.

We say that $E(G)$ is coordinate affine if $E(G)$ is both left and right coordinate affine.

We start with some properties of the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 3.1 related to $E(G)$.

Example 3.2 Let $X = \mathbb{R}^2$ and $C = \mathbb{R} \times \{1\}$. Let $G = (V(G), E(G))$ be a directed graph defined by $V(G) = C$ and $(x,y) \in E(G)$ if $x, y \in \mathbb{R} \times \{1\}$. We will show that $E(G)$ is left coordinate affine. To see this, let (x,y), $(z,y) \in C$ be such that $x = (x_1, 1)$, $y = (y_1, 1)$, and $z = (z_1, 1)$. For all α, $\beta \in \mathbb{R}$ with $\alpha + \beta = 1$,

$$\begin{aligned} \alpha(x,y) + \beta(z,y) &= \alpha\big((x_1,1),(y_1,1)\big) + \beta\big((z_1,1),(y_1,1)\big) \\ &= \big((\alpha x_1, \alpha),(\alpha y_1, \alpha)\big) + \big((\beta z_1, \beta),(\beta y_1, \beta)\big) \\ &= \big((\alpha x_1 + \beta z_1, \alpha + \beta),(\alpha y_1 + \beta y_1, \alpha + \beta)\big) \\ &= \big((\alpha x_1 + \beta z_1, 1),(y_1, 1)\big). \end{aligned}$$

Then $\alpha(x,y) + \beta(z,y) \in E(G)$, and hence $E(G)$ is left coordinate affine.

Proposition 3.3 *Let $\breve{q} \in \Lambda$ and $x_0, x_1 \in C$ be such that $(x_0, \breve{q})$, $(x_1, \breve{q}) \in E(G)$. Let $\{x_n\}$ be a sequence generated by Algorithm* 3.1. *Suppose $E(G)$ is left coordinate affine. Then $(x_n, \breve{q})$ and $(y_n, \breve{q}) \in E(G)$ for all $n \in \mathbb{N}$.*

Proof We will prove the results by using mathematical induction. From Algorithm 3.1 we obtain

$$(y_1, \breve{q}) = \big(x_1 + \mu_1(x_1 - x_0), \breve{q}\big)$$

$$\begin{aligned} &= \bigl((1+\mu_1)x_1 - \mu_1 x_0, \breve{q}\bigr) \\ &= (1+\mu_1)(x_1,\breve{q}) - \mu_1(x_0,\breve{q}). \end{aligned}$$

Since $(x_0,\breve{q})$, $(x_1,\breve{q}) \in E(G)$ and $E(G)$ is left coordinate affine, we get $(y_1,\breve{q}) \in E(G)$. Next, suppose that $(x_k,\breve{q})$, $(y_k,\breve{q}) \in E(G)$. Notice that

$$\begin{aligned} (x_{k+1},\breve{q}) &= \bigl((1-\rho_k)y_k + \rho_k T_k y_k, \breve{q}\bigr) \\ &= (1-\rho_k)(y_k,\breve{q}) + \rho_k(T_k y_k,\breve{q}). \end{aligned}$$

Since $(y_k,\breve{q}) \in E(G)$ and T_k is edge-preserving, we obtain $(x_{k+1},\breve{q}) \in E(G)$. Then

$$\begin{aligned} (y_{k+1},\breve{q}) &= \bigl(x_{k+1} + \mu_{k+1}(x_{k+1} - x_k), \breve{q}\bigr) \\ &= (1+\mu_{k+1})(x_{k+1},\breve{q}) - \mu_{k+1}(x_k,\breve{q}). \end{aligned}$$

Since $(x_{k+1},\breve{q})$, $(x_k,\breve{q}) \in E(G)$ and $E(G)$ is left coordinate affine, we have that $(y_{k+1},\breve{q}) \in E(G)$. By mathematical induction we obtain $(x_n,\breve{q})$, $(y_n,\breve{q}) \in E(G)$ for all $n \in \mathbb{N}$. □

Theorem 3.4 *Let C be a nonempty closed and convex subset of a real Hilbert space H with a directed graph $G = (V(G),E(G))$ with $V(G) = C$ and left coordinate affine $E(G)$. Let $x_0, x_1 \in C$, and let $\{x_n\}$ be the sequence in H defined by Algorithm 3.1. Suppose $\{T_n\}$ satisfies the NST*-condition with $\Lambda \neq \emptyset$ and $(x_0,\breve{q})$, $(x_1,\breve{q}) \in E(G)$ for all $\breve{q} \in \Lambda$. Then $\{x_n\}$ converges weakly to a common fixed point of Λ.*

Proof Let $\breve{q} \in \Lambda$. By Algorithm 3.1 we obtain

$$\begin{aligned} \|y_n - \breve{q}\| &= \bigl\|x_n + \mu_n(x_n - x_{n-1}) - \breve{q}\bigr\| \\ &\leq \|x_n - \breve{q}\| + \mu_n\|x_n - x_{n-1}\| \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \|x_{n+1} - \breve{q}\| &= \bigl\|(1-\rho_n)y_n - \breve{q} + \rho_n\breve{q} + \rho_n T_n y_n - \rho_n\breve{q}\bigr\| \\ &= \bigl\|(1-\rho_n)(y_n - \breve{q}) + \rho_n(T_n y_n - \breve{q})\bigr\| \\ &\leq (1-\rho_n)\|y_n - \breve{q}\| + \rho_n\|T_n y_n - \breve{q}\| \\ &= (1-\rho_n)\|y_n - \breve{q}\| + \rho_n\|T_n y_n - T_n\breve{q}\| \\ &\leq (1-\rho_n)\|y_n - \breve{q}\| + \rho_n\|y_n - \breve{q}\| \\ &= \|y_n - \breve{q}\|. \end{aligned} \tag{3.2}$$

From (3.1) and (3.2) we get

$$\|x_{n+1} - \breve{q}\| \leq \|x_n - \breve{q}\| + \mu_n\|x_n - x_{n-1}\|. \tag{3.3}$$

Then we have

$$\|x_{n+1} - \breve{q}\| \leq (1+\mu_n)\|x_n - \breve{q}\| + \mu_n\|x_{n-1} - \breve{q}\|. \tag{3.4}$$

Applying Lemma 2.6, we get $\|x_{n+1} - \breve{q}\| \leq M \cdot \prod_{j=1}^{n}(1 + 2\mu_j)$, where $M = \max\{\|x_1 - \breve{q}\|, \|x_2 - \breve{q}\|\}$. Since $\sum_{n=1}^{\infty} \mu_n \leq \infty$, we obtain that $\{x_n\}$ is bounded. Thus

$$\sum_{n=1}^{\infty} \mu_n \|x_n - x_{n-1}\| \leq \infty. \tag{3.5}$$

By Lemma 2.5 and (3.3) we get that $\lim_{n\to\infty} \|x_n - \breve{q}\|$ exists. By Algorithm 3.1 and Lemma 2.3(i) we obtain

$$\begin{aligned}
\|x_{n+1} - \breve{q}\|^2 &= \big\|(1-\rho_n)(y_n - \breve{q}) + \rho_n(T_n y_n - \breve{q})\big\|^2 \\
&= (1-\rho_n)\|y_n - \breve{q}\|^2 + \rho_n\|T_n y_n - \breve{q}\|^2 - (1-\rho_n)\rho_n\|y_n - T_n y_n\|^2 \\
&\leq (1-\rho_n)\|y_n - \breve{q}\|^2 + \rho_n\|y_n - \breve{q}\|^2 - (1-\rho_n)\rho_n\|y_n - T_n y_n\|^2 \\
&= \|y_n - \breve{q}\|^2 - (1-\rho_n)\rho_n\|y_n - T_n y_n\|^2 \\
&\leq \big(\|x_n - \breve{q}\| + \mu_n\|x_n - x_{n-1}\|\big)^2 - (1-\rho_n)\rho_n\|y_n - T_n y_n\|^2 \\
&= \|x_n - \breve{q}\|^2 + 2\mu_n\|x_n - \breve{q}\|\|x_n - x_{n-1}\| + \mu_n^2\|x_n - x_{n-1}\|^2 \\
&\quad - (1-\rho_n)\rho_n\|y_n - T_n y_n\|^2.
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$\|y_n - T_n y_n\| \to 0. \tag{3.7}$$

Since

$$\|x_n - y_n\| = \mu_n\|x_n - x_{n-1}\|,$$

it follows that

$$\|x_n - y_n\| \to 0. \tag{3.8}$$

By (3.7) and (3.8) from

$$\|x_{n+1} - x_n\| \leq \|y_n - x_n\| + \rho_n\|T_n y_n - y_n\| \to 0 \tag{3.9}$$

we obtain

$$\|x_{n+1} - x_n\| \to 0. \tag{3.10}$$

Next, we will show that $\|y_n - y_{n+1}\| \to 0$. By Algorithm 3.1 we obtain

$$\begin{aligned}
\|y_n - y_{n+1}\| &= \big\|x_n + \mu_n(x_n - x_{n-1}) - x_{n+1} - \mu_{n+1}(x_{n+1} - x_n)\big\| \\
&\leq \|x_n - x_{n+1}\| + \mu_n\|x_n - x_{n-1}\| + \mu_{n+1}\|x_n - x_{n+1}\|.
\end{aligned}$$

From (3.5) and (3.10) we get

$$\|y_n - y_{n+1}\| \to 0. \tag{3.11}$$

Since $\{T_n\}$ satisfies the *NST*-condition*, by (3.7) and (3.11) we obtain

$$\omega_w(y_n) \subset \Lambda.$$

Finally, we will show that $\omega_w(x_n) \subset \Lambda$. To see this, let $x \in \omega_w(x_n)$. By the definition of $\omega_w(x_n)$ there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$. From (3.8) we obtain that $y_{n_k} \rightharpoonup x$. Then $x \in \omega_w(y_n)$. It follows that $\omega_w(x_n) \subset \omega_w(y_n) \subset \Lambda$. Thus $\omega_w(x_n) \subset \Lambda$. By Lemma 2.5 we get $x_n \rightharpoonup \breve{q}$ in Λ. The proof is now complete. □

4 Application on convex minimization problems

In this section, we are interested in applying our proposed method for solving a convex minimization problem of the sum of two convex and lower semicontinuous functions $f, g : \mathbb{R}^n \to (-\infty, +\infty]$. So we consider the following convex minimization problem: $\min(f(x) + g(x))x \in \mathbb{R}^n$. Combettes and Wajs [17] proved that $\breve{q}$ is a minimizer of (4.1) if and only if $\breve{q} = T\breve{q}$, where $T = prox_{\rho g}(I - \rho\nabla f)$; see [17, Prop. 3.1(iii)]. It is also known that T is nonexpansive if $\rho \in (0, 2/L)$ where L is a Lipschitz constant of ∇f. For the past two decades, several algorithms were introduced for solving problem (4.1). A simple and classical algorithm is the forward–backward algorithm (FBA) introduced by Lions and Mercier [25].

The forward–backward algorithm (FBA) is defined by

$$\begin{cases} y_n = x_n - \gamma \nabla f x_n, \\ x_{n+1} = x_n + \rho_n (J_{\gamma \partial g} y_n - x_n), \end{cases} \tag{4.1}$$

where $n \geq 1$, $x_0 \in H$, L is a Lipschitz constant of ∇f, $\gamma \in (0, 2/L)$, $\delta = 2 - (\gamma L/2)$, and $\{\rho_n\}$ is a sequence in $[0, \delta]$ such that $\sum_{n\in\mathbb{N}} \rho_n(\delta - \rho_n) = +\infty$. The technique for improving speed and giving a better convergence behavior of the algorithms was first introduced by Polyak [26] by adding an inertial step. Since then, many authors employed the inertial technique to accelerate their algorithms for using in various problems [20, 24, 27–31]. The following iterative method with an inertial step can be used for improving performance of FBA.

A fast iterative shrinkage-thresholding algorithm (FISTA) [30] is defined by

$$\begin{cases} y_n = Tx_n, \\ t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}, \\ \mu_n = \frac{t_n - 1}{t_{n+1}}, \\ x_{n+1} = y_n + \mu_n(y_n - y_{n-1}), \end{cases} \tag{4.2}$$

where $n \geq 1$, $x_1 = y_0 \in \mathbb{R}^n$, $t_1 = 1$, $T := prox_{\frac{1}{L}g}(I - \frac{1}{L}\nabla f)$, and μ_n is a so-called inertial step size. The FISTA was suggested by Beck and Teboulle [30]. They proved the convergence rate of the FISTA and applied the FISTA to image restoration problem [30]. The inertial step size μ_n of the FISTA was first introduced by Nesterov [32].

A *new accelerated proximal gradient algorithm* (nAGA) [31] is defined by

$$\begin{cases} y_n = x_n + \mu_n(x_n - x_{n-1}), \\ x_{n+1} = T_n[(1 - \rho_n)y_n + \rho_n T_n y_n], \end{cases} \tag{4.3}$$

where $n \geq 1$, T_n is the forward–backward operator of f and g with respect to $a_n \in (0, 2/L)$, $\{\mu_n\}$ and $\{\rho_n\}$ are sequences in $(0, 1)$, and $\frac{\|x_n - x_{n-1}\|_2}{\mu_n} \to 0$. The nAGA was introduced for proving a convergence theorem by Verma and Shukla [31]. They also applied this method for solving the nonsmooth convex minimization problem with sparsity inducing regularizers for the multitask learning framework.

Theorem 4.1 *Let $f, g : \mathbb{R}^n \to (-\infty, \infty]$ be such that g is a convex function and f is a smooth convex function with a gradient having a Lipschitz constant L. Let $a_n \in (0, 2/L)$ be such that $\{a_n\}$ converges to a, let $T := prox_{ag}(I - a\nabla f)$ and $T_n := prox_{a_n g}(I - a_n \nabla f)$, and let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then*:

(i) $\|x_{n+1} - \breve{q}\| \leq K \cdot \prod_{j=1}^{n}(1 + 2\mu_j)$, *where* $K = \max\{\|x_1 - \breve{q}\|, \|x_2 - \breve{q}\|\}$ *and* $\breve{q} \in Argmin(f + g)$;

(ii) $\{x_n\}$ *converges weakly to a point in* $\mathrm{Argmin}(f + g)$.

Proof It is known that T and $\{T_n\}$ are nonexpansive operators for all n and that $F(T) = \bigcap_{n=1}^{\infty} F(T_n) = Argmin(f + g)$; see [16, Prop. 26.1]. By Lemma 2.2 we obtain that $\{T_n\}$ satisfies the *NST***-condition*. From Theorem 3.4 we get the required result directly by putting $G = \mathbb{R}^n \times \mathbb{R}^n$, the complete graph on $\mathbb{R}^n$. □

5 Application on the image restoration problem

We can describe the image restoration problem as a simple linear model

$$Ax = c + u, \tag{5.1}$$

where $A \in \mathbb{R}^{m\times n}$ is the blurring operation, an image $x \in \mathbb{R}^{n\times 1}$, $c \in \mathbb{R}^{m\times 1}$ is the observed image, and u is an additive noise. The image restoration problem is finding the original image $x^* \in \mathbb{R}^{n\times 1}$ that satisfies (5.1). To find the solution of problem (5.1), we minimize the additive noise to approximate the original image by using the method knowing as the least squares (LS) problem

$$\min_x \|Ax - c\|_2^2, \tag{5.2}$$

where $\|\cdot\|_2$ is an l_2-norm. The solution of (5.2) can be estimated by many iterations such as the Richardson iteration; see [33] for details. However, the number of unknown variables is much greater than that of observations, which causes (5.2) to be an ill-posed problem because of a huge norm result, which is thus meaningless; see [34] and [35]. Therefore, to improve the ill-conditioned least squares problem, several regularization methods were introduced. One of the most popular regularization methods is the Tikhonov regularization suggested by Tikhonov; see [36]. It is defined to solve the following minimization problem:

$$\min_x \left\{ \|Ax - c\|_2^2 + \lambda \|Lx\|_2^2 \right\}, \tag{5.3}$$

where $\lambda > 0$ is called a regularization parameter, and $L \in \mathbb{R}^{m\times n}$ is called the Tikhonov matrix. In the standard form, L is set to be the identity. In statistics, (5.3) is known as a ridge regression.

A new method for estimation a solution of (5.1) called the least absolute shrinkage and selection operator (LASSO), was proposed by Tibshirani [37] as follows:

$$\min_x \{ \|Ax - c\|_2^2 + \lambda \|x\|_1 \}, \tag{5.4}$$

where $\|\cdot\|_1$ is the l_1-norm defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$. This method improved the original LS (5.2) and the classical regularization such as the subset selection and the ridge regression (5.3). Moreover, the LASSO can also be applied to image and regression problems [30, 37], etc.

For solving image restoration problem, especially the true RGB images, the model (5.4) is highly costly to compute the multiplication Ax and $\|x\|_1$ because of the size of matrix A and x as well as their members. To overcome this problem, most of researchers in this area employ the 2-D fast Fourier transform for transformation of the true RGB images, and the model (5.4) is slightly modified by using the 2-D fast Fourier transform of the following form:

$$\min_x \{ \|\mathcal{A}x - \mathcal{C}\|_2^2 + \lambda \|Wx\|_1 \}, \tag{5.5}$$

where $\mathcal{A}$ is the blurring operation, often chosen as $\mathcal{A} = RW$, R is the blurring matrix, W is the 2-D fast Fourier transform, $\mathcal{C} \in \mathbb{R}^{m\times n}$ is the observed blurred and noisy image of size $m \times n$, and λ is a positive regularization parameter.

In this section, we apply Algorithm 3.1 to solving the image restoration problem (5.5) by using Theorem 4.1 when $f(x) = \|\mathcal{A}x - \mathcal{C}\|_2^2$ and $g(x) = \lambda\|Wx\|_1$ and compare the deblurring efficiency of Algorithm 3.1 with FISTA and FBA. In this experiment the true RGB images, Wat Chedi Luang and Wat Boonyawad of size 256^2 are considered as the original images. We blur the images with a Gaussian blur of size 9^2 and $\sigma = 4$, where σ is a standard deviation. After that, we use the peak signal-to-noise ratio (PSNR) [38] to measure the performance of those three algorithms when $\mathrm{PSNR}(x_n)$ is defined by

$$PSNR(x_n) = 10\log_{10}\left(\frac{255^2}{MSE}\right),$$

where 255 is the maximum gray level of an 8 bits/pixel monotonic image, and $MSE = \frac{1}{N}\|x_n - x^*\|_2^2 = \frac{1}{N}\sum_{i=1}^N |x_n(i) - x^*(i)|^2$, $x_n(i)$ and $x^*(i)$ are the ith samples in images x_n and x^*, respectively, N is the number of image samples, and x^* is the original image. We note that a higher PSNR shows a higher quality for deblurring image. For these experiments, we set $\lambda = 5 \times 10^{-5}$, and the original image was the blurred image. Then we compute the Lipschitz constant L by using the maximum eigenvalues of the matrix A^TA.

The parameters for Algorithm 3.1, FISTA and FBA, are set as in Table 1.

Table 1 Algorithms and their setting controls

Methods	Setting
Algorithm 3.1	$\rho_n = 0.9$, $c = 1/L$, $\mu_n = n/(n+1)$ if $1 \le n \le 300$ and $1/2^n$ otherwise
FISTA	$t_1 = 1$, $t_{n+1} = (1 + \sqrt{1 + 4t_n^2})/2$, $\mu_n = (t_n - 1)/t_{n+1}$
FBA	$\rho_n = 0.9$, $\gamma = 1/L$

Note that all parameters in Table 1 satisfy those convergence theorems for each algorithm. Theorem 4.1 guarantees the convergence of the sequence $\{x_n\}$ generated by Algorithm 3.1 to the original image x^*. However, convergence behavior of this sequence is measured by the value of PSNR. It is known that PSNR is an appropriate measurement for image restoration problems.

The following experiments show the performance of the proposed algorithm and compare efficiency of deblurring images with FISTA and FBA by using PSNR as our measurement.

The results of a deblurring image of Wat Chedi Luang and Wat Boonyawad with the 300th iteration of the proposed algorithm, FISTA, and FBA are shown in Figs. 1, 2, 3, 4 and Tables 2, 3, 4.

We observe from Figs. 1 and 2 that the graph of PSNR of our proposed algorithm (Algorithm 3.1) is higher than that of FISTA and FBA, which shows that our proposed algorithm gives a better performance than the others.

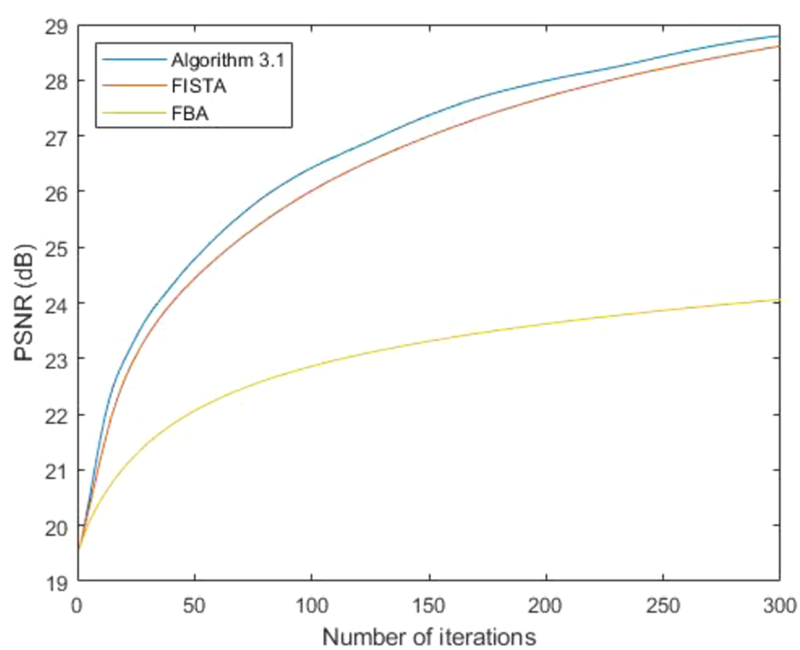

Figure 1 The graphs of PSNR of each algorithm for Wat Chedi Luang

Table 2 The values of PSNR at $x_1, x_5, x_{10}, x_{25}, x_{50}, x_{100}, x_{175}, x_{300}$ (Wat Chedi Luang)

No. Iterations	Algorithm 3.1	FISTA	FBA
1	19.5755	19.5902	19.5755
5	20.4385	20.3075	20.0509
10	21.5938	21.2134	20.4533
25	23.3781	23.0475	21.2725
50	24.7835	24.4344	22.0603
100	26.4226	26.0114	22.8598
175	27.7330	27.3716	23.4775
300	28.7993	28.6113	24.0577

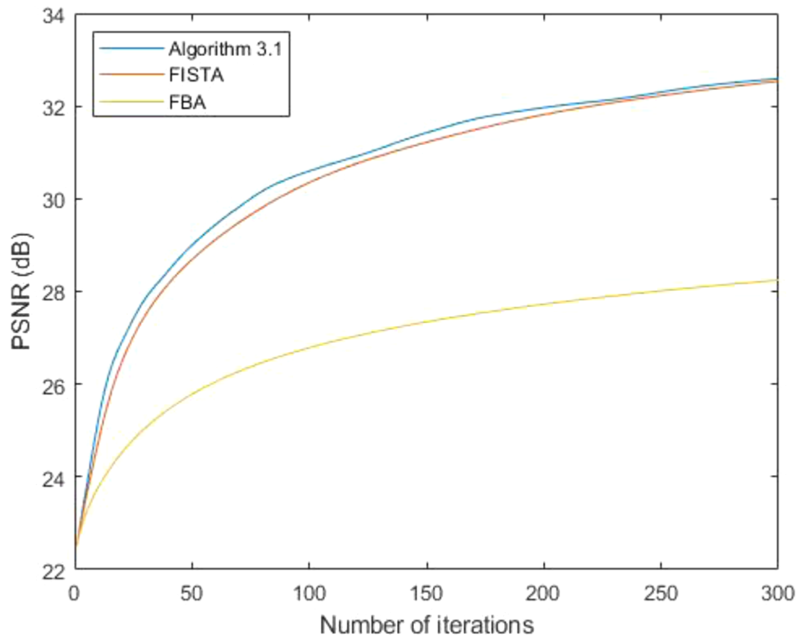

Figure 2 The graphs of PSNR of each algorithm for Wat Boonyawad

Table 3 The values of PSNR at $x_1, x_5, x_{10}, x_{25}, x_{50}, x_{100}, x_{175}, x_{300}$ (Wat Boonyawad)

No. Iterations	Algorithm 3.1	FISTA	FBA
1	22.5136	22.5376	22.5136
5	23.7544	23.5827	23.2316
10	25.1962	24.7313	23.7690
25	27.4028	27.0259	24.8007
50	28.9958	28.6856	25.7830
100	30.5940	30.3467	26.7844
175	31.7665	31.5387	27.5487
300	32.5920	32.5308	28.2435

Table 4 Comparison of image restorations at the 300th iteration of Algorithm 3.1, FISTA, and FBA

Algorithms	Wat Chedi Luang PSNR	Wat Boonyawad PSNR
Algorithm 3.1	28.79933	32.5920
FISTA	28.6113	32.5308
FBA	24.0577	28.7993

Tables 2 and 3 show the efficiency for restoration images of each algorithm under different number of iterations. We found that Algorithm 3.1 gives a higher PSNR than that of FISTA and FBA. Thus our algorithm has a better convergence behavior than the others.

We observed from Table 4 that at the 300th iteration, the value of PSNR of our proposed algorithm is higher than that of FISTA and FBA. This shows that the performance of Algorithm 3.1 is better than the others.

In Figs. 3 and 4, we present the original images, blurred images, and deblurring images by Algorithm 3.1, FISTA, and FBA.

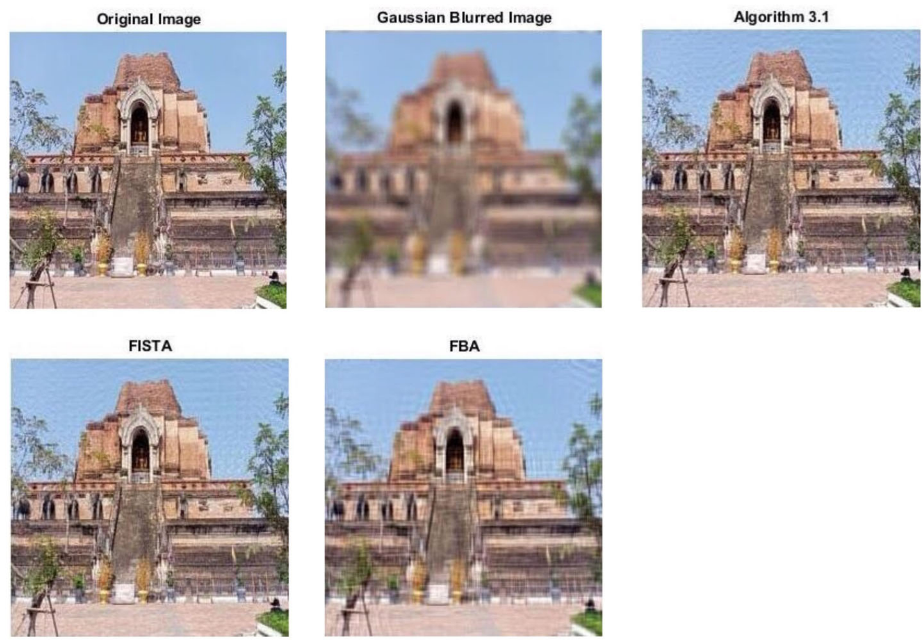

Figure 3 Results for Wat Chedi Luang's deblurring image

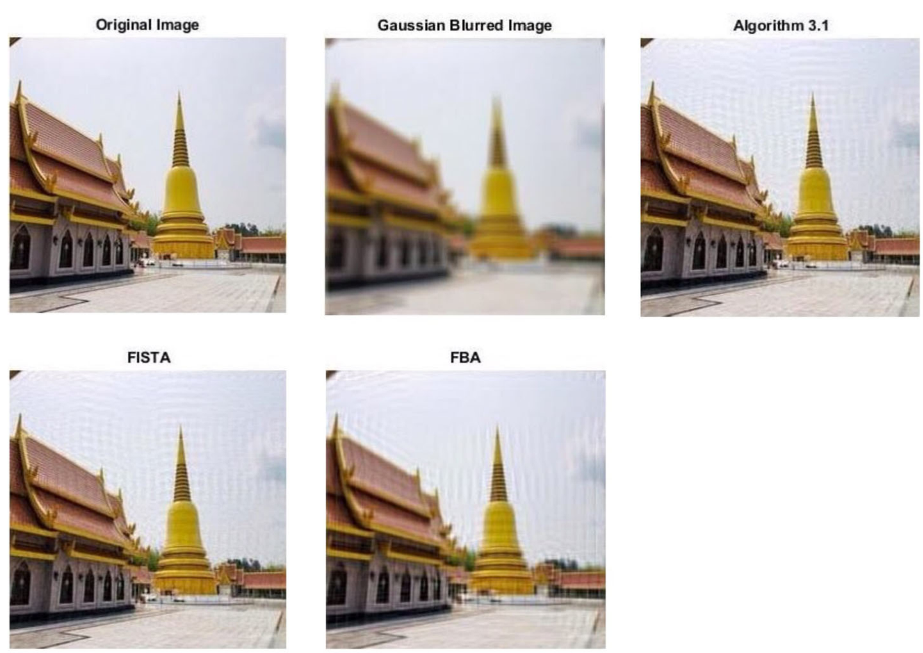

Figure 4 Results for Wat Boonyawad's deblurring image

6 Conclusion

In this study, we introduced a new concept of left and right coordinate affine of a directed graph and used it to introduce a new accelerated common fixed point algorithm for a countable family of G-nonexpansive mappings in a real Hilbert space with a directed graph. The weak convergence theorem of our suggested method, Theorem 3.4, has been established and proven under certain reasonable conditions. Then we compared the convergence behavior of our proposed algorithm with that of FISTA and FBA. We also gave an application of our results to image restoration problems with FISTA and FBA. We found that Algorithm 3.1 gives better results.

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Declarations

Competing interests
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