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The inertial iterative extragradient methods for solving pseudomonotone equilibrium programming in Hilbert spaces

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Abstract

In this paper, we present new iterative techniques for approximating the solution of an equilibrium problem involving a pseudomonotone and a Lipschitz-type bifunction in Hilbert spaces. These techniques consist of two computing steps of a proximal-type mapping with an inertial term. Improved simplified stepsize rules that do not involve line search are investigated, allowing the method to be implemented more quickly without knowing the Lipschitz-type constants of a bifunction. The iterative sequences converge weakly on a specific solution to the problem when the control parameter conditions are properly specified. The numerical tests were carried out, and the results demonstrated the applicability and quick convergence of innovative approaches over earlier ones.

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Keywords: Equilibrium problem; Iterative method; Pseudomonotone bifunction; Weak convergence theorem

1 Introduction

Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{H}$. The main objective here is to study different iterative methods for solving equilibrium problems ((EP), to put it short) involving pseudomonotone and a Lipschitz-type bifunction. Let $\mathcal{F} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction with $\mathcal{F}(u_1, u_1) = 0$, for each $u_1 \in \mathcal{K}$. An *equilibrium problem* for $\mathcal{F}$ on $\mathcal{K}$ is described in the following manner: Find $\eth^* \in \mathcal{K}$ such that

$$\mathcal{F}(\eth^*, u_1) \geq 0, \quad \forall u_1 \in \mathcal{K}. \tag{EP}$$

Let us denote the solution set of a problem (EP) as $\mathrm{Sol}(\mathcal{F}, \mathcal{K})$, and we will assume in the following text that this solution set is nonempty. The numerical evaluation of the equilibrium problem under the following conditions is the focus of this study. We will assume that the following conditions have been met:

($\mathcal{F}1$) The solution set of a problem (EP) is denoted by $\mathrm{Sol}(\mathcal{F}, \mathcal{K})$, and it is nonempty;

($\mathcal{F}2$) The bifunction $\mathcal{F}$ is said to be *pseudomonotone* [5, 7], i.e.,

$$\mathcal{F}(u_1,u_2) \geq 0 \quad \Longrightarrow \quad \mathcal{F}(u_2,u_1) \leq 0, \quad \forall u_1,u_2 \in \mathcal{K};$$

($\mathcal{F}3$) The bifunction $\mathcal{F}$ is said to be *Lipschitz-type continuous* [18] on $\mathcal{K}$ if there exist two constants $k_1, k_2 > 0$, such that

$$\begin{aligned}\mathcal{F}(u_1,u_3) \leq \mathcal{F}(u_1,u_2) + \mathcal{F}(u_2,u_3) + k_1\|u_1-u_2\|^2 \\ + k_2\|u_2-u_3\|^2, \quad \forall u_1,u_2,u_3 \in \mathcal{K};\end{aligned}$$

($\mathcal{F}4$) For any sequence $\{v_k\} \subset \mathcal{K}$ satisfying $v_k \rightharpoonup v^*$, then the following inequality holds

$$\limsup_{k\to+\infty} \mathcal{F}(v_k,u_1) \leq \mathcal{F}\big(v^*,u_1\big), \quad \forall u_1 \in \mathcal{K};$$

($\mathcal{F}5$) $\mathcal{F}(u_1,\cdot)$ is convex and subdifferentiable on $\mathcal{H}$ for each fixed $u_1 \in \mathcal{H}$.

The equilibrium problem is of tremendous interest among researchers these days since it connects numerous mathematical problems, including vector and scalar minimization problems, fixed point problems, variational inequalities, the saddle point problems, the complementarity problems, and Nash equilibrium problems in non-cooperative games (for more details, see [6, 7, 12, 15, 20]). It also has various applications in economics [11], the dynamics of offer and demand [1] and continues to utilize the theoretical framework of non-cooperative games and Nash's equilibrium models [21, 22]. The phrase "equilibrium problem" in its precise design was first introduced in the literature in 1992 by Muu and Oettli [20] and since then has been studied by Blum and Oettli [7]. More precisely, we consider two applications for the problem (EP). (i) The *variational inequality problem* for $\mathcal{A}: \mathcal{K} \to \mathcal{H}$ is stated as follows: Find $\eth^* \in \mathcal{K}$ such that

$$\big\langle \mathcal{A}\big(\eth^*\big), u_1 - \eth^*\big\rangle \geq 0, \quad \forall u_1 \in \mathcal{K}. \tag{VIP}$$

Let us define a bifunction $\mathcal{F}$ define as follows:

$$\mathcal{F}(u_1,u_2) := \big\langle \mathcal{A}(u_1), u_2 - u_1\big\rangle, \quad \forall u_1,u_2 \in \mathcal{K}. \tag{1.1}$$

Then, problem (EP) is converted into the problem of variational inequalities defined in (VIP), and the Lipschitz constants of the mapping $\mathcal{A}$ are $L = 2k_1 = 2k_2$. (ii) Let a mapping $\mathcal{B}: \mathcal{K} \to \mathcal{K}$ is said to κ-strict pseudocontraction [8] with $\kappa \in (0,1)$ such that

$$\|\mathcal{B}u_1 - \mathcal{B}u_2\|^2 \leq \|u_1-u_2\|^2 + \kappa\big\|(u_1-\mathcal{B}u_1)-(u_2-\mathcal{B}u_2)\big\|^2, \quad \forall u_1,u_2 \in \mathcal{K}. \tag{1.2}$$

A fixed point problem (FPP) for $\mathcal{B}: \mathcal{K} \to \mathcal{K}$ is to find $\eth^* \in \mathcal{K}$ such that $\mathcal{B}(\eth^*) = \eth^*$. Let us define a bifunction $\mathcal{F}$ as follows:

$$\mathcal{F}(u_1,u_2) = \langle u_1 - \mathcal{B}u_1, u_2 - u_1\rangle, \quad \forall u_1,u_2 \in \mathcal{K}. \tag{1.3}$$

It can be easily seen in [35] that the expression (1.3) satisfies the conditions $\mathcal{F}1$-$\mathcal{F}5$ as well as the value of Lipschitz constants are $k_1 = k_2 = \frac{3-2\kappa}{2-2\kappa}$.

The extragradient method developed by Lyashko and Semenov [17] is one of the useful methods to solve equilibrium problems. The following is how this approach was constructed. Take an arbitrary initial points $u_0, v_0 \in \mathcal{H}$; using the current iteration u_k, take the next iteration as continues to follow:

$$\begin{cases} u_0, v_0 \in \mathcal{K}, \\ u_{k+1} = \arg\min_{v\in\mathcal{K}}\{\varkappa\mathcal{F}(v_k, v) + \frac{1}{2}\|u_k - v\|^2\}, \\ v_{k+1} = \arg\min_{v\in\mathcal{K}}\{\varkappa\mathcal{F}(v_k, v) + \frac{1}{2}\|u_{k+1} - v\|^2\}, \end{cases} \tag{1.4}$$

where $0 < \varkappa < \frac{1}{2k_2+4k_1}$ and k_1, k_2 are two Lipschitz-type constants. The iterative techniques in [17] are also acknowledged as Popov's extragradient method because of Popov's first contribution in the work [27] to solve saddle point problems. Recently, Yang [36] combined Popov's extragradient method (1.4) with a non-monotonic stepsize rule. This method requires the solution of one optimization program on $\mathcal{K}$ as well as a minimization problem on a half-space with a non-monotonic stepsize rule.

The main goal is to develop inertial-type methods in the case of [36] that will be designed to increase the rate of convergence of the iterative sequence. Such methods have already been established as a result of the oscillator equation, damping, and conservative force restoration. This second-order dynamical scheme represents a heavy friction ball, which Polyak first viewed in [26]. The main characteristic of this method is that the next iteration is composed of two previous iterations. In this context, numerical results indicate that inertial terms increase the method's efficiency in terms of the number of iterations and elapsed time. In recent years, such methods have been extensively studied for specific types of equilibrium problems [2, 4, 13, 14, 19, 29–33] and others in [9, 16, 23, 34, 37–40].

As a result, a natural question arises:

> *Is it possible to develop new inertial-like weakly convergent extragradient-type methods for solving equilibrium problems using monotone and non-monotone stepsize rules?*

In our study, we provide a positive answer to this question, namely, the gradient approach still generates a weak convergence sequence when solving equilibrium problems involving pseudomonotone bifunctions using a monotone and nonmonotone variable stepsize rule. Inspired by the work by Censor et al. [10] and Yang [36], we will describe new inertial extragradient-like approaches to solving the problem (EP) in the setting of real Hilbert spaces.

Our important contributions to this work are as follows: (i) We build an inertial subgradient extragradient method to solving equilibrium problems in Hilbert spaces using a monotone variable stepsize rule and show that the resulting sequence is weakly convergent. (ii) To solve equilibrium problems, we develop a new inertial subgradient extragradient strategy that makes use of a variable nonmonotone stepsize rule that is independent of the Lipschitz constants. (iii) Some conclusions are drawn in order to address various types of equilibrium problems in real Hilbert space. (iv) We provide more mathematical demonstrations of the proposed approaches for the verification of theoretical findings and compare them to the results in Algorithm 3.1 in [36]. The mathematical findings suggest that the proposed methods are advantageous and perform better than the already existed.

The paper is structured as follows: In Sect. 2, preliminary results were presented. Section 3 gives all new methods and their convergence theorems. Finally, Sect. 4 gives certain numerical results to highlight the practical effectiveness of the proposed approaches.

2 Preliminaries

In this section, we will go over some elementary identities as well as key lemmas and definitions. A *metric projection* $P_{\mathcal{K}}(u_1)$ of $u_1 \in \mathcal{H}$ is defined as follows:

$$P_{\mathcal{K}}(u_1) = \arg\min\bigl\{\|u_1 - u_2\| : u_2 \in \mathcal{K}\bigr\}.$$

Lemma 2.1 ([3]) *Let $P_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ be a metric projection. Then*

(i)

$$\bigl\|u_1 - P_{\mathcal{K}}(u_2)\bigr\|^2 + \bigl\|P_{\mathcal{K}}(u_2) - u_2\bigr\|^2 \le \|u_1 - u_2\|^2, \quad u_1 \in \mathcal{K}, u_2 \in \mathcal{H};$$

(ii) $u_3 = P_{\mathcal{K}}(u_1)$ *if and only if*

$$\langle u_1 - u_3, u_2 - u_3\rangle \le 0, \quad \forall u_2 \in \mathcal{K};$$

(iii)

$$\bigl\|u_1 - P_{\mathcal{K}}(u_1)\bigr\| \le \|u_1 - u_2\|, \quad u_2 \in \mathcal{K}, u_1 \in \mathcal{H}.$$

Lemma 2.2 ([3]) *For any $u_1, u_2 \in \mathcal{H}$ and $\digamma \in \mathbb{R}$. Then the following conditions are satisfied*:

(i)

$$\bigl\|\digamma u_1 + (1-\digamma)u_2\bigr\|^2 = \digamma\|u_1\|^2 + (1-\digamma)\|u_2\|^2 - \digamma(1-\digamma)\|u_1 - u_2\|^2;$$

(ii)

$$\|u_1 + u_2\|^2 \le \|u_1\|^2 + 2\langle u_2, u_1 + u_2\rangle.$$

A normal cone of $\mathcal{K}$ at $u_1 \in \mathcal{K}$ is defined as follows:

$$N_{\mathcal{K}}(u_1) = \bigl\{u_3 \in \mathcal{H} : \langle u_3, u_2 - u_1\rangle \le 0, \forall u_2 \in \mathcal{K}\bigr\}.$$

Let $\mho : \mathcal{K} \to \mathbb{R}$ be a convex function and *subdifferential of* $\mho$ at $u_1 \in \mathcal{K}$ is defined by

$$\partial\mho(u_1) = \bigl\{u_3 \in \mathcal{H} : \mho(u_2) - \mho(u_1) \ge \langle u_3, u_2 - u_1\rangle, \forall u_2 \in \mathcal{K}\bigr\}.$$

Lemma 2.3 ([25], Proposition 3.61) *Let $\mho : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function on $\mathcal{H}$. Assume either that $\mho$ is continuous at some point of $\mathcal{K}$, or that there is an interior point of $\mathcal{K}$ where $\mho$ is finite. An element $u \in \mathcal{K}$ is a minimizer of a* *function $\mho$ if and only if*

$$0 \in \partial\mho(u) + N_{\mathcal{K}}(u),$$

while $\partial\mho(u)$ represent subdifferential of $\mho$ at $u \in \mathcal{K}$, and $N_{\mathcal{K}}(u)$ is the normal cone of $\mathcal{K}$ at u.

Lemma 2.4 ([2]) *Suppose that b_k, $c_{k,}$ and d_k are three sequences in $[0, +\infty)$ that meet the inequality below*

$$b_{k+1} \leq b_k + d_k(b_k - b_{k-1}) + c_k,$$

for all $k \geq 1$ and $\sum_{k=1}^{+\infty} c_k < +\infty$. Thus, there exists number d satisfying $0 \leq d_k \leq d < 1$, $\forall k \in \mathbb{N}$. Then

(i) *$\sum_{k=1}^{+\infty} [b_k - b_{k-1}]_+ < +\infty$, while $[t]_+ := \max\{t, 0\}$;*

(ii) *there exists $b^* \in [0, +\infty)$ such that $\lim_{k\to+\infty} b_k = b^*$.*

Lemma 2.5 ([24]) *Let $\mathcal{K}$ be a nonempty subset of $\mathcal{H}$ and $\{u_k\}$ be a sequence in $\mathcal{H}$ satisfying*

(i) *for each $u \in \mathcal{K}$, $\lim_{k\to\infty} \|u_k - u\|$ exists;*

(ii) *each weak sequentially cluster point of $\{u_k\}$ inside $\mathcal{K}$.*

Then, sequence $\{u_k\}$ weakly converges to an element in $\mathcal{K}$.

3 Main results

In this section, we provide a numerical iterative method that comprises two strong convex optimization problems linked by an inertial term to accelerate the rate of convergence of an iterative sequence. We offer the following method for solving equilibrium problems.

Lemma 3.1 *From Algorithm 1, can derive the following useful inequality*

$$\varkappa_k \mathcal{F}(v_k, v) - \varkappa_k \mathcal{F}(v_k, u_{k+1}) \geq \langle \daleth_k - u_{k+1}, v - u_{k+1}\rangle, \quad \forall v \in \mathcal{H}_k.$$

Proof Due to the use of Lemma 2.3, we have

$$0 \in \partial_2\left\{\varkappa_k \mathcal{F}(v_k, \cdot) + \frac{1}{2}\|\daleth_k - \cdot\|^2\right\}(u_{k+1}) + N_{\mathcal{H}_k}(u_{k+1}).$$

Thus, $\upsilon \in \partial\mathcal{F}(v_k, u_{k+1})$, and there exists a vector $\overline{\upsilon} \in N_{\mathcal{H}_k}(u_{k+1})$ such that

$$\varkappa_k \upsilon + u_{k+1} - \daleth_k + \overline{\upsilon} = 0.$$

As a result, we have

$$\langle \daleth_k - u_{k+1}, v - u_{k+1}\rangle = \varkappa_k\langle \upsilon, v - u_{k+1}\rangle + \langle \overline{\upsilon}, v - u_{k+1}\rangle, \quad \forall v \in \mathcal{H}_k.$$

Due to $\overline{\upsilon} \in N_{\mathcal{H}_k}(u_{k+1})$, ensure that $\langle \overline{\upsilon}, v - u_{k+1}\rangle \leq 0$ for all $v \in \mathcal{H}_k$. Thus, we have

$$\langle \daleth_k - u_{k+1}, v - u_{k+1}\rangle \leq \varkappa_k\langle \upsilon, v - u_{k+1}\rangle, \quad \forall v \in \mathcal{H}_k. \tag{3.1}$$

Since $\upsilon \in \partial \mathcal{F}(\nu_k, u_{k+1})$, we have

$$\mathcal{F}(\nu_k, \nu) - \mathcal{F}(\nu_k, u_{k+1}) \geq \langle \upsilon, \nu - u_{k+1} \rangle, \quad \forall \nu \in \mathcal{H}. \tag{3.2}$$

Combining the formulas (3.1) and (3.2), we obtain

$$\varkappa_k \mathcal{F}(\nu_k, \nu) - \varkappa_k \mathcal{F}(\nu_k, u_{k+1}) \geq \langle \gimel_k - u_{k+1}, \nu - u_{k+1} \rangle, \quad \forall \nu \in \mathcal{H}_k. \qquad \square$$

Algorithm 1 Explicit Popov's subgradient method using the monotone stepsize rule

STEP 0: Select $\varkappa_0 = \varkappa_1 > 0$, $u_{-1}, u_0, \nu_0 \in \mathcal{H}$, $\varrho \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$ and α_k to be a decreasing sequence such that $0 \leq \underline{\alpha} \leq \alpha_k \leq \overline{\alpha} < \sqrt{5}-2$. First, we have to compute

$$u_1 = \arg\min_{\nu \in \mathcal{K}} \left\{ \varkappa_0 \mathcal{F}(\nu_0, \nu) + \frac{1}{2} \|\gimel_0 - \nu\|^2 \right\},$$
$$\nu_1 = \arg\min_{\nu \in \mathcal{K}} \left\{ \varkappa_1 \mathcal{F}(\nu_0, \nu) + \frac{1}{2} \|\gimel_1 - \nu\|^2 \right\},$$

where $\gimel_0 = u_0 + \alpha_0(u_0 - u_{-1})$ and $\gimel_1 = u_1 + \alpha_1(u_1 - u_0)$.

STEP 1: Given $u_{k-1}, \nu_{k-1}, u_k, \nu_k$. Firstly choose $\omega_{k-1} \in \partial_2 \mathcal{F}(\nu_{k-1}, \nu_k)$ satisfying $\gimel_k - \varkappa_k \omega_{k-1} - \nu_k \in N_{\mathcal{K}}(\nu_k)$ and generate a half-space

$$\mathcal{H}_k = \left\{ z \in \mathcal{H} : \langle \gimel_k - \varkappa_k \omega_{k-1} - \nu_k, z - \nu_k \rangle \leq 0 \right\}.$$

Compute

$$u_{k+1} = \arg\min_{\nu \in \mathcal{H}_k} \left\{ \varkappa_k \mathcal{F}(\nu_k, \nu) + \frac{1}{2} \|\gimel_k - \nu\|^2 \right\},$$

where $\gimel_k = u_k + \alpha_k(u_k - u_{k-1})$.

STEP 2: Compute

$$\varkappa_{k+1} = \begin{cases} \min\{\varkappa_k, \frac{(2-\sqrt{2}-\theta)\frac{1}{2}\varrho\|\nu_{k-1}-\nu_k\|^2 + (2-\sqrt{2}-\theta)\varrho\|u_{k+1}-\nu_k\|^2}{2[\mathcal{F}(\nu_{k-1},u_{k+1}) - \mathcal{F}(\nu_{k-1},\nu_k) - \mathcal{F}(\nu_k,u_{k+1})]}\}, \\ \quad \text{if } \mathcal{F}(\nu_{k-1}, u_{k+1}) - \mathcal{F}(\nu_{k-1}, \nu_k) - \mathcal{F}(\nu_k, u_{k+1}) > 0, \\ \varkappa_k, \quad \text{otherwise.} \end{cases} \tag{3.3}$$

STEP 3: Compute

$$\nu_{k+1} = \arg\min_{\nu \in \mathcal{K}} \left\{ \varkappa_{k+1} \mathcal{F}(\nu_k, \nu) + \frac{1}{2} \|\gimel_{k+1} - \nu\|^2 \right\},$$

where $\gimel_{k+1} = u_{k+1} + \alpha_{k+1}(u_{k+1} - u_k)$.

STEP 4: If $u_{k+1} = \gimel_k$ and $\nu_k = \nu_{k-1}$, then complete the computation. Otherwise, set $k := k+1$ and go back **STEP 1**.

Lemma 3.2 *From Algorithm 1, can derive the following useful inequality*

$$\varkappa_k \mathcal{F}(\nu_{k-1}, \nu) - \varkappa_k \mathcal{F}(\nu_{k-1}, \nu_k) \geq \langle \beth_k - \nu_k, \nu - \nu_k \rangle, \quad \forall \nu \in \mathcal{K}.$$

Proof The proof is identical to the proof of Lemma 3.1. By substituting $\nu = u_{k+1}$, we have

$$\varkappa_k \big\{ \mathcal{F}(\nu_{k-1}, u_{k+1}) - \mathcal{F}(\nu_{k-1}, \nu_k) \big\} \geq \langle \beth_k - \nu_k, u_{k+1} - \nu_k \rangle. \tag{3.4}$$

□

Lemma 3.3 *Suppose that $\mathcal{F} : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ satisfies the conditions $(\mathcal{F}1)$–$(\mathcal{F}5)$. For any $\eth^* \in \mathrm{Sol}(\mathcal{F}, \mathcal{K})$, we have*

$$\begin{aligned} \|u_{k+1} - \eth^*\|^2 \leq \|\beth_k - \eth^*\|^2 &- \left(1 - \frac{(2 - \sqrt{2} - \theta)\varrho \varkappa_k}{\varkappa_{k+1}}\right) \|\beth_k - \nu_k\|^2 \\ &- \left(1 - \frac{(2 - \sqrt{2} - \theta)\varrho \varkappa_k}{\varkappa_{k+1}}\right) \|u_{k+1} - \nu_k\|^2 \\ &+ \frac{(2 - \sqrt{2} - \theta)\varrho \varkappa_k}{\varkappa_{k+1}} \|\beth_k - \nu_{k-1}\|^2. \end{aligned}$$

Proof By letting $\nu = \eth^*$ in Lemma 3.1, we have

$$\varkappa_k \mathcal{F}\big(\nu_k, \eth^*\big) - \varkappa_k \mathcal{F}(\nu_k, u_{k+1}) \geq \big\langle \beth_k - u_{k+1}, \eth^* - u_{k+1} \big\rangle. \tag{3.5}$$

Using condition $(\mathcal{F}2)$, we obtain

$$\big\langle \beth_k - u_{k+1}, u_{k+1} - \eth^* \big\rangle \geq \varkappa_k \mathcal{F}(\nu_k, u_{k+1}). \tag{3.6}$$

From expression (3.3), we obtain

$$\mathcal{F}(\nu_{k-1}, u_{k+1}) - \mathcal{F}(\nu_{k-1}, \nu_k) - \mathcal{F}(\nu_k, u_{k+1}) \leq \frac{(2 - \sqrt{2} - \theta)\varrho(\frac{1}{2}\|\nu_{k-1} - \nu_k\|^2 + \|u_{k+1} - \nu_k\|^2)}{2\varkappa_{k+1}}$$

with $\varkappa_k > 0$ gives that

$$\begin{aligned} \varkappa_k \mathcal{F}(\nu_k, u_{k+1}) \geq\ & \varkappa_k \mathcal{F}(\nu_{k-1}, u_{k+1}) - \varkappa_k \mathcal{F}(\nu_{k-1}, \nu_k) \\ & - \frac{(2 - \sqrt{2} - \theta)\varkappa_k \varrho(\frac{1}{2}\|\nu_{k-1} - \nu_k\|^2 + \|u_{k+1} - \nu_k\|^2)}{2\varkappa_{k+1}}. \end{aligned} \tag{3.7}$$

Combining expressions (3.6) and (3.7), we obtain

$$\begin{aligned} \big\langle \beth_k - u_{k+1}, u_{k+1} - \eth^* \big\rangle \geq\ & \varkappa_k \big\{ \mathcal{F}(\nu_{k-1}, u_{k+1}) - \mathcal{F}(\nu_{k-1}, \nu_k) \big\} \\ & - \frac{(2 - \sqrt{2} - \theta)\varkappa_k \varrho(\frac{1}{2}\|\nu_{k-1} - \nu_k\|^2 + \|u_{k+1} - \nu_k\|^2)}{2\varkappa_{k+1}}. \end{aligned} \tag{3.8}$$

Using expression (3.4), we have

$$\varkappa_k \big\{ \mathcal{F}(\nu_{k-1}, u_{k+1}) - \mathcal{F}(\nu_{k-1}, \nu_k) \big\} \geq \langle \beth_k - \nu_k, u_{k+1} - \nu_k \rangle. \tag{3.9}$$

Combining expressions (3.8) and (3.9), we have

$$\begin{aligned}\left\langle \beth_k - u_{k+1}, u_{k+1} - \eth^*\right\rangle &\geq \langle \beth_k - v_k, u_{k+1} - v_k\rangle \\ &\quad - \frac{(2-\sqrt{2}-\theta)\varkappa_k\varrho(\frac{1}{2}\|v_{k-1}-v_k\|^2 + \|u_{k+1}-v_k\|^2)}{2\varkappa_{k+1}}. \end{aligned} \tag{3.10}$$

The following facts are available to us:

$$\begin{aligned}&2\left\langle \beth_k - u_{k+1}, u_{k+1} - \eth^*\right\rangle = \left\|\beth_k - \eth^*\right\|^2 - \|u_{k+1} - \beth_k\|^2 - \left\|u_{k+1} - \eth^*\right\|^2, \\ &2\langle v_k - \beth_k, v_k - u_{k+1}\rangle = \|\beth_k - v_k\|^2 + \|u_{k+1} - v_k\|^2 - \|\beth_k - u_{k+1}\|^2.\end{aligned}$$

As a result, we have

$$\begin{aligned}\left\|u_{k+1} - \eth^*\right\|^2 &\leq \left\|\beth_k - \eth^*\right\|^2 - \|\beth_k - v_k\|^2 - \|u_{k+1} - v_k\|^2 \\ &\quad + \frac{(2-\sqrt{2}-\theta)\varkappa_k\varrho(\frac{1}{2}\|v_{k-1}-v_k\|^2 + \|u_{k+1}-v_k\|^2)}{\varkappa_{k+1}}. \end{aligned} \tag{3.11}$$

There are additional inequities to consider

$$\|v_{k-1} - v_k\|^2 \leq \left(\|v_{k-1} - \beth_k\| + \|\beth_k - v_k\|\right)^2 \leq 2\|v_{k-1} - \beth_k\|^2 + 2\|\beth_k - v_k\|^2.$$

The above expressions implies that

$$\begin{aligned}\left\|u_{k+1} - \eth^*\right\|^2 &\leq \left\|\beth_k - \eth^*\right\|^2 - \|\beth_k - v_k\|^2 - \|u_{k+1} - v_k\|^2 \\ &\quad + \frac{(2-\sqrt{2}-\theta)\varkappa_k\varrho(\|v_{k-1}-\beth_k\|^2 + \|\beth_k - v_k\|^2 + \|u_{k+1}-v_k\|^2)}{\varkappa_{k+1}}. \end{aligned} \tag{3.12}$$

Finally, the previous expression implies that

$$\begin{aligned}&\left\|u_{k+1} - \eth^*\right\|^2 \\ &\quad \leq \left\|\beth_k - \eth^*\right\|^2 - \left(1 - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\right)\|\beth_k - v_k\|^2 \\ &\qquad - \left(1 - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\right)\|u_{k+1} - v_k\|^2 + \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\|\beth_k - v_{k-1}\|^2. \end{aligned} \tag{3.13}$$

□

Let us now establish the main convergence result for the Algorithm 1.

Theorem 3.4 *Let* $\{u_k\}$ *be a sequence generated by Algorithm* 1 *and*

$$0 < \varrho < \frac{\frac{1}{2} - 2\alpha - \frac{1}{2}\alpha^2}{(2-\sqrt{2}-\theta)(1 - \frac{1}{2}\alpha + \alpha^2 + \frac{1}{2}\alpha^3)} \quad \textit{and} \quad 0 \leq \alpha_k \leq \alpha < \sqrt{5} - 2.$$

Then, the sequence $\{u_k\}$ *weakly converges to* $\eth^* \in \mathrm{Sol}(\mathcal{F}, \mathcal{K})$.

Proof Adding both sides $\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\|\beth_{k+1}-v_k\|^2$ in Lemma 3.3, we have

$$\begin{aligned}
&\|u_{k+1}-\eth^*\|^2+\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\|\beth_{k+1}-v_k\|^2\\
&\quad\le\|\beth_k-\eth^*\|^2-\left(1-\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\right)\|\beth_k-v_k\|^2\\
&\qquad-\left(1-\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\right)\|u_{k+1}-v_k\|^2+\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\|\beth_k-v_{k-1}\|^2\\
&\qquad+\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\|\beth_{k+1}-v_k\|^2.
\end{aligned}\tag{3.14}$$

Due to the term $\beth_k$ in Algorithm 1, we obtain

$$\begin{aligned}
\|\beth_k-\eth^*\|^2&=\|u_k+\alpha_k(u_k-u_{k-1})-\eth^*\|^2\\
&=\|(1+\alpha_k)(u_k-\eth^*)-\alpha_k(u_{k-1}-\eth^*)\|^2\\
&=(1+\alpha_k)\|u_k-\eth^*\|^2-\alpha_k\|u_{k-1}-\eth^*\|^2+\alpha_k(1+\alpha_k)\|u_k-u_{k-1}\|^2.
\end{aligned}\tag{3.15}$$

Due to the term $\beth_k$ and using Cauchy inequality, we have

$$\begin{aligned}
\|\beth_{k+1}-v_k\|^2&=\|u_{k+1}+\alpha_{k+1}(u_{k+1}-u_k)-v_k\|^2\\
&=\|(1+\alpha_{k+1})(u_{k+1}-v_k)-\alpha_{k+1}(u_k-v_k)\|^2\\
&=(1+\alpha_{k+1})\|u_{k+1}-v_k\|^2-\alpha_{k+1}\|u_k-v_k\|^2\\
&\quad+\alpha_{k+1}(1+\alpha_{k+1})\|u_{k+1}-u_k\|^2\\
&\le(1+\alpha)\|u_{k+1}-v_k\|^2+\alpha(1+\alpha)\|u_{k+1}-u_k\|^2.
\end{aligned}\tag{3.16}$$

Thus, we have

$$\begin{aligned}
&\|u_{k+1}-\eth^*\|^2+\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\|\beth_{k+1}-v_k\|^2\\
&\quad\le(1+\alpha_k)\|u_k-\eth^*\|^2-\alpha_k\|u_{k-1}-\eth^*\|^2+\alpha_k(1+\alpha_k)\|u_k-u_{k-1}\|^2\\
&\qquad-\left(1-\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\right)\|\beth_k-v_k\|^2\\
&\qquad-\left(1-\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\right)\|u_{k+1}-v_k\|^2+\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\|\beth_k-v_{k-1}\|^2\\
&\qquad+\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\left[(1+\alpha)\|u_{k+1}-v_k\|^2+\alpha(1+\alpha)\|u_{k+1}-u_k\|^2\right]\qquad(3.17)\\
&\quad\le(1+\alpha_{k+1})\|u_k-\eth^*\|^2-\alpha_k\|u_{k-1}-\eth^*\|^2+\alpha_k(1+\alpha_k)\|u_k-u_{k-1}\|^2\\
&\qquad+\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\|\beth_k-v_{k-1}\|^2-\left(1-\frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\right)\|\beth_k-v_k\|^2
\end{aligned}$$

$$+ \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\alpha(1+\alpha)\|u_{k+1}-u_k\|^2 - \left(1 - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}} - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}(1+\alpha)\right)\|u_{k+1}-\nu_k\|^2. \tag{3.18}$$

The above expression implies that

$$\begin{aligned} &\|u_{k+1}-\eth^*\|^2 - \alpha_{k+1}\|u_k-\eth^*\|^2 + \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\|\beth_{k+1}-\nu_k\|^2 \\ &\quad\le \|u_k-\eth^*\|^2 - \alpha_k\|u_{k-1}-\eth^*\|^2 + \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\|\beth_k-\nu_{k-1}\|^2 \\ &\qquad + \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\alpha(1+\alpha)\|u_{k+1}-u_k\|^2 + \alpha_k(1+\alpha_k)\|u_k-u_{k-1}\|^2 \\ &\qquad - \frac{1}{2}\left(1 - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}} - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}(1+\alpha)\right)\|u_{k+1}-\beth_k\|^2. \end{aligned} \tag{3.19}$$

By the use $\beth_{k+1}$ and using Cauchy inequality, we have

$$\begin{aligned} \|u_{k+1}-\beth_k\|^2 &= \|u_{k+1}-u_k-\alpha_k(u_k-u_{k-1})\|^2 \\ &= \|u_{k+1}-u_k\|^2 + \alpha_k^2\|u_k-u_{k-1}\|^2 - 2\alpha_k\langle u_{k+1}-u_k, u_k-u_{k-1}\rangle \qquad (3.20) \\ &\ge \|u_{k+1}-u_k\|^2 + \alpha_k^2\|u_k-u_{k-1}\|^2 - 2\alpha_k\|u_{k+1}-u_k\|\|u_k-u_{k-1}\| \\ &\ge \|u_{k+1}-u_k\|^2 + \alpha_k^2\|u_k-u_{k-1}\|^2 - \alpha_k\|u_{k+1}-u_k\|^2 - \alpha_k\|u_k-u_{k-1}\|^2 \\ &= (1-\alpha_k)\|u_{k+1}-u_k\|^2 + (\alpha_k^2-\alpha_k)\|u_k-u_{k-1}\|^2. \qquad (3.21) \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \Psi_{k+1} &\le \Psi_k + \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\alpha(1+\alpha)\|u_{k+1}-u_k\|^2 + \alpha_k(1+\alpha_k)\|u_k-u_{k-1}\|^2 \\ &\quad - \rho_k\left[(1-\alpha_k)\|u_{k+1}-u_k\|^2 + (\alpha_k^2-\alpha_k)\|u_k-u_{k-1}\|^2\right], \end{aligned} \tag{3.22}$$

where

$$\Psi_k = \|u_k-\eth^*\|^2 - \alpha_k\|u_{k-1}-\eth^*\|^2 + \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}}\|\beth_k-\nu_{k-1}\|^2$$

and

$$\rho_k = \frac{1}{2}\left(1 - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}} - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}(1+\alpha)\right).$$

Furthermore, we can write

$$\Psi_{k+1} \le \Psi_k - Q_k\|u_{k+1}-u_k\|^2 + R_k\|u_k-u_{k-1}\|^2, \tag{3.23}$$

where

$$Q_k = \left[\rho_k(1-\alpha_k) - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\alpha(1+\alpha)\right]$$

and

$$R_k = \left[\alpha_k(1+\alpha_k) - \rho_k\left(\alpha_k^2 - \alpha_k\right)\right].$$

Next, we substitute

$$\Gamma_k = \Psi_k + R_k\|u_k - u_{k-1}\|^2.$$

Thus, we have

$$\begin{aligned}\Gamma_{k+1} - \Gamma_k &= \Psi_{k+1} + R_{k+1}\|u_{k+1} - u_k\|^2 - \Psi_k - R_k\|u_k - u_{k-1}\|^2 \\ &\le -(Q_k - R_{k+1})\|u_{k+1} - u_k\|^2. \end{aligned} \tag{3.24}$$

Following that, we must compute

$$\begin{aligned} &Q_k - R_{k+1} \\ &\quad= \rho_k(1-\alpha_k) - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\alpha(1+\alpha) - \alpha_{k+1}(1+\alpha_{k+1}) + \rho_{k+1}\left(\alpha_{k+1}^2 - \alpha_{k+1}\right) \\ &\quad\ge \rho_k(1-\alpha) - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\alpha(1+\alpha) - \alpha(1+\alpha) + \rho_{k+1}\left(\alpha^2 - \alpha\right) \\ &\quad= \left[\frac{1}{2} - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{2\varkappa_{k+1}} - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{2\varkappa_{k+2}} - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{2\varkappa_{k+2}}\alpha\right](1-\alpha) \\ &\qquad- \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\left(\alpha+\alpha^2\right) - \alpha(1+\alpha) \\ &\qquad+ \left[\frac{1}{2} - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{2\varkappa_{k+2}} - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+2}}{2\varkappa_{k+3}} - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+2}}{2\varkappa_{k+3}}\alpha\right] \\ &\qquad\times\left(\alpha^2 - \alpha\right). \end{aligned} \tag{3.25}$$

It is given that $\varkappa_k \to \varkappa$ with ϱ such that

$$0 < \varrho < \frac{\frac{1}{2} - 2\alpha - \frac{1}{2}\alpha^2}{(2-\sqrt{2}-\theta)(1 - \frac{1}{2}\alpha + \alpha^2 + \frac{1}{2}\alpha^3)} \quad \text{and} \quad 0 \le \alpha_k \le \alpha < \sqrt{5} - 2.$$

From above arguments with expression (3.25) through $k_0 \in \mathbb{N}$

$$Q_k - R_{k+1} \ge \epsilon > 0, \quad \forall k \ge k_0. \tag{3.26}$$

From expressions (3.24) and (3.26) with $k \ge k_0$, the following relation is true

$$\Gamma_{k+1} - \Gamma_k \le -\epsilon\|u_{k+1} - u_k\|^2 \le 0. \tag{3.27}$$

Therefore, the sequence $\{\Gamma_k\}$ is nonincreasing for $k \geq k_0$. Using Γ_{k+1} for $k \geq k_0$, we have

$$\begin{aligned} \Gamma_{k+1} &= \left\| u_{k+1} - \eth^* \right\|^2 - \alpha_{k+1} \left\| u_k - \eth^* \right\|^2 \\ &\quad + \frac{(2 - \sqrt{2} - \theta)\varrho \varkappa_{k+1}}{\varkappa_{k+2}} \|\gimel_{k+1} - v_k\|^2 + R_{k+1} \|u_{k+1} - u_k\|^2 \\ &\geq -\alpha_{k+1} \left\| u_k - \eth^* \right\|^2. \end{aligned} \tag{3.28}$$

By the definition of Γ_k for $k \geq k_0$, we obtain

$$\begin{aligned} \Gamma_k &= \left\| u_k - \eth^* \right\|^2 - \alpha_k \left\| u_{k-1} - \eth^* \right\|^2 \\ &\quad + \frac{(2 - \sqrt{2} - \theta)\varrho \varkappa_k}{\varkappa_{k+1}} \|\gimel_k - v_{k-1}\|^2 + R_k \|u_k - u_{k-1}\|^2 \\ &\geq \left\| u_k - \eth^* \right\|^2 - \alpha_k \left\| u_{k-1} - \eth^* \right\|^2. \end{aligned} \tag{3.29}$$

The above expression for $k \geq k_0$ implies that

$$\begin{aligned} \left\| u_k - \eth^* \right\|^2 &\leq \Gamma_k + \alpha_k \left\| u_{k-1} - \eth^* \right\|^2 \\ &\leq \Gamma_{k_0} + \alpha \left\| u_{k-1} - \eth^* \right\|^2 \\ &\leq \cdots \leq \Gamma_{k_0} \bigl(\alpha^{k-k_0} + \cdots + 1\bigr) + \alpha^{k-k_0} \left\| u_{k_0} - \eth^* \right\|^2 \\ &\leq \frac{\Gamma_{k_0}}{1-\alpha} + \alpha^{k-k_0} \left\| u_{k_0} - \eth^* \right\|^2. \end{aligned} \tag{3.30}$$

From expressions (3.28) and (3.30), we obtain

$$\begin{aligned} -\Gamma_{k+1} &\leq \alpha_{k+1} \left\| u_k - \eth^* \right\|^2 \\ &\leq \alpha \left\| u_k - \eth^* \right\|^2 \\ &\leq \alpha \frac{\Gamma_{k_0}}{1-\alpha} + \alpha^{k-k_0+1} \left\| u_{k_0} - \eth^* \right\|^2. \end{aligned} \tag{3.31}$$

It follows from expressions (3.27) and (3.31) that

$$\begin{aligned} \epsilon \sum_{k=k_0}^{j} \|u_{k+1} - u_k\|^2 &\leq \Gamma_{k_0} - \Gamma_{k+1} \\ &\leq \Gamma_{k_0} + \alpha \frac{\Gamma_{k_0}}{1-\alpha} + \alpha^{k-k_0+1} \left\| u_{k_0} - \eth^* \right\|^2 \\ &\leq \frac{\Gamma_{k_0}}{1-\alpha} + \left\| u_{k_0} - \eth^* \right\|^2. \end{aligned} \tag{3.32}$$

Allowing $j \to +\infty$ in (3.32) states that

$$\sum_{k=1}^{+\infty} \|u_{k+1} - u_k\| < +\infty \quad \text{implies that} \quad \|u_{k+1} - u_k\| \to 0 \quad \text{as } k \to +\infty. \tag{3.33}$$

Due to expressions (3.20) and (3.33), we obtain

$$\|u_{k+1} - \beth_k\| \to 0 \quad \text{as } k \to +\infty. \tag{3.34}$$

Moreover, we obtain

$$0 \le \|u_k - \beth_k\| \le \|u_k - u_{k+1}\| + \|u_{k+1} - \beth_k\| \longrightarrow 0 \quad \text{as } k \to +\infty. \tag{3.35}$$

We have the following substitution:

$$\Gamma_{k+1} = \Psi_{k+1} + R_{k+1}\|u_{k+1} - u_k\|^2.$$

From expression (3.31) with above Ψ_k substitution, we have

$$-\Psi_{k+1} \le \alpha \frac{\Gamma_{k_0}}{1-\alpha} + \alpha^{k-k_0+1}\left\|u_{k_0} - \eth^*\right\|^2 + R_{k+1}\|u_{k+1} - u_k\|^2. \tag{3.36}$$

By expression (3.18), we can rewrite

$$\begin{aligned}
&\left(1 - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}} - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}(1+\alpha)\right)\left[\|\beth_k - v_k\|^2 + \|u_{k+1} - v_k\|^2\right] \\
&\quad \le \Psi_k - \Psi_{k+1} + \alpha_k(1+\alpha_k)\|u_k - u_{k-1}\|^2 + \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k+1}}{\varkappa_{k+2}}\alpha(1+\alpha)\|u_{k+1} - u_k\|^2 \\
&\quad \le \Psi_k - \Psi_{k+1} + \alpha(1+\alpha)\|u_k - u_{k-1}\|^2 \\
&\qquad + \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_0}{\varkappa}\alpha(1+\alpha)\|u_{k+1} - u_k\|^2.
\end{aligned} \tag{3.37}$$

Due to the condition on ϱ, we have

$$\begin{aligned}
&\left(1 - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+1}} - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_k}{\varkappa_{k+2}}(1+\alpha)\right) \\
&\quad \longrightarrow 1 - (2-\sqrt{2}-\theta)\varrho(2+\alpha) > \epsilon, \quad \forall k \ge k_0.
\end{aligned}$$

Let fix for some $j \ge k_0$ with expressions (3.36) and (3.37) for $k = 1, 2, \ldots, j$. Thus, we have

$$\begin{aligned}
&\epsilon \sum_{k=k_0}^{j} \|\beth_k - v_k\|^2 + \epsilon \sum_{k=k_0}^{j} \|u_{k+1} - v_k\|^2 \\
&\quad \le \Psi_{k_0} - \Psi_{j+1} + \alpha(1+\alpha)\sum_{k=k_0}^{j} \|u_k - u_{k-1}\|^2 \\
&\qquad + \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_0}{\varkappa}\alpha(1+\alpha)\sum_{k=k_0}^{j} \|u_{k+1} - u_k\|^2 \\
&\quad \le \Psi_{k_0} + \alpha\frac{\Gamma_{k_0}}{1-\alpha} + \alpha^{k-k_0+1}\left\|u_{k_0} - \eth^*\right\|^2 + R\|u_j - u_{j-1}\|^2 \\
&\qquad + \alpha(1+\alpha)\sum_{k=k_0}^{j} \|u_k - u_{k-1}\|^2 + \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_0}{\varkappa}\alpha(1+\alpha)\sum_{k=k_0}^{j} \|u_{k+1} - u_k\|^2,
\end{aligned} \tag{3.38}$$

where $R = \alpha(1+\alpha) + \frac{1}{2}\alpha(1-\alpha)$, and allowing $j \to +\infty$ provides that

$$\sum_{k=1}^{+\infty} \|\gimel_k - v_k\|^2 = \sum_{k=1}^{+\infty} \|u_{k+1} - v_k\|^2 < +\infty. \tag{3.39}$$

Thus, we have

$$\lim_{k\to+\infty} \|\gimel_k - v_k\| = \lim_{k\to+\infty} \|u_{k+1} - v_k\| = 0. \tag{3.40}$$

From expressions (3.33) and (3.40), we can infer that

$$0 \le \|u_k - v_k\| \le \|u_k - u_{k+1}\| + \|u_{k+1} - v_k\| \longrightarrow 0 \quad \text{as } k \to +\infty, \tag{3.41}$$

$$\lim_{k\to+\infty} \|\gimel_k - v_{k-1}\| \le \lim_{k\to+\infty} \|u_k - \gimel_k\| + \lim_{k\to+\infty} \|u_k - v_{k-1}\| \longrightarrow 0, \tag{3.42}$$

$$\lim_{k\to+\infty} \|v_k - v_{k-1}\| \le \lim_{k\to+\infty} \|u_k - v_k\| + \lim_{k\to+\infty} \|u_k - v_{k-1}\| \longrightarrow 0. \tag{3.43}$$

By definition $\gimel_k$ and using Cauchy inequality, we have

$$\begin{aligned}\|\gimel_k - v_{k-1}\|^2 &= \big\|u_k + \alpha_k(u_k - u_{k-1}) - v_{k-1}\big\|^2 \\ &= \big\|(1+\alpha_k)(u_k - v_{k-1}) - \alpha_k(u_{k-1} - v_{k-1})\big\|^2 \\ &= (1+\alpha_k)\|u_k - v_{k-1}\|^2 - \alpha_k\|u_{k-1} - v_{k-1}\|^2 + \alpha_k(1+\alpha_k)\|u_k - u_{k-1}\|^2 \\ &\le (1+\alpha_k)\|u_k - v_{k-1}\|^2 + \alpha_k(1+\alpha_k)\|u_k - u_{k-1}\|^2 \\ &\le (1+\alpha)\|u_k - v_{k-1}\|^2 + \alpha(1+\alpha)\|u_k - u_{k-1}\|^2. \end{aligned} \tag{3.44}$$

Now, summing up expression (3.44) for $k = k_0, \dots, j$ where $j > k_0$, we obtain

$$\sum_{k=k_0}^{j} \|\gimel_k - v_{k-1}\|^2 \le (1+\alpha)\sum_{k=k_0}^{j} \|u_k - v_{k-1}\|^2 + \alpha(1+\alpha)\sum_{k=k_0}^{j} \|u_k - u_{k-1}\|^2. \tag{3.45}$$

Letting $j \to +\infty$ in expression (3.45) implies that

$$\sum_{k=1}^{+\infty} \|\gimel_k - v_{k-1}\|^2 < +\infty. \tag{3.46}$$

Rewriting the expression (3.17), we have

$$\begin{aligned}\big\|u_{k+1} - \eth^*\big\|^2 \le{}& (1+\alpha_k)\big\|u_k - \eth^*\big\|^2 - \alpha_k\big\|u_{k-1} - \eth^*\big\|^2 + \alpha(1+\alpha)\|u_k - u_{k-1}\|^2 \\ &+ \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_0}{\varkappa}\|\gimel_k - v_{k-1}\|^2 - \left(1 - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_0}{\varkappa}\right)\|\gimel_k - v_k\|^2 \\ &- \left(1 - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_0}{\varkappa}\right)\|u_{k+1} - v_k\|^2. \end{aligned} \tag{3.47}$$

Thus, the above expression with (3.33), (3.39), (3.45) through Lemma 2.4 provides limit of $\|u_k - \eth^*\|$ exists. Hence, all $\{u_k\}$, $\{\beth_k\}$ and $\{v_k\}$ sequences are bounded. Consider z to be a weak cluster point of $\{u_k\}$, i.e., there is a subsequence that is indicated by $\{u_{k_m}\}$ of $\{u_k\}$ that is weakly convergent to z. Then $\{v_{k_m}\}$ also weakly convergent to $z \in \mathcal{K}$. We require to prove that $z \in \operatorname{Sol}(\mathcal{F}, \mathcal{K})$. Using Lemma 3.1 with expressions (3.7) and (3.4), we have

$$\begin{aligned}
\varkappa_{k_m}\mathcal{F}(v_{k_m}, v) &\geq \varkappa_{k_m}\mathcal{F}(v_{k_m}, u_{k_m+1}) + \langle \beth_{k_m} - u_{k_m+1}, v - u_{k_m+1}\rangle \\
&\geq \varkappa_{k_m}\mathcal{F}(v_{k_m-1}, u_{k_{m+1}}) - \varkappa_{k_m}\mathcal{F}(v_{k_m-1}, v_{k_m}) \\
&\quad - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k_m}}{2\varkappa_{k_m+1}}\|v_{k_m} - v_{k_m-1}\|^2 \\
&\quad - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k_m}}{2\varkappa_{k_m+1}}\|v_{k_m} - u_{k_m+1}\|^2 + \langle \beth_{k_m} - u_{k_m+1}, v - u_{k_m+1}\rangle \\
&\geq \langle \beth_{k_m} - v_{k_m}, u_{k_m+1} - v_{k_m}\rangle - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k_m}}{2\varkappa_{k_m+1}}\|v_{k_m} - v_{k_m-1}\|^2 \\
&\quad - \frac{(2-\sqrt{2}-\theta)\varrho\varkappa_{k_m}}{2\varkappa_{k_m+1}}\|v_{k_m} - u_{k_m+1}\|^2 + \langle \beth_{k_m} - u_{k_m+1}, v - u_{k_m+1}\rangle,
\end{aligned} \tag{3.48}$$

where v is any member in $\mathcal{H}_k$. It adopts from expressions (3.34) and (3.40)–(3.43) and the boundedness of $\{u_k\}$ that last inequality turns to zero. By employing $\varkappa_{k_m} \geq \varkappa > 0$ with item ($\mathcal{F}$4) and $v_{k_m} \rightharpoonup z$, such as

$$0 \leq \limsup_{m\to+\infty} \mathcal{F}(v_{k_m}, v) \leq \mathcal{F}(z, v), \quad \forall v \in \mathcal{H}_k.$$

Since $\mathcal{K} \subset \mathcal{H}_k$ and $\mathcal{F}(z, v) \geq 0$, $\forall v \in \mathcal{K}$. The above illustrates that $z \in \operatorname{Sol}(\mathcal{F}, \mathcal{K})$. Thus, Lemma 2.5 guarantees that $\{\beth_k\}$, $\{u_k\}$, and $\{v_k\}$ weakly converge to $\eth^*$ as $k \to +\infty$. □

We now provide an iterative method (see Algorithm 2) that consists of a variable nonmonotone stepsize rule and two strongly convex minimization problems. The details of the second main result are presented as follows.

In this section, we solve variational inequalities and fixe point problems using the results from our main results. The expressions (1.1) and (1.3) are employed to obtain the following conclusions. All the methods are based on our main findings, which are interpreted below.

Corollary 3.5 *Assume that* $\mathcal{A} : \mathcal{K} \to \mathcal{H}$ *is a pseudomonotone, weakly continuous and L-Lipschitz continuous operator and the solution set* $\operatorname{Sol}(\mathcal{A}, \mathcal{K}) \neq \emptyset$. *Choose* $\varkappa_0 = \varkappa_1 > 0$, $u_{-1}, u_0, v_0 \in \mathcal{H}$, $\varrho \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$ *and* α_k *to be a decreasing sequence such that* $0 \leq \underline{\alpha} \leq \alpha_k \leq \overline{\alpha} < \sqrt{5} - 2$. *First, we have to compute*

$$\begin{cases} u_1 = P_{\mathcal{K}}(\beth_0 - \varkappa_0\mathcal{A}(v_0)), \\ v_1 = P_{\mathcal{K}}(\beth_1 - \varkappa_1\mathcal{A}(v_0)), \end{cases}$$

Algorithm 2 Explicit Popov's subgradient method using the non-monotone stepsize rule

STEP 0: Choose $\varkappa_0 = \varkappa_1 > 0$, $u_{-1}, u_0, v_0 \in \mathcal{H}$, $\varrho \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$ and let α_k be a decreasing sequence such that $0 \le \underline{\alpha} \le \alpha_k \le \overline{\alpha} < \sqrt{5}-2$. Choose a non-negative real sequence $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$. First, we have to compute

$$u_1 = \arg\min_{v \in \mathcal{K}} \left\{ \varkappa_0 \mathcal{F}(v_0, v) + \frac{1}{2}\|\gimel_0 - v\|^2 \right\},$$
$$v_1 = \arg\min_{v \in \mathcal{K}} \left\{ \varkappa_1 \mathcal{F}(v_0, v) + \frac{1}{2}\|\gimel_1 - v\|^2 \right\},$$

where $\gimel_0 = u_0 + \alpha_0(u_0 - u_{-1})$ and $\gimel_1 = u_1 + \alpha_1(u_1 - u_0)$.

STEP 1: Given $u_{k-1}, v_{k-1}, u_k, v_k$. Firstly choose $\omega_{k-1} \in \partial_2 \mathcal{F}(v_{k-1}, v_k)$ satisfying $\gimel_k - \varkappa_k \omega_{k-1} - v_k \in N_{\mathcal{K}}(v_k)$ and generate a half-space

$$\mathcal{H}_k = \left\{ z \in \mathcal{H} : \langle \gimel_k - \varkappa_k \omega_{k-1} - v_k, z - v_k \rangle \le 0 \right\}.$$

Compute

$$u_{k+1} = \arg\min_{v \in \mathcal{H}_k} \left\{ \varkappa_k \mathcal{F}(v_k, v) + \frac{1}{2}\|\gimel_k - v\|^2 \right\},$$

where $\gimel_k = u_k + \alpha_k(u_k - u_{k-1})$.

STEP 2: Calculate

$$\varkappa_{k+1} = \begin{cases} \min\{\varkappa_k + p_k, \frac{(2-\sqrt{2}-\theta)\frac{1}{2}\varrho\|v_{k-1}-v_k\|^2 + (2-\sqrt{2}-\theta)\varrho\|u_{k+1}-v_k\|^2}{2[\mathcal{F}(v_{k-1},u_{k+1}) - \mathcal{F}(v_{k-1},v_k) - \mathcal{F}(v_k,u_{k+1})]}\}, \\ \quad \text{if } \mathcal{F}(v_{k-1}, u_{k+1}) - \mathcal{F}(v_{k-1}, v_k) - \mathcal{F}(v_k, u_{k+1}) > 0, \\ \varkappa_k + p_k, \quad \text{otherwise.} \end{cases} \tag{3.49}$$

STEP 3: Compute

$$v_{k+1} = \arg\min_{v \in \mathcal{K}} \left\{ \varkappa_{k+1} \mathcal{F}(v_k, v) + \frac{1}{2}\|\gimel_{k+1} - v\|^2 \right\},$$

where $\gimel_{k+1} = u_{k+1} + \alpha_{k+1}(u_{k+1} - u_k)$.

STEP 4: If $u_{k+1} = \gimel_k$ and $v_k = v_{k-1}$, then complete the computation. Otherwise, set $k := k+1$ and go back **STEP 1**.

where $\gimel_0 = u_0 + \alpha_0(u_0 - u_{-1})$ *and* $\gimel_1 = u_1 + \alpha_1(u_1 - u_0)$. *Given* $u_{k-1}, v_{k-1}, u_k, v_k$, *and construct a half-space*

$$\mathcal{H}_k = \left\{ z \in \mathcal{H} : \left\langle \gimel_k - \varkappa_k \mathcal{A}(v_{k-1}) - v_k, z - v_k \right\rangle \le 0 \right\} \quad \textit{for each } k \ge 0.$$

Compute

$$\begin{cases} \gimel_k = u_k + \alpha_k(u_k - u_{k-1}), \\ u_{k+1} = P_{\mathcal{H}_k}(\gimel_k - \varkappa_k \mathcal{A}(v_k)). \end{cases}$$

The stepsize should be updated as follows:

$$\varkappa_{k+1} = \begin{cases} \min\{\varkappa_k, \frac{(2-\sqrt{2}-\theta)\frac{1}{2}\varrho\|v_{k-1}-v_k\|^2+(2-\sqrt{2}-\theta)\varrho\|u_{k+1}-v_k\|^2}{2\langle \mathcal{A}(v_{k-1})-\mathcal{A}(v_k),u_{k+1}-v_k\rangle}\}, \\ \quad \textit{if } \langle \mathcal{A}(v_{k-1}) - \mathcal{A}(v_k), u_{k+1} - v_k\rangle > 0, \\ \varkappa_k, \quad \textit{otherwise.} \end{cases}$$

Compute

$$\begin{cases} \beth_{k+1} = u_{k+1} + \alpha_{k+1}(u_{k+1} - u_k), \\ v_{k+1} = P_{\mathcal{K}}(\beth_{k+1} - \varkappa_{k+1}\mathcal{A}(v_k)). \end{cases}$$

Then, the sequences $\{u_k\}$ *converge weakly to* $\eth^* \in \mathrm{Sol}(\mathcal{A},\mathcal{K})$.

Corollary 3.6 *Assume that* $\mathcal{A}:\mathcal{K}\to\mathcal{H}$ *is a pseudomonotone, weakly continuous and L-Lipschitz continuous operator and the solution set* $\mathrm{Sol}(\mathcal{A},\mathcal{K}) \neq \emptyset$. *Choose* $\varkappa_0 = \varkappa_1 > 0$, $u_{-1}, u_0, v_0 \in \mathcal{H}$, $\varrho \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$ *and* α_k *to be a decreasing sequence such that* $0 \le \underline{\alpha} \le \alpha_k \le \overline{\alpha} < \sqrt{5}-2$. *Select a real sequence that is* $\{p_k\}$ *such that* $\sum_{k=1}^{+\infty} p_k < +\infty$. *First, we have to compute*

$$\begin{cases} u_1 = P_{\mathcal{K}}(\beth_0 - \varkappa_0\mathcal{A}(v_0)), \\ v_1 = P_{\mathcal{K}}(\beth_1 - \varkappa_1\mathcal{A}(v_0)), \end{cases}$$

where $\beth_0 = u_0 + \alpha_0(u_0 - u_{-1})$ *and* $\beth_1 = u_1 + \alpha_1(u_1 - u_0)$. *Given* $u_{k-1}, v_{k-1}, u_k, v_k$, *and construct a half-space*

$$\mathcal{H}_k = \{z \in \mathcal{H} : \langle \beth_k - \varkappa_k\mathcal{A}(v_{k-1}) - v_k, z - v_k\rangle \le 0\} \quad \textit{for each } k \ge 0.$$

Compute

$$\begin{cases} \beth_k = u_k + \alpha_k(u_k - u_{k-1}), \\ u_{k+1} = P_{\mathcal{H}_k}(\beth_k - \varkappa_k\mathcal{A}(v_k)). \end{cases}$$

Update the stepsize in the following way:

$$\varkappa_{k+1} = \begin{cases} \min\{\varkappa_k + p_k, \frac{(2-\sqrt{2}-\theta)\frac{1}{2}\varrho\|v_{k-1}-v_k\|^2+(2-\sqrt{2}-\theta)\varrho\|u_{k+1}-v_k\|^2}{2\langle \mathcal{A}(v_{k-1})-\mathcal{A}(v_k),u_{k+1}-v_k\rangle}\}, \\ \quad \textit{if } \langle \mathcal{A}(v_{k-1}) - \mathcal{A}(v_k), u_{k+1} - v_k\rangle > 0, \\ \varkappa_k + p_k, \quad \textit{otherwise.} \end{cases}$$

Compute

$$\begin{cases} \beth_{k+1} = u_{k+1} + \alpha_{k+1}(u_{k+1} - u_k), \\ v_{k+1} = P_{\mathcal{K}}(\beth_{k+1} - \varkappa_{k+1}\mathcal{A}(v_k)). \end{cases}$$

Then, $\{u_k\}$ *sequence weakly converges to* $\eth^* \in \mathrm{Sol}(\mathcal{A},\mathcal{K})$.

Corollary 3.7 *Let $\mathcal{B} : \mathcal{K} \to \mathcal{H}$ be a weakly continuous and κ-strict pseudocontraction with the solution set* $\mathrm{Sol}(\mathcal{B}, \mathcal{K}) \neq \emptyset$. *Choose $\varkappa_0 = \varkappa_1 > 0$, $u_{-1}, u_0, v_0 \in \mathcal{H}$, $\varrho \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$ and α_k to be a decreasing sequence such that $0 \le \underline{\alpha} \le \alpha_k \le \overline{\alpha} < \sqrt{5}-2$. First, we have to compute*

$$\begin{cases} u_1 = P_{\mathcal{K}}[\beth_0 - \varkappa_0(v_0 - \mathcal{B}(v_0))], \\ v_1 = P_{\mathcal{K}}[\beth_1 - \varkappa_1(v_0 - \mathcal{B}(v_0))], \end{cases}$$

where $\beth_0 = u_0 + \alpha_0(u_0 - u_{-1})$ and $\beth_1 = u_1 + \alpha_1(u_1 - u_0)$. Given u_{k-1}, v_{k-1}, u_k, v_k, construct a half-space

$$\mathcal{H}_k = \left\{ z \in \mathcal{E} : \left\langle (1-\varkappa_k)\beth_k + \varkappa_k \mathcal{B}(v_{k-1}) - v_k, z - v_k \right\rangle \le 0 \right\}.$$

Compute

$$\begin{cases} \beth_k = u_k + \alpha_k(u_k - u_{k-1}), \\ u_{k+1} = P_{\mathcal{H}_k}[\beth_k - \varkappa_k(v_k - \mathcal{B}(v_k))]. \end{cases}$$

Evaluate stepsize rule for the next iteration is evaluated as follows:

$$\varkappa_{k+1} = \begin{cases} \min\{\varkappa_k, \frac{(2-\sqrt{2}-\theta)\frac{1}{2}\varrho\|v_{k-1}-v_k\|^2 + (2-\sqrt{2}-\theta)\varrho\|u_{k+1}-v_k\|^2}{2\langle (v_{k-1}-v_k) - [\mathcal{B}(v_{k-1}) - \mathcal{B}(v_k)], u_{k+1}-v_k\rangle}\}, \\ \quad \textit{if } \langle (v_{k-1} - v_k) - [\mathcal{B}(v_{k-1}) - \mathcal{B}(v_k)], u_{k+1} - v_k\rangle > 0, \\ \varkappa_k, \quad \textit{otherwise}. \end{cases}$$

Compute

$$\begin{cases} \beth_{k+1} = u_{k+1} + \alpha_{k+1}(u_{k+1} - u_k), \\ v_{k+1} = P_{\mathcal{K}}[\beth_{k+1} - \varkappa_{k+1}(v_k - \mathcal{B}(v_k))]. \end{cases}$$

Then, the sequence $\{u_k\}$ converges weakly to $\eth^* \in \mathrm{Sol}(\mathcal{B}, \mathcal{K})$.

Corollary 3.8 *Let $\mathcal{B} : \mathcal{K} \to \mathcal{H}$ be a weakly continuous and κ-strict pseudocontraction with the solution set* $\mathrm{Sol}(\mathcal{B}, \mathcal{K}) \neq \emptyset$. *Choose $\varkappa_0 = \varkappa_1 > 0$, $u_{-1}, u_0, v_0 \in \mathcal{H}$, $\varrho \in (0,1)$, $\theta \in (0, 2-\sqrt{2})$ and α_k to be a decreasing sequence such that $0 \le \underline{\alpha} \le \alpha_k \le \overline{\alpha} < \sqrt{5}-2$. Select a real sequence that is $\{p_k\}$ such that $\sum_{k=1}^{+\infty} p_k < +\infty$. First, we have to compute*

$$\begin{cases} u_1 = P_{\mathcal{K}}[\beth_0 - \varkappa_0(v_0 - \mathcal{B}(v_0))], \\ v_1 = P_{\mathcal{K}}[\beth_1 - \varkappa_1(v_0 - \mathcal{B}(v_0))], \end{cases}$$

where $\beth_0 = u_0 + \alpha_0(u_0 - u_{-1})$ and $\beth_1 = u_1 + \alpha_1(u_1 - u_0)$. Given u_{k-1}, v_{k-1}, u_k, v_k, and

$$\mathcal{H}_k = \left\{ z \in \mathcal{E} : \left\langle (1-\varkappa_k)\beth_k + \varkappa_k \mathcal{B}(v_{k-1}) - v_k, z - v_k \right\rangle \le 0 \right\}.$$

Compute

$$\begin{cases} \beth_k = u_k + \alpha_k(u_k - u_{k-1}), \\ u_{k+1} = P_{\mathcal{H}_k}[\beth_k - \varkappa_k(v_k - \mathcal{B}(v_k))]. \end{cases}$$

Evaluate stepsize rule for the next iteration is evaluated as follows:

$$\varkappa_{k+1} = \begin{cases} \min\{\varkappa_k + p_k, \frac{(2-\sqrt{2}-\theta)\frac{1}{2}\varrho\|v_{k-1}-v_k\|^2+(2-\sqrt{2}-\theta)\varrho\|u_{k+1}-v_k\|^2}{2\langle(v_{k-1}-v_k)-[\mathcal{B}(v_{k-1})-\mathcal{B}(v_k)],u_{k+1}-v_k\rangle}\}, \\ \quad \textit{if } \langle(v_{k-1}-v_k)-[\mathcal{B}(v_{k-1})-\mathcal{B}(v_k)], u_{k+1}-v_k\rangle > 0, \\ \varkappa_k + p_k, \quad \textit{otherwise}. \end{cases}$$

Compute

$$\begin{cases} \beth_{k+1} = u_{k+1} + \alpha_{k+1}(u_{k+1} - u_k), \\ v_{k+1} = P_{\mathcal{K}}[\beth_{k+1} - \varkappa_{k+1}(v_k - \mathcal{B}(v_k))]. \end{cases}$$

Then, the sequence $\{u_k\}$ *converges weakly to* $\eth^* \in \mathrm{Sol}(\mathcal{B},\mathcal{K})$.

4 Numerical illustrations

This section describes a number of computational experiments conducted to demonstrate the efficacy of the proposed methods. Some of these numerical illustrations provide a thorough understanding of how to select effective control parameters. Some of them demonstrate how proactive approaches outperform current ones in the literature. All MATLAB codes were run in MATLAB 9.5 (R2018b) on an Intel(R) Core(TM) i5-6200 Processor CPU at 2.30 GHz, 2.40 GHz, and 8.00 GB RAM.

Example 4.1 The first test problem here is taken from the Nash-Cournot Oligopolistic Equilibrium model in [28]. In this case, the bifunction $\mathcal{F}$ can be defined as follows:

$$\mathcal{F}(u,v) = \langle Pu + Qv + c, v - u\rangle,$$

where $c \in \mathbb{R}^M$, and P, Q are matrices of order M. The matrix $Q - P$ is symmetric negative semidefinite, and the matrix P is symmetric positive semidefinite, having Lipschitz-type constants that are $k_1 = k_2 = \frac{1}{2}\|P - Q\|$ (see [28] for more details).

Experiment 1: In the first experiment, we take an Example 4.1 to examine how Algorithm 2 performs numerically when alternative control sequence ϱ options are used. This experiment assisted us in determining the best potential control parameter ϱ. The starting points for these numerical studies are $u_{-1} = v_{-1} = u_0 = (1,1,\dots,1)$, $M = 5$, and error term $D_k = \|u_{k+1} - u_k\|$. Two matrices P, Q, and vector c are written as

$$P = \begin{pmatrix} 3.1 & 2 & 0 & 0 & 0 \\ 2 & 3.6 & 0 & 0 & 0 \\ 0 & 0 & 3.5 & 2 & 0 \\ 0 & 0 & 2 & 3.3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1.6 & 1 & 0 & 0 & 0 \\ 1 & 1.6 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ -2 \\ -1 \\ 2 \\ -1 \end{pmatrix}.$$

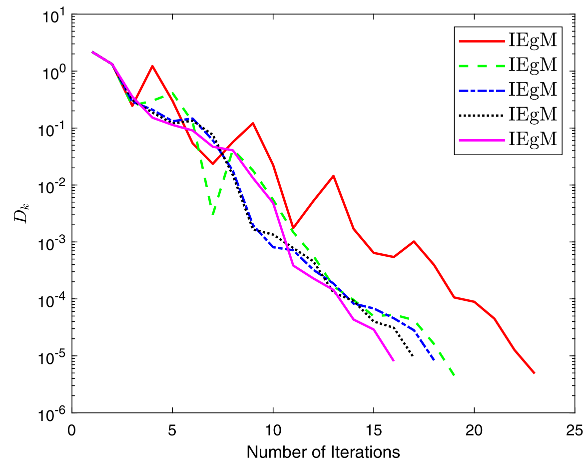

Figure 1 Computational performance of Algorithm 2 for different values of $\varrho = 0.22, 0.43, 0.63, 0.84, 0.98$, respectively

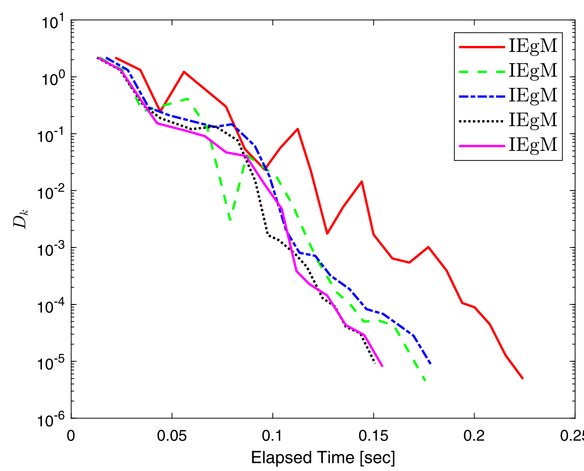

Figure 2 Computational performance of Algorithm 2 for different values of $\varrho = 0.22, 0.43, 0.63, 0.84, 0.98$, respectively

The feasible set $\mathcal{K} \subset \mathbb{R}^M$ is defined by

$$\mathcal{K} := \{u \in \mathbb{R}^M : -2 \leq u_i \leq 5\}.$$

Figures 1 and 2 demonstrate a numerical results using an error $D_k = \|u_{k+1} - u_k\| \leq 10^{-5}$. The following information about control settings should be considered: (i) Algorithm 2 (shortly, IEgM):

$$\varkappa_0 = \frac{1}{2c}, \qquad \theta = 0.050, \qquad \alpha_k = 0.210, \qquad p_k = \frac{100}{(k+1)^2}.$$

Experiment 2: In the second experiment, we look at Example 4.1 to examine how Algorithm 2 performs numerically when alternative control sequence θ options are used. This experiment assisted us in determining the best potential control parameter θ. The

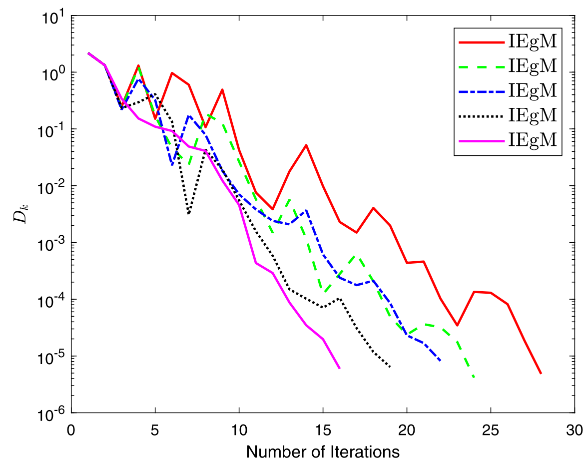

Figure 3 Computational performance of Algorithm 2 using different values of $\theta = 0.45, 0.35, 0.25, 0.15, 0.05$, respectively

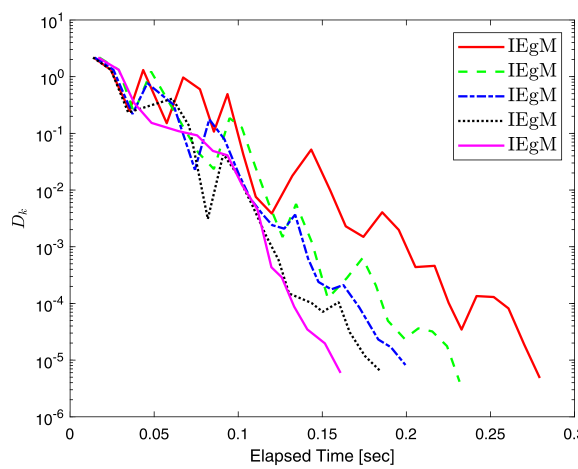

Figure 4 Computational performance of Algorithm 2 using different values of $\theta = 0.45, 0.35, 0.25, 0.15, 0.05$, respectively

starting points for these numerical studies are $u_{-1} = v_{-1} = u_0 = (1, 1, \ldots, 1)$, $M = 5$, and error term $D_k = \|u_{k+1} - u_k\|$. Figures 3 and 4 show a number of results for the error term $D_k = \|u_{k+1} - u_k\| \le 10^{-5}$. Information concerning the control parameters shall be considered as follows: (i) Algorithm 2 (shortly, IEgM):

$$\varkappa_0 = \frac{1}{2c}, \qquad \varrho = 0.55, \qquad \alpha_k = 0.20, \qquad p_k = \frac{100}{(k+1)^2}.$$

Experiment 3: In the third experiment, we consider Example 4.1 to see the computational performance of Algorithm 2 with Algorithm 3.1 in [36] using different choices for the dimension M. Two P, Q matrices are taken randomly [Two diagonal matrices randomly A_1 and A_2 with elements from $[0, 2]$ and $[-2, 0]$, respectively. Two random orthogonal matrices $O_1 = \mathrm{RandOrthMat}(M)$ and $O_2 = \mathrm{RandOrthMat}(M)$ are generated. Thus, a positive semi-definite matrix $B_1 = O_1 A_1 O_1^T$ and a negative semi-definite matrix $B_2 = O_2 A_2 O_2^T$ are achieved. Finally, set $Q = B_1 + B_1^T$, $S = B_2 + B_2^T$ and $P = Q - S]$. A set of constraints $\mathcal{K} \subset \mathbb{R}^M$ is illustrated by

$$\mathcal{K} := \{u \in \mathbb{R}^M : -10 \leq u_i \leq 10\}.$$

For these numerical studies, starting points are $u_{-1} = v_{-1} = u_0 = (1, 1, \dots, 1)$, and error term $D_k = \|u_{k+1} - u_k\|$. Figures 5, 6, 7, 8, 9 and Table 1 show a number of results for the error term

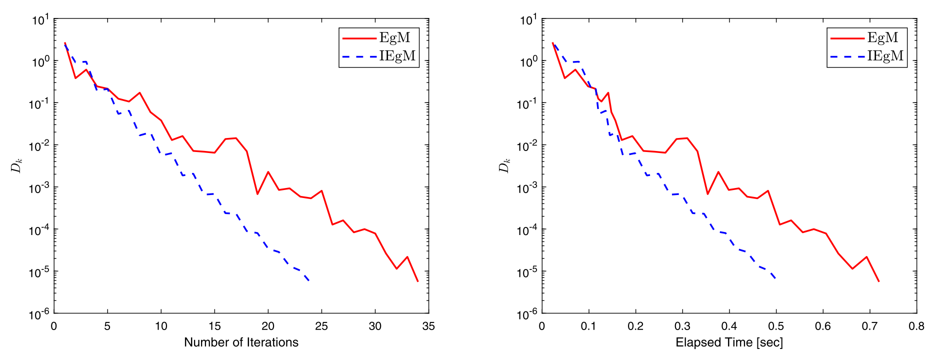

Figure 5 Computational comparability of Algorithm 2 with Algorithm 3.1 in [36] for $M = 5$

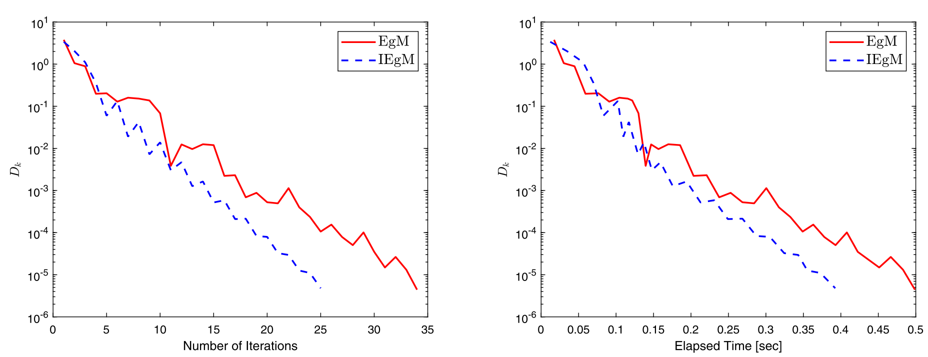

Figure 6 Computational comparability of Algorithm 2 with Algorithm 3.1 in [36] for $M = 10$

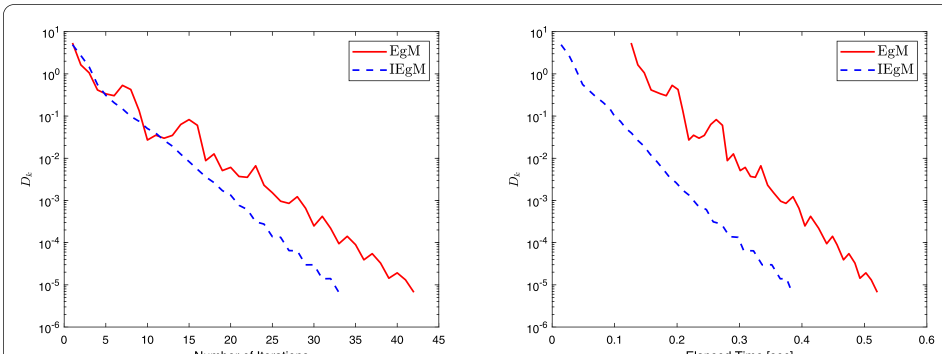

Figure 7 Computational comparability of Algorithm 2 with Algorithm 3.1 in [36] for $M = 20$

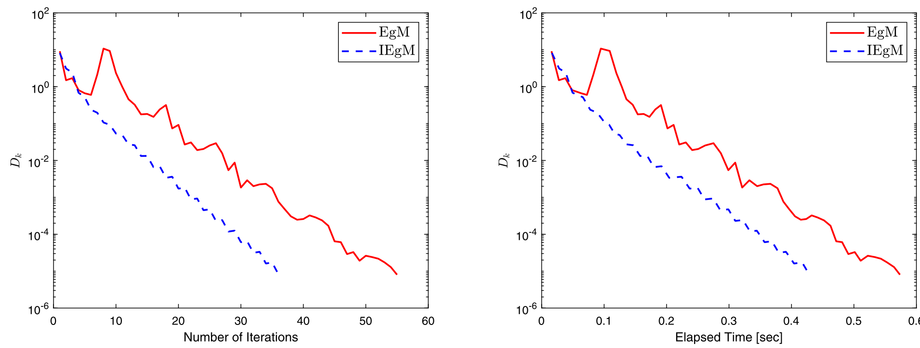

Figure 8 Computational comparability of Algorithm 2 with Algorithm 3.1 in [36] for $M = 50$

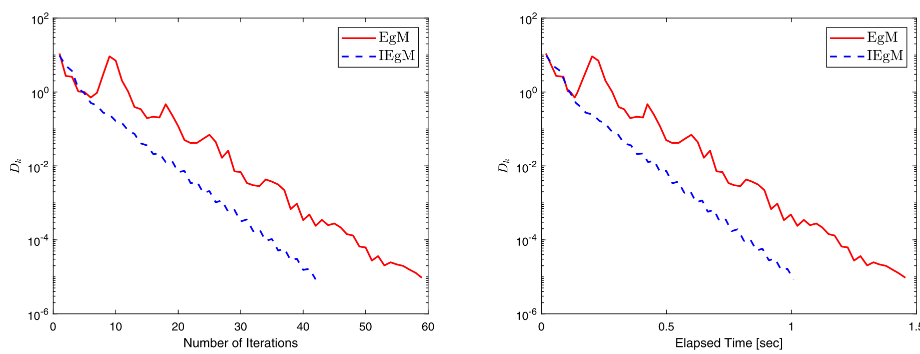

Figure 9 Computational comparability of Algorithm 2 with Algorithm 3.1 in [36] for $M = 100$

Table 1 Numerical data for Figs. 5, 6, 7, 8, 9

M	Number of iterations		Execution time in seconds	
	Algorithm 2	Algorithm 3.1 in [36]	Algorithm 2	Algorithm 3.1 in [36]
5	34	24	0.719490100000000	0.506414100000000
10	34	25	0.498847400000000	0.392540700000000
20	42	33	0.520615800000000	0.384156300000000
50	55	36	0.573747200000000	0.428239300000000
100	59	42	1.45613430000000	1.00880530000000

$D_k = \|u_{k+1} - u_k\| \leq 10^{-5}$. Information regarding the control parameters shall be considered as follows:

(i) Algorithm 3.1 in [36] (shortly, EgM):

$$\varkappa_0 = \frac{1}{2c}, \qquad \varrho = 0.45, \qquad \theta = 0.05, \qquad p_k = \frac{100}{(k+1)^2};$$

(ii) Algorithm 2 (shortly, IEgM):

$$\varkappa_0 = \frac{1}{2c}, \qquad \varrho = 0.45, \qquad \alpha_k = 0.18, \qquad \theta = 0.05, \qquad p_k = \frac{100}{(k+1)^2}.$$

Example 4.2 Let a bifunction $\mathcal{F}$ is defined by

$$\mathcal{F}(\breve{u},\breve{v}) = \langle I(\breve{u}), \breve{v}-\breve{u}\rangle, \quad \forall \breve{u},\breve{v}\in\mathcal{K},$$

where

$$\mathcal{K} = \{(u_1,\dots,u_M)\in\mathbb{R}^M : u_i\geq 1, i=1,2,\dots,M\}.$$

Consider that $I(\breve{u}) = G(\breve{u}) + H(\breve{u})$ in the following manner:

$$G(\breve{u}) = (g_1(\breve{u}), g_2(\breve{u}),\dots,g_M(\breve{u})), \qquad H(\breve{u}) = E\breve{u}+c,$$

where $c = (-1,-1,\dots,-1)$ and

$$g_i(\breve{u}) = \breve{u}_{i-1}^2 + \breve{u}_i^2 + \breve{u}_{i-1}\breve{u}_i + \breve{u}_i\breve{u}_{i+1}, \quad i=1,2,\dots,M, \qquad \breve{u}_0 = \breve{u}_{M+1} = 0.$$

The matrix E entries are taken as follows:

$$e_{i,j} = \begin{cases} 4, & j=i, \\ 1, & i-j=1, \\ -2, & i-j=-1, \\ 0, & \text{otherwise}. \end{cases}$$

In this experiment, we consider Example 4.2 to see the numerical illustration of Algorithm 2 in comparison with Algorithm 3.1 in [36] using different choices for different values of the dimension M. For these numerical studies, starting points are $u_{-1} = v_{-1} = u_0 = (1,1,\dots,1)$, $M = 5$, and the error term $D_k = \|u_{k+1}-u_k\|$. Figures 10, 11, 12, 13 and Table 2 show a number of results for the error term $D_k = \|u_{k+1}-u_k\| \leq 10^{-5}$. Information concerning the control parameters shall be considered as follows:

(i) Algorithm 3.1 in [36] (shortly, EgM):

$$\varkappa_0 = \frac{1}{2c}, \qquad \varrho = 0.45, \qquad \theta = 0.05, \qquad p_k = \frac{100}{(k+1)^2}.$$

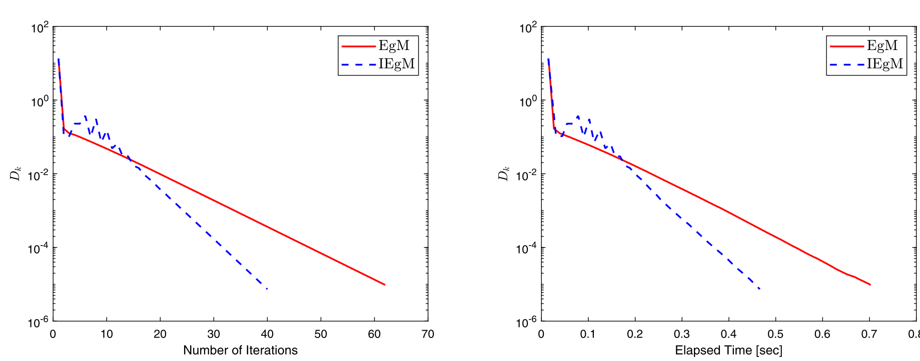

Figure 10 Computational comparability of Algorithm 2 with Algorithm 3.1 in [36] for $M = 20$

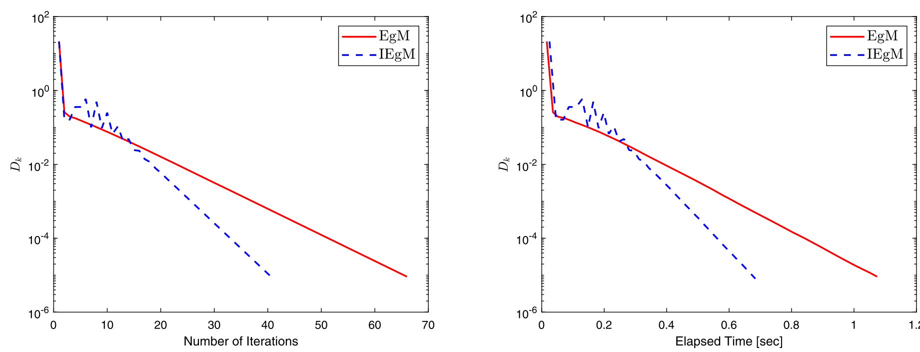

Figure 11 Computational comparability of Algorithm 2 with Algorithm 3.1 in [36] for $M = 50$

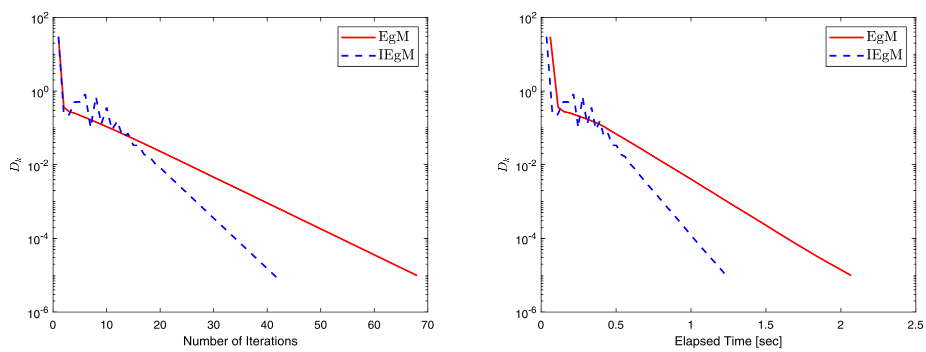

Figure 12 Computational comparability of Algorithm 2 with Algorithm 3.1 in [36] for $M = 100$

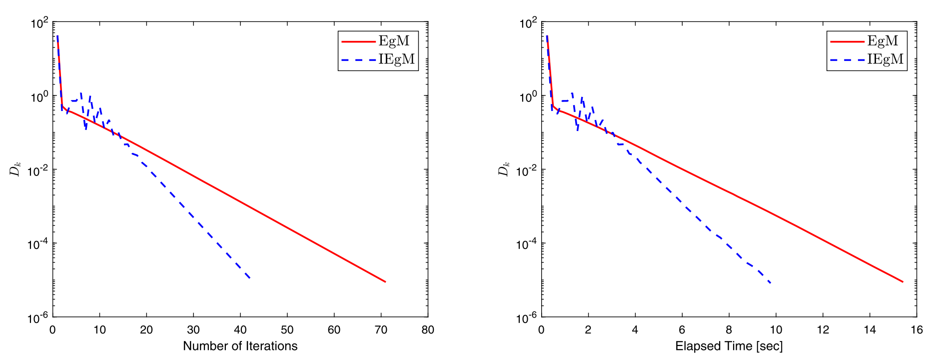

Figure 13 Computational comparability of Algorithm 2 with Algorithm 3.1 in [36] for $M = 200$

Table 2 Numerical data for Figs. 10, 11, 12, 13

M	Number of iterations		Execution time in seconds	
	Algorithm 2	Algorithm 3.1 in [36]	Algorithm 2	Algorithm 3.1 in [36]
20	62	40	0.702655300000000	0.466009800000000
50	66	41	1.07402510000000	0.684056100000000
100	68	42	2.06877200000000	1.25467120000000
200	71	43	15.4223303000000	9.76697390000000

(ii) Algorithm 2 (shortly, IEgM):

$$\varkappa_0 = \frac{1}{2c}, \qquad \varrho = 0.45, \qquad \alpha_k = 0.18, \qquad \theta = 0.05, \qquad p_k = \frac{100}{(k+1)^2}.$$

5 Conclusion

This research presents two explicit extragradient-like methods for solving an equilibrium problem in a real Hilbert space, which include a pseudomonotone and a Lipschitz-type bifunction. A novel stepsize rule has been given that does not become reliant on the information provided by Lipschitz-type constants. For the given methods, convergence theorems have been established. Several experiments are detailed in order to show the numerical behavior of algorithms and compare them to other well-known methods in the literature.

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Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HR was a major contributor in writing the manuscript and conceptualization. PK dealt with the conceptualization, supervision, and funding acquisition. IKA dealt with the methodology, investigation, and edition original draft preparation. WK performed the validation, formal analysis, and funding acquisition. MS performed conceptualization, formal analysis and writing revised version. All authors read and approved the final manuscript.

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