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A new subgradient extragradient method for solving the split modified system of variational inequality problems and fixed point problem

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Abstract

We introduce a new subgradient extragradient algorithm utilizing the concept of the set of solutions of the split modified system of variational inequality problems (SMSVIP). Our main theorem is weak convergence theorem for such an algorithm for approximating the fixed point problem in a real Hilbert space. We also apply these results to approximate the split minimization problem. In the last section, we provide an example to illustrate the potential of our main theorem.

MSC: 47H09; 47H10; 90C33

Keywords: Fixed point; Subgradient extragradient; The split modified system of variational inequality problems; Variational inequality

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. The mapping $T: C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. An element $x \in C$ is said to be a fixed point of T if $Tx = x$ and $F(T) = \{x \in C : Tx = x\}$ denotes the set of fixed points of T. Fixed point problem has been widely studied and developed in the literature; see [5, 11, 26, 27, 29] and the references therein.

We now recall some well-known concepts and results in a real Hilbert space H.

The variational inequality problem (VIP) is to find a point $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0$$

for all $y \in C$. The set of all solutions of the variational inequality is denoted by $VI(C, A)$. Since its inception by Stampacchia [24] in 1964, the variational inequality problem has become interesting in several topics arising in structural analysis, physic, economics, optimization, and applied sciences; see [1, 3, 6, 8, 11–13, 15, 18, 20, 30, 32] and the references therein.

Several algorithms for solving the VIP are projection algorithms that employ projections onto the feasible set C of the VIP, or onto some related set, in order to iteratively reach a solution. In 1976, Korpelevich [19] proposed an algorithm for solving the VIP in a Euclidean space, known as *the extragradient method*. In each iteration of her algorithm, in order to get the next iterate x^{k+1}, two orthogonal projections onto C are calculated, according to the following iterative step. Given the current iterate x^k, calculate

$$y^k = P_C\big(x^k - \tau f\big(x^k\big)\big), \tag{1}$$

$$x^{k+1} = P_C\big(x^k - \tau f\big(y^k\big)\big) \tag{2}$$

for all $k \in \mathbb{N}$, where τ is some positive number and P_C denotes the Euclidean least distance projection onto C.

The convergence was proved in [19] under the assumptions of Lipschitz continuity and pseudo-monotonicity. However, there is still the need to calculate two projections onto C. If the set C is simple enough so that projections onto it can be easily computed, but if C is a general closed and convex set, a minimal distance problem has to be solved twice in order to obtain the next iterate. This might seriously affect the efficiency of the extragradient method. Korpelevich's extragradient method has been widely studied in the literature; see [2, 4, 7, 9, 14, 16, 17, 22, 28, 31] and the references therein.

In the past decade years, Censor et al. [10] developed the subgradient extragradient algorithm in a Euclidean space, in which they replaced the (second) projection (2) onto C by a projection onto a specific constructible half-space as follows:

Algorithm 1 (The subgradient extragradient algorithm)

Step 0 : *Select a starting point* $x^0 \in H$ *and* $\tau > 0$, *and set* $k = 0$.

Step 1 : *Given the current iterate* x^k, *compute*

$$y^k = P_C\big(x^k - \tau f\big(x^k\big)\big),$$

construct the half-space T_k *the bounding hyperplane of which supports* C *at* y^k,

$$T_k := \big\{w \in H | \big\langle \big(x^k - \tau f\big(x^k\big)\big) - y^k, w - y^k \big\rangle \le 0\big\}, \tag{3}$$

and calculate the next iterate

$$x^{k+1} = P_{T_k}\big(x^k - \tau f\big(y^k\big)\big).$$

Step 2 : *If* $x^k = y^k$ *then stop. Otherwise, set* $k \leftarrow (k+1)$ *and return to step* 1.

Furthermore, under some control conditions, they proved weak convergence theorems for their algorithms.

Very recently, Sripattanet and Kangtunyakarn [23] introduced the following *split modified system of variational inequality problems (SMSVIP)*, which involves finding

$(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \zeta D_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \zeta D_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \zeta D_3)x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{4}$$

and finding $(\bar{x^*} = Ax^*, \bar{y^*} = Ay^*, \bar{z^*} = Az^*) \in Q \times Q \times Q$ such that

$$\begin{cases} \langle \bar{x^*} - (I - \bar{\zeta}\bar{D_1})(a\bar{x^*} + (1-a)\bar{y^*}), \bar{x} - \bar{x^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{y^*} - (I - \bar{\zeta}\bar{D_2})(a\bar{x^*} + (1-a)\bar{z^*}), \bar{x} - \bar{y^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{z^*} - (I - \bar{\zeta}\bar{D_3})\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, & \forall \bar{x} \in Q, \end{cases} \tag{5}$$

where $D_1, D_2, D_3 : C \to H_1$, $\bar{D_1}, \bar{D_2}, \bar{D_3} : Q \to H_2$ are six different mappings, $\zeta, \bar{\zeta} > 0$, and $a \in [0,1]$. The sets of all solutions of (4) and (5) are denoted by Ψ_{D_1,D_2,D_3} and $\Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}$, respectively. The set of all solutions of the SMSVIP is denoted by $\Psi^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3}}$, that is,

$$\Psi^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3}} = \left\{ \left(x^*, y^*, z^*\right) \in \Psi_{D_1,D_2,D_3} : \left(\bar{x^*}, \bar{y^*}, \bar{z^*}\right) \in \Psi_{\bar{D_1},\bar{D_2},\bar{D_3}} \right\}.$$

If we put $a = 0$ in (4) and (5), we have

$$\begin{cases} \langle x^* - (I - \zeta D_1)y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \zeta D_2)z^*, x - y^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \zeta D_3)x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases}$$

and

$$\begin{cases} \langle \bar{x^*} - (I - \bar{\zeta}\bar{D_1})\bar{y^*}, \bar{x} - \bar{x^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{y^*} - (I - \bar{\zeta}\bar{D_2})\bar{z^*}, \bar{x} - \bar{y^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{z^*} - (I - \bar{\zeta}\bar{D_3})\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, & \forall \bar{x} \in Q, \end{cases}$$

which is a modified the split general system of variational inequalities (SVIP) [21].

Based on the above works and observation of a half-space in Algorithm 1 related to the VIP, we introduce a new half-space related to the SMSVIP and prove weak convergence theorems of the sequence $\{x_n\}$ generated by our new half-space for approximating the solutions of the SMSVIP. Moreover, using our main result, we obtain the additional results involving the split minimization problem. Finally, we perform numerical examples to illustrate the computational performance of the proposed algorithms.

2 Preliminaries

We denote the weak convergence and the strong convergence by "$\rightharpoonup$" and "$\to$", respectively. For every $x \in \mathcal{H}$, there exists a unique nearest point $P_C x$ in C such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called the metric projection of $\mathcal{H}$ onto C.

The metric projection P_C is characterized by the following two properties:

1. $P_C x \in C$,

2. $\langle x - P_C x, P_C x - y\rangle \geq 0$, $\forall x \in \mathcal{H}$, $y \in C$,
   and if C is a hyperplane, it follows that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{6}$$

$\forall x \in \mathcal{H}$, $y \in C$.

Definition 2.1 A mapping $A : C \to H$ is called α-*inversestronglymonotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$.

The following lemmas are needed to prove the main theorem.

Lemma 2.2 *Let $\mathcal{H}$ be a real Hilbert space, and let C be a nonempty closed convex subset of $\mathcal{H}$. Let $\{x^k\}_{k=0}^{\infty} \subset \mathcal{H}$ be Fejer-monotone with respect to C, i.e., for every $u \in C$,*

$$\left\|x^{k+1} - u\right\| \leq \left\|x^k - u\right\|, \quad \forall k \geq 0.$$

Then $\{P_C x^k\}_{k=0}^{\infty}$ converges strongly to some $z \in C$.

Lemma 2.3 *Each Hilbert space $\mathcal{H}$ satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset \mathcal{H}$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in \mathcal{H}$ with $y \neq x$.

Lemma 2.4 ([23]) *Let H_1 and H_2 be real Hilbert spaces, and let C, Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3-inverse strongly monotone, respectively, where $\zeta \in (0, 2d^*)$ with $d^* = \min\{d_1, d_2, d_3\}$. Let $\bar{D_1}, \bar{D_2}, \bar{D_3} : Q \to H_2$ be $\bar{d}_1, \bar{d}_2, \bar{d}_3$-inverse strongly monotone, respectively, where $\bar{\zeta} \in (0, 2\hat{d})$ with $\hat{d} = \min\{\bar{d}_1, \bar{d}_2, \bar{d}_3\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C : C \to C$ by*

$$M_C(x) = P_C(I - \zeta D_1)\big(ax + (1 - a)P_C(I - \zeta D_2)\big(ax + (1 - a)P_C(I - \zeta D_3)x\big)\big),$$

$\forall x \in C$, and define $M_Q : Q \to Q$ by

$$M_Q(\hat{x}) = P_Q(I - \bar{\zeta}\bar{D_1})\big(a\hat{x} + (1 - a)P_Q(I - \bar{\zeta}\bar{D_2})\big(a\hat{x} + (1 - a)P_Q(I - \bar{\zeta}\bar{D_3})\hat{x}\big)\big),$$

$\forall \hat{x} \in Q$. Define $M : C \to C$ by $M(x) = M_C(x - \eta A^*(I - M_Q)Ax)$ for all $x \in C$. Then M is a nonexpansive mapping for all $x \in C$.

Remark 1 From the study of Lemma 2.4, we have

$$\begin{aligned}&\left\|\left(x-\eta A^*(I-M_Q)Ax\right)-\left(y-\eta A^*(I-M_Q)Ay\right)\right\|^2\\&\quad\le\|x-y\|^2-\eta(1-\eta L)\left\|(I-M_Q)Ax-(I-M_Q)Ay\right\|^2\end{aligned}$$

for all $x,y\in H_1$.

Lemma 2.5 ([23]) *Let H_1 and H_2 be real Hilbert spaces, and let C,Q be nonempty closed convex subsets of H_1,H_2, respectively. Define the mappings $D_1,D_2,D_3,\bar{D_1},\bar{D_2},\bar{D_3},M_C$, and M_Q as in Lemma 2.4, where $\zeta\in(0,2d^*)$ with $d^*=\min\{d_1,d_2,d_3\}$, $\bar{\zeta}\in(0,2\hat{d})$ with $\hat{d}=\min\{\bar{d_1},\bar{d_2},\bar{d_3}\}$. Let $A:H_1\to H_2$ be a bounded linear operator with adjoint A^* and $\eta\in(0,\frac{1}{L})$ with L being the spectral radius of the operator A^*A.*

Assume

$$\Psi^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3}}=\left\{\left(x^*,y^*,z^*\right)\in\Psi_{D_1,D_2,D_3}:\left(\bar{x^*},\bar{y^*},\bar{z^*}\right)\in\Psi_{\bar{D_1},\bar{D_2},\bar{D_3}}\right\}\neq\emptyset.$$

The following statements are equivalent:

(i) $(x^*,y^*,z^*)\in\Psi^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3}}$,

(ii) $x^*=M_C(x^*-\eta A^*(I-M_Q)Ax^*)$, *where* $y^*=P_C(I-\zeta D_2)(ax^*+(1-a)z^*)$, $z^*=P_C(I-\zeta D_3)x^*$, $\bar{x^*}=Ax^*=P_Q(I-\bar{\zeta}\bar{D_1})(a\bar{x^*}+(1-a)\bar{y^*})$, $\bar{y^*}=Ay^*=P_Q(I-\bar{\zeta}\bar{D_2})(a\bar{x^*}+(1-a)\bar{z^*})$, *and* $\bar{z^*}=Az^*=P_Q(I-\bar{\zeta}\bar{D_3})\bar{x^*}$.

Lemma 2.6 ([23]) *Let H_1 and H_2 be real Hilbert spaces, and let C,Q be nonempty closed convex subsets of H_1,H_2, respectively. Define the mappings $D_1,D_2,D_3,\bar{D_1},\bar{D_2},\bar{D_3},M_C$, and M_Q as in Lemma 2.4 where $\zeta\in(0,2d^*)$ with $d^*=\min\{d_1,d_2,d_3\}$, $\bar{\zeta}\in(0,2\hat{d})$ with $\hat{d}=\min\{\bar{d_1},\bar{d_2},\bar{d_3}\}$ and $a\in[0,1]$. Let $A:H_1\to H_2$ be a bounded linear operator with adjoint A^* and $\eta\in(0,\frac{1}{L})$ with L being the spectral radius of the operator A^*A. Let $\bigcap_{i=1}^3\Phi_i\neq\emptyset$ and $\Phi_i=\{w\in VI(C,D_i)|Aw=\bar{w}\in VI(Q,\bar{D}_i)\}$ for all $i=1,2,3$. Then*

$$\bigcap_{i=1}^{3}\Phi_i=F\left(M_C\left(I-\eta A^*(I-M_Q)A\right)\right).$$

In order to prove our main result, we need to prove the lemmas involving the split variational inequality problem.

Lemma 2.7 *Let H_1 and H_2 be real Hilbert spaces, and let C,Q be nonempty closed convex subsets of H_1,H_2, respectively. Define the mappings $D_1,D_2,D_3,\bar{D_1},\bar{D_2},\bar{D_3},M_C$, and M_Q as in Lemma 2.4 where $\zeta\in(0,2d^*)$ with $d^*=\min\{d_1,d_2,d_3\}$, $\bar{\zeta}\in(0,2\hat{d})$ with $\hat{d}=\min\{\bar{d_1},\bar{d_2},\bar{d_3}\}$ and $a\in[0,1]$. Let $\{x_n\}$ be a sequence in H_1, and let $A:H_1\to H_2$ be a bounded linear operator with adjoint A^* and $\eta\in(0,\frac{1}{L})$ with L being the spectral radius of the operator A^*A. For every $n\in\mathbb{N}$, let $T_n=aW_n+(1-a)P_C(I-\zeta D_2)(aW_n+(1-a)P_C(I-\zeta D_3)W_n))$ and $W_n=(I-\eta A^*(I-M_Q)A)x_n$. If $x^*\in\bigcap_{i=1}^3\Phi_i$, then*

$$\left\|T_n-x^*\right\|^2\le\left\|x_n-x^*\right\|^2-\eta(1-\eta L)\left\|(I-M_Q)Ax_n\right\|^2$$

for all $n\in\mathbb{N}$.

Proof Let $x^* \in \bigcap_{i=1}^3 \Phi_i$. From Lemma 2.6, we have

$$x^* \in F\bigl(M_C\bigl(I - \eta A^*(I - M_Q)A\bigr)\bigr).$$

It implies that $x^* = M_C(I - \eta A^*(I - M_Q)A)x^*$, $y^* = P_C(I - \zeta D_2)(ax^* + (1-a)z^*)$, and $z^* = P_C(I - \zeta D_3)x^*$, where $\bar{x^*} = Ax^* = P_Q(I - \bar{\zeta}\bar{D_1})(a\bar{x^*} + (1-a)\bar{y^*})$, $\bar{y^*} = Ay^* = P_Q(I - \bar{\zeta}\bar{D_2})(a\bar{x^*} + (1-a)\bar{z^*})$, and $\bar{z^*} = Az^* = P_Q(I - \bar{\zeta}\bar{D_3})\bar{x^*}$. From Lemma 2.5, we have $(x^*, y^*, z^*) \in \Omega^{D_1,D_2,D_3}_{\bar{D_1},\bar{D_2},\bar{D_3}}$. That is, $(x^*, y^*, z^*) \in \Omega_{D_1,D_2,D_3}$ and $(\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Omega_{\bar{D_1},\bar{D_2},\bar{D_3}}$. From $(\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Omega_{\bar{D_1},\bar{D_2},\bar{D_3}}$, we obtain that

$$\begin{aligned}
\bar{x^*} &= P_Q(I - \bar{\zeta}\bar{D_1})\bigl(a\bar{x^*} + (1-a)\bar{y^*}\bigr),\\
\bar{y^*} &= P_Q(I - \bar{\zeta}\bar{D_2})\bigl(a\bar{x^*} + (1-a)\bar{z^*}\bigr),\\
\bar{z^*} &= P_Q(I - \bar{\zeta}\bar{D_3})\bar{x^*}.
\end{aligned}$$

It implies that

$$\begin{aligned}
Ax^* = \bar{x^*} &= P_Q(I - \bar{\zeta}\bar{D_1})\bigl(a\bar{x^*} + (1-a)P_Q(I - \bar{\zeta}\bar{D_2})\bigl(a\bar{x^*} + (1-a)P_Q(I - \bar{\zeta}\bar{D_3})\bar{x^*}\bigr)\bigr)\\
&= M_Q\bar{x^*} = M_Q Ax^*.
\end{aligned}$$

From the definition of x^*, we get $x^* = P_C(I - \zeta D_1)T^*_x$, where $T^*_x = aW^*_x + (1-a)P_C(I - \zeta D_2)(aW^*_x + (1-a)P_C(I - \zeta D_3)W^*_x))$ and $W^*_x = (I - \eta A^*(I - M_Q)A)x^*) = x^*$.

From Lemma 2.6, we have that $P_C(I - \zeta D_1), P_C(I - \zeta D_2)$ and $P_C(I - \zeta D_3)$ are nonexpansive.

By the definition of T_n, Lemma 2.4, and Remark 1, we have

$$\begin{aligned}
\|T_n - x^*\|^2 &= \bigl\|aW_n + (1-a)P_C(I - \zeta D_2)\bigl(aW_n + (1-a)\\
&\quad \times P_C(I - \zeta D_3)W_n\bigr)\bigr) - \bigl(aW_{x^*} + (1-a)P_C(I - \zeta D_2)\bigl(aW_{x^*}\\
&\quad + (1-a)P_C(I - \zeta D_3)W_{x^*}\bigr)\bigr)\bigr\|^2\\
&= \bigl\|a(W_n - W_{x^*}) + (1-a)\bigl[P_C(I - \zeta D_2)\bigl(aW_n + (1-a)P_C(I - \zeta D_3)W_n\bigr)\\
&\quad - P_C(I - \zeta D_2)\bigl(aW_{x^*} + (1-a)P_C(I - \zeta D_3)W_{x^*}\bigr)\bigr]\bigr\|^2\\
&\le a\|W_n - W_{x^*}\|^2 + (1-a)\bigl\|P_C(I - \zeta D_2)\bigl(aW_n + (1-a)P_C(I - \zeta D_3)W_n\bigr)\\
&\quad - P_C(I - \zeta D_2)\bigl(aW_{x^*} + (1-a)P_C(I - \zeta D_3)W_{x^*}\bigr)\bigr\|^2\\
&\le a\|W_n - W_{x^*}\|^2 + (1-a)\bigl\|aW_n + (1-a)P_C(I - \zeta D_3)W_n\\
&\quad - \bigl(aW_{x^*} + (1-a)P_C(I - \zeta D_3)W_{x^*}\bigr)\bigr\|^2\\
&= a\|W_n - W_{x^*}\|^2 + (1-a)\bigl\|a(W_n - W_{x^*}) + (1-a)\\
&\quad \times \bigl[P_C(I - \zeta D_3)W_n - x^*\bigr]\bigr\|^2\\
&\le a\|W_n - W_{x^*}\|^2 + a(1-a)\|W_n - W_{x^*}\|^2 + (1-a)^2\\
&\quad \times \bigl\|P_C(I - \zeta D_3)W_n - x^*\bigr\|^2\\
&= \bigl(2a - a^2\bigr)\|W_n - W_{x^*}\|^2 + (1-a)^2\bigl\|P_C(I - \zeta D_3)W_n - x^*\bigr\|^2
\end{aligned}$$

$$
\begin{aligned}
&\le \|W_n - x^*\|^2 \\
&= \|x_n - \eta A^*(I - M_Q)Ax_n - x^*\|^2 \\
&\le \|x_n - x^*\|^2 - \eta(1 - \eta L)\|(I - M_Q)Ax_n\|^2. \qquad (7)
\end{aligned}
$$

□

3 Main results

Theorem 3.1 *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $S : C \to C$ be a nonexpansive mapping. Let $D_1, D_2, D_3 : C \to H_1$ be d_1, d_2, d_3-inverse strongly monotone, respectively, with $d^* = \min\{d_1, d_2, d_3\}$. Let $\bar{D}_1, \bar{D}_2, \bar{D}_3 : Q \to H_2$ be $\bar{d}_1, \bar{d}_2, \bar{d}_3$-inverse strongly monotone, respectively, with $\hat{d} = \min\{\bar{d}_1, \bar{d}_2, \bar{d}_3\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C : H_1 \to C$ by*

$$M_C(x) = P_C(I - \zeta D_1)\big(ax + (1 - a)P_C(I - \zeta D_2)\big(ax + (1 - a)P_C(I - \zeta D_3)x\big)\big),$$

$\forall x \in H_1$, where $a \in [0, 1)$, $\zeta \in (0, 2d^*)$, and define $M_Q : H_2 \to Q$ by

$$M_Q(x) = P_Q(I - \bar{\zeta}\bar{D}_1)\big(a\hat{x} + (1 - a)P_Q(I - \bar{\zeta}\bar{D}_2)\big(a\hat{x} + (1 - a)P_Q(I - \bar{\zeta}\bar{D}_3)\hat{x}\big)\big),$$

$\forall \hat{x} \in H_1$, where $a \in [0, 1)$, $\bar{\zeta} \in (0, 2\hat{d})$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1 \in H_1$ and

$$y_n = M_C W_n = P_C(I - \zeta D_1)T_n,$$

where $W_n = (I - \eta A^*(I - M_Q)A)x_n$ and $T_n = aW_n + (1 - a)P_C(I - \zeta D_2)(aW_n + (1 - a)P_C(I - \zeta D_3)W_n))$.

$$
\begin{aligned}
&Q_n = \big\{z \in H : \big\langle (I - \zeta D_1)T_n - y_n, y_n - z\big\rangle \ge 0\big\}, \\
&x_{n+1} = \alpha_n T_n + (1 - \alpha_n)SP_{Q_n}\big(T_n - \zeta D_1(y_n)\big)
\end{aligned}
$$

for all $n \in \mathbb{N}$.

Assume that the following conditions hold:

(i) $\Im = F(S) \bigcap \bigcap_{i=1}^3 \Phi_i \neq \emptyset$, *where $\Phi_i = \{w \in VI(C, D_i) | Aw \in VI(Q, \bar{D}_i)\}$ for all $i = 1, 2, 3$.*

(ii) $\alpha_n \in [c, d] \subset (0, 1)$.

Then $\{x_n\}$ converges weakly to $x_0 = P_\Im x_n$, which $(x_0, y_0, z_0) \in \Omega^{D_1, D_2, D_3}_{\bar{D}_1, \bar{D}_2, \bar{D}_3}$, $y_0 = P_C(I - \zeta D_2)(ax_0 + (1 - a)z_0)$, and $z_0 = P_C(I - \zeta D_3)x_0$ with $\bar{x_0} = Ax_0$, $\bar{y_0} = Ay_0$ and $\bar{z_0} = Az_0$.

Proof Denote $k_n := P_{Q_n}(T_n - \zeta D_1(y_n))$ for all $n \ge 0$. Let $x^* \in \Im$. From the definition of P_{Q_n}, we have $y_n = P_{Q_n}(I - \zeta D_1)T_n$. Let $M_n = T_n - \zeta D_1(y_n)$. From $C \subseteq Q_n$, and applying (6), we have

$$
\begin{aligned}
\|k_n - x^*\|^2 &= \|P_{Q_n}M_n - x^*\|^2 \\
&\le \|M_n - x^*\|^2 - \|M_n - P_{Q_n}M_n\|^2 \\
&= \|T_n - \zeta D_1(y_n) - x^*\|^2 - \|T_n - \zeta D_1(y_n) - P_{Q_n}M_n\|^2
\end{aligned}
$$

$$
\begin{aligned}
&= \|T_n - x^*\|^2 - 2\zeta\langle T_n - x^*, D_1(y_n)\rangle + \zeta^2\|D_1(y_n)\|^2 \\
&\quad - \|T_n - P_{Q_n}M_n\|^2 + 2\zeta\langle T_n - P_{Q_n}M_n, D_1(y_n)\rangle - \zeta^2\|D_1(y_n)\|^2 \\
&= \|T_n - x^*\|^2 - \|T_n - P_{Q_n}M_n\|^2 - 2\zeta\langle P_{Q_n}M_n - x^*, D_1(y_n)\rangle. \qquad (8)
\end{aligned}
$$

From the monotonicity of D_1, we have

$$
\begin{aligned}
0 &\le \langle D_1y_n - D_1x^*, y_n - x^*\rangle \\
&= \langle D_1y_n, y_n - x^*\rangle - \langle D_1x^*, y_n - x^*\rangle \\
&\le \langle D_1y_n, y_n - x^*\rangle \\
&= \langle D_1y_n, y_n - P_{Q_n}M_n\rangle - \langle D_1y_n, x^* - P_{Q_n}M_n\rangle,
\end{aligned}
$$

which implies that

$$
\langle D_1y_n, x^* - P_{Q_n}M_n\rangle \le \langle D_1y_n, y_n - P_{Q_n}M_n\rangle. \qquad (9)
$$

From (8) and (9), we have

$$
\|k_n - x^*\|^2 \le \|T_n - x^*\|^2 - \|T_n - P_{Q_n}M_n\|^2 + 2\zeta\langle D_1y_n, y_n - P_{Q_n}M_n\rangle. \qquad (10)
$$

From (10) and Lemma 2.7, we have

$$
\begin{aligned}
\|k_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I - M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - T_n\|^2 \\
&\quad + 2\zeta\langle D_1y_n, y_n - P_{Q_n}M_n\rangle \\
&= \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I - M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - y_n\|^2 \\
&\quad - \|y_n - T_n\|^2 - 2\langle P_{Q_n}M_n - y_n, y_n - T_n\rangle \\
&\quad + 2\zeta\langle D_1y_n, y_n - P_{Q_n}M_n\rangle \\
&= \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I - M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - y_n\|^2 \\
&\quad - \|y_n - T_n\|^2 + 2\langle P_{Q_n}M_n - y_n, T_n - y_n - \zeta D_1y_n\rangle \\
&= \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I - M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - y_n\|^2 \\
&\quad - \|y_n - T_n\|^2 + 2\langle (I - \zeta D_1)T_n - y_n, P_{Q_n}M_n - y_n\rangle \\
&\quad + 2\langle \zeta D_1T_n - \zeta D_1y_n, P_{Q_n}M_n - y_n\rangle \\
&\le \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I - M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - y_n\|^2 \\
&\quad - \|y_n - T_n\|^2 + 2\zeta\|D_1T_n - D_1y_n\|\|P_{Q_n}M_n - y_n\| \\
&\le \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I - M_Q)Ax_n\|^2 - \|P_{Q_n}M_n - y_n\|^2 \\
&\quad - \|y_n - T_n\|^2 + \frac{\zeta}{d_1}\left[\|T_n - y_n\|^2 + \|P_{Q_n}M_n - y_n\|^2\right] \\
&= \|x_n - x^*\|^2 - \eta(1-\eta L)\|(I - M_Q)Ax_n\|^2
\end{aligned}
$$

$$-\left(1-\frac{\zeta}{d_1}\right)\|P_{Q_n}M_n-y_n\|^2-\left(1-\frac{\zeta}{d_1}\right))\|T_n-y_n\|^2. \tag{11}$$

By the definition of x_{n+1}, (11), and Lemma 2.7, we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n(T_n-x^*)+(1-\alpha_n)(Sk_n-x^*)\|^2 \\
&\leq \alpha_n\|T_n-x^*\|^2+(1-\alpha_n)\|Sk_n-x^*\|^2 \\
&= \alpha_n\|T_n-x^*\|^2+(1-\alpha_n)\|Sk_n-x^*\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|T_n-Sk_n\|^2 \qquad (12)\\
&= \alpha_n\|T_n-x^*\|^2+(1-\alpha_n)\|k_n-x^*\|^2 \\
&\leq \alpha_n\|T_n-x^*\|^2+(1-\alpha_n)\Big[\|x_n-x^*\|^2 \\
&\quad -\eta(1-\eta L)\|(I-M_Q)Ax_n\|^2 \\
&\quad -\left(1-\frac{\zeta}{d_1}\right)\|P_{Q_n}M_n-y_n\|^2-\left(1-\frac{\zeta}{d_1}\right))\|T_n-y_n\|^2\Big] \\
&\leq \alpha_n\big[\|x_n-x^*\|^2-\alpha_n\eta(1-\eta L)\|(I-M_Q)Ax_n\|^2\big] \\
&\quad +(1-\alpha_n)\Big[\|x_n-x^*\|^2-\eta(1-\eta L)\|(I-M_Q)Ax_n\|^2 \\
&\quad -\left(1-\frac{\zeta}{d_1}\right)\|P_{Q_n}M_n-y_n\|^2-\left(1-\frac{\zeta}{d_1}\right))\|T_n-y_n\|^2\Big] \\
&= \|x_n-x^*\|^2-\eta(1-\eta L)(1+\alpha_n)\|(I-M_Q)Ax_n\|^2 \\
&\quad -(1-\alpha_n)\left(1-\frac{\zeta}{d_1}\right)\big[\|T_n-y_n\|^2+\|y_n-k_n\|^2\big]. \qquad (13)
\end{aligned}$$

So,

$$\|x_{n+1}-x^*\|^2 \leq \|x_n-x^*\|^2.$$

Therefore $\lim_{n\to\infty}\|x_{n+1}-x^*\|$ exists, $\forall x^*\in\Im$. So, we have $\{x_n\}_{n=0}^{\infty}$ and $\{k_n\}_{n=0}^{\infty}$ are bounded. From the last relations it follows that

$$\eta(1-\eta L)(1+\alpha_n)\|(I-M_Q)Ax_n\|^2 \leq \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2$$

or

$$\|(I-M_Q)Ax_n\|^2 \leq \frac{\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2}{\eta(1-\eta L)(1+\alpha_n)}.$$

Thus

$$\lim_{n\to\infty}\|(I-M_Q)Ax_n\|=0. \tag{14}$$

By using the same method as above, we have

$$\lim_{n\to\infty}\|T_n-y_n\|=0. \tag{15}$$

From (12), we get

$$\begin{aligned}\left\|x_{n+1}-x^*\right\|^2 &\le \alpha_n\left\|T_n-x^*\right\|^2+(1-\alpha_n)\left\|Sk_n-x^*\right\|^2\\&\quad-\alpha_n(1-\alpha_n)\|T_n-Sk_n\|^2\\&\le \alpha_n\left\|T_n-x^*\right\|^2+(1-\alpha_n)\left\|x_n-x^*\right\|^2\\&\quad-\alpha_n(1-\alpha_n)\|T_n-Sk_n\|^2\\&\le \alpha_n\left\|x_n-x^*\right\|^2-\alpha_n(1-\alpha_n)\|T_n-Sk_n\|^2,\end{aligned}$$

so

$$\|T_n-Sk_n\|^2\le\frac{\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2}{\alpha_n(1-\alpha_n)},$$

which implies that

$$\lim_{n\to\infty}\|T_n-Sk_n\|=0. \tag{16}$$

Consider

$$W_n-x_n=-\eta A^*(I-M_Q)Ax_n,$$

and by (14), we have

$$\lim_{n\to\infty}\|W_n-x_n\|=0. \tag{17}$$

From the property of P_C, we have

$$\begin{aligned}&\left\|P_C(I-\zeta D_3)W_n-x^*\right\|^2\\&\quad=\left\|P_C(I-\zeta D_3)W_n-P_C(I-\zeta D_3)x^*\right\|^2\\&\quad\le\left\|(I-\zeta D_3)W_n-(I-\zeta D_3)x^*\right\|^2\\&\quad=\left\|\left(W_n-x^*\right)-\zeta\left(D_3W_n-D_3x^*\right)\right\|^2\\&\quad=\left\|W_n-x^*\right\|^2-2\zeta\left\langle W_n-x^*,D_3W_n-D_3x^*\right\rangle\\&\qquad+\zeta^2\left\|D_3W_n-D_3x^*\right\|^2\\&\quad\le\left\|W_n-x^*\right\|^2-2\zeta d_3\left\|D_3W_n-D_3x^*\right\|^2\\&\qquad+\zeta^2\left\|D_3W_n-D_3x^*\right\|^2\\&\quad=\left\|W_n-x^*\right\|^2-\zeta(2d_3-\zeta)\left\|D_3W_n-D_3x^*\right\|^2\\&\quad\le\left\|x_n-x^*\right\|^2-\zeta(2d_3-\zeta)\left\|D_3W_n-D_3x^*\right\|^2.\end{aligned} \tag{18}$$

By the definition of T_n, (7), Remark 1, and (18), we have

$$\left\|T_n-x^*\right\|^2\le a\|W_n-W_{x^*}\|^2+a(1-a)\|W_n-W_{x^*}\|^2$$

$$
\begin{aligned}
&+ (1-a)^2 \| P_C(I-\zeta D_3)W_n - x^* \|^2 \\
&\le a \| x_n - x^* \|^2 + a(1-a) \| x_n - x^* \|^2 \\
&\quad + (1-a)^2 \| P_C(I-\zeta D_3)W_n - x^* \|^2 \\
&\le (2a-a^2) \| x_n - x^* \|^2 + (1-a)^2 \| P_C(I-\zeta D_3)W_n - x^* \|^2 \\
&\le (2a-a^2) \| x_n - x^* \|^2 + (1-a)^2 \big[\| x_n - x^* \|^2 \\
&\quad - \zeta(2d_3-\zeta) \| D_3 W_n - D_3 x^* \|^2 \big] \\
&= \| x_n - x^* \|^2 - \zeta(2d_3-\zeta)(1-a)^2 \| D_3 W_n - D_3 x^* \|^2. \qquad (19)
\end{aligned}
$$

In addition, by the definition of x_{n+1} and (19), we have

$$
\begin{aligned}
\| x_{n+1} - x^* \|^2 &\le \alpha_n \| T_n - x^* \|^2 + (1-\alpha_n) \| k_n - x^* \|^2 \\
&\le \alpha_n \big[\| x_n - x^* \|^2 - \zeta(2d_3-\zeta)(1-a)^2 \| D_3 W_n - D_3 x^* \|^2 \big] \\
&\quad + (1-\alpha_n) \| k_n - x^* \|^2 \\
&= \alpha_n \| x_n - x^* \|^2 - \alpha_n \zeta(2d_3-\zeta)(1-a)^2 \| D_3 W_n - D_3 x^* \|^2 \\
&\quad + (1-\alpha_n) \| x_n - x^* \|^2 \\
&= \| x_n - x^* \|^2 - \alpha_n \zeta(2d_3-\zeta)(1-a)^2 \| D_3 W_n - D_3 x^* \|^2,
\end{aligned}
$$

so

$$
\| D_3 W_n - D_3 x^* \|^2 \le \frac{\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2}{\alpha_n \zeta(2d_3-\zeta)(1-a)^2},
$$

which implies that

$$
\lim_{n\to\infty} \| D_3 W_n - D_3 x^* \| = 0. \qquad (20)
$$

From the property of P_C, we have

$$
\begin{aligned}
&\| P_C(I-\zeta D_3)W_n - x^* \|^2 \\
&\le \langle (I-\zeta D_3)W_n - (I-\zeta D_3)x^*, P_C(I-\zeta D_3)W_n - x^* \rangle \\
&= \frac{1}{2} \big[\| (I-\zeta D_3)W_n - (I-\zeta D_3)x^* \|^2 \big] + \| P_C(I-\zeta D_3)W_n - x^* \|^2 \\
&\quad - \| (I-\zeta D_3)W_n - (I-\zeta D_3)x^* - (P_C(I-\zeta D_3)W_n - x^*) \|^2 \big] \\
&\le \frac{1}{2} \big[\| W_n - x^* \|^2 + \| P_C(I-\zeta D_3)W_n - x^* \|^2 \\
&\quad - \| (I-\zeta D_3)W_n - (I-\zeta D_3)x^* - (P_C(I-\zeta D_3)W_n - x^*) \|^2 \big] \\
&= \frac{1}{2} \big[\| W_n - x^* \|^2 + \| P_C(I-\zeta D_3)W_n - x^* \|^2 \\
&\quad - \| (W_n - P_C(I-\zeta D_3)W_n) - \zeta(D_3 W_n - D_3 x^*) \|^2 \big]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\big[\|W_n - x^*\|^2 + \|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\quad - \|W_n - P_C(I-\zeta D_3)W_n\|^2 - \zeta^2\|D_3W_n - D_3x^*\|^2 \\
&\quad + 2\zeta\langle W_n - P_C(I-\zeta D_3)W_n, D_3W_n - D_3x^*\rangle\big],
\end{aligned}
$$

so

$$
\begin{aligned}
\|P_C(I-\zeta D_3)W_n - x^*\|^2 \leq{}& \|W_n - x^*\|^2 - \|W_n - P_C(I-\zeta D_3)W_n\|^2 \\
&+ 2\zeta\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\|. \qquad (21)
\end{aligned}
$$

By the definition of T_n, (7), Remark 1, and (21), we have

$$
\begin{aligned}
&\|T_n - x^*\|^2 \\
&\quad\leq a\|W_n - W_{x^*}\|^2 + a(1-a)\|W_n - W_{x^*}\|^2 \\
&\qquad + (1-a)^2\|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\quad\leq a\|x_n - x^*\|^2 + a(1-a)\|x_n - x^*\|^2 \\
&\qquad + (1-a)^2\|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\quad\leq (2a-a^2)\|x_n - x^*\|^2 + (1-a)^2\|P_C(I-\zeta D_3)W_n - x^*\|^2 \\
&\quad\leq (2a-a^2)\|x_n - x^*\|^2 + (1-a)^2\big[\|W_n - x^*\|^2 - \|W_n - P_C \\
&\qquad \times (I-\zeta D_3)W_n\|^2 + 2\zeta\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\|\big] \\
&\quad= (2a-a^2)\|x_n - x^*\|^2 + (1-a)^2\|x_n - x^*\|^2 \\
&\qquad - (1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|^2 \\
&\qquad + 2\zeta(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\| \\
&\quad= \|x_n - x^*\|^2 - (1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|^2 \\
&\qquad + 2\zeta(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\|. \qquad (22)
\end{aligned}
$$

In addition, by the definition of x_{n+1}, (11), and (22), we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|k_n - x^*\|^2 \\
&\leq \alpha_n\big[\|x_n - x^*\|^2 - (1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|^2 \\
&\quad + 2\zeta(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\|\big] \\
&\quad + (1-\alpha_n)\|k_n - x^*\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 - \alpha_n(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|^2 \\
&\quad + 2\alpha_n\zeta(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\| \\
&\quad + (1-\alpha_n)\|x_n - x^*\|^2 \\
&= \|x_n - x^*\|^2 - \alpha_n(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|^2
\end{aligned}
$$

$$+ 2\alpha_n\zeta(1-a)^2\|W_n - P_C(I-\zeta D_3)W_n\|\|D_3W_n - D_3x^*\|. \tag{23}$$

From (20) and (23), we get

$$\lim_{n\to\infty}\|W_n - P_C(I-\zeta D_3)W_n\| = 0. \tag{24}$$

Let $G_n = aW_n + (1-a)P_C(I-\lambda_3 D_3)W_n$. From the property of P_C, we have

$$\begin{aligned}
&\|P_C(I-\zeta D_2)G_n - x^*\|^2 \\
&\quad= \|P_C(I-\zeta D_2)G_n - P_C(I-\zeta D_2)x^*\|^2 \\
&\quad\le \|(I-\zeta D_2)G_n - (I-\zeta D_2)x^*\|^2 \\
&\quad= \|(G_n - x^*) - \zeta(D_2G_n - D_2x^*)\|^2 \\
&\quad= \|G_n - x^*\|^2 - 2\zeta\langle G_n - x^*, D_2G_n - D_2x^*\rangle \\
&\qquad+ \zeta^2\|D_2G_n - D_2x^*\|^2 \\
&\quad\le \|x_n - x^*\|^2 - 2\zeta d_2\|D_2G_n - D_2x^*\|^2 \\
&\qquad+ \zeta^2\|D_2G_n - D_2x^*\|^2 \\
&\quad= \|x_n - x^*\|^2 - \zeta(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2.
\end{aligned} \tag{25}$$

By the definition of T_n and (25), we have

$$\begin{aligned}
\|T_n - x^*\|^2 &\le a\|W_n - W_{x^*}\|^2 + (1-a)\|P_C(I-\zeta D_2)G_n - x^*\|^2 \\
&\le a\|x_n - x^*\|^2 + (1-a)\big[\|x_n - x^*\|^2 \\
&\quad- \zeta(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2\big] \\
&= \|x_n - x^*\|^2 - \zeta(1-a)(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2.
\end{aligned} \tag{26}$$

In addition, by the definition of x_{n+1} and (26), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n\|T_n - x^*\|^2 + (1-\alpha_n)\|k_n - x^*\|^2 \\
&\le \alpha_n\big[\|x_n - x^*\|^2 - \zeta(1-a)(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2\big] \\
&\quad+ (1-\alpha_n)\|x_n - x^*\|^2 \\
&= \|x_n - x^*\|^2 - \zeta\alpha_n(1-\alpha_n)(2d_2-\zeta)\|D_2G_n - D_2x^*\|^2,
\end{aligned}$$

so

$$\|D_2G_n - D_2x^*\|^2 \le \frac{\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2}{\zeta\alpha_n(1-\alpha_n)(2d_2-\zeta)}.$$

It implies that

$$\lim_{n\to\infty}\|D_2G_n - D_2x^*\| = 0. \tag{27}$$

From the property of P_C, we have

$$\begin{aligned}
&\left\|P_C(I-\zeta D_2)G_n - x^*\right\|^2 \\
&\quad = \left\langle (I-\zeta D_2)G_n - (I-\zeta D_2)x^*, P_C(I-\zeta D_2)G_n - x^*\right\rangle \\
&\quad = \frac{1}{2}\Big[\left\|(I-\zeta D_2)G_n - (I-\zeta D_2)x^*\right\|^2 + \left\|P_C(I-\zeta D_2)G_n - x^*\right\|^2 \\
&\qquad - \left\|(I-\zeta D_2)G_n - (I-\zeta D_2)x^* - \big((I-\zeta D_2)G_n - x^*\big)\right\|^2\Big] \\
&\quad \le \frac{1}{2}\Big[\left\|G_n - x^*\right\|^2 + \left\|P_C(I-\zeta D_2)G_n - x^*\right\|^2 \\
&\qquad - \left\|(I-\zeta D_2)G_n - (I-\zeta D_2)x^* - \big((I-\zeta D_2)G_n - x^*\big)\right\|^2\Big] \\
&\quad = \frac{1}{2}\Big[\left\|G_n - x^*\right\|^2 + \left\|P_C(I-\zeta D_2)G_n - x^*\right\|^2 \\
&\qquad - \left\|\big(G_n - P_C(I-\zeta D_2)G_n\big) - \zeta\big(D_2G_n - D_2x^*\big)\right\|^2\Big] \\
&\quad = \frac{1}{2}\Big[\left\|G_n - x^*\right\|^2 + \left\|P_C(I-\zeta D_2)G_n - x^*\right\|^2 \\
&\qquad - \left\|G_n - P_C(I-\zeta D_2)G_n\right\|^2 \\
&\qquad + 2\zeta\left\langle G_n - P_C(I-\zeta D_2)G_n, D_2G_n - D_2x^*\right\rangle \\
&\qquad - \zeta^2\left\|D_2G_n - D_2x^*\right\|^2\Big].
\end{aligned}$$

It implies that

$$\begin{aligned}
&\left\|P_C(I-\zeta D_2)G_n - x^*\right\|^2 \\
&\quad \le \left\|G_n - x^*\right\|^2 - \left\|G_n - P_C(I-\zeta D_2)G_n\right\|^2 \\
&\qquad + 2\zeta\left\langle G_n - P_C(I-\zeta D_2)G_n, D_2G_n - D_2x^*\right\rangle \\
&\quad \le \left\|G_n - x^*\right\|^2 - \left\|G_n - P_C(I-\zeta D_2)G_n\right\|^2 \\
&\qquad + 2\zeta\left\|G_n - P_C(I-\zeta D_2)G_n\right\|\left\|D_2G_n - D_2x^*\right\|.
\end{aligned} \tag{28}$$

By the definition of T_n and (28), we have

$$\begin{aligned}
\left\|T_n - x^*\right\|^2 &\le a\|W_n - W_{x^*}\|^2 + (1-a)\left\|P_C(I-\zeta D_2)G_n - x^*\right\|^2 \\
&\le a\left\|x_n - x^*\right\|^2 + (1-a)\Big[\left\|G_n - x^*\right\|^2 - \big\|G_n - P_C \\
&\quad \times (I-\zeta D_2)G_n\big\|^2 + 2\zeta\left\|G_n - P_C(I-\zeta D_2)G_n\right\|\left\|D_2G_n - D_2x^*\right\|\Big] \\
&\le a\left\|x_n - x^*\right\|^2 + (1-a)\left\|x_n - x^*\right\|^2 \\
&\quad - (1-a)\left\|G_n - P_C(I-\zeta D_2)G_n\right\|^2 \\
&\quad + 2\zeta\left\|G_n - P_C(I-\zeta D_2)G_n\right\|\left\|D_2G_n - D_2x^*\right\|] \\
&= \left\|x_n - x^*\right\|^2 - (1-a)\left\|G_n - P_C(I-\zeta D_2)G_n\right\|^2 \\
&\quad + 2\zeta\left\|G_n - P_C(I-\zeta D_2)G_n\right\|\left\|D_2G_n - D_2x^*\right\|].
\end{aligned} \tag{29}$$

In addition, by the definition of x_{n+1} and (29), we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|T_n-x^*\|^2+(1-\alpha_n)\|k_n-x^*\|^2\\ &\le \alpha_n\big[\|x_n-x^*\|^2-(1-a)\|G_n-P_C(I-\zeta D_2)G_n\|^2\\ &\quad +2\zeta\|G_n-P_C(I-\zeta D_2)G_n\|\|D_2G_n-D_2x^*\|\big]\\ &\quad +(1-\alpha_n)\|x_n-x^*\|^2\\ &=\|x_n-x^*\|^2-\alpha_n(1-a)\|G_n-P_C(I-\zeta D_2)G_n\|^2\\ &\quad +2\zeta\alpha_n(1-a)\|G_n-P_C(I-\zeta D_2)G_n\|\|D_2G_n-D_2x^*\|,\end{aligned}\tag{30}$$

by (30) and (27), we get

$$\lim_{n\to\infty}\|G_n-P_C(I-\zeta D_2)G_n\|=0.\tag{31}$$

Since

$$T_n-W_n=(1-a)\big(P_C(I-\zeta D_2)\big(aW_n+(1-a)P_C(I-\zeta D_3)W_n\big)-W_n\big).$$

From the property of norm, we have

$$\begin{aligned}&\|P_C(I-\zeta D_2)\big(aW_n+(1-a)P_C(I-\zeta D_3)W_n\big)-W_n\|\\ &\quad\le\|P_C(I-\zeta D_2)\big(aW_n+(1-a)P_C(I-\zeta D_3)W_n\big)\\ &\qquad-\big(aW_n+(1-a)P_C(I-\zeta D_3)W_n\big)\|\\ &\qquad+\|(aW_n+(1-a)P_C(I-\zeta D_3)W_n-W_n\|\\ &\quad=\|P_C(I-\zeta D_2)G_n-G_n\|+(1-a)\|P_C(I-\zeta D_3)W_n-W_n\|.\end{aligned}\tag{32}$$

Then we have

$$\begin{aligned}\|T_n-W_n\|\le(1-a)\big[&\|P_C(I-\zeta D_2)G_n-G_n\|\\ &+(1-a)\|P_C(I-\zeta D_3)W_n-W_n\|\big].\end{aligned}$$

From (24) and (31), it implies that

$$\lim_{n\to\infty}\|T_n-W_n\|=0.\tag{33}$$

From (15), (17), (33), and

$$\|y_n-x_n\|\le\|y_n-T_n\|+\|T_n-W_n\|+\|W_n-x_n\|,$$

we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0.\tag{34}$$

Moreover, from (16), (15), (34), and

$$\|x_n - Sk_n\| \leq \|x_n - y_n\| + \|y_n - T_n\| + \|T_n - Sk_n\|,$$

we have

$$\lim_{n\to\infty} \|x_n - Sk_n\| = 0. \tag{35}$$

Since $\{x_n\}_{n=0}^{\infty}$ is bounded, it has a subsequence $\{x_{n_k}\}_{k=0}^{\infty}$ which weakly converges to some $\bar{x} \in C$.

Assume $\bar{x} \notin F(S)$. By the nonexpansiveness of S and Opial's property and (35), we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| &< \liminf_{k\to\infty} \|x_{n_k} - S\bar{x}\| \\
&\leq \liminf_{k\to\infty} \bigl[\|x_{n_k} - Sk_{n_k}\| + \|Sk_{n_k} - S\bar{x}\|\bigr] \\
&\leq \liminf_{k\to\infty} \bigl[\|x_{n_k} - Sk_{n_k}\| + \|k_{n_k} - \bar{x}\|\bigr] \\
&= \liminf_{k\to\infty} \|k_{n_k} - \bar{x}\| \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\|.
\end{aligned}$$

This is a contradiction, then we have

$$\bar{x} \in F(S).$$

Assume $\bar{x} \notin \bigcap_{i=1}^{3} \Phi_i$. From Lemma 2.6, we have $\bar{x} \notin F(M_C(I - \eta A^*(I - M_Q)A))$. By Opial's condition, (34), and Remark 1, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| &< \liminf_{k\to\infty} \bigl\|x_{n_k} - M_C\bigl(I - \eta A^*(I - M_Q)A\bigr)\bar{x}\bigr\| \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - y_{n_k}\| + \liminf_{k\to\infty} \bigl\|M_C\bigl(x_{n_k} - \eta A^* \\
&\quad \times (I - M_Q)Ax_{n_k}\bigr) - M_C\bigl(I - \eta A^*(I - M_Q)A\bigr)\bar{x}\bigr\| \\
&\leq \liminf_{k\to\infty} \bigl(\|x_{n_k} - y_{n_k}\| + \|x_{n_k} - \bar{x}\|\bigr) \\
&= \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\|.
\end{aligned} \tag{36}$$

This is a contradiction, then we have

$$\bar{x} \in F\bigl(M_C\bigl(I - \eta A^*(I - M_Q)A\bigr)\bigr).$$

It implies that

$$\bar{x} \in \bigcap_{i=1}^{3} \Phi_i.$$

Hence

$$\bar{x} \in \Im.$$

In order to show that the entire sequence $\{x_n\}$ weakly converges to $\bar{x}$, assume $\{x\}_{n_k} \rightharpoonup \hat{x}$ as $k \to \infty$, with $\bar{x} \neq \hat{x}$ and $\hat{x} \in \Im$. By Opial's condition, we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - \bar{x}\| &= \lim_{k\to\infty} \inf \|x_{n_k} - \bar{x}\| \\
&< \lim_{k\to\infty} \inf \|x_{n_k} - \hat{x}\| \\
&= \lim_{n\to\infty} \|x_n - \hat{x}\| \\
&= \lim_{n\to\infty} \inf \|x_{n_k} - \hat{x}\| \\
&< \lim_{n\to\infty} \inf \|x_{n_k} - \bar{x}\| \\
&= \lim_{n\to\infty} \|x_n - \bar{x}\|.
\end{aligned}$$

This is a contradiction, thus

$$\bar{x} \doteq \hat{x}.$$

It implies that the sequence $\{x_n\}_{n=0}^{\infty}$ weakly converges to $\bar{x} \in \Im$.

From (34), we have $\{y_n\}_{n=0}^{\infty}$ weakly converges to $\bar{x} \in \Im$.

Finally, if we take

$$U_n = P_{\Im} x_n, \tag{37}$$

by Lemma 2.2, we see that $\{P_{\Im} x_n\}_{n=0}^{\infty}$ converges strongly to some $z \in \Im$. From (37), we get

$$\langle \bar{x} - U_n, U_n - x_n \rangle \geq 0, \quad \forall \bar{x} \in \Im.$$

Take $n \to \infty$, we also have

$$\langle \bar{x} - z, z - \bar{x} \rangle \geq 0,$$

and hence $\bar{x} = z$. Therefore U_n converges strongly to $\bar{x} \in \Im$, this completes the proof. □

4 Application

Let C be a closed convex subset of H. The standard constrained convex optimization problem is to find $x^* \in C$ such that

$$\Im\bigl(x^*\bigr) = \min_{x\in C} \Im(x), \tag{38}$$

where $\Im : C \to \mathbb{R}$ is a convex, Frechet differentiable function. The set of all solution of (38) is denoted by $\Phi_{\Im}$.

Lemma 4.1 ([25] Optimality condition) *A necessary condition of optimality for a point $x^* \in C$ to be a solution of the minimization problem (38) is that x^* solves the variational inequality*

$$\langle \nabla \Im(x^*), x - x^* \rangle \geq 0 \tag{39}$$

for all $x \in C$. Equivalently, $x^* \in C$ solves the fixed point equation

$$x^* = P_C(I - \zeta \nabla \Im)x^*$$

for every $\zeta > 0$. If, in addition, $\Im$ is convex, then the optimality condition (39) is also sufficient.

By using the concept of the split modified system of variational inequalities problem (SMSVIP), we consider the problem for finding $(x^*, y^*, z^*) \in C \times C \times C$ such that

$$\begin{cases} \langle x^* - (I - \zeta \nabla \Im_1)(ax^* + (1-a)y^*), x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle y^* - (I - \zeta \nabla \Im_2)(ax^* + (1-a)z^*), x - y^* \rangle \geq 0, & \forall x \in C, \\ \langle z^* - (I - \zeta \nabla \Im_3)x^*, x - z^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{40}$$

and finding $(\bar{x^*} = Ax^*, \bar{y^*} = Ay^*, \bar{z^*} = Az^*) \in Q \times Q \times Q$ such that

$$\begin{cases} \langle \bar{x^*} - (I - \bar{\zeta} \nabla \bar{\Im_1})(a\bar{x^*} + (1-a)\bar{y^*}), \bar{x} - \bar{x^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{y^*} - (I - \bar{\zeta} \nabla \bar{\Im_2})(a\bar{x^*} + (1-a)\bar{z^*}), \bar{x} - \bar{y^*} \rangle \geq 0, & \forall \bar{x} \in Q, \\ \langle \bar{z^*} - (I - \bar{\zeta} \nabla \bar{\Im_3})\bar{x^*}, \bar{x} - \bar{z^*} \rangle \geq 0, & \forall \bar{x} \in Q, \end{cases} \tag{41}$$

where $\Im_1, \Im_2, \Im_3 : C \to \mathbb{R}$ with $\nabla \Im_1, \nabla \Im_2, \nabla \Im_3$ are the gradients of $\Im_1, \Im_2, \Im_3$, respectively, and $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3 : Q \to \mathbb{R}$ with $\nabla \bar{\Im}_1, \nabla \bar{\Im}_2, \nabla \bar{\Im}_3$ are the gradients of $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3$, respectively, $\zeta, \bar{\zeta} > 0$ and $a \in [0,1]$. The sets of all solution of (40) and (41) are denoted by $\Psi_{\nabla \Im_1, \nabla \Im_2, \nabla \Im_3}$ and $\Psi_{\nabla \bar{\Im_1}, \nabla \bar{\Im_2}, \nabla \bar{\Im_3}}$, respectively. The set of all solutions of the split modified system of variational inequalities (SMSVIP) is denoted by $\Psi^{\nabla \Im_1, \nabla \Im_2, \nabla \Im_3}_{\nabla \bar{\Im_1}, \nabla \bar{\Im_2}, \nabla \bar{\Im_3}}$, that is,

$$\Psi^{\nabla \Im_1, \nabla \Im_2, \nabla \Im_3}_{\nabla \bar{\Im_1}, \nabla \bar{\Im_2}, \nabla \bar{\Im_3}} = \left\{ (x^*, y^*, z^*) \in \Psi_{\nabla \Im_1, \nabla \Im_2, \nabla \Im_3} : (\bar{x^*}, \bar{y^*}, \bar{z^*}) \in \Psi_{\nabla \bar{\Im_1}, \nabla \bar{\Im_2}, \nabla \bar{\Im_3}} \right\}.$$

Lemma 4.2 ([23]) *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $\Im_1, \Im_2, \Im_3 : C \to \mathbb{R}$ be real-valued convex functions with the gradients $\nabla \Im_1, \nabla \Im_2, \nabla \Im_3$ being $\frac{1}{L_{\Im_1}}, \frac{1}{L_{\Im_2}}, \frac{1}{L_{\Im_3}}$-inverse strongly monotone and continuous, respectively, where $\zeta \in (0, \frac{2}{L_\Im})$ with $\frac{1}{L_\Im} = \min\{\frac{1}{L_{\Im_1}}, \frac{1}{L_{\Im_2}}, \frac{1}{L_{\Im_3}}\}$. Let $\bar{\Im}_1, \bar{\Im}_2, \bar{\Im}_3 : Q \to \mathbb{R}$ be real-valued convex functions with the gradients $\nabla \bar{\Im}_1$, $\nabla \bar{\Im}_2$, $\nabla \bar{\Im}_3$ being $\frac{1}{L_{\bar{\Im}_1}}, \frac{1}{L_{\bar{\Im}_2}}, \frac{1}{L_{\bar{\Im}_3}}$-inverse strongly monotone and continuous, respectively, where $\bar{\zeta} \in (0, \frac{2}{L_{\bar{\Im}}})$ with $\frac{1}{L_{\bar{\Im}}} = \min\{\frac{1}{L_{\bar{\Im}_1}}, \frac{1}{L_{\bar{\Im}_2}}, \frac{1}{L_{\bar{\Im}_3}}\}$. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $\eta \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C : H_1 \to C$ by $M_C(x) = P_C(I - \zeta \nabla \Im_1)(ax + (1-a)P_C(I - \zeta \nabla \Im_2)(ax + (1-a)P_C(I - \zeta \nabla \Im_3)x))$, $\forall x \in H_1$, and define $M_Q : H_2 \to Q$ by $M_Q(\hat{x}) = P_Q(I - \bar{\zeta} \nabla \bar{\Im}_1)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta} \nabla \bar{\Im}_2)(a\hat{x} + (1-a)P_Q(I - \bar{\zeta} \nabla \bar{\Im}_3)\hat{x}))$, $\forall \hat{x} \in H_2$.*

Let $\bigcap_{i=1}^{3}\Phi_{\Im_i}\neq\emptyset$ *and* $\Phi_{\Im_i}=\{\Im_i(x)=\min_{x^*\in C}\Im_i(x^*):\bar{\Im}_i(Ax)=\min_{Ax^*\in Q}\bar{\Im}_i(Ax^*)\}$ *for all* $i=1,2,3$. *Then*

$$\bigcap_{i=1}^{3}\Phi_{\Im_i}=F\bigl(M_C\bigl(I-\eta A^*(I-M_Q)A\bigr)\bigr).$$

Theorem 4.3 *Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $S:C\to C$ be a nonexpansive mapping. Let $\Im_1,\Im_2,\Im_3:C\to\mathbb{R}$ be real-valued convex functions with the gradients $\nabla\Im_1,\nabla\Im_2,\nabla\Im_3$ being $\frac{1}{L_{\Im_1}},\frac{1}{L_{\Im_2}},\frac{1}{L_{\Im_3}}$-inverse strongly monotone and continuous, respectively, where $\zeta\in(0,\frac{2}{L_\Im})$ with $\frac{1}{L_\Im}=\min\{\frac{1}{L_{\Im_1}},\frac{1}{L_{\Im_2}},\frac{1}{L_{\Im_3}}\}$. Let $\bar{\Im}_1,\bar{\Im}_2,\bar{\Im}_3:Q\to\mathbb{R}$ be real-valued convex functions with the gradients $\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3$ being $\frac{1}{L_{\bar{\Im}_1}},\frac{1}{L_{\bar{\Im}_2}},\frac{1}{L_{\bar{\Im}_3}}$-inverse strongly monotone and continuous, respectively, where $\bar{\zeta}\in(0,\frac{2}{L_{\bar{\Im}}})$ with $\frac{1}{L_{\bar{\Im}}}=\min\{\frac{1}{L_{\bar{\Im}_1}},\frac{1}{L_{\bar{\Im}_2}},\frac{1}{L_{\bar{\Im}_3}}\}$. Let $A:H_1\to H_2$ be a bounded linear operator with adjoint A^* and $\eta\in(0,\frac{1}{L})$ with L being the spectral radius of the operator A^*A. Define $M_C:H_1\to C$ by $M_C(x)=P_C(I-\zeta\nabla\Im_1)(ax+(1-a)P_C(I-\zeta\nabla\Im_2)(ax+(1-a)P_C(I-\zeta\nabla\Im_3)x))$, $\forall x\in H_1$, and define $M_Q:H_2\to Q$ by $M_Q(\hat{x})=P_Q(I-\bar{\zeta}\nabla\bar{\Im}_1)(a\hat{x}+(1-a)P_Q(I-\bar{\zeta}\nabla\bar{\Im}_2)(a\hat{x}+(1-a)P_Q(I-\bar{\zeta}\nabla\bar{\Im}_3)\hat{x}))$, $\forall\hat{x}\in H_2$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1\in H_1$ and*

$$y_n=M_CW_n=P_C(I-\zeta\nabla\Im_1)T_n,$$

where $W_n=(I-\eta A^*(I-M_Q)A)x_n$ and $T_n=aW_n+(1-a)P_C(I-\zeta\nabla\Im_2)(aW_n+(1-a)P_C(I-\zeta\nabla\Im_3)W_n))$.

$$Q_n=\bigl\{z\in H:\bigl\langle(I-\zeta\nabla\Im_1)T_n-y_n,y_n-z\bigr\rangle\geq 0\bigr\},$$
$$x_{n+1}=\alpha_nT_n+(1-\alpha_n)SP_{Q_n}\bigl(T_n-\zeta\nabla\Im_1(y_n)\bigr),\quad\forall n\in\mathbb{N}.$$

Assume that the following conditions hold:

(i) $\Im=F(S)\bigcap\bigcap_{i=1}^{3}\Phi_{\Im_i}\neq\emptyset$, *where* $\Phi_{\Im_i}=\{\Im_i(x)=\min_{x^*\in C}\Im_i(x^*):\bar{\Im}_i(Ax)=\min_{Ax^*\in Q}\bar{\Im}_i(Ax^*)\}$ *for all* $i=1,2,3$.

(ii) $\alpha_n\in[c,d]\subset(0,1)$.

Then $\{x_n\}$ converges weakly to $x_0=P_\Im x_n$, which $(x_0,y_0,z_0)\in\Omega^{\nabla\Im_1,\nabla\Im_2,\nabla\Im_3}_{\nabla\bar{\Im}_1,\nabla\bar{\Im}_2,\nabla\bar{\Im}_3}$, where $y_0=P_C(I-\zeta\nabla\Im_2)(ax_0+(1-a)z_0)$ and $z_0=P_C(I-\zeta\nabla\Im_3)x_0$ with $\bar{x_0}=Ax_0$, $\bar{y_0}=Ay_0$, and $\bar{z_0}=Az_0$.

Proof By using Theorem 3.1 and Lemma 4.2, we obtain the conclusion. □

5 Example and numerical results

In this section, we give the following example to support our main theorem.

Example 5.1 Let $\mathbb{R}$ be the set of real numbers, $C:=\{x\in H|1\leq 2x_1+x_2\leq 7\}$, $Q:=\{x\in H|-10\leq 3x_1-x_2\leq 20\}$, $H_1=H_2=\mathbb{R}^2$. Let $D_1,D_2,D_3:C\to\mathbb{R}^2$ be defined by $D_1(x_1,x_2)=(x_1-2,x_2+1)$, $D_2(x_1,x_2)=(x_1-3,x_2-\frac{5}{2})$, and $D_3(x_1,x_2)=(x_1+2,x_2-6)$ for all $(x_1,x_2)\in C$. Let $\bar{D_1},\bar{D_2},\bar{D_3}:Q\to\mathbb{R}^2$ be defined by $\bar{D_1}(\bar{x_1},\bar{x_2})=(\bar{x_1}-4,\bar{x_2}+8)$, $\bar{D_2}(\bar{x_1},\bar{x_2})=(\bar{x_1}-12,\bar{x_2}-8)$, and $\bar{D_3}(\bar{x_1},\bar{x_2})=(\bar{x_1}+16,\bar{x_2}-30)$ for all $(\bar{x_1},\bar{x_2})\in Q$. Let $A:\mathbb{R}^2\to\mathbb{R}^2$ be defined by

$A(x_1,x_2) = (2x_1,2x_2)$ and $A^* : \mathbb{R}^2 \to \mathbb{R}$ be defined by $A^*(x_1,x_2) = (2x_1,2x_2)$. Define $M_C : H_1 \to C$ by $M_C(x) = P_C(I-\frac{1}{2}D_1)(\frac{1}{2}x+\frac{1}{2}P_C(I-\frac{1}{2}D_2)(\frac{1}{2}x+\frac{1}{2}P_C(I-\frac{1}{2}D_3)x))$, $\forall x = (x_1,x_2) \in H_1$, define $M_Q : H_2 \to Q$ by $M_Q(\hat{x}) = P_Q(I - \frac{1}{5}\bar{D}_1)(\frac{1}{2}\hat{x} + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D}_2)(\frac{1}{2}\hat{x} + \frac{1}{2}P_Q(I - \frac{1}{5}\bar{D}_3)\hat{x}))$, $\forall \hat{x} = (\hat{x_1},\hat{x_2}) \in H_2$, and define $S : C \to C$ by $S(x_1,x_2) = (\frac{x_1}{2} + 1, \frac{x_2}{2})$. Let the sequences $\{x_n\}$ and $\{y_n\}$ be generated by $x_1 \in H_1$ and

$$y_n = M_C W_n = P_C\left(I - \frac{1}{2}(x_1 - 2, x_2 + 1)\right)T_n,$$

where $W_n = (I - \frac{1}{8}A^*(I-M_Q)A)x_n$ and $T_n = \frac{1}{2}W_n + \frac{1}{2}P_C(I-\frac{1}{2}(x_1-3,x_2-\frac{5}{2}))(\frac{1}{2}W_n + \frac{1}{2}P_C(I-\frac{1}{2}(x_1+2,x_2-6))W_n))$,

$$Q_n = \left\{z \in H : \left\langle \left(I - \frac{1}{2}(x_1 - 2, x_2 + 1)\right)T_n - y_n, y_n - z\right\rangle \geq 0\right\},$$

and

$$x_{n+1} = \frac{n+1}{5n}T_n + \left(1 - \frac{n+1}{5n}\right)SP_{Q_n}\left(T_n - \frac{1}{2}(x_1 - 2, x_2 + 1)(y_n)\right), \quad \forall n \in \mathbb{N},$$

where

$$P_C x = \begin{cases} (x_1,x_2) - \frac{[2x_1+x_2-7](2,1)}{5} & \text{if } 2x_1 + x_2 > 7, \\ (x_1,x_2) & \text{if } 1 \leq 2x_1 + x_2 \leq 7, \\ (x_1,x_2) - \frac{[2x_1+x_2-1](2,1)}{5} & \text{if} 2x_1 + x_2 < 1, \end{cases}$$

for every $x = (x_1,x_2) \in H_1$ and

$$P_Q\hat{x} = \begin{cases} (x_1,x_2) - \frac{[3x_1-x_2-20](3,-1)}{10} & \text{if } 3x_1 - x_2 > 20, \\ (x_1,x_2) & \text{if } -10 \leq 3x_1 - x_2 \leq 20, \\ (x_1,x_2) - \frac{[3x_1-x_2+10](3,-1)}{10} & \text{if } 3x_1 - x_2 < -10, \end{cases}$$

for every $\hat{x} = (x_1,x_2) \in H_2$. By the definition of $S, D_i, \bar{D}_i, M_C, M_Q$ for every $i = 1,2,3$, we have $(2,0) \in F(M_C(I - \frac{1}{8}A^*(I-M_Q)A))$. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $(2,0)$.

Table 1 and Fig. 1 show the numerical results of sequences $\{x_n\}$ and $\{y_n\}$ where $x_1 = (-5,5)$ and $n = N = 30$.

Table 1 The values of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = (-5,5)$ and $n = N = 30$

n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$
1	(−5.000000, 5.000000)	(0.034028, 1.404266)
2	(−0.457465, 2.305332)	(1.309813, 0.647460)
3	(1.223540, 1.203392)	(1.781929, 0.337977)
⋮	⋮	⋮
15	(2.000000, 0.000000)	(2.000000, 0.000000)
⋮	⋮	⋮
28	(2.000000, 0.000000)	(2.000000, 0.0000000)
29	(2.000000, 0.000000)	(2.000000, 0.000000)
30	(2.000000, 0.000000)	(2.000000, 0.000000)

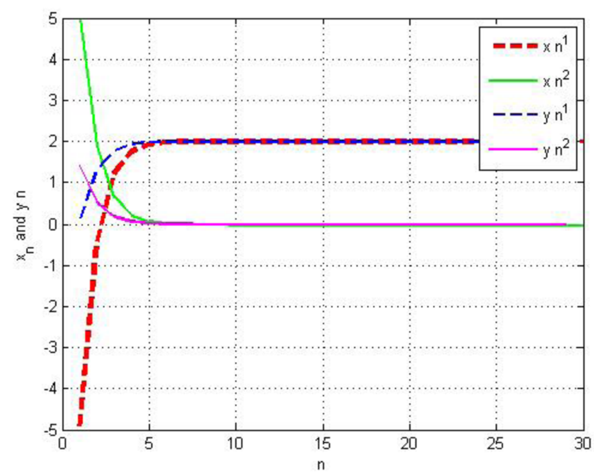

Figure 1 The convergence of $\{x_n\}$ and $\{y_n\}$ with initial values $x_1 = (-5, 5)$ and $n = N = 30$

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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AK dealt with the conceptualization, formal analysis, supervision, writing—review and editing. AS writing—original draft, formal analysis, writing—review and editing. Both authors have read and approved the manuscript.

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