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Degree of convergence of the functions of trigonometric series in Sobolev spaces and its applications

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Abstract

In this paper, we study the degree of convergence of the functions of Fourier series and conjugate Fourier series in Sobolev spaces using Riesz means. We also study some applications of our main results and observe that our results are much better than earlier results.

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1 Introduction

Sobolev spaces are vector spaces whose elements are functions defined on domains in an $\mathbb{N}$-dimensional Euclidean space $\mathbb{R}^{\mathbb{N}}$ and whose partial derivatives satisfy certain integrability conditions. In order to develop and elucidate the properties of these spaces and mappings between them, we require some machinery of general topology and real and functional analysis.

In one of the classical approximation theories, the properties of approximation of orthogonal function systems, polynomials, and trigonometric have been studied in L^q-norm, and mostly in the maximum norm by [3–5, 25, 27, 28, 30, 31].

The L^q-norm for $q < \infty$ captures the "*height*" and "*width*" of a function. In mathematical terms "*width*" is same as the measure of support of the function. The Sobolev norms capture "*height*", "*width*", and "*oscillations*". The Fourier transform measures oscillation (or frequency or wavelength) by decay of the Fourier transform i.e. the "*oscillation*" of a function is translated to "*decay*" of its Fourier transform. Sobolev norms measure "*oscillation*" via its derivatives (or regularity).

The idea of the best approximation of a function by a polynomial was aggravated by P. L. Chebyshev. This idea chronologically pioneered the discovery of Weierstrass theorem and formed the basis of the modern constructive theory of functions.

The quantity

$$E_\nu(f) = E_\nu(f;a,b) = \inf_{P_\nu(t)} \text{vrai} \sup_{a\le t\le b} \left|f(t) - P_\nu(t)\right| \quad [29],$$

which gives a measure of the deviation(error) of $f(t)$ from the polynomial $P_\nu(t) = c_0 + c_1 t + \cdots + c_\nu t^\nu$ corresponding to it, has been given the title of the best approximation of order ν of this function. If the polynomial P_ν is a trigonometric polynomial T_ν of degree ν, then the best approximation of a function $f \in C^*$ is given by

$$E_\nu(f) = \min_{T_\nu} \|f - T_\nu\|, \|f - T_\nu\| = \max_t \left|f(t) - T_\nu(t)\right|.$$

In this paper, we study the degree of convergence of the functions of Fourier series and conjugate Fourier series in Sobolev norms using Riesz means. However, detailed objectives of this paper will be presented in Sect. 3. Organization of the paper is as follows: In Sect. 2, we give important definitions and known results related to our work. In Sect. 3, we mention detailed objectives of the proposed problems and obtain their results. Applications and their numerical results are discussed in Sect. 4, while conclusion is given in Sect. 5.

2 Notations and preliminaries

In this section, we present notations, definitions, and known results.

2.1 Notations

(i) C^* —$C[K]$ with the continuous 2π-periodic functions on $\mathbb{R}$.

(ii) vrai sup —The essential upper bound vrai sup $f(t)$ is the lower bound of all the numbers M, for which $f(t) > M$ on a set of measure zero.

2.2 Sobolev spaces

For $1 \le q < \infty$, the space $L^q[0,2\pi]$ consists of all measurable functions on $[0,2\pi]$ such that

$$\int_0^{2\pi} \left|f(t)\right|^q dt < \infty,$$

and the norm is defined by

$$\|f\|_q = \begin{cases} (\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^q \, dt)^{\frac{1}{q}}, & 1 \le q < \infty; \\ ess\sup_{f\in(0,2\pi)} |f(t)|, & q = \infty. \end{cases}$$

When $q = 2$,

$$\|f\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

The νth order modulus of smoothness of a function $f : A \to \mathbb{R}$ is defined by

$$\omega_\nu(f,t) = \sup_{0\le h\le t} \left\{\sup\left\{\left|\Delta_h^\nu f(t)\right| : t, t+\nu h \in A\right\}\right\}, \quad t \ge 0, \tag{1}$$

where

$$\Delta_h^\nu f(t) = \sum_{j=0}^{\nu} (-1)^{\nu-j} \binom{\nu}{j} f(t+jh), \quad \nu \in \mathbb{N}.$$

For $\nu = 1$, $\omega_1(f,t)$ is called the modulus of continuity of f [8].

Assume that $\mathbf{X}$ is an open subset of $\mathbb{R}^{\mathbb{N}}$. The Sobolev space $W^{\nu,q}(\mathbf{X})$, $\nu = 1, 2, 3, \dots$, consists of functions $f \in L^q(\mathbf{X})$ such that, for every multi-index β with $|\beta| \le \nu$, the weak derivative $D^\beta f$ exists and $D^\beta f \in L^q(\mathbf{X})$.

Thus,

$$W^{\nu,q}(\mathbf{X}) = \{f \in L^q(\mathbf{X}) : D^\beta f \in L^q(\mathbf{X}), |\beta| \le \nu\} \quad [1]. \tag{2}$$

The norm of (2) is defined by

$$\|f\|_{W^{\nu,q}(\mathbf{X})} = \left(\sum_{|\beta|\le\nu} \|D^\beta f\|^q_{L^q(\mathbf{X})}\right)^{\frac{1}{q}}, \quad 1 \le q < \infty, \tag{3}$$

and

$$\|f\|_{W^{\nu,\infty}(\mathbf{X})} = \max_{|\beta|\le\nu} \|D^\beta f\|_{L^\infty(\mathbf{X})}. \tag{4}$$

The semi-norm of (2) is defined by

$$|f|_{W^{\nu,q}(\mathbf{X})} = \left(\sum_{|\beta|=\nu} \|D^\beta f\|^q_{L^q(\mathbf{X})}\right)^{\frac{1}{q}}, \quad 1 \le q < \infty, \tag{5}$$

and

$$|f|_{W^{\nu,\infty}(\mathbf{X})} = \max_{|\beta|=\nu} \|D^\beta f\|_{L^\infty(\mathbf{X})}. \tag{6}$$

When $q = 2$, the Sobolev space $W^{\nu,2}(\mathbf{X})$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{W^{\nu,2}(\mathbf{X})} = \sum_{|\beta|\le\nu} \langle D^\beta f, D^\beta g\rangle_{L^2(\mathbf{X})},$$

where

$$\langle D^\beta f, D^\beta g\rangle_{L^2(\mathbf{X})} = \int_{\mathbf{X}} D^\beta f D^\beta g \, dt$$

and

$$\|f\|_{W^{\nu,2}(\mathbf{X})} = \langle f, f\rangle^{\frac{1}{2}}_{W^{\nu,2}(\mathbf{X})}.$$

For $\nu = 1$, $q = 2$, the Sobolev space is defined by

$$W^{1,2}(\mathbf{X}) = \{f \in L^2(\mathbf{X}) : D^\beta f \in L^2(\mathbf{X}), |\beta| \le 1\}, \tag{7}$$

and its norm is defined by

$$\|f\|_{W^{1,2}(\mathbf{X})} = \left(\sum_{|\beta|\leq 1} \|D^{\beta}f\|^{2}_{L^{2}(\mathbf{X})}\right)^{\frac{1}{2}}. \tag{8}$$

Example 2.1 ([2]) For $1 \leq q \leq \infty$, the function $f(t) = |t|$ belongs to $W^{1,q}(\mathbf{X})$, where $\mathbf{X} = (-1,+1)$ and

$$f'(t) = \begin{cases} +1 & \text{if } 0 < t < 1, \\ -1 & \text{if } -1 < t < 0. \end{cases}$$

Remark 2.2 Here, we discuss some important properties of the Sobolev space.

(i) For $1 \leq q \leq \infty$ and $\nu = 1,2,\dots,$ the Sobolev space $W^{\nu,q}(\mathbf{X})$ is a Banach space.
(ii) For $1 \leq q < \infty$ and $\nu = 1,2,\dots,$ the Sobolev space $W^{\nu,q}(\mathbf{X})$ is separable.

Remark 2.3

(i) For $\nu = 0$, the Sobolev space reduces in L^q space i.e. $W^{0,q}(\mathbf{X}) = L^q(\mathbf{X})$.
(ii) For $\nu = 1,2,3,\dots,$ $W^{\nu,q}(\mathbf{X}) = Lip(\nu,q)$.
(iii) For $\beta = \nu$, we have $W^{1,q}(\mathbf{X}) = Lip(1,q)$.
(iv) For $\nu = 1$, $q \to \infty$, $Lip(1,q) = Lip(1)$.

2.3 Fourier and derived Fourier series

Let f be a 2π-periodic Lebesgue integrable function defined on $[-\pi,\pi]$. The Fourier series of f is given by

$$f(t) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty}(a_\nu \cos \nu t + b_\nu \sin \nu t). \tag{9}$$

The ν^{th} partial sum of (9) is given by

$$s_\nu(f;t) = s_\nu(t) - f(t) = \frac{1}{2\pi}\int_0^{\pi} \phi_t(s) D_\nu(s)\,ds, \tag{10}$$

where

$$\phi_t(s) = f(t+s) + f(t-s) - 2f(t), \tag{11}$$

and $D_\nu(s)$ (Dirichlet kernel) is defined by

$$D_\nu(s) = \frac{\sin(\nu + \frac{1}{2})s}{\sin \frac{s}{2}}. \tag{12}$$

The derived Fourier series of (9) is given by

$$f'(t) \sim \sum_{\nu=1}^{\infty} \nu(b_\nu \cos \nu t - a_\nu \sin \nu t), \tag{13}$$

which is obtained by differentiating (9) term by term.

The ν^{th} partial sum of (13) is given by

$$s'_\nu\big(f';t\big) = s'_\nu(t) - f'(t) = \frac{1}{2\pi}\int_0^\pi D_\nu(s)\,dg_t(s), \tag{14}$$

where

$$g_t(s) = f(t+s) - f(t-s) - 2sf'(t)$$

and

$$dg_t(s) = d\big(f(t+s) - f(t-s)\big) - 2f'(t)\,ds.$$

2.4 Conjugate Fourier and conjugate derived Fourier series

The conjugate series of (9) is given by

$$\tilde{f}(t) \sim \sum_{\nu=1}^{\infty}(a_\nu \sin \nu t - b_\nu \cos \nu t), \tag{15}$$

which is said to be a conjugate Fourier series.

The ν^{th} partial sum of (15) is given by

$$\tilde{s}_\nu(\tilde{f};t) = \tilde{s}_{\nu(t)} - \tilde{f}(t) = -\frac{1}{2\pi}\int_0^\pi \varphi_t(s)\frac{\cos(\nu+\frac{1}{2})s}{\sin\frac{s}{2}}\,ds, \tag{16}$$

where the function $\tilde{f}$, the conjugate to a 2π-periodic function f, is given by

$$\tilde{f}(t) = -\frac{1}{2\pi}\int_0^\pi \varphi_t(s)\cot\left(\frac{s}{2}\right)ds, \tag{17}$$

where

$$\varphi_t(s) = f(t+s) - f(t-s). \tag{18}$$

The derived series of (15) is given by

$$\tilde{f}'(t) \sim \sum_{\nu=1}^{\infty}\nu(a_\nu \cos \nu t + b_\nu \sin \nu t), \tag{19}$$

which is said to be a conjugate derived Fourier series.

The ν^{th} partial sum of (19) is given by

$$\begin{aligned}\tilde{s}'_\nu\big(f';t\big) &= \tilde{s}'_\nu(t) - \tilde{f}'(t)\\ &= -\frac{2(\nu+\frac{1}{2})}{\pi}\int_0^\pi \frac{\rho_t(s)\sin(\nu+\frac{1}{2})s}{4\sin\frac{s}{2}}\,ds - \frac{1}{\pi}\int_0^\pi \frac{\rho_t(s)}{4\sin\frac{s}{2}}\frac{\cos(\nu+\frac{1}{2})s}{\tan\frac{s}{2}}\,ds\\ &= -\frac{2\nu}{\pi}\int_0^\pi \frac{\rho_t(s)\sin(\nu+\frac{1}{2})s}{4\sin\frac{s}{2}}\,ds - \frac{1}{\pi}\int_0^\pi \frac{\rho_t(s)\cos\nu s}{4\sin^2\frac{s}{2}}\,ds,\end{aligned} \tag{20}$$

where the function $\tilde{f}'$, the conjugate to a 2π-periodic function $\tilde{f}$, is given by

$$\tilde{f}'(t) = -\frac{1}{4\pi}\int_0^{\pi} \rho_t(s)\operatorname{cosec}^2\frac{s}{2}\,ds, \tag{21}$$

where

$$\rho_t(s) = f(t+s) + f(t-s). \tag{22}$$

The following result is relevant to our discussion.

Theorem 2.4 ([10]) *Let $f \in L^q(\mathbb{R})$ with $1 < q \leq \infty$. The following properties are equivalent*:

(i) $f \in W^{1,q}(\mathbb{R})$;

(ii) $\exists$ *a constant C such that for all $s \in (\mathbb{R})$*

$$\|\tau_s f - f\|_{L^q}(\mathbb{R}) \leq C|s|.$$

Moreover, one can choose $C = \|f'\|_{L^q}(\mathbb{R})$ in (ii) *and $(\tau_s(f))(t) = f(t+s)$.*

2.5 Riesz means

Let $\sum_{\nu=0}^{\infty} u_\nu$ be an infinite series such that $s_k = \sum_{\nu=0}^{k} u_\nu$. Let p_ν be a nonnegative, nondecreasing sequence of numbers such that

$$P_\nu = \sum_{k=0}^{\nu} p_k \neq 0 \quad \forall\, \nu \geq 0, \qquad P_{-1} = p_{-1} = 0 \quad \text{and} \quad P_\nu \to \infty \quad \text{as } \nu \to \infty.$$

The sequence-to-sequence transformation defined by

$$t_\nu = \frac{1}{P_\nu}\sum_{k=0}^{\nu} p_k s_k$$

is called Riesz means or (R, p_ν) means of the sequence $\{s_\nu\}$. The series $\sum_{\nu=0}^{\infty} u_\nu$ is said to be summable to the sum s by Riesz method if we can write $t_\nu \to s$ as $\nu \to \infty$.

The necessary and sufficient conditions for the (R, p_ν) method to be regular are given by

$$\sum_{k=0}^{\nu} |p_k| < c|P_\nu|, \qquad |P_\nu| \to \infty.$$

2.6 Degree of convergence

The degree of convergence of a summation method to a given function f is a measure how fast T_ν converges to f, which is given by

$$\|f - T_\nu\| = \mathcal{O}\left(\frac{1}{\lambda_\nu}\right) \quad [14],$$

where $\lambda_\nu \to \infty$ as $\nu \to \infty$.

3 Main results

In this section, we study the following results.

3.1 Degree of convergence of a function of Fourier series

The degree of approximation of a function in function spaces, viz. Lipschitz, Hölder, generalized Hölder, generalized Zygmund, and Besov spaces, using different means of Fourier series, has been studied by the authors [7, 12, 13, 15, 17–19, 21, 22, 24] etc.

Since the degree of approximation of a function of Fourier series in the above mentioned spaces only gives the degree of the polynomial with respect to the function, but the degree of convergence of a function of Fourier series gives the convergence of the polynomial with respect to the function. The degree of convergence of a function of Fourier series in Sobolev spaces gives a much better result than that of the earlier results obtained using the spaces other than Sobolev spaces.

Therefore, in this subsection, we study the degree of convergence of a function in Sobolev spaces using the Riesz means of Fourier series and establish the following theorem.

Theorem 3.1 *Let f be a 2π-period and Lebesgue integrable function belonging to Sobolev spaces $W^{1,2}$, then the degree of convergence of a function f of Fourier series using Riesz means is given by*

$$\|T_\nu(t)\|_{1,2} = \mathcal{O}\biggl[\left(\frac{p_\nu}{P_\nu(\nu+1)}\right) + \left(\frac{p_\nu \log \pi(\nu+1)}{P_\nu}\right) + \left(\frac{p_{\nu(\nu+1)}}{P_\nu}\right)\int_0^{\frac{1}{\nu+1}} |dg_t(s)| + \left(\frac{p_\nu}{P_\nu}\right)\int_{\frac{1}{\nu+1}}^{\pi} \frac{1}{s^2}|dg_t(s)|\biggr].$$

The following lemmas are required for the proof of Theorem 3.1.

Lemma 3.2 *Let $\{p_n\}$ be a nonnegative and nondecreasing sequence, then for $0 < s \le \frac{1}{\nu+1}$, $M_\nu(s) = \mathcal{O}(\frac{p_\nu(\nu+1)}{P_\nu})$.*

Proof For $0 < s \le \frac{1}{\nu+1}$, $\sin(\frac{s}{2}) \ge \frac{s}{\pi}$ and $\sin(k+\frac{1}{2})s \le (k+\frac{1}{2})s$.

$$\begin{aligned}
|M_\nu(s)| &= \left|\frac{1}{2\pi P_\nu}\sum_{k=0}^{\nu} p_k \frac{\sin(k+\frac{1}{2})s}{\sin\frac{s}{2}}\right| \\
&\le \frac{1}{4\pi P_\nu}\left|\sum_{k=0}^{\nu} p_k \frac{(2k+1)s}{\frac{s}{\pi}}\right| \\
&\le \frac{1}{4P_\nu}\left|\sum_{k=0}^{\nu} p_k(2k+1)\right| \\
&\le \frac{1}{4P_\nu}\mathcal{O}\bigl(p_\nu(\nu+1)\bigr).
\end{aligned}$$

Thus,

$$M_\nu(s) = \mathcal{O}\left(\frac{p_\nu(\nu+1)}{P_\nu}\right). \qquad \square$$

Lemma 3.3 *Let $\{p_n\}$ be a nonnegative and nondecreasing sequence, then for $\frac{1}{\nu+1} < s \leq \pi$, $M_\nu(s) = \mathcal{O}(\frac{p_\nu}{s^2 P_\nu})$.*

Proof For $\frac{1}{\nu+1} < s \leq \pi$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$, $|\sin s| \leq 1$.

$$\begin{aligned}|M_\nu(s)| &= \left|\frac{1}{2\pi P_\nu}\sum_{k=0}^{\nu} p_k \frac{\sin(k+\frac{1}{2})s}{\sin\frac{s}{2}}\right| \\ &\leq \frac{1}{2sP_\nu}\left|\sum_{k=0}^{\nu} p_k \sin\left(k+\frac{1}{2}\right)s\right|.\end{aligned}$$

Now, using Abel's transformation, we have

$$\begin{aligned}\left|\sum_{k=0}^{\nu} p_k \sin\left(k+\frac{1}{2}\right)s\right| &= \left|\sum_{k=0}^{\nu-1}(p_k - p_{k+1})\sum_{r=0}^{k}\sin\left(r+\frac{1}{2}\right)s + p_\nu\sum_{k=0}^{\nu}\sin\left(k+\frac{1}{2}\right)s\right| \\ &= \mathcal{O}\left(\frac{1}{s}\right)\left[\sum_{k=0}^{\nu-1}|p_k - p_{k+1}| + |p_\nu|\right] \\ &= \mathcal{O}\left(\frac{p_\nu}{s}\right).\end{aligned}$$

Thus,

$$M_\nu(s) = \mathcal{O}\left(\frac{p_\nu}{s^2 P_\nu}\right).$$ □

Proof of Theorem 3.1 Using (10), the Riesz transform of the sequence $\{s_\nu(t)\}$ is given by

$$T_\nu(t) = t_\nu^R(t) - f(t) = \frac{1}{P_\nu}\sum_{k=0}^{\nu} p_k\{s_k(t) - f(t)\} = \frac{1}{P_\nu}\sum_{k=0}^{\nu} p_k\left[\frac{1}{2\pi}\int_0^\pi \frac{\phi_t(s)\sin(k+\frac{1}{2})s}{\sin\frac{s}{2}}\,ds\right].$$

Thus,

$$T_\nu(t) = \frac{1}{2\pi P_\nu}\int_0^\pi \phi_t(s)\sum_{k=0}^{\nu} p_k\frac{\sin(k+\frac{1}{2})s}{\sin\frac{s}{2}}\,ds \tag{23}$$

$$= \int_0^\pi \phi_t(s) M_\nu(s)\,ds. \tag{24}$$

Using (14), the Riesz transform of the sequence $\{s'_\nu(t)\}$ is given by

$$\begin{aligned}T'_\nu(t) = t'^R_\nu(t) - f'(t) &= \frac{1}{P_\nu}\sum_{k=0}^{\nu} p_k\{s'_k(t) - f'(t)\} \\ &= \frac{1}{P_\nu}\sum_{k=0}^{\nu} p_k\left[\frac{1}{2\pi}\int_0^\pi \frac{\sin(k+\frac{1}{2})s}{\sin\frac{s}{2}}\,dg_t(s)\right].\end{aligned} \tag{25}$$

Thus,

$$T'_\nu(t) = \frac{1}{2\pi P_\nu}\int_0^\pi \sum_{k=0}^{\nu} p_k\frac{\sin(k+\frac{1}{2})s}{\sin\frac{s}{2}}\,dg_t(s) \tag{26}$$

$$= \int_0^{\pi} M_\nu(s)\, dg_t(s). \tag{27}$$

Now, using the definition of Sobolev norm given in (8), we have

$$\|T_\nu(t)\|_{1,2} = \|T_\nu(t)\|_2 + \|T'_\nu(t)\|_2. \tag{28}$$

Using the definition of L^2 norm, we have

$$\begin{aligned}\|T_\nu(t)\|_2 &= \left\{\frac{1}{2\pi}\int_0^{2\pi} |T_\nu(t)|^2\, dt\right\}^{\frac{1}{2}} \\ &= \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\int_0^{\pi} \phi_t(s) M_\nu(s)\, ds\right|^2 dt\right\}^{\frac{1}{2}}.\end{aligned}$$

Using generalized Minkowski's inequality [6], we have

$$\begin{aligned}\|T_\nu(t)\|_2 &\le \int_0^{\pi} \left\{\frac{1}{2\pi}\int_0^{2\pi} |\phi_t(s)|^2\, dt\right\}^{\frac{1}{2}} |M_\nu(s)|\, ds \\ &\le \int_0^{\pi} \|\phi_t(s)\|_2 |M_\nu(s)|\, ds.\end{aligned} \tag{29}$$

Using Theorem 2.4, we get

$$\begin{aligned}\|T_\nu(t)\|_2 &\le \int_0^{\pi} 2Cs|M_\nu(s)|\, ds \\ &\le 2C\int_0^{\pi} s|M_\nu(s)|\, ds \\ &= 2C\left[\int_0^{\frac{1}{\nu+1}} s|M_\nu(s)|\, ds + \int_{\frac{1}{\nu+1}}^{\pi} s|M_\nu(s)|\, ds\right] \\ &= 2C[I_1 + I_2].\end{aligned} \tag{30}$$

Now, using Lemma 3.2, we get

$$\begin{aligned}I_1 &= \int_0^{\frac{1}{\nu+1}} s|M_\nu(s)|\, ds \\ &\le \frac{p_{\nu(\nu+1)}}{P_\nu}\int_0^{\frac{1}{\nu+1}} s\, ds \\ &= \mathcal{O}\left(\frac{p_\nu}{P_\nu(\nu+1)}\right).\end{aligned} \tag{31}$$

Now, using Lemma 3.3, we get

$$\begin{aligned}I_2 &= \int_{\frac{1}{\nu+1}}^{\pi} s|M_\nu(s)|\, ds \\ &\le \int_{\frac{1}{\nu+1}}^{\pi} \frac{p_\nu}{sP_\nu}\, ds\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{P_\nu} \int_{\frac{1}{\nu+1}}^{\pi} \frac{P_m}{s}\, ds \\
&= \mathcal{O}\left(\frac{p_\nu \log \pi (\nu+1)}{P_\nu} \right).
\end{aligned} \tag{32}$$

From (31) and (32), we have

$$\| T_\nu(t) \|_2 = \mathcal{O}\left[\left(\frac{p_\nu}{P_\nu(\nu+1)} \right) + \left(\frac{p_\nu \log \pi (\nu+1)}{P_\nu} \right) \right]. \tag{33}$$

Using the definition of L^2 norm, we get

$$\begin{aligned}
\| T'_\nu(t) \|_2 &= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| T'_\nu(t) \right|^2 dt \right\}^{\frac{1}{2}} \\
&= \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \int_0^{\pi} M_\nu(s)\, dg_t(s) \right|^2 dt \right\}^{\frac{1}{2}}.
\end{aligned} \tag{34}$$

Using generalized Minkowski's inequality [6], we get

$$\begin{aligned}
\| T'_\nu(t) \|_2 &\le \int_0^{\pi} \left| M_\nu(s) \right| \left| dg_t(s) \right| \\
&\le \left[\int_0^{\frac{1}{\nu+1}} \left| M_\nu(s) \right| \left| dg_t(s) \right| + \int_{\frac{1}{\nu+1}}^{\pi} \left| M_\nu(s) \right| \left| dg_t(s) \right| \right] \\
&= I_3 + I_4.
\end{aligned}$$

Now, using Lemma 3.2, we get

$$\begin{aligned}
I_3 &= \int_0^{\frac{1}{\nu+1}} \left| M_\nu(s) \right| \left| dg_t(s) \right| \\
&= \mathcal{O}\left(\frac{p_{\nu(\nu+1)}}{P_\nu} \right) \int_0^{\frac{1}{\nu+1}} \left| dg_t(s) \right|.
\end{aligned} \tag{35}$$

Now, using Lemma 3.3, we get

$$\begin{aligned}
I_4 &= \int_{\frac{1}{\nu+1}}^{\pi} \left| M_\nu(s) \right| \left| dg_t(s) \right| \\
&= \int_{\frac{1}{\nu+1}}^{\pi} \frac{p_\nu}{s^2 P_\nu} \left| dg_t(s) \right| \\
&= \mathcal{O}\left(\frac{p_\nu}{P_\nu} \right) \int_{\frac{1}{\nu+1}}^{\pi} \frac{1}{s^2} \left| dg_t(s) \right|.
\end{aligned} \tag{36}$$

From (35) and (36), we have

$$\| T'_\nu(t) \|_2 = \mathcal{O}\left[\left(\frac{p_{\nu(\nu+1)}}{P_\nu} \right) \int_0^{\frac{1}{\nu+1}} \left| dg_t(s) \right| + \left(\frac{p_\nu}{P_\nu} \right) \int_{\frac{1}{\nu+1}}^{\pi} \frac{1}{s^2} \left| dg_t(s) \right| \right]. \tag{37}$$

From (33) and (37), we have

$$\|T_\nu(t)\|_{1,2} = \mathcal{O}\biggl[\biggl(\frac{p_\nu}{P_\nu(\nu+1)}\biggr) + \biggl(\frac{p_\nu \log \pi(\nu+1)}{P_\nu}\biggr) + \biggl(\frac{p_{\nu(\nu+1)}}{P_\nu}\biggr)\int_0^{\frac{1}{\nu+1}} |dg_t(s)| + \biggl(\frac{p_\nu}{P_\nu}\biggr)\int_{\frac{1}{\nu+1}}^{\pi} \frac{1}{s^2}|dg_t(s)|\biggr]. \qquad \square$$

3.2 Degree of convergence of a function of conjugate Fourier series

Consider a series

$$\sum_{\nu=2}^{\infty} \frac{\sin(\nu t)}{\log \nu}. \tag{38}$$

We note that (38) is a conjugate series of a Fourier series $\sum_{\nu=2}^{\infty} \frac{\cos(\nu t)}{\log \nu}$, but it is not a Fourier series that can be easily observed by the following theorem.

Theorem 3.4 ([9]) *If $a_\nu > 0$, $\sum \frac{a_\nu}{\nu} = \infty$, then $\sum a_\nu \sin \nu t$ is not a Fourier series. Hence, there exists a trigonometric series with coefficients tending to zero which are not Fourier series.*

One can see [9] for more details on conjugate Fourier series.

The degree of approximation of a conjugate function in function spaces, viz. Lipschitz, Hölder, generalized Hölder, generalized Zygmund, and Besov spaces, using different means of conjugate Fourier series, has been studied by the authors [7, 11, 12, 16, 17, 19, 20, 23, 26] etc.

As discussed in Sect. 3.1, the degree of convergence of a function of conjugate Fourier series also gives the convergence of the polynomial with respect to the function. The degree of convergence of a function of conjugate Fourier series in Sobolev spaces gives a much better result than that of the results using the spaces other than Sobolev spaces.

Therefore, in this subsection, we study the degree of convergence of conjugate of a function in Sobolev spaces using the Riesz means of conjugate Fourier series and establish a following theorem.

Theorem 3.5 *Let $\tilde{f}$ be a 2π-period and Lebesgue integrable function belonging to Sobolev spaces $W^{1,2}$, then the degree of convergence of a function $\tilde{f}$ of conjugate Fourier series using Riesz means is given by*

$$\begin{aligned}\|\tilde{T}_\nu(t)\|_{1,2} = \mathcal{O}\biggl[&\biggl(\frac{1}{\nu+1}\biggr) + \biggl(\frac{p_\nu \log \pi(\nu+1)}{P_\nu}\biggr) + \biggl(\frac{(\nu+1)p_\nu}{P_\nu}\biggr)\int_0^{\frac{1}{\nu+1}} \|\rho_t(s)\|_2\, ds \\ &+ \biggl(\frac{p_\nu}{P_\nu}\biggr)\int_{\frac{1}{\nu+1}}^{\pi} \frac{1}{s^2}\|\rho_t(s)\|_2\, ds + \biggl(\int_0^{\frac{1}{\nu+1}} \|\rho_t(s)\|_2 \frac{1}{s^2}\, ds\biggr) \\ &+ \biggl(\frac{p_\nu}{P_\nu}\biggr)\int_{\frac{1}{\nu+1}}^{\pi} \|\rho_t(s)\|_2 \frac{1}{s^3}\, ds\biggr].\end{aligned}$$

The following lemmas are required for the proof of Theorem 3.5.

Lemma 3.6 *Let $\{p_n\}$ be a nonnegative and nondecreasing sequence, then for $0 < s \le \frac{1}{\nu+1}$, $\tilde{M}_\nu(s) = \mathcal{O}(\frac{1}{s})$.*

Proof For $0 < s \leq \frac{1}{\nu+1}$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$ and $|\cos ks| \leq 1$.

$$\begin{aligned}
\left|\tilde{M}_\nu(s)\right| &= \left|\frac{1}{2\pi P_\nu}\sum_{k=0}^{\nu} p_k \frac{\cos(k+\frac{1}{2})s}{\sin\frac{s}{2}}\right| \\
&\leq \frac{1}{2\pi P_\nu}\left|\sum_{k=0}^{\nu} p_k \frac{\cos(k+\frac{1}{2})s}{\frac{s}{\pi}}\right| \\
&\leq \frac{1}{2P_\nu}\left|\sum_{k=0}^{\nu} p_k \frac{\cos(k+\frac{1}{2})s}{s}\right| \\
&\leq \frac{1}{2sP_\nu}\left|\sum_{k=0}^{\nu} p_k \cos\left(k+\frac{1}{2}\right)s\right|.
\end{aligned}$$

Thus,

$$\tilde{M}_\nu(s) = \mathcal{O}\left(\frac{1}{s}\right). \qquad \square$$

Lemma 3.7 *Let* $\{p_n\}$ *be a nonnegative and nondecreasing sequence, then for* $\frac{1}{\nu+1} < s \leq \pi$, $\tilde{M}_\nu(s) = \mathcal{O}(\frac{p_\nu}{s^2 P_\nu})$.

Proof For $\frac{1}{\nu+1} < s \leq \pi$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$.

$$\begin{aligned}
\left|\tilde{M}_\nu(s)\right| &= \left|\frac{1}{2\pi P_\nu}\sum_{k=0}^{\nu} p_k \frac{\cos(k+\frac{1}{2})s}{\sin\frac{s}{2}}\right|, \\
\left|\tilde{M}_\nu(s)\right| &\leq \frac{1}{2\pi P_\nu}\left|\sum_{k=0}^{\nu} p_k \frac{\cos(k+\frac{1}{2})s}{\frac{s}{\pi}}\right| \\
&\leq \frac{1}{2sP_\nu}\left|\sum_{k=0}^{\nu} p_k \cos\left(k+\frac{1}{2}\right)s\right|.
\end{aligned}$$

Now, using Abel's transformation, we have

$$\left|\sum_{k=0}^{\nu} p_k \cos\left(k+\frac{1}{2}\right)s\right| = \mathcal{O}\left(\frac{p_\nu}{s}\right).$$

Thus,

$$\tilde{M}_\nu(s) = \mathcal{O}\left(\frac{p_\nu}{s^2 P_\nu}\right). \qquad \square$$

Lemma 3.8 *Let* $\{p_n\}$ *be a nonnegative and nondecreasing sequence, then for* $0 < s \leq \frac{1}{\nu+1}$, $\tilde{M}'_{\nu_1}(s) = \mathcal{O}(\frac{(\nu+1)p_\nu}{P_\nu})$.

Proof For $0 < s \leq \frac{1}{\nu+1}$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$ and $\sin(k+\frac{1}{2})s \leq (k+\frac{1}{2})s$.

$$\begin{aligned}
\left|\tilde{M}'_{\nu_1}(s)\right| &= \left|\frac{-2k}{\pi P_\nu}\sum_{k=0}^{\nu} p_k \frac{\sin(k+\frac{1}{2})s}{4\sin\frac{s}{2}}\right| \\
&\leq \frac{k}{2\pi P_\nu}\left|\sum_{k=0}^{\nu} p_k \frac{\sin(k+\frac{1}{2})s}{4\sin\frac{s}{2}}\right| \\
&= \frac{k}{4\pi P_\nu}\left|\sum_{k=0}^{\nu} p_k (2k+1)\right| \\
&\leq \frac{k}{4\pi P_\nu}\mathcal{O}\big((\nu+1)p_\nu\big).
\end{aligned}$$

Thus,

$$\tilde{M}'_{\nu_1}(s) = \mathcal{O}\left(\frac{(\nu+1)p_\nu}{P_\nu}\right). \qquad \square$$

Lemma 3.9 *Let* $\{p_n\}$ *be nonnegative and nondecreasing, then for* $\frac{1}{\nu+1} < s \leq \pi$, $\tilde{M}'_{\nu_1}(s) = \mathcal{O}(\frac{p_\nu}{s^2 P_\nu})$.

Proof For $\frac{1}{\nu+1} < s \leq \pi$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$.

$$\begin{aligned}
\left|\tilde{M}'_{\nu_1}(s)\right| &= \left|\frac{-2k}{\pi P_\nu}\sum_{k=0}^{\nu} p_k \frac{\sin(k+\frac{1}{2})s}{4\sin\frac{s}{2}}\right| \\
&\leq \frac{k}{2\pi P_\nu}\left|\sum_{k=0}^{\nu} p_k \frac{\sin(k+\frac{1}{2})s}{4\sin\frac{s}{2}}\right| \\
&\leq \frac{k}{4\pi P_\nu}\left|\sum_{k=0}^{\nu} p_k \frac{\sin(k+\frac{1}{2})s}{\frac{s}{\pi}}\right| \\
&= \frac{k}{2s\pi P_\nu}\left|\sum_{k=0}^{\nu} p_k \sin\left(k+\frac{1}{2}\right)\right|.
\end{aligned}$$

Now, using Abel's transformation, we have

$$\begin{aligned}
\left|\sum_{k=0}^{\nu} p_k \sin\left(k+\frac{1}{2}\right)\right| &= \left|\sum_{k=0}^{\nu-1}(p_k - p_{k+1})\sum_{r=0}^{k}\sin\left(r+\frac{1}{2}\right)s + p_\nu \sum_{k=0}^{\nu}\sin\left(k+\frac{1}{2}\right)s\right| \\
&\leq \mathcal{O}\left(\frac{1}{s}\right)\left[\sum_{k=0}^{\nu-1}\left|(p_k - p_{k+1})\right| + |p_\nu|\right] \\
&= \mathcal{O}\left(\frac{p_\nu}{s}\right).
\end{aligned}$$

Thus,

$$\tilde{M}'_{\nu_1}(s) = \mathcal{O}\left(\frac{p_\nu}{s^2 P_\nu}\right). \qquad \square$$

Lemma 3.10 *Let $\{p_n\}$ be a nonnegative and nondecreasing sequence, then for* $0 < s \leq \frac{1}{\nu+1}$, $\tilde{M}'_{\nu_2}(s) = \mathcal{O}(\frac{1}{s^2})$.

Proof For $0 < s \leq \frac{1}{\nu+1}$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$ and $|\cos ks| \leq 1$.

$$\begin{aligned}
\left|\tilde{M}'_{\nu_2}(s)\right| &= \left|-\frac{1}{\pi P_\nu}\sum_{k=0}^{\nu} p_k \frac{\cos ks}{4\sin^2\frac{s}{2}}\right| \\
&\leq \frac{\pi}{2s^2 P_\nu}\left|\sum_{k=0}^{\nu} p_k \cos ks\right|.
\end{aligned}$$

Thus,

$$\tilde{M}'_{\nu_2}(s) = \mathcal{O}\left(\frac{1}{s^2}\right). \qquad \square$$

Lemma 3.11 *Let $\{p_n\}$ be nonnegative and nondecreasing, then for* $\frac{1}{\nu+1} < s \leq \pi$, $\tilde{M}'_{\nu_2}(s) = \mathcal{O}(\frac{p_\nu}{s^3 P_\nu})$.

Proof For $\frac{1}{\nu+1} < s \leq \pi$, $\sin(\frac{s}{2}) \geq \frac{s}{\pi}$.

$$\begin{aligned}
\left|\tilde{M}'_{\nu_2}(s)\right| &= \left|-\frac{1}{\pi P_\nu}\sum_{k=0}^{\nu} p_k \frac{\cos ks}{4\sin^2\frac{s}{2}}\right| \\
&\leq \frac{\pi}{2s^2 P_\nu}\left|\sum_{k=0}^{\nu} p_k \cos ks\right|.
\end{aligned}$$

Now, using Abel's transformation, we have

$$\left|\sum_{k=0}^{\nu} p_k \cos\left(k+\frac{1}{2}\right)s\right| = \mathcal{O}\left(\frac{p_\nu}{s}\right).$$

Thus,

$$\tilde{M}'_{\nu_2}(s) = \mathcal{O}\left(\frac{p_\nu}{s^3 P_\nu}\right). \qquad \square$$

Proof of Theorem 3.5 Using (16), the Riesz transform of the sequence $\{\tilde{s}_\nu(t)\}$ is given by

$$\begin{aligned}
\tilde{T}_\nu(t) = \tilde{t}^R_\nu(t) - \tilde{f}(t) &= \frac{1}{P_\nu}\sum_{k=0}^{\nu} p_k\left\{\tilde{s}_k(t) - \tilde{f}(t)\right\} \\
&= \frac{1}{P_\nu}\sum_{k=0}^{\nu} p_k\left[-\frac{1}{2\pi}\int_0^\pi \frac{\varphi_t(s)\cos(k+\frac{1}{2})s}{\sin\frac{s}{2}}\,ds\right].
\end{aligned}$$

Thus,

$$\tilde{T}_\nu(t) = -\frac{1}{2\pi P_\nu}\int_0^\pi \varphi_t(s)\sum_{k=0}^{\nu} p_k \frac{\cos(k+\frac{1}{2})s}{\sin\frac{s}{2}}\,ds$$

$$= \int_0^\pi \varphi_t(s)\tilde{M}_\nu(s)\,ds. \tag{39}$$

Using (20), the Riesz transform of the sequence $\{\tilde{s}'_\nu(t)\}$ is given by

$$\begin{aligned}
\tilde{T}'_\nu(t) &= \tilde{t}'^R_\nu(t) - \tilde{f}'(t) = \frac{1}{P_\nu}\sum_{k=0}^{\nu} p_k\{\tilde{s}'_k(t) - \tilde{f}'(t)\} \\
&= \frac{1}{P_\nu}\sum_{k=0}^{\nu} p_k\left(-\frac{2k}{\pi}\int_0^\pi \frac{\rho_t(s)\sin(k+\frac{1}{2})s}{4\sin\frac{s}{2}}\,ds - \frac{1}{\pi}\int_0^\pi \frac{\rho_t(s)\cos ks}{4\sin^2\frac{s}{2}}\,ds\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\tilde{T}'_\nu(t) &= -\frac{2}{\pi P_\nu}\left(\int_0^\pi k\rho_t(s)\sum_{k=0}^{\nu} p_k \frac{\sin(k+\frac{1}{2})s}{4\sin\frac{s}{2}}\,ds\right) \\
&\quad - \frac{1}{\pi P_\nu}\left(\int_0^\pi \rho_t(s)\sum_{k=0}^{\nu} p_k \frac{\cos ks}{4\sin^2\frac{s}{2}}\,ds\right) \\
&= \int_0^\pi \rho_t(s)\big(\tilde{M}'_{\nu_1}(s) + \tilde{M}'_{\nu_2}(s)\big)\,ds \\
&= \int_0^\pi \rho_t(s)\tilde{M}'_\nu(s)\,ds, \qquad (40)
\end{aligned}$$

where

$$\tilde{M}'_\nu(s) = \tilde{M}'_{\nu_1}(s) + \tilde{M}'_{\nu_2}(s). \tag{41}$$

Now, using the definition of Sobolev norm given in (8), we have

$$\|\tilde{T}_\nu(t)\|_{1,2} = \|\tilde{T}_\nu(t)\|_2 + \|\tilde{T}'_\nu(t)\|_2. \tag{42}$$

Using the definition of L^2 norm, we have

$$\begin{aligned}
\|\tilde{T}_\nu(t)\|_2 &= \left\{\frac{1}{2\pi}\int_0^{2\pi} |\tilde{T}_\nu(t)|^2\,ds\right\}^{\frac{1}{2}} \\
&= \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\int_0^\pi \varphi_t(s)\tilde{M}_\nu(s)\,ds\right|^2 dt\right\}^{\frac{1}{2}}.
\end{aligned}$$

Using generalized Minkowski's inequality [6], we have

$$\begin{aligned}
\|\tilde{T}_\nu(t)\|_2 &\le \int_0^\pi \left\{\frac{1}{2\pi}\int_0^{2\pi} |\varphi_t(s)|^2\,dt\right\}^{\frac{1}{2}} |\tilde{M}_\nu(s)|\,ds \\
&\le \int_0^\pi \|\varphi_t(s)\|_2 |\tilde{M}_\nu(s)|\,ds. \qquad (43)
\end{aligned}$$

Using Theorem 2.4, we get

$$\|\tilde{T}_\nu(t)\|_2 \le \int_0^\pi 2Cs|\tilde{M}_\nu(s)|\,ds$$

$$\begin{aligned}
&\leq 2C \int_0^\pi s\left|\tilde{M}_\nu(s)\right| ds \\
&= 2C\left[\int_0^{\frac{1}{\nu+1}} s\left|\tilde{M}_\nu(s)\right| ds + \int_{\frac{1}{\nu+1}}^\pi s\left|\tilde{M}_\nu(s)\right| ds\right] \\
&= 2C[\tilde{I}_1 + \tilde{I}_2]. \qquad (44)
\end{aligned}$$

Now, using Lemma 3.6, we get

$$\begin{aligned}
\tilde{I}_1 &= \int_0^{\frac{1}{\nu+1}} s\left|\tilde{M}_\nu(s)\right| ds \\
&\leq \int_0^{\frac{1}{\nu+1}} ds \\
&= \mathcal{O}\left(\frac{1}{\nu+1}\right). \qquad (45)
\end{aligned}$$

Now, using Lemma 3.7, we get

$$\begin{aligned}
\tilde{I}_2 &= \int_{\frac{1}{\nu+1}}^\pi s\left|\tilde{M}_\nu(s)\right| ds \\
&\leq \frac{p_\nu}{P_\nu}\int_{\frac{1}{\nu+1}}^\pi \frac{1}{s}\, ds \\
&= \frac{p_\nu}{P_\nu}\left[\log \pi(\nu+1)\right] \\
&= \mathcal{O}\left(\frac{p_\nu \log \pi(\nu+1)}{P_\nu}\right). \qquad (46)
\end{aligned}$$

From (45) and (46), we have

$$\left\|\tilde{T}_\nu(t)\right\|_2 = \mathcal{O}\left[\left(\frac{1}{\nu+1}\right) + \left(\frac{p_\nu \log \pi(\nu+1)}{P_\nu}\right)\right]. \qquad (47)$$

Using the definition of L^2 norm and generalized Minkowski's inequality [6], we get

$$\begin{aligned}
\left\|\tilde{T}'_\nu(t)\right\|_2 &\leq \int_0^\pi \left\|\rho_t(s)\right\|_2\left|\tilde{M}'_\nu(s)\right| ds \\
&= \int_0^\pi \left\|\rho_t(s)\right\|_2\left|\tilde{M}'_{\nu_1}(s)\right| ds + \int_0^\pi \left\|\rho_t(s)\right\|_2\left|\tilde{M}'_{\nu_2}(s)\right| ds \\
&= \tilde{I}'_3 + \tilde{I}'_4. \qquad (48)
\end{aligned}$$

Now, using Lemmas 3.8 and 3.9, we get

$$\begin{aligned}
\tilde{I}'_3 &= \int_0^\pi \left\|\rho_t(s)\right\|_2\left|\tilde{M}'_{\nu_1}(s)\right| ds \\
&= \int_0^{\frac{1}{\nu+1}} \left\|\rho_t(s)\right\|_2\left|\tilde{M}'_{\nu_1}(s)\right| ds + \int_{\frac{1}{\nu+1}}^\pi \left\|\rho_t(s)\right\|_2\left|\tilde{M}'_{\nu_1}(s)\right| ds \\
&= \mathcal{O}\left[\left(\frac{(\nu+1)p_\nu}{P_\nu}\right)\int_0^{\frac{1}{\nu+1}} \left\|\rho_t(s)\right\|_2 ds + \left(\frac{p_\nu}{P_\nu}\right)\int_{\frac{1}{\nu+1}}^\pi \frac{1}{s^2}\left\|\rho_t(s)\right\|_2 ds\right]. \qquad (49)
\end{aligned}$$

Now, using Lemmas 3.10 and 3.11, we get

$$\begin{aligned}\tilde{I}_4' &= \int_0^{\pi} \|\rho_t(s)\|_2 \left|\tilde{M}_{\nu_2}'(s)\right| ds \\ &= \int_0^{\frac{1}{\nu+1}} \|\rho_t(s)\|_2 \left|\tilde{M}_{\nu_2}'(s)\right| ds + \int_{\frac{1}{\nu+1}}^{\pi} \|\rho_t(s)\|_2 \left|\tilde{M}_{\nu_2}'(s)\right| ds \\ &= \mathcal{O}\left[\left(\int_0^{\frac{1}{\nu+1}} \|\rho_t(s)\|_2 \frac{1}{s^2}\, ds\right) + \left(\frac{p_\nu}{P_\nu}\right)\int_{\frac{1}{\nu+1}}^{\pi} \|\rho_t(s)\|_2 \frac{1}{s^3}\, ds\right]. \end{aligned} \tag{50}$$

From (49) and (50), we have

$$\begin{aligned}\|\tilde{T}_\nu'(t)\|_2 = \mathcal{O}\Bigg[&\left(\frac{(\nu+1)p_\nu}{P_\nu}\right)\int_0^{\frac{1}{\nu+1}} \|\rho_t(s)\|_2\, ds \\ &+ \left(\frac{p_\nu}{P_\nu}\right)\int_{\frac{1}{\nu+1}}^{\pi} \frac{1}{s^2}\|\rho_t(s)\|_2\, ds + \left(\int_0^{\frac{1}{\nu+1}} \|\rho_t(s)\|_2 \frac{1}{s^2}\, ds\right) \\ &+ \left(\frac{p_\nu}{P_\nu}\right)\int_{\frac{1}{\nu+1}}^{\pi} \|\rho_t(s)\|_2 \frac{1}{s^3}\, ds\Bigg]. \end{aligned} \tag{51}$$

From (47) and (51), we have

$$\begin{aligned}\|\tilde{T}_\nu(t)\|_{1,2} = \mathcal{O}\Bigg[&\left(\frac{1}{\nu+1}\right) + \left(\frac{p_\nu \log \pi(\nu+1)}{P_\nu}\right) + \left(\frac{(\nu+1)p_\nu}{P_\nu}\right)\int_0^{\frac{1}{\nu+1}} \|\rho_t(s)\|_2\, ds \\ &+ \left(\frac{p_\nu}{P_\nu}\right)\int_{\frac{1}{\nu+1}}^{\pi} \frac{1}{s^2}\|\rho_t(s)\|_2\, ds + \left(\int_0^{\frac{1}{\nu+1}} \|\rho_t(s)\|_2 \frac{1}{s^2}\, ds\right) \\ &+ \left(\frac{p_\nu}{P_\nu}\right)\int_{\frac{1}{\nu+1}}^{\pi} \|\rho_t(s)\|_2 \frac{1}{s^3}\, ds\Bigg]. \end{aligned}$$

□

4 Applications

In this section, we study some applications of our main results.

4.1 Application on the degree of convergence of a function of Fourier series in Sobolev norm using Riesz means

Consider a function $f(t) = t^3$ and $P_{-1} = p_{-1} = 0$ and $p_\nu = 1\ \forall\ \nu \geq 0$ and $P_\nu = 1 + \nu$.

Then $\phi_t(s) = 0$ and $dg_t(s) = 6s^2\, ds$.

Therefore, $M_\nu(s) = \mathcal{O}(1)$ for $0 < s \leq \frac{1}{\nu+1}$ and $M_\nu(s) = \mathcal{O}(\frac{1}{s^2(\nu+1)})$ for $\frac{1}{\nu+1} < s \leq \pi$.

Then, we have

$$\|T_\nu'(t)\|_2 = \mathcal{O}\left(\frac{1}{(\nu+1)^3} + \frac{1}{(\nu+1)}\left[\pi - \frac{1}{(\nu+1)}\right]\right). \tag{52}$$

Since $\|T_\nu(t)\|_2 = 0$, the degree of convergence of $f(t) = t^3$ is obtained by

$$\|T_\nu(t)\|_{1,2} = \mathcal{O}\left(\frac{1}{(\nu+1)^3} + \frac{1}{(\nu+1)}\left[\pi - \frac{1}{(\nu+1)}\right]\right).$$

Now, we draw the graphs of $T_\nu(f)$ for different values of ν (see Fig. 1).

Table 1 Degree of convergence of $f(t) = t^3$

ν	$T_\nu(t) = \frac{1}{(\nu+1)^3} + \frac{1}{(\nu+1)}[\pi - \frac{1}{(\nu+1)}]$
100	0.031105
1000	0.003139
10,000	0.00031413
50,000	0.00006283
75,000	0.00004189
100,000	0.00003142
.	.
.	.
.	.
∞	0

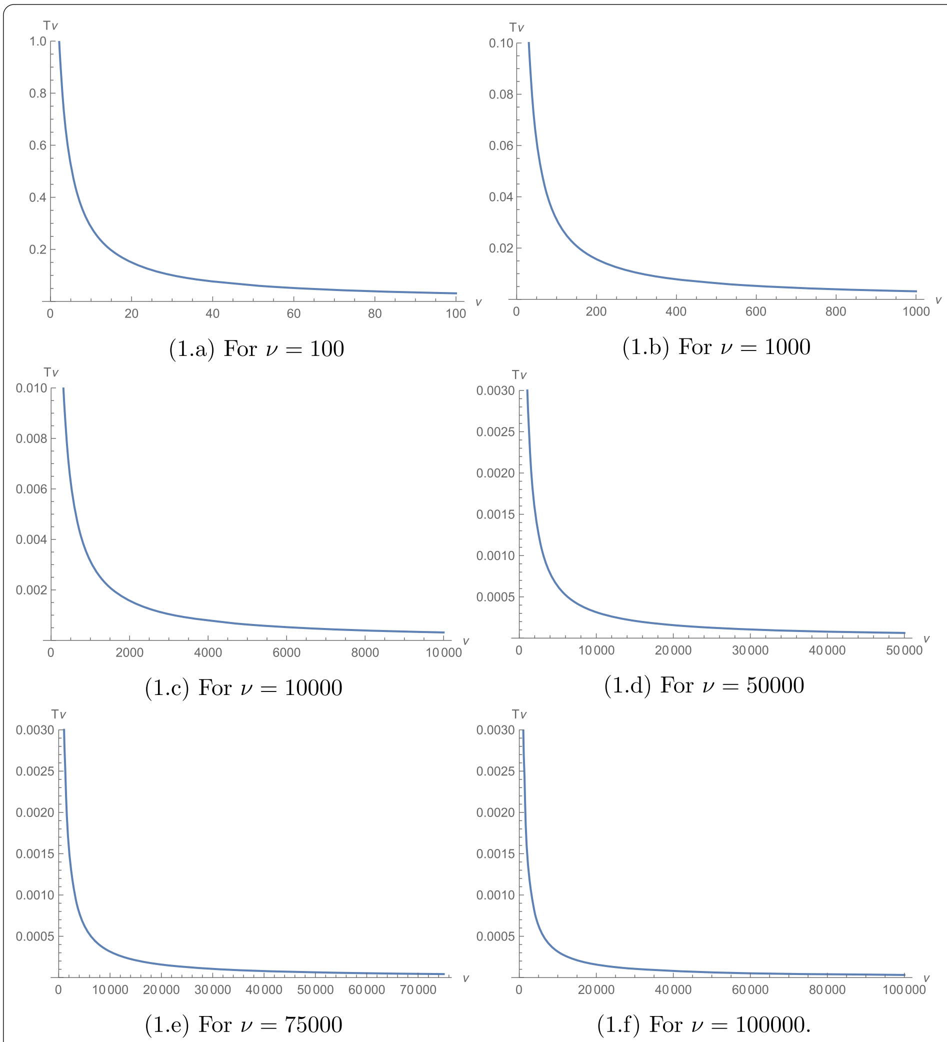

Figure 1 Degree of convergence of $f(t) = t^3$

Remark 4.1 From Table 1 and Figs. 1(a) to 1(f), we observe that the result obtained in Theorem 3.1 is much better than earlier results.

4.2 Application on the degree of convergence of a function of conjugate Fourier series in Sobolev norm using Riesz means

Consider a conjugate function $\tilde{f}(t) = \sum_{\nu=2}^{\infty} \frac{\sin \nu t}{\log \nu}$ for $\nu \geq 2$ and $P_{-1} = p_{-1} = 0$ and $p_\nu = 1$ $\forall\ \nu \geq 0$ and $P_\nu = 1 + \nu$.

Then $\varphi_t(s) = \sum_{\nu=2}^{\infty} \frac{2\cos \nu t \sin \nu s}{\log \nu}$, $\|\varphi_t(s)\|_2 = \sum_{\nu=2}^{\infty} \frac{s}{\log \nu}$ and $\rho_t(s) = \sum_{\nu=2}^{\infty} \frac{2\sin \nu t \cos \nu s}{\log \nu}$, $\|\rho_t(s)\|_2 = \sum_{\nu=2}^{\infty} \frac{1}{\log \nu}$.

Therefore, $\tilde{M}_\nu(s) = \mathcal{O}(\frac{1}{s})$ for $0 < s \leq \frac{1}{\nu+1}$, $\tilde{M}_\nu(s) = \mathcal{O}(\frac{1}{s^2(\nu+1)})$ for $\frac{1}{\nu+1} < s \leq \pi$, $\tilde{M}_{\nu_1}(s) = \mathcal{O}(1)$ for $0 < s \leq \frac{1}{\nu+1}$, $\tilde{M}_{\nu_1}(s) = \mathcal{O}(\frac{1}{(\nu+1)s^2})$ for $\frac{1}{\nu+1} < s \leq \pi$, $\tilde{M}_{\nu_2}(s) = \mathcal{O}(\frac{1}{s^2})$ for $0 < s \leq \frac{1}{\nu+1}$, $\tilde{M}_{\nu_2}(s) = \mathcal{O}(\frac{1}{s^3})$ for $\frac{1}{\nu+1} < s \leq \pi$.

Then, we have

$$\|\tilde{T}_\nu(t)\|_2 = \mathcal{O}\left[\left(\sum_{\nu=2}^{\infty} \frac{1}{\log \nu}\right)\left[\left(\frac{1 + \log \pi(\nu+1)}{\nu+1}\right)\right]\right]$$

and

$$\|\tilde{T}'_\nu(t)\|_2 = \mathcal{O}\left[\left(\sum_{\nu=2}^{\infty} \frac{1}{\log \nu}\right)\left[\left(\frac{1}{(\nu+1)^2}\right) + \left(\frac{1}{(\nu+1)^2}\left(\frac{1}{\pi} - (\nu+1)\right)\right) + \left(\frac{1}{(\nu+1)^2}\left(\frac{1}{\pi^2} - (\nu+1)^2\right)\right)\right]\right].$$

Thus, the degree of convergence of $\tilde{f}(t) = \sum_{\nu=2}^{\infty} \frac{\sin \nu t}{\log \nu}$ for $\nu \geq 2$ is obtained by

$$\|\tilde{T}_\nu(t)\|_{1,2} = \mathcal{O}\left[\left(\sum_{\nu=2}^{\infty} \frac{1}{\log \nu}\right)\left[\left(\frac{1 + \log \pi(\nu+1)}{\nu+1}\right) + \left(\frac{1}{(\nu+1)^2}\right) + \left(\frac{1}{(\nu+1)^2}\left(\frac{1}{\pi} - (\nu+1)\right)\right) + \left(\frac{1}{(\nu+1)^2}\left(\frac{1}{\pi^2} - (\nu+1)^2\right)\right)\right]\right].$$

Now, we draw the graphs of $\tilde{T}_\nu(f)$ for different values of ν (see Fig. 2).

Remark 4.2 From Table 2 and Figs. 2(a) to 2(f), we observe that the result obtained in Theorem 3.5 is much better than earlier results.

Remark 4.3 From Table 1 and Table 2, we also observe that the convergence of Fourier series is faster than the convergence of conjugate Fourier series.

Table 2 Degree of convergence of $f(t) = \sum_{\nu=2}^{\infty} \frac{\sin \nu t}{\log \nu}$

ν	$\tilde{T}_\nu(t) = (\sum_{\nu=2}^{\infty} \frac{1}{\log \nu})[\frac{1+\log \pi(\nu+1)}{\nu+1} + \frac{(\frac{1}{\pi}-(\nu+1))(\frac{1}{\pi^2}-(\nu+1)^2)}{(\nu+1)^2}]$
100	−0.15357
1000	−0.15841
10,000	−0.15906
50,000	−0.15913
75,000	−0.15914
100,000	−0.15914
.	.
.	.
.	.

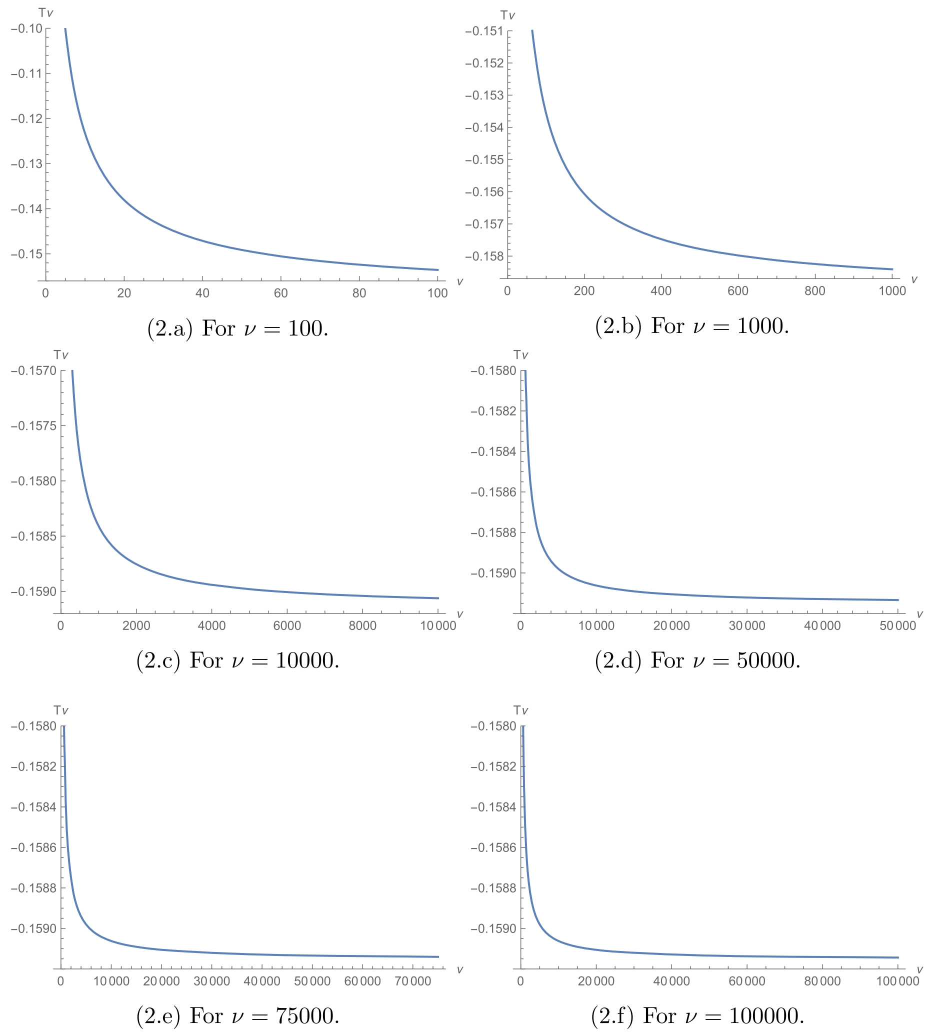

Figure 2 Degree of convergence of $f(t) = \sum_{\nu=2}^{\infty} \frac{\sin \nu t}{\log \nu}$

5 Conclusion

From Table 1 and Figs. 1(a) to 1(f), we observe that the degree of convergence of Fourier series $f(t) = t^3$ is much better than that of earlier results, and from Table 2 and Figs. 2(a) to 2(f), we observe that the degree of convergence of conjugate Fourier series $\tilde{f}(t) = \sum_{\nu=2}^{\infty} \frac{\sin \nu t}{\log \nu}$ for $\nu \geq 2$ is much better than that of earlier results. Also, from Table 1 and Table 2, we observe that the convergence of Fourier series is faster than the convergence of conjugate Fourier series.

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Competing interests
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Authors' contributions
HKN framed the problems. HKN and SY carried out the results and wrote the manuscripts. Both the authors contributed equally to the writing of this paper. All the authors read and approved the final manuscripts.

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