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On the domain of generalized mean in Nakano sequence space, and its prequasi ideal with some applications

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Abstract

Some topological and geometric behavior of the space of all sequences whose generalized mean transforms are in Nakano sequence space, the multiplication mappings acting on it, and the eigenvalue distribution of mappings ideal generated by this space and s-numbers are discussed. We construct the existence of a fixed point of Kannan contraction mapping on these spaces. Several numerical experiments are presented to illustrate our results. Moreover, some successful applications to the existence of solutions of nonlinear difference equations are explained. The strength here is that the current results are constructed under flexible setups given by controlling the weight and power of these spaces.

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1 Introduction

Since the principle of variable exponent function spaces have built upon the boundedness of the Hardy–Little-wood maximal mapping, this explains its technique in image processing, differential equations, and approximation theory. We will use the following conventions in this article, if others are used we will notate them.

Conventions 1.1

$\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. $\mathfrak{C}$: The space of all complex numbers.

$[a]$: The integral part of a.

R : The set of real numbers.

$\mathfrak{C}^{\mathbb{Z}^+}$: The space of all sequences of complex numbers.

$(0, \infty)^{\mathbb{Z}^+}$: The space of all sequences of positive reals.

ℓ_∞ : The space of bounded sequences of complex numbers.

ℓ_r : The space of r-absolutely summable sequences of complex numbers.

c_0 : The space of null sequences of complex numbers.

$e_l = (0, 0, \ldots, 1, 0, 0, \ldots)$, as 1 lies at the lth coordinate, for all $l \in \mathbb{Z}^+$.

F : The space of each sequences with finite nonzero coordinates.

$\mathfrak{I}$: The space of all sets with a finite number of elements.

$\mathfrak{I}_{\nearrow}$: The space of all monotonic increasing sequences of positive reals.

$\mathfrak{B}$: The ideal of all bounded linear mappings between any arbitrary Banach spaces.

$\mathfrak{F}$: The ideal of finite-rank mappings between any arbitrary Banach spaces.

$\mathfrak{A}$: The ideal of approximable mappings between any arbitrary Banach spaces.

$\mathfrak{K}$: The ideal of compact mappings between any arbitrary Banach spaces.

$\mathfrak{B}(\mathcal{N}, \mathcal{M})$: The space of all bounded linear mappings from a Banach space $\mathcal{N}$ into a Banach space $\mathcal{M}$.

$\mathfrak{B}(\mathcal{N})$: The space of all bounded linear mappings from a Banach space $\mathcal{N}$ into itself.

$\mathfrak{F}(\mathcal{N}, \mathcal{M})$: The space of finite-rank mappings from a Banach space $\mathcal{N}$ into a Banach space $\mathcal{M}$.

$\mathfrak{F}(\mathcal{N})$: The space of finite-rank mappings from a Banach space $\mathcal{N}$ into itself.

$\mathfrak{A}(\mathcal{N}, \mathcal{M})$: The space of approximable mappings from a Banach space $\mathcal{N}$ into a Banach space $\mathcal{M}$.

$\mathfrak{A}(\mathcal{N})$: The space of approximable mappings from a Banach space $\mathcal{N}$ into itself.

$\mathfrak{K}(\mathcal{N}, \mathcal{M})$: The space of compact mappings from a Banach space $\mathcal{N}$ into a Banach space $\mathcal{M}$.

$\mathfrak{K}(\mathcal{N})$: The space of compact mappings from a Banach space $\mathcal{N}$ into itself.

Definition 1.2 ([1]) A mapping $s : \mathfrak{B}(\mathcal{N}, \mathcal{M}) \to [0, \infty)^{\mathbb{Z}^+}$ is named an s-number, if the sequence $(s_j(H))_{j=0}^{\infty}$, for all $H \in \mathfrak{B}(\mathcal{N}, \mathcal{M})$, verifies the following conditions:

(a) $\|H\| = s_0(H) \geq s_1(H) \geq s_2(H) \geq \cdots \geq 0$, with $H \in \mathfrak{B}(\mathcal{N}, \mathcal{M})$,
(b) $s_{j+l-1}(H_1 + H_2) \leq s_j(H_1) + s_l(H_2)$, with $H_1, H_2 \in \mathfrak{B}(\mathcal{N}, \mathcal{M})$ and $j, l \in \mathbb{Z}^+$,
(c) $s_j(ZYH) \leq \|Z\| s_j(Y) \|H\|$, for every $H \in \mathfrak{B}(\mathcal{N}_0, \mathcal{N})$, $Y \in \mathfrak{B}(\mathcal{N}, \mathcal{M})$ and $Z \in \mathfrak{B}(\mathcal{M}, \mathcal{M}_0)$, where $\mathcal{N}_0$ and $\mathcal{M}_0$ are any two Banach spaces,
(d) suppose $G \in \mathfrak{B}(\mathcal{N}, \mathcal{M})$ and $\gamma \in \mathfrak{C}$, hence $s_j(\gamma G) = |\gamma| s_j(G)$,
(e) assume rank$(H) \leq j$, then $s_j(H) = 0$, for all $H \in \mathfrak{B}(\mathcal{N}, \mathcal{M})$,
(f) $s_{l \geq j}(I_j) = 0$ or $s_{l<j}(I_j) = 1$, where I_j marks the identity mapping on the j-dimensional Hilbert space ℓ_2^j.

We explain a few examples of s-numbers as follows:

(1) The jth Kolmogorov number, $d_j(H)$, where
$d_j(H) = \inf_{\dim J \le j} \sup_{\|\lambda\| \le 1} \inf_{\beta \in J} \|H\lambda - \beta\|$.

(2) The jth approximation number, $\alpha_j(H)$, where
$\alpha_j(H) = \inf\{\|H - Z\| : Z \in \mathfrak{B}(\mathcal{N}, \mathcal{M}) \text{ and } \operatorname{rank}(Z) \le j\}$.

Notations 1.3 ([2])

$$\mathfrak{B}^s_{\mathcal{K}} := \{\mathfrak{B}^s_{\mathcal{K}}(\mathcal{N}, \mathcal{M}); \mathcal{N} \text{ and } \mathcal{M} \text{ are Banach Spaces}\}, \quad \text{where}$$
$$\mathfrak{B}^s_{\mathcal{K}}(\mathcal{N}, \mathcal{M}) := \{H \in \mathfrak{B}(\mathcal{N}, \mathcal{M}) : ((s_j(H))_{j=0}^{\infty} \in \mathcal{K}\}.$$
$$\mathfrak{B}^\alpha_{\mathcal{K}} := \{\mathfrak{B}^\alpha_{\mathcal{K}}(\mathcal{N}, \mathcal{M}); \mathcal{N} \text{ and } \mathcal{M} \text{ are Banach Spaces}\}, \quad \text{where}$$
$$\mathfrak{B}^\alpha_{\mathcal{K}}(\mathcal{N}, \mathcal{M}) := \{H \in \mathfrak{B}(\mathcal{N}, \mathcal{M}) : ((\alpha_j(H))_{j=0}^{\infty} \in \mathcal{K}\}.$$
$$\mathfrak{B}^d_{\mathcal{K}} := \{\mathfrak{B}^d_{\mathcal{K}}(\mathcal{N}, \mathcal{M})\mathcal{N} \text{ and } \mathcal{M} \text{ are Banach Spaces}\}, \quad \text{where}$$
$$\mathfrak{B}^d_{\mathcal{K}}(\mathcal{N}, \mathcal{M}) := \{H \in \mathfrak{B}(\mathcal{N}, \mathcal{M}) : ((d_j(H))_{j=0}^{\infty} \in \mathcal{K}\}.$$

Functional analysis places a high value on the operator ideal theory. Operators ideal can be constructed using s-numbers, one of the most important ways. The theory of s-numbers of linear bounded operators between Banach spaces was introduced and investigated by Pietsch [3–6]. He offered and explained some geometric and topological structure of the quasi-ideals $\mathfrak{B}^\alpha_{\ell_b}$. Then, Constantin [7] generalized the class of ℓ_p-type operators to the class of ces_p-type operators by using Cesàro sequence spaces. Makarov and Faried [8], showed that for any infinite-dimensional Banach spaces $\mathcal{N}$, $\mathcal{M}$ and $l > j > 0$, then $\mathfrak{B}^\alpha_{\ell_j}(\mathcal{N}, \mathcal{M}) \subsetneqq \mathfrak{B}^\alpha_{\ell_l}(\mathcal{N}, \mathcal{M}) \subsetneqq \mathfrak{B}(\mathcal{N}, \mathcal{M})$. As a generalization of ℓ_p-type operators, Stolz mappings and operators ideal were examined by Tita [9, 10]. In [11], Maji and Srivastava studied the class $A^{(s)}_p$ of s-type ces_p operators using s-number sequence and Cesàro sequence spaces and they introduced a new class $A^{(s)}_{p,q}$ of s-type $\mathrm{ces}(p, q)$ operators by using a weighted Cesàro sequence space for $1 < p < \infty$. In [12], the class of s-type $Z(u, v; \ell_p)$ operators was defined and some of their properties were explained. Yaying et al. [13], defined and studied the sequence space, χ^η_r, whose r-Cesàro matrix is in ℓ_η, with $r \in (0, 1]$ and $1 \le \eta \le \infty$. They explained the quasi-Banach ideal of type χ^η_r, with $r \in (0, 1]$ and $1 < \eta < \infty$. They offered its Schauder basis, $\alpha-$, $\beta-$ and γ-duals, and evaluated certain matrix classes connected to this sequence space. The compact mappings were studied by many authors for different sequence spaces, for this see [14–20]. Komal et al. [21], explained the multiplication mappings defined on Cesàro sequence spaces equipped with the Luxemburg norm. The multiplication mappings acting on Cesàro second-order function spaces was discussed by İlkhan et al. [22]. The nonabsolute-type sequence spaces are a generalization of the corresponding absolute type. Hence, there is great interest in studying these sequence spaces. Recently, many authors in the literature have explained some nonabsolute-type sequence spaces and published new, exciting papers, for example, see Mursaleen and Noman [23, 24] and Mursaleen and Başar [25]. In the field of the Banach fixed-point theorem [26], Kannan [27] discussed an example of a class of mappings with the same fixed-point actions as contractions, although it failed to be continuous. Ghoncheh [28] was the only one who investigated Kannan mappings in modular vector spaces. He proved the existence of a fixed point of Kannan mapping in complete modular spaces with the Fatou property. Bakery and Mohamed [29] offered the concept of the prequasinorm on a Nakano sequence space such that its variable exponent is in $(0,1]$. They discussed enough setup on it equipped with the known prequasinorm to form prequasi-Banach and closed space, and investigated the Fatou property of different prequasinorms on it. They proved the existence of a fixed point of Kannan prequasinorm contraction mappings on it and on the prequasi-Banach mappings ideal constructed by s-numbers that lie in this sequence space.

Lemma 1.4 ([30]) *Assume $\eta_j > 0$ and $\lambda_j, \beta_j \in \mathfrak{C}$, for all $j \in \mathbb{Z}^+$, and $\hbar = \max\{1, \sup_j \eta_j\}$, then*

$$|\lambda_j + \beta_j|^{\eta_j} \leq 2^{\hbar-1}\left(|\lambda_j|^{\eta_j} + |\beta_j|^{\eta_j}\right). \tag{1}$$

The aim of this article is confirmed as follows: In Sect. 3, we offer the definition and some inclusion relations of the sequence space $(\Pi(\zeta,\eta))_\mu$ equipped with the function μ. In Sect. 4, we explain the sufficient conditions on $\Pi(\zeta,\eta)$ with definite μ to construct premodular private sequence space ($\mathfrak{pss}$). This investigates whether $(\Pi(\zeta,\eta))_\mu$ is a prequasinormed $\mathfrak{pss}$. In Sect. 5, we examine multiplication mappings on $(\Pi(\zeta,\eta))_\mu$, and introduce the necessity and enough setups on this sequence space so that the multiplication mapping is bounded, approximable, invertible, Fredholm and closed range. In Sect. 6, first we introduce the enough conditions (not necessary) on $(\Pi(\zeta,\eta))_\mu$, so that $\overline{\mathfrak{F}} = \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$. This offers a negative answer to Rhoades [31] open problem about the linearity of s-type $(\Pi(\zeta,\eta))_\mu$ spaces. Secondly, we investigate the setups on $(\Pi(\zeta,\eta))_\mu$ such that the elements of prequasi ideal $\mathfrak{B}^s_{\Pi(\zeta,\eta)}$ are complete and closed. Thirdly, we explain the enough conditions on $(\Pi(\zeta,\eta))_\mu$ so that $\mathfrak{B}^\alpha_{(\Pi(\zeta,\eta))_\mu}$ is strictly included for different weights and powers. We introduce the setups for which the prequasi ideal $\mathfrak{B}^\alpha_{(\Pi(\zeta,\eta))_\mu}$ is minimum. Fourthly, we suggest the conditions for which the Banach prequasi ideal $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$ is simple. Fifthly, we explain the enough conditions on $(\Pi(\zeta,\eta))_\mu$ so that the class $\mathfrak{B}$ which sequence of eigenvalues in $(\Pi(\zeta,\eta))_\mu$ equals $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$. In Sect. 7, the existence of a fixed point of Kannan prequasinorm contraction mapping on this sequence space and on its prequasi-mappings ideal generated by $(\Pi(\zeta,\eta))_\mu$ and s-numbers with several numerical experiments to illustrate our results are presented. Moreover, in Sect. 8, some successful applications to the existence of solutions of nonlinear difference equations are explained. Finally, we give our conclusions in Sect. 9.

2 Definitions and preliminaries

Lemma 2.1 ([5]) *If $H \in \mathfrak{B}(\mathcal{N},\mathcal{M})$ and $H \notin \mathfrak{A}(\mathcal{N},\mathcal{M})$, we have $X \in \mathfrak{B}(\mathcal{N})$ and $Y \in \mathfrak{B}(\mathcal{M})$ so that $YHXe_j = e_j$, for all $j \in \mathbb{Z}^+$.*

Definition 2.2 ([5]) A Banach space $\mathcal{K}$ is said to be simple if the algebra $\mathfrak{B}(\mathcal{K})$ contains one and only one nontrivial closed ideal.

Theorem 2.3 ([5]) *Let $\mathcal{K}$ be a Banach space with* $\dim(\mathcal{K}) = \infty$, *one has*

$$\mathfrak{F}(\mathcal{K}) \subsetneqq \mathfrak{A}(\mathcal{K}) \subsetneqq \mathfrak{K}(\mathcal{K}) \subsetneqq \mathfrak{B}(\mathcal{K}).$$

Definition 2.4 ([32]) A mapping $U \in \mathfrak{B}(\mathcal{K})$ is said to be Fredholm if $\dim(\mathrm{Range}(U))^c < \infty$, $\dim(\ker(U)) < \infty$ and $\mathrm{Range}(U)$ is closed, where $(\mathrm{Range}(U))^c$ denotes the complement of $\mathrm{Range}(U)$.

Definition 2.5 ([33]) A class $\mathbb{W} \subseteq \mathfrak{B}$ is said to be a mappings ideal if every component $\mathbb{W}(\mathcal{N},\mathcal{M}) = \mathbb{W} \cap \mathfrak{B}(\mathcal{N},\mathcal{M})$ satisfies the next setups:

(i) $I_\Omega \in \mathbb{W}$, if Ω explains a Banach space of one dimension.

(ii) $\mathbb{W}(\mathcal{N},\mathcal{M})$ is a linear space on $\mathfrak{C}$.

(iii) Let $X \in \mathfrak{B}(\mathcal{N}_0,\mathcal{N})$, $Y \in \mathbb{W}(\mathcal{N},\mathcal{M})$ and $Z \in \mathfrak{B}(\mathcal{M},\mathcal{M}_0)$, then $ZYX \in \mathbb{W}(\mathcal{N}_0,\mathcal{M}_0)$, where $\mathcal{N}_0$ and $\mathcal{M}_0$ are normed spaces.

Definition 2.6 ([2]) A mapping $\Lambda : \mathbb{W} \to [0,\infty)$ is called a prequasinorm on the mappings ideal $\mathbb{W}$, if it verifies the next setups:

(1) For all $X \in \mathbb{W}(\mathcal{N},\mathcal{M})$, $\Lambda(X) \geq 0$ and $\Lambda(X) = 0 \Longleftrightarrow X = 0$,

(2) we have $E_0 \geq 1$ so that $\Lambda(\kappa X) \leq E_0|\kappa|\Lambda(X)$, with $X \in \mathbb{W}(\mathcal{N},\mathcal{M})$ and $\kappa \in \mathfrak{C}$,

(3) there are $G_0 \geq 1$ so that $\Lambda(Z_1 + Z_2) \leq G_0[\Lambda(Z_1) + \Lambda(Z_2)]$, for all $Z_1, Z_2 \in \mathbb{W}(\mathcal{N},\mathcal{M})$,

(4) there are $D_0 \geq 1$ so that if $X \in \mathfrak{B}(\mathcal{N}_0,\mathcal{N})$, $Y \in \mathbb{W}(\mathcal{N},\mathcal{M})$ and $Z \in \mathfrak{B}(\mathcal{M},\mathcal{M}_0)$, then $\Lambda(ZYX) \leq D_0\|Z\|\Lambda(Y)\|X\|$.

Theorem 2.7 ([2]) *Every quasinorm on the ideal $\mathbb{W}$ is a prequasinorm on the same ideal.*

Definition 2.8 ([34]) The linear space of sequences $\mathcal{K}$ is said to be a private sequence space ($\mathfrak{pss}$), if it satisfies the following conditions:

(1) $e_j \in \mathcal{K}$, with $j \in \mathbb{Z}^+$,

(2) $\mathcal{K}$ is solid, i.e., for $f = (f_j) \in \mathfrak{C}^{\mathbb{Z}^+}$, $|g| = (|g_j|) \in \mathcal{K}$ and $|f_j| \leq |g_j|$, with $j \in \mathbb{Z}^+$, then $|f| \in \mathcal{K}$,

(3) $(|f_{[\frac{j}{2}]}|)_{j=0}^\infty \in \mathcal{K}$, if $(|f_j|)_{j=0}^\infty \in \mathcal{K}$.

Theorem 2.9 ([34]) *Let the linear sequence space $\mathcal{K}$ be a $\mathfrak{pss}$, then $\mathfrak{B}^s_{\mathcal{K}}$ is a mapping ideal.*

Definition 2.10 ([34]) A subspace of the $\mathfrak{pss}$ is said to be a premodular $\mathfrak{pss}$, if there is a mapping $\mu : \mathcal{K} \to [0,\infty)$ that satisfies the following conditions:

(i) For every $\lambda \in \mathcal{K}$, $\lambda = \theta \Longleftrightarrow \mu(|\lambda|) = 0$, and $\mu(\lambda) \geq 0$, where θ is the zero vector of $\mathcal{K}$,

(ii) suppose $\lambda \in \mathcal{K}$ and $\rho \in \mathfrak{C}$, there are $E_0 \geq 1$ with $\mu(\rho\lambda) \leq |\rho|E_0\mu(\lambda)$,

(iii) $\mu(\lambda + \beta) \leq G_0(\mu(\lambda) + \mu(\beta))$ holds for some $G_0 \geq 1$, with $\lambda, \beta \in \mathcal{K}$,

(iv) if $j \in \mathbb{Z}^+$, $|\lambda_j| \leq |\beta_j|$, we have $\mu((|\lambda_j|)) \leq \mu((|\beta_j|))$,

(v) the inequality, $\mu((|\lambda_j|)) \leq \mu((|\lambda_{[\frac{j}{2}]}|)) \leq D_0\mu((|\lambda_j|))$ holds, for $D_0 \geq 1$,

(vi) $\overline{\mathrm{F}} = \mathcal{K}_\mu$,

(vii) one has $\varpi > 0$ such that $\mu(\rho,0,0,0,\dots) \geq \varpi|\rho|\mu(1,0,0,0,\dots)$, with $\rho \in \mathfrak{C}$.

Definition 2.11 ([34]) The $\mathfrak{pss}$ $\mathcal{K}_\mu$ is called a prequasinormed $\mathfrak{pss}$, if μ satisfies the setups (i)–(iii) of Definition 2.10. When $\mathcal{K}$ is complete and equipped with μ, then $\mathcal{K}_\mu$ is called a prequasi-Banach $\mathfrak{pss}$.

Theorem 2.12 ([34]) *Every premodular $\mathfrak{pss}$ $\mathcal{K}_\mu$ is a prequasinormed $\mathfrak{pss}$.*

Theorem 2.13 ([34]) *The function Λ is a prequasinorm on $\mathfrak{B}^s_{(\mathcal{K})_\mu}$, where $\Lambda(Y) = \mu(s_j(Y))_{j=0}^\infty$, for every $Y \in \mathfrak{B}^s_{(\mathcal{K})_\mu}(\mathcal{N},\mathcal{M})$, if $(\mathcal{K})_\mu$ is a premodular $\mathfrak{pss}$.*

Definition 2.14 ([29]) A prequasinorm μ on $\mathcal{K}$ satisfies the Fatou property, if for every sequence $\{\lambda^a\} \subseteq \mathcal{K}_\mu$ with $\lim_{a\to\infty} \mu(\lambda^a - \lambda) = 0$ and every $\beta \in \mathcal{K}_\mu$ then $\mu(\beta - \lambda) \leq \sup_j \inf_{a\geq j} \mu(\beta - \lambda^a)$.

Definition 2.15 ([29]) A prequasinorm Λ on the ideal $\mathfrak{B}^s_{\mathcal{K}}$, where $\Lambda(W) = \mu((s_a(W))_{a=0}^\infty)$, satisfies the Fatou property if for every sequence $\{W_a\}_{a\in\mathbb{Z}^+} \subseteq \mathfrak{B}^s_{\mathcal{K}}(\mathcal{N},\mathcal{M})$ with $\lim_{a\to\infty}\Lambda(W_a - W) = 0$ and every $V \in \mathfrak{B}^s_{\mathcal{K}}(\mathcal{N},\mathcal{M})$, then

$$\Lambda(V-W) \le \sup_a \inf_{i\ge a} \Lambda(V-W_i).$$

Definition 2.16 ([29]) A mapping $W:\mathcal{K}_\mu \to \mathcal{K}_\mu$ is called a Kannan μ-contraction, if there is $\beta \in [0,\frac{1}{2})$, such that $\mu(Wp-Wq) \le \beta(\mu(Wp-p)+\mu(Wq-q))$, for every $p,q\in\mathcal{K}_\mu$.

An element $p\in\mathcal{K}_\mu$ is called a fixed point of W, if $W(p)=p$.

Definition 2.17 ([29]) A mapping $W:\mathfrak{B}^s_{\mathcal{K}}(\mathcal{N},\mathcal{M}) \to \mathfrak{B}^s_{\mathcal{K}}(\mathcal{N},\mathcal{M})$ is called a Kannan Λ-contraction, if there is $\beta\in[0,\frac{1}{2})$, so that $\Lambda(WV-WT) \le \beta(\Lambda(WV-V)+\Lambda(WT-T))$, for every $V,T\in\mathfrak{B}^s_{\mathcal{K}}(\mathcal{N},\mathcal{M})$.

Definition 2.18 ([29]) Suppose $\mathcal{K}_\mu$ is a prequasinormed (sss), $W:\mathcal{K}_\mu\to\mathcal{K}_\mu$ and $b\in\mathcal{K}_\mu$. The mapping W is termed μ-sequentially continuous at b, if and only if, when $\lim_{a\to\infty}\mu(t_a-b)=0$, then $\lim_{a\to\infty}\mu(Wt_a-Wb)=0$.

Definition 2.19 ([29]) For the prequasinorm Λ on the ideal $\mathfrak{B}^s_{\mathcal{K}}$, where $\Lambda(W) = \mu((s_a(W))_{a=0}^\infty)$, $G:\mathfrak{B}^s_{\mathcal{K}}(\mathcal{N},\mathcal{M})\to\mathfrak{B}^s_{\mathcal{K}}(\mathcal{N},\mathcal{M})$ and $B\in\mathfrak{B}^s_{\mathcal{K}}(\mathcal{N},\mathcal{M})$. The mapping G is termed Λ-sequentially continuous at B, if and only if, when $\lim_{p\to\infty}\Lambda(W_p-B)=0$, then $\lim_{p\to\infty}\Lambda(GW_p-GB)=0$.

Definition 2.20 ([34]) If $\vartheta=(\vartheta_j)\in\mathfrak{C}^{\mathbb{Z}^+}$ and $\mathcal{K}_\mu$ is a prequasinormed $\mathfrak{pss}$. The mapping $L_\vartheta:\mathcal{K}_\mu\to\mathcal{K}_\mu$ is termed a multiplication mapping on $\mathcal{K}_\mu$, if $L_\vartheta\lambda=(\vartheta_j\lambda_j)\in\mathcal{K}_\mu$, with $\lambda\in\mathcal{K}_\mu$. The multiplication mapping is termed created by ϑ, if $L_\vartheta\in\mathfrak{B}(\mathcal{K}_\mu)$.

Theorem 2.21 ([35]) *Suppose s-type* $\mathcal{K}_\mu := \{\lambda=(s_j(X))\in \mathrm{R}^{\mathbb{Z}^+} : X\in\mathfrak{B}(\mathcal{N},\mathcal{M}) \text{and} \mu(\lambda)<\infty\}$. *If* $\mathfrak{B}^s_{\mathcal{K}_\mu}$ *is a mappings ideal, then the following conditions are satisfied*:

1. $\mathrm{F}\subset$ *s-type* $\mathcal{K}_\mu$.
2. *Let* $(s_j(X_1))_{j=0}^\infty\in$ *s-type* $\mathcal{K}_\mu$ *and* $(s_j(X_2))_{j=0}^\infty\in$ *s-type* $\mathcal{K}_\mu$, *then* $(s_j(X_1+X_2))_{j=0}^\infty\in$ *s-type* $\mathcal{K}_\mu$.
3. *Suppose* $\varepsilon\in\mathfrak{C}$ *and* $(s_j(X))_{j=0}^\infty\in$ *s-type* $\mathcal{K}_\mu$, *then* $|\varepsilon|(s_j(X))_{j=0}^\infty\in$ *s-type* $\mathcal{K}_\mu$.
4. *The sequence space* $\mathcal{K}_\mu$ *is solid, i.e., if* $(s_j(Y))_{j=0}^\infty\in$ *s-type* $\mathcal{K}_\mu$ *and* $s_j(X)\le s_j(Y)$, *for every* $j\in\mathbb{Z}^+$ *and* $X,Y\in\mathfrak{B}(\mathcal{N},\mathcal{M})$, *then* $(s_j(X))_{j=0}^\infty\in$ *s-type* $\mathcal{K}_\mu$.

3 The sequence space $(\Pi(\zeta,\eta))_\mu$

In this section, the definition and some inclusion relations of the sequence space $(\Pi(\zeta,\eta))_\mu$ equipped with the function μ are considered.

Definition 3.1 Let $(\zeta_j),(\eta_j)\in(0,\infty)^{\mathbb{Z}^+}$. The sequence space $(\Pi(\zeta,\eta))_\mu$ with the function μ is defined by: $(\Pi(\zeta,\eta))_\mu=\{\lambda=(\lambda_j)\in\mathfrak{C}^{\mathbb{Z}^+}:\mu(\rho\lambda)<\infty, \text{for some } \rho>0\}$, where $\mu(\lambda)=\sum_{j=0}^\infty(\zeta_j|\sum_{l=0}^j\lambda_l|)^{\eta_j}$.

Theorem 3.2 *Let* $(\eta_j)\in(0,\infty)^{\mathbb{Z}^+}\cap\ell_\infty$, *then we have*

$$\big(\Pi(\zeta,\eta)\big)_\mu=\big\{\lambda=(\lambda_j)\in\mathfrak{C}^{\mathbb{Z}^+}:\mu(\rho\lambda)<\infty, \textit{for any } \rho>0\big\}.$$

Proof Suppose $(\eta_j) \in (0,\infty)^{\mathbb{Z}^+} \cap \ell_\infty$, we have

$$\begin{aligned}
\big(\Pi(\zeta,\eta)\big)_\mu &= \big\{\lambda = (\lambda_j) \in \mathfrak{C}^{\mathbb{Z}^+} : \mu(\rho\lambda) < \infty, \text{for some } \rho > 0\big\} \\
&= \left\{\lambda = (\lambda_j) \in \mathfrak{C}^{\mathbb{Z}^+} : \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\rho\lambda_l\right|\right)^{\eta_j} < \infty, \text{for some } \rho > 0\right\} \\
&= \left\{\lambda = (\lambda_j) \in \mathfrak{C}^{\mathbb{Z}^+} : \inf_j \rho^{\eta_j}\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l\right|\right)^{\eta_j} < \infty, \text{for some } \rho > 0\right\} \\
&= \left\{\lambda = (\lambda_j) \in \mathfrak{C}^{\mathbb{Z}^+} : \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l\right|\right)^{\eta_j} < \infty\right\} \\
&= \big\{\lambda = (\lambda_j) \in \mathfrak{C}^{\mathbb{Z}^+} : \mu(\rho\lambda) < \infty, \text{for any } \rho > 0\big\}.
\end{aligned}$$

□

Remark 3.3 Assume $\eta_j = \eta$, $\zeta_j = \frac{1}{j+1}$, for every $j \in \mathbb{Z}^+$ and $\eta \geq 1$, then $\Pi(\zeta,\eta) = \mathrm{ces}^\eta$, as defined and studied by Ng and Lee [36].

Theorem 3.4 *If* $(\eta_j) \in (0,\infty)^{\mathbb{Z}^+} \cap \ell_\infty$, *one has that* $(\Pi(\zeta,\eta))_\mu$ *is a nonabsolute type.*

Proof Let $\lambda = (1,-1,0,0,0,\dots)$, then $|\lambda| = (1,1,0,0,0,\dots)$. One has

$$\mu(\lambda) = \zeta_0^{\eta_0} \neq \zeta_0^{\eta_0} + (2\zeta_1)^{\eta_1} + (2\zeta_2)^{\eta_2} + \dots = \mu\big(|\lambda|\big).$$

Therefore, the sequence space $(\Pi(\zeta,\eta))_\mu$ is a nonabsolute type. □

Definition 3.5 ([37]) Assume $(\zeta_j),(\eta_j) \in (0,\infty)^{\mathbb{Z}^+}$. The generalized Nakano sequence space, $(\ell(\zeta,\eta))_\varphi$, is defined as: $(\ell(\zeta,\eta))_\varphi = \{\lambda = (\lambda_j) \in \mathfrak{C}^{\mathbb{Z}^+} : \varphi(\rho\lambda) < \infty, \text{for some} \rho > 0\}$, where $\varphi(\lambda) = \sum_{j=0}^{\infty}(\zeta_j \sum_{l=0}^{j} |\lambda_l|)^{\eta_j}$.

Theorem 3.6 *If* $(\zeta_j),(\eta_j) \in (0,\infty)^{\mathbb{Z}^+} \cap \ell_\infty$ *with* $(\zeta_j) \in \ell_{((\eta_j))}$ *and* $((j+1)\zeta_j) \notin \ell_{(\eta_j)}$, *one has* $(\ell(\zeta,\eta))_\varphi \subsetneqq (\Pi(\zeta,\eta))_\mu$.

Proof Suppose $\lambda \in (\ell(\zeta,\eta))_\varphi$, as

$$\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l\right|\right)^{\eta_j} \leq \sum_{j=0}^{\infty}\left(\zeta_j\sum_{l=0}^{j}|\lambda_l|\right)^{\eta_j} < \infty.$$

Therefore, $\lambda \in (\Pi(\zeta,\eta))_\mu$. We take $\beta = ((-1)^j)_{j\in\mathbb{Z}^+}$, we have $\beta \in (\Pi(\zeta,\eta))_\mu$ and $\beta \notin (\ell(\zeta,\eta))_\varphi$. □

4 Premodular private sequence space

In this section, we explain the enough setups on $\Pi(\zeta,\eta)$ with definite function μ to be premodular $\mathfrak{pss}$. This implies that $\Pi(\zeta,\eta)$ is a prequasinormed $\mathfrak{pss}$.

Theorem 4.1 $\Pi(\zeta,\eta)$ *is a* $\mathfrak{pss}$, *if the next setups are verified*:

($f1$) $(\eta_j) \in \Im_\nearrow \cap \ell_\infty$ *with* $\eta_0 > 0$.
($f2$) $(\zeta_j)_{j=0}^{\infty} \in (0,\infty)^{\mathbb{Z}^+} \cap \ell_{((\eta_j))}$.

Proof (1-i) Assume $\lambda, \beta \in \Pi(\zeta,\eta)$. One obtains

$$\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l+\beta_l\right|\right)^{\eta_j} \le 2^{\hbar-1}\left(\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l\right|\right)^{\eta_j}+\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\beta_l\right|\right)^{\eta_j}\right)<\infty,$$

hence, $\lambda+\beta \in \Pi(\zeta,\eta)$.

(1-ii) Let $\rho \in \mathfrak{C}$, $\lambda \in \Pi(\zeta,\eta)$ and as $(\eta_j) \in \Im_{\nearrow}\cap \ell_\infty$, we have

$$\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\rho\lambda_l\right|\right)^{\eta_j} \le \sup_j |\rho|^{\eta_j}\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l\right|\right)^{\eta_j}<\infty.$$

Then, $\rho\lambda \in \Pi(\zeta,\eta)$. From setups (1-i) and (1-ii), we have that $\Pi(\zeta,\eta)$ is a linear space.

As $(\eta_j) \in \Im_{\nearrow}\cap \ell_\infty$ and $\eta_0>0$, one has

$$\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}(e_b)_l\right|\right)^{\eta_j}=\sum_{j=b}^{\infty}\zeta_j^{\eta_j}<\infty.$$

Therefore, $e_b \in \Pi(\zeta,\eta)$, for every $b \in \mathbb{Z}^+$.

(2) Suppose $|\lambda_b| \le |\beta_b|$, with $b \in \mathbb{Z}^+$ and $|\beta| \in \Pi(\zeta,\eta)$. One has

$$\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}|\lambda_l|\right|\right)^{\eta_j} \le \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}|\beta_l|\right|\right)^{\eta_j}<\infty,$$

hence $|\lambda| \in \Pi(\zeta,\eta)$.

(3) Let $(|\lambda_j|) \in \Pi(\zeta,\eta)$, with $(\eta_j) \in \Im_{\nearrow}\cap \ell_\infty$, we have

$$\begin{aligned}
&\sum_{j=0}^{\infty}\left(\zeta_j\sum_{l=0}^{j}|\lambda_{[\frac{l}{2}]}|\right)^{\eta_j}\\
&\quad=\sum_{j=0}^{\infty}\left(\zeta_{2j}\sum_{l=0}^{2j}|\lambda_{[\frac{l}{2}]}|\right)^{\eta_{2j}}+\sum_{j=0}^{\infty}\left(\zeta_{2j+1}\sum_{l=0}^{2j+1}|\lambda_{[\frac{l}{2}]}|\right)^{\eta_{2j+1}}\\
&\quad\le\sum_{j=0}^{\infty}\left(\zeta_j\left(|\lambda_j|+\sum_{l=0}^{j}2|\lambda_l|\right)\right)^{\eta_j}+\sum_{j=0}^{\infty}\left(\zeta_j\left(\sum_{l=0}^{j}2|\lambda_l|\right)\right)^{\eta_j}\\
&\quad\le 2^{\hbar-1}\left(\sum_{j=0}^{\infty}\left(\zeta_j\sum_{l=0}^{j}|\lambda_l|\right)^{\eta_j}+\sum_{j=0}^{\infty}\left(2\zeta_j\sum_{l=0}^{j}|\lambda_l|\right)^{\eta_j}\right)+\sum_{j=0}^{\infty}\left(2\zeta_j\sum_{l=0}^{j}|\lambda_l|\right)^{\eta_j}\\
&\quad\le\left(2^{2\hbar-1}+2^{\hbar-1}+2^{\hbar}\right)\sum_{j=0}^{\infty}\left(\zeta_j\sum_{l=0}^{j}|\lambda_l|\right)^{\eta_j}<\infty,
\end{aligned}$$

therefore $(|\lambda_{[\frac{j}{2}]}|) \in \Pi(\zeta,\eta)$. □

In view of Theorem 2.9, we conclude the next Theorem.

Theorem 4.2 *Suppose the setups* $(f1)$ *and* $(f2)$ *are verified, one has* $\mathfrak{B}^s_{\Pi(\zeta,\eta)}$ *is a mappings ideal.*

Theorem 4.3 $(\Pi(\zeta,\eta))_\mu$ *is a premodular* $\mathfrak{pss}$, *if the setups* $(f1)$ *and* $(f2)$ *are confirmed.*

Proof (i) Definitely, $\mu(\lambda) \geq 0$ and $\mu(|\lambda|) = 0 \Leftrightarrow \lambda = \theta$.

(ii) There are $E_0 = \max\{1, \sup_j |\rho|^{\eta_j - 1}\} \geq 1$ with $\mu(\rho\lambda) \leq E_0|\rho|\mu(\lambda)$, for each $\lambda \in \Pi(\zeta,\eta)$ and $\rho \in \mathfrak{C}$.

(iii) The inequality $\mu(\lambda + \beta) \leq 2^{\hbar-1}(\mu(\lambda) + \mu(\beta))$ explains this, with $\lambda, \beta \in \Pi(\zeta,\eta)$.

(iv) Clearly, from the proof part (2) of Theorem 4.1.

(v) Obviously, from the proof part (3) of Theorem $D_0 \geq 2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar} \geq 1$.

(vi) It is clear that $\overline{\mathrm{F}} = \Pi(\zeta,\eta)$.

(vii) One has $0 < \varpi \leq \sup_j |\rho|^{\eta_j-1}$ with $\mu(\rho,0,0,0,\ldots) \geq \varpi|\rho|\mu(1,0,0,0,\ldots)$, for each $\rho \neq 0$ and $\varpi > 0$, if $\rho = 0$. □

Theorem 4.4 *Assume the setups* $(f1)$ *and* $(f2)$ *are satisfied, then* $(\Pi(\zeta,\eta))_\mu$ *is a prequasi-Banach* $\mathfrak{pss}$.

Proof According to Theorem 4.3, the space $(\Pi(\zeta,\eta))_\mu$ is a premodular $\mathfrak{pss}$. According to Theorem 2.12, the space $(\Pi(\zeta,\eta))_\mu$ is a prequasinormed $\mathfrak{pss}$. To show that $(\Pi(\zeta,\eta))_\mu$ is a prequasi-Banach $\mathfrak{pss}$, assume $\lambda^a = (\lambda_j^a)_{j=0}^\infty$ is a Cauchy sequence in $(\Pi(\zeta,\eta))_\mu$, one has for all $\varepsilon \in (0,1)$ that there is $a_0 \in \mathbb{Z}^+$ so that for all $a, b \geq a_0$, one has

$$\mu\left(\lambda^a - \lambda^b\right) = \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l^a - \lambda_l^b\right|\right)^{\eta_j} < \varepsilon^{\hbar}.$$

Hence, for $a, b \geq a_0$ and $j \in \mathbb{Z}^+$, we have $|\lambda_j^a - \lambda_j^b| < \varepsilon$. Hence, (λ_j^b) is a Cauchy sequence in $\mathfrak{C}$, for fixed $j \in \mathbb{Z}^+$, which gives $\lim_{b\to\infty}\lambda_j^b = \lambda_j^0$, for fixed $j \in \mathbb{Z}^+$. Therefore, $\mu(\lambda^a - \lambda^0) < \varepsilon^{\hbar}$, for all $a \geq a_0$. Moreover, to show that $\lambda^0 \in (\Pi(\zeta,\eta))_\mu$, one obtains $\mu(\lambda^0) \leq 2^{\hbar-1}(\mu(\lambda^a - \lambda^0) + \mu(\lambda^a)) < \infty$, then $\lambda^0 \in (\Pi(\zeta,\eta))_\mu$, which implies that $(\Pi(\zeta,\eta))_\mu$ is a prequasi-Banach $\mathfrak{pss}$. □

In view of Theorem 2.21, we conclude the following behaviors of the s-type $(\Pi(\zeta,\eta))_\mu$.

Theorem 4.5 *Let* s*-type* $(\Pi(\zeta,\eta))_\mu := \{\lambda = (s_j(V)) \in \mathrm{R}^{\mathbb{Z}^+} : V \in \mathfrak{B}(\mathcal{N},\mathcal{M}) and \mu(\lambda) < \infty\}$. *The next conditions are confirmed*:

1. *One has* s*-type* $(\Pi(\zeta,\eta))_\mu \supset \mathrm{F}$.
2. *Suppose* $(s_j(V_1))_{j=0}^\infty \in s$*-type* $(\Pi(\zeta,\eta))_\mu$ *and* $(s_j(V_2))_{j=0}^\infty \in s$*-type* $(\Pi(\zeta,\eta))_\mu$, *then* $(s_j(V_1 + V_2))_{j=0}^\infty \in s$*-type* $(\Pi(\zeta,\eta))_\mu$.
3. *For every* $r \in \mathfrak{C}$ *and* $(s_j(V))_{j=0}^\infty \in s$*-type* $(\Pi(\zeta,\eta))_\mu$, *then* $|r|(s_j(V))_{j=0}^\infty \in s$*-type* $(\Pi(\zeta,\eta))_\mu$.
4. *The* s*-type* $(\Pi(\zeta,\eta))_\mu$ *is solid.*

5 Multiplication mappings on $(\Pi(\zeta,\eta))_\mu$

In this section, we perform the multiplication mapping on $\mathfrak{pss}$, $(\Pi(\zeta,\eta))_\mu$, and explain the necessity and enough setups on $(\Pi(\zeta,\eta))_\mu$ so that the multiplication mapping is bounded, invertible, approximable, Fredholm, and closed range.

Theorem 5.1 *Fix* $\vartheta \in \mathfrak{C}^{\mathbb{Z}^+}$, *the setups* $(f1)$ *and* $(f2)$ *are satisfied, and one has*

$$\vartheta \in \ell_\infty \quad \Longleftrightarrow \quad L_\vartheta \in \mathfrak{B}\left(\left(\Pi(\zeta,\eta)\right)_\mu\right).$$

Proof Let $\vartheta \in \ell_\infty$. Hence, there is $\nu > 0$ so that $|\vartheta_j| \le \nu$, for every $j \in \mathbb{Z}^+$. Assume $\lambda \in (\Pi(\zeta,\eta))_\mu$, one obtains

$$\begin{aligned}\mu(L_\vartheta \lambda) &= \mu(\vartheta\lambda)\\ &= \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\vartheta_l\lambda_l\right|\right)^{\eta_j}\\ &\le \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\nu\lambda_l\right|\right)^{\eta_j}\\ &\le \sup_j \nu^{\eta_j}\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l\right|\right)^{\eta_j}\\ &= \sup_j \nu^{\eta_j}\mu(\lambda).\end{aligned}$$

Therefore, $L_\vartheta \in \mathfrak{B}((\Pi(\zeta,\eta))_\mu)$.

Next, assume $L_\vartheta \in \mathfrak{B}((\Pi(\zeta,\eta))_\mu)$ and $\vartheta \notin \ell_\infty$. Hence, for all $b \in \mathbb{Z}^+$, there are $j_b \in \mathbb{Z}^+$ so that $\vartheta_{j_b} > b$. We have

$$\mu(L_\vartheta e_{j_b}) = \mu(\vartheta e_{j_b}) = \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\vartheta_l(e_{j_b})_l\right|\right)^{\eta_j} = \sum_{j=j_b}^{\infty}\left(\zeta_j|\vartheta_{j_b}|\right)^{\eta_j} > \sum_{j=j_b}^{\infty}(\zeta_j b)^{\eta_j} > b^{\eta_0}\mu(e_{j_b}).$$

Then, $L_\vartheta \notin \mathfrak{B}((\Pi(\zeta,\eta))_\mu)$. Hence, $\vartheta \in \ell_\infty$. □

Theorem 5.2 *Suppose $\vartheta \in \mathfrak{C}^{\mathbb{Z}^+}$ and $(\Pi(\zeta,\eta))_\mu$ is a prequasinormed $\mathfrak{pss}$. Hence $\vartheta_j = g$, for every $j \in \mathbb{Z}^+$ and $g \in \mathfrak{C}$ with $|g| = 1$, if and only if, L_ϑ is an isometry.*

Proof Let the enough setup be confirmed. One obtains

$$\mu(L_\vartheta \lambda) = \mu(\vartheta\lambda) = \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{k=0}^{j}\vartheta_k\lambda_k\right|\right)^{\eta_j} = \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{k=0}^{j}|g|\lambda_k\right|\right)^{\eta_j} = \mu(\lambda),$$

for every $\lambda \in (\Pi(\zeta,\eta))_\mu$. Therefore, L_ϑ is an isometry.

Suppose the necessity setup is verified and $|\vartheta_j| < 1$, for some $j = j_0$. We obtain

$$\mu(L_\vartheta e_{j_0}) = \mu(\vartheta e_{j_0}) = \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{k=0}^{j}\vartheta_k(e_{j_0})_k\right|\right)^{\eta_j} = \sum_{j=j_0}^{\infty}\left(\zeta_j|\vartheta_{j_0}|\right)^{\eta_j} < \sum_{j=j_0}^{\infty}\zeta_j^{\eta_j} = \mu(e_{j_0}).$$

Also, when $|\vartheta_{j_0}| > 1$, obviously, $\mu(L_\vartheta e_{j_0}) > \mu(e_{j_0})$. This gives a contradiction for the two cases. Therefore, $|\vartheta_j| = 1$, for all $j \in \mathbb{Z}^+$. □

Theorem 5.3 *Assume $\vartheta \in \mathfrak{C}^{\mathbb{Z}^+}$, the setups $(f1)$ and $(f2)$ are confirmed. Hence, $L_\vartheta \in \mathfrak{A}((\Pi(\zeta,\eta))_\mu)$, if and only if, $(\vartheta_b)_{b=0}^{\infty} \in c_0$.*

Proof Let $L_\vartheta \in \mathfrak{A}((\Pi(\zeta,\eta))_\mu)$, then $L_\vartheta \in \mathfrak{K}((\Pi(\zeta,\eta))_\mu)$. Assume $\lim_{j\to\infty}\vartheta_j \ne 0$. Therefore, we have $\varrho > 0$ such that the set $K_\varrho = \{j \in \mathbb{Z}^+ : |\vartheta_j| \ge \varrho\} \nsubseteq \mathfrak{I}$. Let $\{\alpha_j\}_{j\in\mathbb{Z}^+} \subset K_\varrho$. Hence,

$\{e_{\alpha_j} : \alpha_j \in K_\varrho\} \in \ell_\infty$ is an infinite set in $(\Pi(\zeta,\eta))_\mu$. Since

$$\begin{aligned}\mu(L_\vartheta e_{\alpha_a} - L_\vartheta e_{\alpha_b}) = \mu(\vartheta e_{\alpha_a} - \vartheta e_{\alpha_b}) &= \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{k=0}^{j}\vartheta_k\bigl((e_{\alpha_a})_k - (e_{\alpha_b})_k\bigr)\right|\right)^{\eta_j} \\ &\geq \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{k=0}^{j}\varrho\bigl((e_{\alpha_a})_k - (e_{\alpha_b})_k\bigr)\right|\right)^{\eta_j} \\ &\geq \inf_j \varrho^{\eta_j}\mu(e_{\alpha_a} - e_{\alpha_b}),\end{aligned}$$

for every $\alpha_a, \alpha_b \in K_\varrho$. Therefore, $\{e_{\alpha_b} : \alpha_b \in K_\varrho\} \in \ell_\infty$, which cannot have a convergent subsequence under L_ϑ. Hence, $L_\vartheta \notin \mathfrak{K}((\Pi(\zeta,\eta))_\mu)$. This gives $L_\vartheta \notin \mathfrak{A}((\Pi(\zeta,\eta))_\mu)$, hence suggesting a contradiction. Hence, $\lim_{j\to\infty}\vartheta_j = 0$. On the contrary, assume $\lim_{j\to\infty}\vartheta_j = 0$. Then, for all $\varrho > 0$, one has $K_\varrho = \{j \in \mathbb{Z}^+ : |\vartheta_j| \geq \varrho\} \subset \mathfrak{I}$. Then, for every $\varrho > 0$, we obtain $\dim(((\Pi(\zeta,\eta))_\mu)_{K_\varrho}) = \dim(\mathfrak{C}^{K_\varrho}) < \infty$. Hence, $L_\vartheta \in \mathfrak{F}(((\Pi(\zeta,\eta))_\mu)_{K_\varrho})$. Let $\vartheta_a \in \mathfrak{C}^{\mathbb{Z}^+}$, for every $a \in \mathbb{Z}^+$, where

$$(\vartheta_a)_b = \begin{cases}\vartheta_b, & b \in K_{\frac{1}{a+1}}, \\ 0, & \text{otherwise.}\end{cases}$$

Obviously, $L_{\vartheta_a} \in \mathfrak{F}(((\Pi(\zeta,\eta))_\mu)_{B_{\frac{1}{a+1}}})$ such as $\dim(((\Pi(\zeta,\eta))_\mu)_{B_{\frac{1}{a+1}}}) < \infty$, for all $a \in \mathbb{Z}^+$. Since $(\eta_j) \in \Im_\nearrow \cap \ell_\infty$ with $\eta_0 > 0$, we obtain

$$\begin{aligned}&\mu\bigl((L_\vartheta - L_{\vartheta_a})\lambda\bigr) \\ &\quad= \mu\Bigl(\bigl(\bigl(\vartheta_b - (\vartheta_a)_b\bigr)\lambda_b\bigr)_{b=0}^{\infty}\Bigr) \\ &\quad= \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{b=0}^{j}\bigl(\vartheta_b - (\vartheta_a)_b\bigr)\lambda_b\right|\right)^{\eta_j} \\ &\quad= \sum_{j=0, j\in K_{\frac{1}{a+1}}}^{\infty}\left(\zeta_j\left|\sum_{b=0}^{j}\bigl(\vartheta_b - (\vartheta_a)_b\bigr)\lambda_b\right|\right)^{\eta_j} + \sum_{j=0, j\notin K_{\frac{1}{a+1}}}^{\infty}\left(\zeta_j\left|\sum_{b=0}^{j}\bigl(\vartheta_b - (\vartheta_a)_b\bigr)\lambda_b\right|\right)^{\eta_j} \\ &\quad= \sum_{j=0, j\notin K_{\frac{1}{a+1}}}^{\infty}\left(\zeta_j\left|\sum_{b=0}^{j}\vartheta_b\lambda_b\right|\right)^{\eta_j} \leq \frac{1}{(a+1)^{\eta_0}}\sum_{j=0, j\notin K_{\frac{1}{a+1}}}^{\infty}\left(\zeta_j\left|\sum_{b=0}^{j}\lambda_b\right|\right)^{\eta_j} \\ &\quad< \frac{1}{(a+1)^{\eta_0}}\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{b=0}^{j}\lambda_b\right|\right)^{\eta_j} = \frac{1}{(a+1)^{\eta_0}}\mu(\lambda).\end{aligned}$$

Hence, $\|L_\vartheta - L_{\vartheta_a}\| \leq \frac{1}{(a+1)^{\eta_0}}$, which explains L_ϑ is a limit of finite-rank mappings. Then, $L_\vartheta \in \mathfrak{A}((\Pi(\zeta,\eta))_\mu)$. □

Theorem 5.4 *Suppose $\vartheta \in \mathfrak{C}^{\mathbb{Z}^+}$, the setups $(f1)$ and $(f2)$ are verified. Hence, $L_\vartheta \in \mathfrak{K}((\Pi(\zeta,\eta))_\mu)$, if and only if, $(\vartheta_j)_{j=0}^{\infty} \in c_0$.*

Proof Obviously, since $\mathfrak{A}((\Pi(\zeta,\eta))_\mu) \subsetneqq \mathfrak{K}((\Pi(\zeta,\eta))_\mu)$. □

Corollary 5.5 *If the setups* $(f1)$ *and* $(f2)$ *are confirmed, then* $\mathfrak{K}((\Pi(\zeta,\eta))_\mu) \subsetneqq \mathfrak{B}((\Pi(\zeta,\eta))_\mu)$.

Proof As $\vartheta = (1,1,\dots)$ creates the multiplication mapping I on $(\Pi(\zeta,\eta))_\mu$, which gives $I \notin \mathfrak{K}((\Pi(\zeta,\eta))_\mu)$ and $I \in \mathfrak{B}((\Pi(\zeta,\eta))_\mu)$. □

Theorem 5.6 *If* $(\Pi(\zeta,\eta))_\mu$ *is a prequasi-Banach* $\mathfrak{pss}$ *and* $L_\vartheta \in \mathfrak{B}((\Pi(\zeta,\eta))_\mu)$. *Hence, there is* $p > 0$ *and* $q > 0$ *such that* $p < |\vartheta_j| < q$, *with* $j \in (\ker(\vartheta))^c$, *if and only if,* $\mathrm{Range}(L_\vartheta)$ *is closed.*

Proof Suppose the enough set-up is verified. Hence, there is $\varrho > 0$ so that $|\vartheta_j| \geq \varrho$, for all $j \in (\ker(\vartheta))^c$. To show that $\mathrm{Range}(L_\vartheta)$ is closed, assume g is a limit point of $\mathrm{Range}(L_\vartheta)$. We have $L_\vartheta\lambda_j \in (\Pi(\zeta,\eta))_\mu$, for every $j \in \mathbb{Z}^+$ so that $\lim_{j\to\infty} L_\vartheta\lambda_j = g$. Obviously, the sequence $L_\vartheta\lambda_j$ is a Cauchy sequence. As $(\eta_j) \in \Im_\nearrow \cap \ell_\infty$ with $\eta_0 > 0$, one has

$$\begin{aligned}
&\mu(L_\vartheta\lambda_a - L_\vartheta\lambda_b)\\
&\quad= \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{k=0}^{j}\bigl(\vartheta_k(\lambda_a)_k - \vartheta_k(\lambda_b)_k\bigr)\right|\right)^{\eta_j}\\
&\quad= \sum_{j=0, j\in(\ker(\vartheta))^c}^{\infty}\left(\zeta_j\left|\sum_{k=0}^{j}\bigl(\vartheta_k(\lambda_a)_k - \vartheta_k(\lambda_b)_k\bigr)\right|\right)^{\eta_j}\\
&\qquad+ \sum_{j=0, j\notin(\ker(\vartheta))^c}^{\infty}\left(\zeta_j\left|\sum_{k=0}^{j}\bigl(\vartheta_k(\lambda_a)_k - \vartheta_k(\lambda_b)_k\bigr)\right|\right)^{\eta_j}\\
&\quad\geq \sum_{j=0, j\in(\ker(\vartheta))^c}^{\infty}\left(\zeta_j\left|\sum_{k=0}^{j}\bigl(\vartheta_k(\lambda_a)_k - \vartheta_k(\lambda_b)_k\bigr)\right|\right)^{\eta_j}\\
&\quad= \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{k=0}^{j}\bigl(\vartheta_k(u_a)_k - \vartheta_k(u_b)_k\bigr)\right|\right)^{\eta_j}\\
&\quad> \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{k=0}^{j}\varrho\bigl((u_a)_k - (u_b)_k\bigr)\right|\right)^{\eta_j} \geq \inf_j \varrho^{\eta_j}\mu(u_a - u_b),
\end{aligned}$$

where

$$(u_a)_j = \begin{cases} (\lambda_a)_j, & j \in (\ker(\vartheta))^c,\\ 0, & j \notin (\ker(\vartheta))^c. \end{cases}$$

Hence, $\{u_a\}$ is a Cauchy sequence in $(\Pi(\zeta,\eta))_\mu$. As $(\Pi(\zeta,\eta))_\mu$ is complete, there is $\lambda \in (\Pi(\zeta,\eta))_\mu$ so that $\lim_{j\to\infty} u_j = \lambda$. Since $L_\vartheta \in \mathfrak{B}((\Pi(\zeta,\eta))_\mu)$, we have $\lim_{j\to\infty} L_\vartheta u_j = L_\vartheta\lambda$. However, $\lim_{j\to\infty} L_\vartheta u_j = \lim_{j\to\infty} L_\vartheta\lambda_j = g$. Then, $L_\vartheta\lambda = g$. Hence, $g \in \mathrm{Range}(L_\vartheta)$. Hence, $\mathrm{Range}(L_\vartheta)$ is closed. Then, assume the necessity setup is satisfied. Hence, there is $\varrho > 0$ so that $\mu(L_\vartheta\lambda) \geq \varrho\mu(\lambda)$, with $\lambda \in ((\Pi(\zeta,\eta))_\mu)_{(\ker(\vartheta))^c}$. If $K = \{j \in (\ker(\vartheta))^c : |\vartheta_j| < \varrho\} \neq \emptyset$, then for $a_0 \in K$, one obtains

$$\begin{aligned}\mu(L_\vartheta e_{a_0}) &= \mu\left(\left(\vartheta_b(e_{a_0})_b\right)_{b=0}^{\infty}\right)\\ &= \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{b=0}^{j}\vartheta_b(e_{a_0})_b\right|\right)^{\eta_j}\\ &< \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{b=0}^{j}(e_{a_0})_b\varrho\right|\right)^{\eta_j} \le \sup_j \varrho^{\eta_j}\mu(e_{a_0}),\end{aligned}$$

which gives a contradiction. Hence, $K = \phi$, and we have $|\vartheta_j| \ge \varrho$, with $j \in (\ker(\vartheta))^c$. This proves the theorem. □

Theorem 5.7 *Assume $\vartheta \in \mathfrak{C}^{\mathbb{Z}^+}$ and $(\Pi(\zeta,\eta))_\mu$ is a prequasi-Banach $\mathfrak{pss}$. Hence, there are $p > 0$ and $q > 0$ so that $p < |\vartheta_j| < q$, for all $j \in \mathbb{Z}^+$, if and only if, $L_\vartheta \in \mathfrak{B}((\Pi(\zeta,\eta))_\mu)$ is invertible.*

Proof Assume the enough setup is verified. Let $\kappa \in \mathfrak{C}^{\mathbb{Z}^+}$ with $\kappa_j = \frac{1}{\vartheta_j}$. In view of Theorem 5.1, the mappings L_ϑ and L_κ are bounded linear. Hence, $L_\vartheta.L_\kappa = L_\kappa.L_\vartheta = I$. Hence, $L_\kappa = L_\vartheta^{-1}$. Now, assume L_ϑ is invertible. Hence, $\mathrm{Range}(L_\vartheta) = ((\Pi(\zeta,\eta))_\mu)_{\mathbb{Z}^+}$. Hence, $\mathrm{Range}(L_\vartheta)$ is closed. From Theorem 5.6, there is $p > 0$ so that $|\vartheta_j| \ge p$, for every $j \in (\ker(\vartheta))^c$. We have $\ker(\vartheta) = \emptyset$, when $\vartheta_{j_0} = 0$, with $j_0 \in \mathbb{Z}^+$, which explains $e_{j_0} \in \ker(L_\vartheta)$, this gives an inconsistency, as $\ker(L_\vartheta)$ is trivial. Therefore, $|\vartheta_j| \ge p$, for every $j \in \mathbb{Z}^+$. Since $L_\vartheta \in \ell_\infty$. From Theorem 5.1, there is $q > 0$ so that $|\vartheta_j| \le q$, for every $j \in \mathbb{Z}^+$. Therefore, one has $p \le |\vartheta_j| \le q$, with $j \in \mathbb{Z}^+$. □

Theorem 5.8 *Let $(\Pi(\zeta,\eta))_\mu$ be a prequasi-Banach $\mathfrak{pss}$ and $L_\vartheta \in \mathfrak{B}((\Pi(\zeta,\eta))_\mu)$. Hence, L_ϑ is a Fredholm mapping, if and only if,* (i) $\ker(\vartheta) \subsetneqq \mathbb{Z}^+$ *is finite and* (ii) $|\vartheta_j| \ge \varrho$, *with $j \in (\ker(\vartheta))^c$.*

Proof Suppose the enough condition is confirmed. Assume $\ker(\vartheta) \subsetneqq \mathbb{Z}^+$ is infinite, hence $e_j \in \ker(L_\vartheta)$, for every $j \in \ker(\vartheta)$. Since e_js are linearly independent, one obtains that $\dim(\ker(L_\vartheta)) = \infty$, which suggests an inconsistency. Hence, $\ker(\vartheta) \subsetneqq \mathbb{Z}^+$ must be finite. The condition (ii) follows from Theorem 5.6. Next, suppose the conditions (i) and (ii) are verified. From Theorem 5.6, the condition (ii) implies that $\mathrm{Range}(L_\vartheta)$ is closed. The condition (i) gives that $\dim(\ker(L_\vartheta)) < \infty$ and $\dim((\mathrm{Range}(L_\vartheta))^c) < \infty$. Therefore, L_ϑ is Fredholm. □

6 Prequasi ideal behavior

In this section, first we investigate the enough (not necessary) setups on $(\Pi(\zeta,\eta))_\mu$ such that $\overline{\mathfrak{F}} = \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$. This gives a negative answer to Rhoades [31] open problem about the linearity of s-type $(\Pi(\zeta,\eta))_\mu$ spaces. Secondly, we ask for which conditions on $(\Pi(\zeta,\eta))_\mu$, are $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$ complete and closed? Thirdly, we explain the enough setups on $(\Pi(\zeta,\eta))_\mu$ such that $\mathfrak{B}^\alpha_{(\Pi(\zeta,\eta))_\mu}$ is strictly contained for different weights and powers. We offer the setups so that $\mathfrak{B}^\alpha_{(\Pi(\zeta,\eta))_\mu}$ is minimum. Fourthly, we explain the conditions so that the class $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$ is simple. Fifthly, we introduce the enough conditions on $(\Pi(\zeta,\eta))_\mu$ such that the space of all bounded linear mappings which sequence of eigenvalues in $(\Pi(\zeta,\eta))_\mu$ equals $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$.

6.1 Finite rank prequasi ideal

Theorem 6.1 $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M}) = \overline{\mathfrak{F}(\mathcal{N},\mathcal{M})}$, *if the setups* $(f1)$ *and* $(f2)$ *are verified. But the converse is not necessarily true.*

Proof To show that $\overline{\mathfrak{F}(\mathcal{N},\mathcal{M})} \subseteq \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$. As $e_j \in (\Pi(\zeta,\eta))_\mu$, for every $j \in \mathbb{Z}^+$ and $(\Pi(\zeta,\eta))_\mu$ is a linear space. Suppose $Z \in \mathfrak{F}(\mathcal{N},\mathcal{M})$, one has $(s_j(Z))_{j=0}^\infty \in$ F. To show that $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M}) \subseteq \overline{\mathfrak{F}(\mathcal{N},\mathcal{M})}$, assume $Z \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$, we have $(s_j(Z))_{j=0}^\infty \in (\Pi(\zeta,\eta))_\mu$. As $\mu(s_j(Z))_{j=0}^\infty < \infty$, assume $\rho \in (0,1)$, then there is $q_0 \in \mathbb{Z}^+ - \{0\}$ with $\mu((s_q(Z))_{q=q_0}^\infty) < \frac{\rho}{2^{\hbar+3}\eta d}$, for some $d \geq 1$, where $\eta = \max\{1, \sum_{q=q_0}^\infty \zeta_q^{\eta_q}\}$. Since $s_j(Z)$ is decreasing, we have

$$\begin{aligned}\sum_{q=q_0+1}^{2q_0}\left(\zeta_q\sum_{j=0}^{q}s_{2q_0}(Z)\right)^{\eta_q} &\leq \sum_{q=q_0+1}^{2q_0}\left(\zeta_q\sum_{j=0}^{q}s_j(Z)\right)^{\eta_q} \\ &\leq \sum_{q=q_0}^{\infty}\left(\zeta_q\sum_{j=0}^{q}s_j(Z)\right)^{\eta_q} < \frac{\rho}{2^{\hbar+3}\eta d}. \end{aligned} \tag{2}$$

Hence, there is $Y \in \mathfrak{F}_{2q_0}(\mathcal{N},\mathcal{M})$ so that $\operatorname{rank}(Y) \leq 2q_0$ and

$$\sum_{q=2q_0+1}^{3q_0}\left(\zeta_q\sum_{j=0}^{q}\|Z-Y\|\right)^{\eta_q} \leq \sum_{q=q_0+1}^{2q_0}\left(\zeta_q\sum_{j=0}^{q}\|Z-Y\|\right)^{\eta_q} < \frac{\rho}{2^{\hbar+3}\eta d}, \tag{3}$$

since $(\eta_q) \in \Im_{\nearrow} \cap \ell_\infty$, we have

$$\sup_{q=q_0}^{\infty}\left(\sum_{j=0}^{q_0}\|Z-Y\|\right)^{\eta_q} < \frac{\rho}{2^{2\hbar+2}\eta}. \tag{4}$$

Therefore, one has

$$\sum_{q=0}^{q_0}\left(\zeta_q\sum_{j=0}^{q}\|Z-Y\|\right)^{\eta_q} < \frac{\rho}{2^{\hbar+3}\eta d}. \tag{5}$$

In view of inequalities (1)–(5), one has

$$\begin{aligned} d(Z,Y) &= \mu\big(s_j(Z-Y)\big)_{j=0}^\infty \\ &= \sum_{q=0}^{3q_0-1}\left(\zeta_q\sum_{j=0}^{q}s_j(Z-Y)\right)^{\eta_q} + \sum_{q=3q_0}^{\infty}\left(\zeta_q\sum_{j=0}^{q}s_j(Z-Y)\right)^{\eta_q} \\ &\leq \sum_{q=0}^{3q_0}\left(\zeta_q\sum_{j=0}^{q}\|Z-Y\|\right)^{\eta_q} + \sum_{q=q_0}^{\infty}\left(\zeta_{q+2q_0}\sum_{j=0}^{q+2q_0}s_j(Z-Y)\right)^{\eta_{q+2q_0}} \\ &\leq \sum_{q=0}^{3q_0}\left(\zeta_q\sum_{j=0}^{q}\|Z-Y\|\right)^{\eta_q} + \sum_{q=q_0}^{\infty}\left(\zeta_q\sum_{j=0}^{q+2q_0}s_j(Z-Y)\right)^{\eta_q} \\ &\leq 3\sum_{q=0}^{q_0}\left(\zeta_q\sum_{j=0}^{i}\|Z-Y\|\right)^{\eta_q} + \sum_{q=q_0}^{\infty}\left(\zeta_q\left(\sum_{j=0}^{2q_0-1}s_j(Z-Y) + \sum_{j=2q_0}^{q+2q_0}s_j(Z-Y)\right)\right)^{\eta_q} \end{aligned}$$

$$
\begin{aligned}
&\le 3\sum_{q=0}^{q_0}\left(\zeta_q\sum_{j=0}^{q}\|Z-Y\|\right)^{\eta_q} \\
&\quad + 2^{\hbar-1}\left[\sum_{q=q_0}^{\infty}\left(\zeta_q\sum_{j=0}^{2q_0-1}s_j(Z-Y)\right)^{\eta_q} + \sum_{q=q_0}^{\infty}\left(\zeta_q\sum_{j=2q_0}^{q+2q_0}s_j(Z-Y)\right)^{\eta_q}\right] \\
&\le 3\sum_{q=0}^{q_0}\left(\zeta_q\sum_{j=0}^{q}\|Z-Y\|\right)^{\eta_q} \\
&\quad + 2^{\hbar-1}\left[\sum_{q=q_0}^{\infty}\left(\zeta_q\sum_{j=0}^{2q_0-1}\|Z-Y\|\right)^{\eta_q} + \sum_{q=q_0}^{\infty}\left(\zeta_q\sum_{j=0}^{q}s_{j+2q_0}(Z-Y)\right)^{\eta_q}\right] \\
&\le 3\sum_{q=0}^{q_0}\left(\zeta_q\sum_{j=0}^{q}\|Z-Y\|\right)^{\eta_q} + 2^{\hbar-1}\sup_{q=q_0}^{\infty}\left(\sum_{j=0}^{2q_0-1}\|Z-Y\|\right)^{\eta_q}\sum_{q=q_0}^{\infty}\zeta_q^{\eta_q} \\
&\quad + 2^{\hbar-1}\sum_{q=q_0}^{\infty}\left(\zeta_q\sum_{j=0}^{q}s_j(Z)\right)^{\eta_q} < \rho.
\end{aligned}
$$

Contrarily, one has a counter example as $I_4 \in \mathfrak{B}^s_{(\Xi(\zeta,(0)))_\mu}(\mathcal{N},\mathcal{M})$, but $\eta_0 > 0$ is not verified. This implies the proof. □

6.2 Banach and closed prequasi ideal

Theorem 6.2 *Let the setups* $(f1)$ *and* $(f2)$ *be confirmed, hence* $(\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu},\Lambda)$ *is a prequasi-Banach ideal, where* $\Lambda(X) = \mu((s_j(X))_{j=0}^\infty)$.

Proof As $(\Pi(\zeta,\eta))_\mu$ is a premodular $\mathfrak{pss}$, from Theorem 2.13, Λ is a prequasinorm on $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$. Assume $(X_j)_{j\in\mathbb{Z}^+}$ is a Cauchy sequence in $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$. As $\mathfrak{B}(\mathcal{N},\mathcal{M}) \supseteq \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$, one has

$$
\Lambda(X_a - X_b) = \sum_{j=0}^{\infty}\left(\zeta_j\sum_{l=0}^{j}s_l(X_a - X_b)\right)^{\eta_j} \ge \left(\zeta_0\|X_a - X_b\|\right)^{\eta_0},
$$

hence $(X_b)_{b\in\mathbb{Z}^+}$ is a Cauchy sequence in $\mathfrak{B}(\mathcal{N},\mathcal{M})$. Since $\mathfrak{B}(\mathcal{N},\mathcal{M})$ is a Banach space, then there is $X \in \mathfrak{B}(\mathcal{N},\mathcal{M})$ with $\lim_{b\to\infty}\|X_b - X\| = 0$. Since $(s_j(X_b))_{j=0}^\infty \in (\Pi(\zeta,\eta))_\mu$, for every $b \in \mathbb{Z}^+$. In view of Definition 2.10, setups (ii), (iii), and (v), one obtains

$$
\begin{aligned}
\Lambda(X) &= \sum_{j=0}^{\infty}\left(\zeta_j\sum_{l=0}^{j}s_l(X)\right)^{\eta_j} \\
&\le 2^{\hbar-1}\sum_{j=0}^{\infty}\left(\zeta_j\sum_{l=0}^{j}s_{[\frac{l}{2}]}(X - X_b)\right)^{\eta_j} + 2^{\hbar-1}\sum_{j=0}^{\infty}\left(\zeta_j\sum_{l=0}^{j}s_{[\frac{l}{2}]}(X_b)\right)^{\eta_j} \\
&\le 2^{\hbar-1}\sum_{j=0}^{\infty}\left(\zeta_j\sum_{l=0}^{j}\|X - X_b\|\right)^{\eta_j} + 2^{\hbar-1}D_0\sum_{j=0}^{\infty}\left(\zeta_j\sum_{l=0}^{j}s_l(X_b)\right)^{\eta_j} < \infty.
\end{aligned}
$$

Therefore, $(s_j(X))_{j=0}^\infty \in (\Pi(\zeta,\eta))_\mu$, then $X \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$. □

Theorem 6.3 *Assume $\mathcal{N}$ and $\mathcal{M}$ are normed spaces, the setups $(f1)$ and $(f2)$ are verified, hence $(\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}, \Lambda)$ is a prequasi closed ideal, where $\Lambda(X) = \mu((s_j(X))_{j=0}^\infty)$.*

Proof As $(\Pi(\zeta,\eta))_\mu$ is a premodular $\mathfrak{pss}$, by using Theorem 2.13, then Λ is a prequasinorm on $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$. Suppose $X_b \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$, for every $b \in \mathbb{Z}^+$ and $\lim_{b\to\infty} \Lambda(X_b - X) = 0$. As $\mathfrak{B}(\mathcal{N},\mathcal{M}) \supseteq \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$, we have

$$\Lambda(X - X_b) = \sum_{j=0}^{\infty}\left(\zeta_j \sum_{l=0}^{j} s_l(X - X_b)\right)^{\eta_j} \geq \left(\zeta_0\|X - X_b\|\right)^{\eta_0},$$

hence $(X_b)_{b\in\mathbb{Z}^+}$ is a convergent sequence in $\mathfrak{B}(\mathcal{N},\mathcal{M})$. Since $(s_j(X_b))_{j=0}^\infty \in (\Pi(\zeta,\eta))_\mu$, for every $b \in \mathbb{Z}^+$. By using Definition 2.10, setups (ii), (iii), and (v), one obtains

$$\begin{aligned}
\Lambda(X) &= \sum_{j=0}^{\infty}\left(\zeta_j \sum_{l=0}^{j} s_l(X)\right)^{\eta_j} \\
&\leq 2^{\hbar-1}\sum_{j=0}^{\infty}\left(\zeta_j \sum_{l=0}^{j} s_{[\frac{l}{2}]}(X - X_b)\right)^{\eta_j} + 2^{\hbar-1}\sum_{j=0}^{\infty}\left(\zeta_j \sum_{l=0}^{j} s_{[\frac{l}{2}]}(X_b)\right)^{\eta_j} \\
&\leq 2^{\hbar-1}\sum_{j=0}^{\infty}\left(\zeta_j \sum_{l=0}^{j} \|X - X_b\|\right)^{\eta_j} + 2^{\hbar-1}D_0\sum_{j=0}^{\infty}\left(\zeta_j \sum_{l=0}^{j} s_l(X_b)\right)^{\eta_j} < \infty.
\end{aligned}$$

Then, $(s_j(X))_{j=0}^\infty \in (\Pi(\zeta,\eta))_\mu$, and hence $X \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$. □

6.3 Minimum prequasi ideal

Theorem 6.4 *Suppose $\mathcal{N}$ and $\mathcal{M}$ are Banach spaces with $\dim(\mathcal{N}) = \dim(\mathcal{M}) = \infty$, and the setups $(f1)$ and $(f2)$ are confirmed with $0 < \eta_j^{(1)} < \eta_j^{(2)}$ and $0 < \zeta_j^{(2)} \leq \zeta_j^{(1)}$, for all $j \in \mathbb{Z}^+$, then*

$$\mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M}) \subsetneqq \mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M}) \subsetneqq \mathfrak{B}(\mathcal{N},\mathcal{M}).$$

Proof Let $Z \in \mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M})$, hence $(s_j(Z)) \in (\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu$. One obtains

$$\sum_{j=0}^{\infty}\left(\zeta_j^{(2)} \sum_{q=0}^{j} s_q(Z)\right)^{\eta_j^{(2)}} < \sum_{j=0}^{\infty}\left(\zeta_j^{(1)} \sum_{q=0}^{j} s_q(Z)\right)^{\eta_j^{(1)}} < \infty,$$

then $Z \in \mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M})$. Now, if we choose $(s_j(Z))_{j=0}^\infty$ with $\sum_{q=0}^{j} s_q(Z) = \frac{1}{\zeta_j^{(1)} \sqrt[\eta_j^{(1)}]{j+1}}$, we have $Z \in \mathfrak{B}(\mathcal{N},\mathcal{M})$ such that

$$\sum_{j=0}^{\infty}\left(\zeta_j^{(1)} \sum_{q=0}^{j} s_q(Z)\right)^{\eta_j^{(1)}} = \sum_{j=0}^{\infty}\frac{1}{j+1} = \infty,$$

and

$$\sum_{j=0}^{\infty}\left(\zeta_j^{(2)}\sum_{q=0}^{j}s_q(Z)\right)^{\eta_j^{(2)}} \leq \sum_{j=0}^{\infty}\left(\zeta_j^{(1)}\sum_{q=0}^{j}s_q(Z)\right)^{\eta_j^{(2)}} = \sum_{j=0}^{\infty}\left(\frac{1}{j+1}\right)^{\frac{\eta_j^{(2)}}{\eta_j^{(1)}}} < \infty.$$

Then, $Z \notin \mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M})$ and $Z \in \mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M})$.

Clearly, $\mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M}) \subset \mathfrak{B}(\mathcal{N},\mathcal{M})$. Next, if we put $(s_j(Z))_{j=0}^\infty$ such that $\sum_{q=0}^{j} s_q(Z) = \frac{1}{\zeta_j^{(2)}\ \eta_j^{(2)}\sqrt{j+1}}$. We have $Z \in \mathfrak{B}(\mathcal{N},\mathcal{M})$ such that $Z \notin \mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M})$. This finishes the proof. □

Theorem 6.5 *Assume $\mathcal{N}$ and $\mathcal{M}$ are Banach spaces with* $\dim(\mathcal{N}) = \dim(\mathcal{M}) = \infty$, *and the setups* $(f1)$ *and* $(f2)$ *are satisfied with* $((j+1)\zeta_j)_{j\in\mathbb{Z}^+} \notin \ell_{((\eta_j))}$, *hence* $\mathfrak{B}^\alpha_{(\Pi(\zeta,\eta))_\mu}$ *is minimum.*

Proof Assume the enough setups are confirmed. Then, $(\mathfrak{B}^\alpha_{\Pi(\zeta,\eta)},\Lambda)$, where $\Lambda(Z) = \sum_{j=0}^{\infty}(\zeta_j\sum_{q=0}^{j}\alpha_q(Z))^{\eta_j}$, is a prequasi-Banach ideal. Suppose $\mathfrak{B}^\alpha_{\Pi(\zeta,\eta)}(\mathcal{N},\mathcal{M}) = \mathfrak{B}(\mathcal{N},\mathcal{M})$, then there is $\eta > 0$ with $\Lambda(Z) \leq \eta\|Z\|$, for every $Z \in \mathfrak{B}(\mathcal{N},\mathcal{M})$. By using Dvoretzky's theorem [38], for every $b \in \mathbb{Z}^+$, one has quotient spaces $\mathcal{N}/Y_b$ and subspaces M_b of $\mathcal{M}$ that can be mapped onto ℓ_2^b by isomorphisms V_b and X_b with $\|V_b\|\|V_b^{-1}\| \leq 2$ and $\|X_b\|\|X_b^{-1}\| \leq 2$. If I_b is the identity mapping on ℓ_2^b, T_b is the quotient mapping from $\mathcal{N}$ onto $\mathcal{N}/Y_b$, and J_b is the natural embedding mapping from M_b into $\mathcal{M}$. Suppose m_q is the Bernstein number [4] then

$$\begin{aligned}
1 &= m_q(I_b) = m_q\left(X_bX_b^{-1}I_bV_bV_b^{-1}\right)\\
&\leq \|X_b\|m_q\left(X_b^{-1}I_bV_b\right)\left\|V_b^{-1}\right\|\\
&= \|X_b\|m_q\left(J_bX_b^{-1}I_bV_b\right)\left\|V_b^{-1}\right\|\\
&\leq \|X_b\|d_q\left(J_bX_b^{-1}I_bV_b\right)\left\|V_b^{-1}\right\|\\
&= \|X_b\|d_q\left(J_bX_b^{-1}I_bV_bT_b\right)\left\|V_b^{-1}\right\|\\
&\leq \|X_b\|\alpha_q\left(J_bX_b^{-1}I_bV_bT_b\right)\left\|V_b^{-1}\right\|,
\end{aligned}$$

for $0 \leq j \leq b$. We have

$$\begin{aligned}
&\zeta_j(j+1) \leq \zeta_j\sum_{q=0}^{j}\|X_b\|\alpha_q\left(J_bX_b^{-1}I_bV_bT_b\right)\left\|V_b^{-1}\right\|\\
&\quad\Rightarrow\quad \left(\zeta_j(j+1)\right)^{\eta_j} \leq \left(\|X_b\|\left\|V_b^{-1}\right\|\right)^{\eta_j}\left(\zeta_j\sum_{q=0}^{j}\alpha_q\left(J_bX_b^{-1}I_bV_bT_b\right)\right)^{\eta_j}.
\end{aligned}$$

Hence, for some $\varrho \geq 1$, one obtains

$$\begin{aligned}
&\sum_{j=0}^{b}\left(\zeta_j(j+1)\right)^{\eta_j} \leq \varrho\|X_b\|\left\|V_b^{-1}\right\|\sum_{j=0}^{b}\left(\zeta_j\sum_{q=0}^{j}\alpha_q\left(J_bX_b^{-1}I_bV_bT_b\right)\right)^{\eta_j}\\
&\quad\Rightarrow\quad \sum_{j=0}^{b}\left(\zeta_j(j+1)\right)^{\eta_j} \leq \varrho\|X_b\|\left\|V_b^{-1}\right\|\Lambda\left(J_bX_b^{-1}I_bV_bT_b\right)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad \sum_{j=0}^{b} \bigl(\zeta_j(j+1)\bigr)^{\eta_j} \le \varrho\eta \|X_b\| \bigl\| V_b^{-1} \bigr\| \bigl\| J_b X_b^{-1} I_b V_b T_b \bigr\| \\
&\Rightarrow \quad \sum_{j=0}^{b} \bigl(\zeta_j(j+1)\bigr)^{\eta_j} \le \varrho\eta \|X_b\| \bigl\| V_b^{-1} \bigr\| \bigl\| J_b X_b^{-1} \bigr\| \|I_b\| \|V_b T_b\| \\
&\qquad\qquad = \varrho\eta \|X_b\| \bigl\| V_b^{-1} \bigr\| \bigl\| X_b^{-1} \bigr\| \|I_b\| \|V_b\| \le 4\varrho\eta.
\end{aligned}$$

Hence, there is a contradiction, if $b \to \infty$. Therefore, $\mathcal{N}$ and $\mathcal{M}$ both cannot be infinite dimensional if $\mathfrak{B}^{\alpha}_{\Pi(\zeta,\eta)}(\mathcal{N},\mathcal{M}) = \mathfrak{B}(\mathcal{N},\mathcal{M})$. This completes the proof. □

Theorem 6.6 *Suppose $\mathcal{N}$ and $\mathcal{M}$ are Banach spaces with $\dim(\mathcal{N}) = \dim(\mathcal{M}) = \infty$, and the setups $(f1)$ and $(f2)$ are confirmed with $(\zeta_j(j+1))_{j\in\mathbb{Z}^+} \notin \ell_{((\eta_j))}$, then $\mathfrak{B}^{d}_{\Pi(\zeta,\eta)}$ is minimum.*

6.4 Simple Banach prequasi ideal

Theorem 6.7 *Let $\mathcal{N}$ and $\mathcal{M}$ be Banach spaces with $\dim(\mathcal{N}) = \dim(\mathcal{M}) = \infty$, and the setups $(f1)$ and $(f2)$ be confirmed with $0 < \eta_j^{(1)} < \eta_j^{(2)}$ and $0 < \zeta_j^{(2)} \le \zeta_j^{(1)}$, for all $j \in \mathbb{Z}^+$, hence*

$$\begin{aligned}
&\mathfrak{B}\bigl(\mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M}), \mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M})\bigr) \\
&\quad = \mathfrak{A}\bigl(\mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M}), \mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M})\bigr).
\end{aligned}$$

Proof Assume $X \in \mathfrak{B}(\mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M}), \mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M}))$ and $X \notin \mathfrak{A}(\mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M}), \mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M}))$. By using Lemma 2.1, we have $Y \in \mathfrak{B}(\mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M}))$ and $Z \in \mathfrak{B}(\mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M}))$ with $ZXYI_b = I_b$. Hence, for every $b \in \mathbb{Z}^+$, we obtain

$$\begin{aligned}
\|I_b\|_{\mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M})} &= \sum_{j=0}^{\infty} \left(\zeta_j^{(1)} \sum_{q=0}^{j} s_q(I_b) \right)^{\eta_j^{(1)}} \\
&\le \|ZXY\| \|I_b\|_{\mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M})} \\
&\le \sum_{j=0}^{\infty} \left(\zeta_j^{(2)} \sum_{q=0}^{j} s_q(I_j) \right)^{\eta_j^{(2)}}.
\end{aligned}$$

This fails Theorem 6.4. Hence, $X \in \mathfrak{A}(\mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M}), \mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M}))$, which completes the proof. □

Corollary 6.8 *Assume $\mathcal{N}$ and $\mathcal{M}$ are Banach spaces with $\dim(\mathcal{N}) = \dim(\mathcal{M}) = \infty$, and the setups $(f1)$ and $(f2)$ are satisfied with $0 < \eta_j^{(1)} < \eta_j^{(2)}$ and $0 < \zeta_j^{(2)} \le \zeta_j^{(1)}$, for all $j \in \mathbb{Z}^+$, then*

$$\begin{aligned}
&\mathfrak{B}\bigl(\mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M}), \mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M})\bigr) \\
&\quad = \mathfrak{K}\bigl(\mathfrak{B}^s_{(\Xi((\zeta_j^{(2)}),(\eta_j^{(2)})))_\mu}(\mathcal{N},\mathcal{M}), \mathfrak{B}^s_{(\Xi((\zeta_j^{(1)}),(\eta_j^{(1)})))_\mu}(\mathcal{N},\mathcal{M})\bigr).
\end{aligned}$$

Proof Evidently, as $\mathfrak{A} \subset \mathfrak{K}$. □

Theorem 6.9 *Let $\mathcal{N}$ and $\mathcal{M}$ be Banach spaces with* $\dim(\mathcal{N}) = \dim(\mathcal{M}) = \infty$, *and the setups* $(f1)$ *and* $(f2)$ *are satisfied, hence* $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$ *is simple.*

Proof Assume the closed ideal $\mathfrak{K}(\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M}))$ includes a mapping $X \notin \mathfrak{A}(\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M}))$. From Lemma 2.1, there exist $Y, Z \in \mathfrak{B}(\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M}))$ with $ZXYI_b = I_b$, which gives that $I_{\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})} \in \mathfrak{K}(\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M}))$. Then, $\mathfrak{B}(\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})) = \mathfrak{K}(\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M}))$. Hence, $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$ is a simple Banach space. □

6.5 Eigenvalues of s-type mappings

Notation 6.10

$$\begin{aligned}
&(\mathfrak{B}^s_{\mathcal{K}})^\rho := \{(\mathfrak{B}^s_{\mathcal{K}})^\rho(\mathcal{N},\mathcal{M}); \mathcal{N} \text{ and } \mathcal{M} \text{ are Banach Spaces}\}, \quad \text{where}\\
&(\mathfrak{B}^s_{\mathcal{K}})^\rho(\mathcal{N},\mathcal{M}) := \{X \in \mathfrak{B}(\mathcal{N},\mathcal{M}) : ((\rho_j(X))_{j=0}^\infty \in \mathcal{K} \text{ and } \|X - \rho_j(X)I\| \\
&\qquad\qquad \text{is not invertible, for all } j \in \mathbb{Z}^+\}.
\end{aligned}$$

Theorem 6.11 *Assume $\mathcal{N}$ and $\mathcal{M}$ are Banach spaces with* $\dim(\mathcal{N}) = \dim(\mathcal{M}) = \infty$, *and the setups* $(f1)$ *and* $(f2)$ *are confirmed with* $\inf_j(\zeta_j(j+1))^{\eta_j} > 0$, *hence*

$$(\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu})^\rho(\mathcal{N},\mathcal{M}) = \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M}).$$

Proof Suppose $X \in (\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu})^\rho(\mathcal{N},\mathcal{M})$, hence $(\rho_j(X))_{j=0}^\infty \in (\Pi(\zeta,\eta))_\mu$ and $\|X - \rho_j(X)I\| = 0$, for all $j \in \mathbb{Z}^+$. We have $X = \rho_j(X)I$, for all $j \in \mathbb{Z}^+$, so $s_j(X) = s_j(\rho_j(X)I) = |\rho_j(X)|$, for every $j \in \mathbb{Z}^+$. Therefore, $(s_j(X))_{j=0}^\infty \in (\Pi(\zeta,\eta))_\mu$, hence $X \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$. Next, suppose $X \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$. Then, $(s_j(X))_{j=0}^\infty \in (\Pi(\zeta,\eta))_\mu$. Hence, we have

$$\sum_{j=0}^\infty \left(\zeta_j \sum_{q=0}^j s_q(X)\right)^{\eta_j} \ge \inf_j(\zeta_j(j+1))^{\eta_j} \sum_{j=0}^\infty [s_j(X)]^{\eta_j}.$$

Then, $\lim_{j\to\infty} s_j(X) = 0$. Assume $\|X - s_j(X)I\|^{-1}$ exists, for every $j \in \mathbb{Z}^+$. Hence, $\|X - s_j(X)I\|^{-1}$ exists and is bounded, for every $j \in \mathbb{Z}^+$. Then, $\lim_{j\to\infty}\|X - s_j(X)I\|^{-1} = \|X\|^{-1}$ exists and is bounded. As $(\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}, \Lambda)$ is a prequasi-operator ideal, we have

$$I = XX^{-1} \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M}) \quad\Rightarrow\quad (s_j(I))_{j=0}^\infty \in \Pi(\zeta,\eta) \quad\Rightarrow\quad \lim_{j\to\infty} s_j(I) = 0.$$

This gives a contradiction, as $\lim_{j\to\infty} s_j(I) = 1$. Hence $\|X - s_j(X)I\| = 0$, for every $j \in \mathbb{Z}^+$, which explains $X \in (\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu})^\rho(\mathcal{N},\mathcal{M})$. This completes the proof. □

7 Kannan contraction mapping

Theorem 7.1 *Suppose the setups* $(f1)$ *and* $(f2)$ *are confirmed, then the function* $\mu(\lambda) = [\sum_{j=0}^\infty(\zeta_j|\sum_{l=0}^j \lambda_l|)^{\eta_j}]^{\frac{1}{\hbar}}$ *satisfies the Fatou property, for all* $\lambda \in \Pi(\zeta,\eta)$.

Proof Assume $\{\beta^b\} \subseteq (\Pi(\zeta,\eta))_\mu$ with $\lim_{b\to\infty}\mu(\beta^b-\beta)=0$. As the space $(\Pi(\zeta,\eta))_\mu$ is a prequasiclosed space, then $\beta \in (\Pi(\zeta,\eta))_\mu$. Hence, for all $\lambda \in (\Pi(\zeta,\eta))_\mu$, we have

$$\begin{aligned}\mu(\lambda-\beta) &= \left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l-\beta_l\right|\right)^{\eta_j}\right]^{\frac{1}{\hbar}}\\ &\le \left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l-\beta_l^b\right|\right)^{\eta_j}\right]^{\frac{1}{\hbar}} + \left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\beta_l^b-\beta_l\right|\right)^{\eta_j}\right]^{\frac{1}{\hbar}}\\ &\le \sup_j \inf_{b\ge j}\mu\big(\lambda-\beta^b\big).\end{aligned}$$

□

Theorem 7.2 *Suppose the setups $(f1)$ and $(f2)$ are confirmed with $\eta_j > 1$, for all $j \in \mathbb{Z}^+$, then the function $\mu(\lambda)=\sum_{j=0}^{\infty}(\zeta_j|\sum_{l=0}^{j}\lambda_l|)^{\eta_j}$ does not satisfy the Fatou property, for every $\lambda \in \Pi(\zeta,\eta)$.*

Proof Assume $\{\beta^b\} \subseteq (\Pi(\zeta,\eta))_\mu$ with $\lim_{b\to\infty}\mu(\beta^b-\beta)=0$. As the space $(\Pi(\zeta,\eta))_\mu$ is a prequasiclosed space, then $\beta \in (\Pi(\zeta,\eta))_\mu$. Hence, for all $\lambda \in (\Pi(\zeta,\eta))_\mu$, we have

$$\begin{aligned}\mu(\lambda-\beta) &= \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l-\beta_l\right|\right)^{\eta_j}\\ &\le 2^{\hbar-1}\left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\lambda_l-\beta_l^b\right|\right)^{\eta_j} + \sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\beta_l^b-\beta_l\right|\right)^{\eta_j}\right]\\ &\le 2^{\hbar-1}\sup_j \inf_{b\ge j}\mu\big(\lambda-\beta^b\big).\end{aligned}$$

Therefore, μ does not satisfy the Fatou property.

Next, we offer enough setups on $(\Pi(\zeta,\eta))_\mu$ equipped with μ so that there is only a fixed point of the Kannan contraction mapping. □

Theorem 7.3 *Suppose the setups $(f1)$ and $(f2)$ are satisfied, and $W:(\Pi(\zeta,\eta))_\mu \to (\Pi(\zeta,\eta))_\mu$ is a Kannan μ-contraction mapping, where $\mu(\lambda)=[\sum_{j=0}^{\infty}(\zeta_j|\sum_{l=0}^{j}\lambda_l|)^{\eta_j}]^{\frac{1}{\hbar}}$, for every $\lambda \in \Pi(\zeta,\eta)$, then W has a unique fixed point.*

Proof Let $\lambda \in \Pi(\zeta,\eta)$, then $W^r\lambda \in \Pi(\zeta,\eta)$. Since W is a Kannan μ-contraction mapping, we have

$$\begin{aligned}&\mu\big(W^{r+1}\lambda-W^r\lambda\big) \le \beta\big(\mu\big(W^{r+1}\lambda-W^r\lambda\big)+\mu\big(W^r\lambda-W^{r-1}\lambda\big)\big)\\ &\quad\Rightarrow\quad \mu\big(W^{r+1}\lambda-W^r\lambda\big) \le \frac{\beta}{1-\beta}\mu\big(W^r\lambda-W^{r-1}\lambda\big)\\ &\qquad\qquad\qquad\qquad\qquad \le \left(\frac{\beta}{1-\beta}\right)^2\mu\big(W^{r-1}\lambda-W^{r-2}\lambda\big) \le \cdots\\ &\qquad\qquad\qquad\qquad\qquad \le \left(\frac{\beta}{1-\beta}\right)^r\mu(W\lambda-\lambda).\end{aligned}$$

Therefore, for every $r,q \in \mathbb{Z}^+$ with $q > r$, we obtain

$$\begin{aligned}\mu\left(W^r\lambda - W^q\lambda\right) &\leq \beta\left(\mu\left(W^r\lambda - W^{r-1}\lambda\right) + \mu\left(W^q\lambda - W^{q-1}\lambda\right)\right) \\ &\leq \beta\left(\left(\frac{\beta}{1-\beta}\right)^{r-1} + \left(\frac{\beta}{1-\beta}\right)^{q-1}\right)\mu(W\lambda - \lambda).\end{aligned}$$

Hence, $\{W^r\lambda\}$ is a Cauchy sequence in $(\Pi(\zeta,\eta))_\mu$. Since the space $(\Pi(\zeta,\eta))_\mu$ is a prequasi-Banach space. Then, there exists $g \in (\Pi(\zeta,\eta))_\mu$ so that $\lim_{r\to\infty} W^r\lambda = g$. To show that $Wg = g$, as μ has the Fatou property, we obtain

$$\mu(Wg - g) \leq \sup_i \inf_{r\geq i} \mu\left(W^{r+1}\lambda - W^r\lambda\right) \leq \sup_i \inf_{r\geq i}\left(\frac{\beta}{1-\beta}\right)^r \mu(W\lambda - \lambda) = 0,$$

hence $Wg = g$. So, g is a fixed point of W. To prove that the fixed point is unique, suppose we have two different fixed points $b,g \in (\Pi(\zeta,\eta))_\mu$ of W. Then, one has

$$\mu(b-g) \leq \mu(Wb - Wg) \leq \xi\left(\mu(Wb-b) + \mu(Wg-g)\right) = 0.$$

Therefore, $b = g$. □

Corollary 7.4 *Assume the setups $(f1)$ and $(f2)$ are confirmed, and $W : (\Pi(\zeta,\eta))_\mu \to (\Pi(\zeta,\eta))_\mu$ is a Kannan μ-contraction mapping, where $\mu(\lambda) = [\sum_{j=0}^{\infty}(\zeta_j|\sum_{l=0}^{j}\lambda_l|)^{\eta_j}]^{\frac{1}{\hbar}}$, for all $\lambda \in \Pi(\zeta,\eta)$, then W has a unique fixed point b with $\mu(W^r\lambda - b) \leq \beta(\frac{\beta}{1-\beta})^{r-1}\mu(W\lambda - \lambda)$.*

Proof By using Theorem 7.3, there is a unique fixed point b of W. Then, one has

$$\begin{aligned}\mu\left(W^r\lambda - b\right) &= \mu\left(W^r\lambda - Wb\right) \\ &\leq \beta\left(\mu\left(W^r\lambda - W^{r-1}\lambda\right) + \mu(Wb-b)\right) \\ &= \beta\left(\frac{\beta}{1-\beta}\right)^{r-1}\mu(W\lambda - \lambda).\end{aligned}$$

□

Theorem 7.5 *If the setups $(f1)$ and $(f2)$ are satisfied with $\eta_j > 1$, for all $j \in \mathbb{Z}^+$, and $W : (\Pi(\zeta,\eta))_\mu \to (\Pi(\zeta,\eta))_\mu$, where $\mu(\lambda) = \sum_{j=0}^{\infty}(\zeta_j|\sum_{l=0}^{j}\lambda_l|)^{\eta_j}$, for all $\lambda \in \Pi(\zeta,\eta)$. The point $g \in (\Pi(\zeta,\eta))_\mu$ is the unique fixed point of W, if the next setups are verified*:

(a) *W is a Kannan μ-contraction mapping,*
(b) *W is μ-sequentially continuous at $g \in (\Pi(\zeta,\eta))_\mu$,*
(c) *we have $\lambda \in (\Pi(\zeta,\eta))_\mu$ such that the sequence of iterates $\{W^r\lambda\}$ has a subsequence $\{W^{r_i}\lambda\}$ that converges to g.*

Proof Assume the enough setups are confirmed. Let g be not a fixed point of W, then $Wg \neq g$. By using the setups (b) and (c), one has

$$\lim_{r_i\to\infty}\mu\left(W^{r_i}\lambda - g\right) = 0 \quad \text{and} \quad \lim_{r_i\to\infty}\mu\left(W^{r_i+1}\lambda - Wg\right) = 0.$$

Since the mapping W is a Kannan μ-contraction, we have

$$0 < \mu(Wg - g)$$

$$\begin{aligned} &= \mu\big(\big(Wg - W^{r_i+1}\lambda\big) + \big(W^{r_i}\lambda - g\big) + \big(W^{r_i+1}\lambda - W^{r_i}\lambda\big)\big) \\ &\leq 2^{2\hbar-2}\mu\big(W^{r_i+1}\lambda - Wg\big) + 2^{2\hbar-2}\mu\big(W^{r_i}\lambda - g\big) + 2^{\hbar-1}\beta\left(\frac{\beta}{1-\beta}\right)^{r_i-1}\mu(W\lambda - \lambda). \end{aligned}$$

Since $r_i \to \infty$, one has a contradiction. Hence, g is a fixed point of W. To show that the fixed point g is unique, assume one has two different fixed points $g, b \in (\Pi(\zeta, \eta))_\mu$ of W. Therefore, we have

$$\mu(g-b) \leq \mu(Wg - Wb) \leq \beta\big(\mu(Wg-g) + \mu(Wb-b)\big) = 0.$$

Hence, $g = b$. □

Example 7.6 If $T : (\Pi((\frac{1}{m+5})_{m=0}^\infty, (\frac{2m+3}{m+2})_{m=0}^\infty))_\mu \to (\Pi((\frac{1}{m+5})_{m=0}^\infty, (\frac{2m+3}{m+2})_{m=0}^\infty))_\mu$, where $\mu(p) = \sqrt{\sum_{m=0}^\infty (\frac{|\sum_{j=0}^m p_j|}{m+5})^{\frac{2m+3}{m+2}}}$, with $p \in \Pi((\frac{m+2}{m+1})_{m=0}^\infty, (\frac{2m+3}{m+2})_{m=0}^\infty)$ and

$$T(p) = \begin{cases} \frac{p}{4}, & \mu(p) \in [0,1), \\ \frac{p}{5}, & \mu(p) \in [1,\infty). \end{cases}$$

Since for all $p, q \in (\Pi((\frac{1}{m+5})_{m=0}^\infty, (\frac{2m+3}{m+2})_{m=0}^\infty))_\mu$ with $\mu(p), \mu(q) \in [0,1)$, we have

$$\begin{aligned} \mu(Tp - Tq) &= \mu\left(\frac{p}{4} - \frac{q}{4}\right) \\ &\leq \frac{1}{\sqrt[4]{27}}\left(\mu\left(\frac{3p}{4}\right) + \mu\left(\frac{3q}{4}\right)\right) = \frac{1}{\sqrt[4]{27}}\big(\mu(Tp-p) + \mu(Tq-q)\big). \end{aligned}$$

For all $p, q \in (\Pi((\frac{1}{m+5})_{m=0}^\infty, (\frac{2m+3}{m+2})_{m=0}^\infty))_\mu$ with $\mu(p), \mu(q) \in [1,\infty)$, we obtain

$$\begin{aligned} \mu(Tp - Tq) &= \mu\left(\frac{p}{5} - \frac{q}{5}\right) \\ &\leq \frac{1}{\sqrt[4]{64}}\left(\mu\left(\frac{4p}{5}\right) + \mu\left(\frac{4q}{5}\right)\right) = \frac{1}{\sqrt[4]{64}}\big(\mu(Tp-p) + \mu(Tq-q)\big). \end{aligned}$$

For every $p, q \in (\Pi((\frac{1}{m+5})_{m=0}^\infty, (\frac{2m+3}{m+2})_{m=0}^\infty))_\mu$ with $\mu(p) \in [0,1)$ and $\mu(q) \in [1,\infty)$, we obtain

$$\begin{aligned} \mu(Tp - Tq) &= \mu\left(\frac{p}{4} - \frac{q}{5}\right) \\ &\leq \frac{1}{\sqrt[4]{27}}\mu\left(\frac{3p}{4}\right) + \frac{1}{\sqrt[4]{64}}\mu\left(\frac{4q}{5}\right) \\ &\leq \frac{1}{\sqrt[4]{27}}\left(\mu\left(\frac{3p}{4}\right) + \mu\left(\frac{4q}{5}\right)\right) \\ &= \frac{1}{\sqrt[4]{27}}\big(\mu(Tp-p) + \mu(Tq-q)\big). \end{aligned}$$

Therefore, the mapping T is a Kannan μ-contraction. Since μ satisfies the Fatou property, by using Theorem 7.3, the mapping T has a unique fixed point $\theta \in (\Pi((\frac{1}{m+5})_{m=0}^\infty, (\frac{2m+3}{m+2})_{m=0}^\infty))_\mu$.

Suppose $\{p^{(a)}\} \subseteq (\Pi((\frac{1}{m+5})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $\lim_{a\to\infty} \mu(p^{(a)} - p^{(0)}) = 0$, where $p^{(0)} \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $\mu(p^{(0)}) = 1$. Since the prequasinorm μ is continuous, we obtain

$$\lim_{a\to\infty} \mu\left(Tp^{(a)} - Tp^{(0)}\right) = \lim_{a\to\infty} \mu\left(\frac{p^{(a)}}{4} - \frac{p^{(0)}}{5}\right) = \mu\left(\frac{p^{(0)}}{20}\right) > 0.$$

Hence, T is not μ-sequentially continuous at $p^{(0)}$. Therefore, the mapping T is not continuous at $p^{(0)}$.

Suppose $\mu(p) = \sum_{m=0}^{\infty}(\frac{|\sum_{j=0}^{m} p_j|}{m+5})^{\frac{2m+3}{m+2}}$, for every $p \in \Pi((\frac{m+2}{m+1})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty})$.

Since for all $p, q \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $\mu(p), \mu(q) \in [0, 1)$, we obtain

$$\begin{aligned}\mu(Tp - Tq) &= \mu\left(\frac{p}{4} - \frac{q}{4}\right)\\ &\le \frac{2}{\sqrt{27}}\left(\mu\left(\frac{3p}{4}\right) + \mu\left(\frac{3q}{4}\right)\right) = \frac{2}{\sqrt{27}}\left(\mu(Tp - p) + \mu(Tq - q)\right).\end{aligned}$$

Suppose $p, q \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $\mu(p), \mu(q) \in [1, \infty)$, we have

$$\mu(Tp - Tq) = \mu\left(\frac{p}{5} - \frac{q}{5}\right) \le \frac{1}{4}\left(\mu\left(\frac{4p}{5}\right) + \mu\left(\frac{4q}{5}\right)\right) = \frac{1}{4}\left(\mu(Tp - p) + \mu(Tq - q)\right).$$

For every $p, q \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $\mu(p) \in [0, 1)$ and $\mu(q) \in [1, \infty)$, we obtain

$$\begin{aligned}\mu(Tp - Tq) &= \mu\left(\frac{p}{4} - \frac{q}{5}\right) \le \frac{2}{\sqrt{27}}\mu\left(\frac{3p}{4}\right) + \frac{1}{4}\mu\left(\frac{4q}{5}\right) \le \frac{2}{\sqrt{27}}\left(\mu\left(\frac{3p}{4}\right) + \mu\left(\frac{4q}{5}\right)\right)\\ &= \frac{2}{\sqrt{27}}\left(\mu(Tp - p) + \mu(Tq - q)\right).\end{aligned}$$

Therefore, the mapping T is a Kannan μ-contraction and

$$T^r(p) = \begin{cases} \frac{p}{4^r}, & \mu(p) \in [0, 1),\\ \frac{p}{5^r}, & \mu(p) \in [1, \infty).\end{cases}$$

Evidently, T is μ-sequentially continuous at $\theta \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ and $\{T^r p\}$ has a subsequence $\{T^{r_j} p\}$ that converges to θ. By using Theorem 7.5, the element $\theta \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ is the only fixed point of T.

Example 7.7 Assume $T : (\Pi((\frac{1}{m+5})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu} \to (\Pi((\frac{1}{m+5})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$, with $\mu(p) = \sum_{m=0}^{\infty}(\frac{|\sum_{j=0}^{m} p_j|}{m+5})^{\frac{2m+3}{m+2}}$, for all $p \in \Pi((\frac{m+2}{m+1})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty})$ and

$$T(p) = \begin{cases} \frac{1}{4}(e_1 + p), & p_0 \in (-\infty, \frac{1}{3}),\\ \frac{1}{3}e_1, & p_0 = \frac{1}{3},\\ \frac{1}{4}e_1, & p_0 \in (\frac{1}{3}, \infty).\end{cases}$$

Since for all $p,q \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $p_0,q_0 \in (-\infty,\frac{1}{3})$, we have

$$\begin{aligned}\mu(Tp-Tq) &= \mu\left(\frac{1}{4}(p_0-q_0,p_1-q_1,p_2-q_2,\dots)\right)\\ &\le \frac{2}{\sqrt{27}}\left(\mu\left(\frac{3p}{4}\right)+\mu\left(\frac{3q}{4}\right)\right)\\ &\le \frac{2}{\sqrt{27}}\big(\mu(Tp-p)+\mu(Tq-q)\big).\end{aligned}$$

For all $p,q \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $p_0,q_0 \in (\frac{1}{3},\infty)$, then for all $\varepsilon > 0$ we obtain

$$\mu(Tp-Tq) = 0 \le \varepsilon\big(\mu(Tp-p)+\mu(Tq-q)\big).$$

For all $p,q \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $p_0 \in (-\infty,\frac{1}{3})$ and $q_0 \in (\frac{1}{3},\infty)$, we have

$$\begin{aligned}\mu(Tp-Tq) &= \mu\left(\frac{p}{4}\right)\\ &\le \frac{1}{\sqrt{27}}\mu\left(\frac{3p}{4}\right) = \frac{1}{\sqrt{27}}\mu(Tp-p) \le \frac{1}{\sqrt{27}}\big(\mu(Tp-p)+\mu(Tq-q)\big).\end{aligned}$$

Therefore, the mapping T is a Kannan μ-contraction.

Obviously, T is μ-sequentially continuous at $\frac{1}{3}e_1 \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ and there is $p \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $p_0 \in (-\infty,\frac{1}{3})$ such that the sequence of iterates $\{T^r p\} = \{\sum_{a=1}^{r}\frac{1}{4^a}e_1 + \frac{1}{4^r}p\}$ includes a subsequence $\{T^{r_j}p\} = \{\sum_{a=1}^{r_j}\frac{1}{4^a}e_1 + \frac{1}{4^{r_j}}p\}$ that converges to $\frac{1}{3}e_1$. By using Theorem 7.5, the mapping T has a unique fixed point $\frac{1}{3}e_1 \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$. Note that T is not continuous at $\frac{1}{3}e_1 \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$.

Suppose $\mu(p) = \sqrt{\sum_{m=0}^{\infty}(\frac{|\sum_{j=0}^{m}p_j|}{m+5})^{\frac{2m+3}{m+2}}}$, for all $p \in \Pi((\frac{m+2}{m+1})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty})$. Since for all $p,q \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $p_0,q_0 \in (-\infty,\frac{1}{3})$, we have

$$\begin{aligned}\mu(Tp-Tq) &= \mu\left(\frac{1}{4}(p_0-q_0,p_1-q_1,p_2-q_2,\dots)\right)\\ &\le \frac{1}{\sqrt[4]{27}}\left(\mu\left(\frac{3p}{4}\right)+\mu\left(\frac{3q}{4}\right)\right)\\ &\le \frac{1}{\sqrt[4]{27}}\big(\mu(Tp-p)+\mu(Tq-q)\big).\end{aligned}$$

For all $p,q \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $p_0,q_0 \in (\frac{1}{3},\infty)$, hence for all $\varepsilon > 0$, one has

$$\mu(Tp-Tq) = 0 \le \varepsilon\big(\mu(Tp-p)+\mu(Tq-q)\big).$$

For all $p,q \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}$ with $p_0 \in (-\infty,\frac{1}{3})$ and $q_0 \in (\frac{1}{3},\infty)$, we have

$$\begin{aligned}\mu(Tp-Tq) &= \mu\left(\frac{p}{4}\right)\\ &\le \frac{1}{\sqrt[4]{27}}\mu\left(\frac{3p}{4}\right) = \frac{1}{\sqrt[4]{27}}\mu(Tp-p) \le \frac{1}{\sqrt[4]{27}}\big(\mu(Tp-p)+\mu(Tq-q)\big).\end{aligned}$$

Therefore, the mapping T is a Kannan μ-contraction. Since μ satisfies the Fatou property, by using Theorem 7.3, the mapping T has one fixed point $\frac{1}{3}e_1 \in (\Pi((\frac{1}{m+5})_{m=0}^{\infty}, (\frac{2m+3}{m+2})_{m=0}^{\infty}))_\mu$.

We investigate the existence of a fixed point of a Kannan contraction mapping in the prequasi-Banach mappings ideal generated by $(\Pi(\zeta,\eta))_\mu$ and s-numbers.

Theorem 7.8 *If the setups* $(f1)$ *and* $(f2)$ *are satisfied, then the prequasinorm* $\Lambda(W) = [\sum_{j=0}^{\infty}(\zeta_j|\sum_{l=0}^{j} s_l(W)|)^{\eta_j}]^{\frac{1}{\hbar}}$ *does not satisfy the Fatou property, for every* $W \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$.

Proof Let the conditions be verified and $\{W_r\}_{r\in\mathbb{Z}^+} \subseteq \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$ with $\lim_{r\to\infty}\Lambda(W_r - W) = 0$. As the space $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$ is a prequasiclosed ideal. Hence, $W \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$. Therefore, for all $V \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$, one has

$$\begin{aligned}
\Lambda(V-W) &= \left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j} s_l(V-W)\right|\right)^{\eta_j}\right]^{\frac{1}{\hbar}} \\
&\le \left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j} s_{[\frac{l}{2}]}(V-W_i)\right|\right)^{\eta_j}\right]^{\frac{1}{\hbar}} + \left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j} s_{[\frac{l}{2}]}(W-W_i)\right|\right)^{\eta_j}\right]^{\frac{1}{\hbar}} \\
&\le 2^{\frac{1}{\hbar}}\sup_r\inf_{i\ge r}\left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j} s_l(V-W_i)\right|\right)^{\eta_j}\right]^{\frac{1}{\hbar}}.
\end{aligned}$$

Hence, Λ does not satisfy the Fatou property. □

Theorem 7.9 *Let the setups* $(f1)$ *and* $(f2)$ *be satisfied and* $G : \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M}) \to \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$, *where* $\Lambda(W) = [\sum_{j=0}^{\infty}(\zeta_j|\sum_{l=0}^{j} s_l(W)|)^{\eta_j}]^{\frac{1}{\hbar}}$, *for all* $W \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$. *The element* $A \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$ *is the unique fixed point of* G, *if the next setups are confirmed*:

(a) *G is a Kannan Λ-contraction mapping,*
(b) *G is Λ-sequentially continuous at a point $A \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$,*
(c) *one has $B \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$ so that the sequence of iterates $\{G^rB\}$ has a subsequence $\{G^{r_i}B\}$ that converges to A.*

Proof Let the enough setups be satisfied. Assume A is not a fixed point of G, then $GA \neq A$. By using the conditions (b) and (c), one has

$$\lim_{r_i\to\infty}\Lambda\left(G^{r_i}B - A\right) = 0 \quad \text{and} \quad \lim_{r_i\to\infty}\Lambda\left(G^{r_i+1}B - GA\right) = 0.$$

As G is a Kannan Λ-contraction mapping, we obtain

$$\begin{aligned}
0 &< \Lambda(GA-A) \\
&= \Lambda\left(\left(GA - G^{r_i+1}B\right) + \left(G^{r_i}B - A\right) + \left(G^{r_i+1}B - G^{r_i}B\right)\right) \\
&\le 2^{\frac{1}{\hbar}}\Lambda\left(G^{r_i+1}B - GA\right) + 2^{\frac{2}{\hbar}}\Lambda\left(G^{r_i}B - A\right) + 2^{\frac{2}{\hbar}}\beta\left(\frac{\beta}{1-\beta}\right)^{r_i-1}\Lambda(GB-B).
\end{aligned}$$

Since $r_i \to \infty$, this implies a contradiction. Hence, A is a fixed point of G. To prove that the fixed point A is unique, assume we have two different fixed points $A, D \in \mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}(\mathcal{N},\mathcal{M})$ of G. Then, we have

$$\Lambda(A-D) \le \Lambda(GA-GD) \le \beta\big(\Lambda(GA-A)+\Lambda(GD-D)\big)=0.$$

So, $A = D$. □

Example 7.10 Suppose $M : S_{(\Pi((\frac{1}{m+4})_{m=0}^\infty,(\frac{2m+3}{m+2})_{m=0}^\infty))_\mu}(\mathcal{N},\mathcal{M}) \to S_{(\Pi((\frac{1}{m+4})_{m=0}^\infty,(\frac{2m+3}{m+2})_{m=0}^\infty))_\mu}(\mathcal{N},\mathcal{M})$, where $\Lambda(H)=\sqrt{\sum_{m=0}^\infty(\frac{|\sum_{j=0}^m s_j|}{m+4})^{\frac{2m+3}{m+2}}}$, for all $H \in S_{(\Pi((\frac{1}{m+4})_{m=0}^\infty,(\frac{2m+3}{m+2})_{m=0}^\infty))_\mu}(\mathcal{N},\mathcal{M})$ and

$$M(H)=\begin{cases}\frac{H}{6}, & \Lambda(H)\in[0,1),\\ \frac{H}{7}, & \Lambda(H)\in[1,\infty).\end{cases}$$

Since for all $H_1, H_2 \in S_{(\Pi((\frac{1}{m+4})_{m=0}^\infty,(\frac{2m+3}{m+2})_{m=0}^\infty))_\mu}$ with $\Lambda(H_1), \Lambda(H_2) \in [0,1)$, we obtain

$$\begin{aligned}\Lambda(MH_1-MH_2)&=\Lambda\left(\frac{H_1}{6}-\frac{H_2}{6}\right)\\ &\le \frac{\sqrt{2}}{\sqrt[4]{125}}\left(\Lambda\left(\frac{5H_1}{6}\right)+\Lambda\left(\frac{5H_2}{6}\right)\right)\\ &=\frac{\sqrt{2}}{\sqrt[4]{125}}\big(\Lambda(MH_1-H_1)+\Lambda(MH_2-H_2)\big).\end{aligned}$$

For all $H_1, H_2 \in S_{(\Pi((\frac{1}{m+4})_{m=0}^\infty,(\frac{2m+3}{m+2})_{m=0}^\infty))_\mu}$ with $\Lambda(H_1), \Lambda(H_2) \in [1,\infty)$, we have

$$\begin{aligned}\Lambda(MH_1-MH_2)&=\Lambda\left(\frac{H_1}{7}-\frac{H_2}{7}\right)\\ &\le \frac{\sqrt{2}}{\sqrt[4]{216}}\left(\Lambda\left(\frac{6H_1}{7}\right)+\Lambda\left(\frac{6H_2}{7}\right)\right)\\ &=\frac{\sqrt{2}}{\sqrt[4]{216}}\big(\Lambda(MH_1-H_1)+\Lambda(MH_2-H_2)\big).\end{aligned}$$

For all $H_1, H_2 \in S_{(\Pi((\frac{1}{m+4})_{m=0}^\infty,(\frac{2m+3}{m+2})_{m=0}^\infty))_\mu}$ with $\Lambda(H_1) \in [0,1)$ and $\Lambda(H_2) \in [1,\infty)$, we have

$$\begin{aligned}\Lambda(MH_1-MH_2)&=\Lambda\left(\frac{H_1}{6}-\frac{H_2}{7}\right)\\ &\le \frac{\sqrt{2}}{\sqrt[4]{125}}\Lambda\left(\frac{5H_1}{6}\right)+\frac{\sqrt{2}}{\sqrt[4]{216}}\Lambda\left(\frac{6H_2}{7}\right)\\ &\le\frac{\sqrt{2}}{\sqrt[4]{125}}\big(\Lambda(MH_1-H_1)+\Lambda(MH_2-H_2)\big).\end{aligned}$$

Therefore, the mapping M is a Kannan Λ-contraction and

$$M^r(H)=\begin{cases}\frac{H}{6^r}, & \Lambda(H)\in[0,1),\\ \frac{H}{7^r}, & \Lambda(H)\in[1,\infty).\end{cases}$$

Evidently, the operator M is Λ-sequentially continuous at the zero mapping $\Theta \in S_{(\Pi((\frac{1}{m+4})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}}$ and $\{M^r H\}$ has a subsequence $\{M^{r_j} H\}$ that converges to Θ. By using Theorem 7.9, the zero mapping $\Theta \in S_{(\Pi((\frac{1}{m+4})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}}$ is the unique fixed point of M. Suppose $\{H^{(a)}\} \subseteq S_{(\Pi((\frac{1}{m+4})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}}$ is such that $\lim_{a\to\infty} \Lambda(H^{(a)} - H^{(0)}) = 0$, where $H^{(0)} \in S_{(\Pi((\frac{1}{m+4})_{m=0}^{\infty},(\frac{2m+3}{m+2})_{m=0}^{\infty}))_{\mu}}$ with $\Lambda(H^{(0)}) = 1$. Since the prequasinorm Λ is continuous, we obtain

$$\lim_{a\to\infty} \Lambda\left(MH^{(a)} - MH^{(0)}\right) = \lim_{a\to\infty} \Lambda\left(\frac{H^{(0)}}{6} - \frac{H^{(0)}}{7}\right) = \Lambda\left(\frac{H^{(0)}}{42}\right) > 0.$$

Hence, M is not Λ-sequentially continuous at $H^{(0)}$. Therefore, the mapping M is not continuous at $H^{(0)}$.

8 The existence of solutions of nonlinear difference equations

Summable equations such as (6) were discussed by Salimi et al. [39], Agarwal et al. [40], and Hussain et al. [41]. In this section, we search for a solution to (6) in $(\Pi(\zeta,\eta))_{\mu}$, where the setups $(f1)$ and $(f2)$ are confirmed and $\mu(\lambda) = [\sum_{j=0}^{\infty}(\zeta_j|\sum_{l=0}^{j}\lambda_l|)^{\eta_j}]^{\frac{1}{\hbar}}$, for every $\lambda \in \Pi(\zeta,\eta)$.

Evaluate the summable equations:

$$\lambda_l = y_l + \sum_{m=0}^{\infty} A(l,m)g(m,\lambda_m), \tag{6}$$

and assume $W : (\Pi(\zeta,\eta))_{\mu} \to (\Pi(\zeta,\eta))_{\mu}$ is defined by

$$W(\lambda_l)_{l\in\mathbb{Z}^+} = \left(y_l + \sum_{m=0}^{\infty} A(l,m)g(m,\lambda_m)\right)_{l\in\mathbb{Z}^+}. \tag{7}$$

Theorem 8.1 *The summable equation* (6) *contains a unique solution in* $(\Pi(\zeta,\eta))_{\mu}$, *when* $A : \mathbb{Z}^{+2} \to \mathrm{R}$, $g : \mathbb{Z}^+ \times \mathfrak{C} \to \mathfrak{C}$, $y : \mathbb{Z}^+ \to \mathfrak{C}$, $\lambda : \mathbb{Z}^+ \to \mathfrak{C}$, $\gamma : \mathbb{Z}^+ \to \mathfrak{C}$, *assume there exists* $\beta \in \mathrm{R}$ *so that* $\sup_j \beta^{\frac{\eta_j}{\hbar}} \in [0,\frac{1}{2})$ *and for all* $j \in \mathbb{Z}^+$, *we have*

$$\begin{aligned}
&\left|\sum_{l=0}^{j}\sum_{m\in\mathbb{Z}^+} A(l,m)\left[g(m,\lambda_m) - g(m,\gamma_m)\right]\right| \\
&\quad\le \beta\left[\left|\sum_{l=0}^{j} y_l - \lambda_l + \sum_{m\in\mathbb{Z}^+} A(l,m)g(m,\lambda_m)\right| + \left|\sum_{l=0}^{j} y_l - \gamma_l + \sum_{m\in\mathbb{Z}^+} A(l,m)g(m,\gamma_m)\right|\right].
\end{aligned}$$

Proof Suppose the conditions are confirmed. Assume the mapping $W : (\Pi(\zeta,\eta))_{\mu} \to (\Pi(\zeta,\eta))_{\mu}$ defined by equation (7). We have

$$\begin{aligned}
\mu(W\lambda - W\gamma) &= \left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j} W\lambda_l - W\gamma_l\right|\right)^{\eta_j}\right]^{\frac{1}{\hbar}} \\
&= \left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\sum_{m\in\mathbb{Z}^+} A(l,m)\left[g(m,\lambda_m) - g(m,\gamma_m)\right]\right|\right)^{\eta_j}\right]^{\frac{1}{\hbar}} \\
&\le \sup_j \beta^{\frac{\eta_j}{\hbar}}\left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j} y_l - \lambda_l + \sum_{m\in\mathbb{Z}^+} A(l,m)g(m,\lambda_m)\right|\right)^{\eta_j}\right]^{\frac{1}{\hbar}}
\end{aligned}$$

$$+\sup_j \beta^{\frac{\eta_j}{\hbar}}\left[\sum_{j=0}^{\infty}\left(\zeta_j\left|\sum_{l=0}^{j}\gamma_l-\gamma_l+\sum_{m\in\mathbb{Z}^+}A(l,m)g(m,\gamma_m)\right|\right)^{\eta_j}\right]^{\frac{1}{\hbar}}$$
$$=\sup_j \beta^{\frac{\eta_j}{\hbar}}\big(\mu(W\lambda-\lambda)+\mu(W\gamma-\gamma)\big).$$

By using Theorem 7.3, we obtain a unique solution of equation (6) in $(\Pi(\zeta,\eta))_\mu$. □

Example 8.2 Suppose the sequence space $(\Pi((\frac{1}{p+1})_{p=0}^{\infty},(\frac{2p+3}{p+2})_{p=0}^{\infty}))_\mu$, where $\mu(\lambda)=\sqrt{\sum_{p=0}^{\infty}(\frac{|\sum_{j=0}^{p}\lambda_j|}{p+1})^{\frac{2p+3}{p+2}}}$, for every $\lambda\in(\Pi((\frac{1}{p+1})_{p=0}^{\infty},(\frac{2p+3}{p+2})_{p=0}^{\infty}))_\mu$.

Examine the nonlinear difference equations:

$$\lambda_l=e^{-(3l+6)}+\sum_{m\in\mathbb{Z}^+}(-1)^{l+m}\frac{\lambda_{l-2}^r}{\lambda_{l-1}^q+m^2+1},\tag{8}$$

with $r,q,\lambda_{-2},\lambda_{-1}>0$ and suppose

$$W:\left(\Pi\left(\left(\frac{1}{p+1}\right)_{p=0}^{\infty},\left(\frac{2p+3}{p+2}\right)_{p=0}^{\infty}\right)\right)_\mu\to\left(\Pi\left(\left(\frac{1}{p+1}\right)_{p=0}^{\infty},\left(\frac{2p+3}{p+2}\right)_{p=0}^{\infty}\right)\right)_\mu,$$

defined by

$$W(\lambda_l)_{l=0}^{\infty}=\left(e^{-(3l+6)}+\sum_{m\in\mathbb{Z}^+}(-1)^{l+m}\frac{\lambda_{l-2}^r}{\lambda_{l-1}^q+m^2+1}\right)_{l=0}^{\infty}.\tag{9}$$

Clearly, there is a number β such that $\sup_j\beta^{\frac{2j+3}{2j+4}}\in[0,\frac{1}{2})$ and for all $j\in\mathbb{Z}^+$, we obtain

$$\left|\sum_{l=0}^{j}\sum_{m\in\mathbb{Z}^+}(-1)^l\frac{\lambda_{l-2}^r}{\lambda_{l-1}^q+m^2+1}\big((-1)^m-(-1)^m\big)\right|$$
$$\leq\lambda\left|\sum_{l=0}^{j}e^{-(3l+6)}-\lambda_l+\sum_{m\in\mathbb{Z}^+}(-1)^{l+m}\frac{\lambda_{l-2}^r}{f_{l-1}^q+m^2+1}\right|$$
$$+\lambda\left|\sum_{l=0}^{j}e^{-(3l+6)}-\gamma_l+\sum_{m\in\mathbb{Z}^+}(-1)^{l+m}\frac{\lambda_{l-2}^r}{\lambda_{l-1}^q+m^2+1}\right|.$$

By using Theorem 8.1, the nonlinear difference equations (8) include a unique solution in $(\Pi((\frac{1}{p+1})_{p=0}^{\infty},(\frac{2p+3}{p+2})_{p=0}^{\infty}))_\mu$.

9 Conclusion

In this article, we discuss some topological and geometric structure of $(\Pi(\zeta,\eta))_\mu$, of the multiplication mappings defined on $(\Pi(\zeta,\eta))_\mu$, of the class $\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu}$, and of the class $(\mathfrak{B}^s_{(\Pi(\zeta,\eta))_\mu})^\rho$. We explain the existence of a fixed point of the Kannan contraction mapping acting on these spaces. Interestingly, several numerical experiments are presented to illustrate our results. Additionally, some successful applications to the existence of solutions of nonlinear difference equations are introduced. This article has many advantages for researchers, such as studying the fixed points of any contraction mappings on this prequasinormed sequence space that is a generalization of the quasinormed sequence spaces, a new general space of solutions for many difference equations, the spectrum of any bounded linear operators between any two Banach spaces with s-numbers in this sequence space and recall that the closed mappings ideal are sure to play an influential function in the principle of Banach lattices.

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AAB carried out most of the analysis and methods, as well as contributing significantly to the writing of the manuscript. MHE was in charge of research and the provision of study materials. All the authors read and approved the final manuscript.

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