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A new reverse Hardy–Hilbert inequality with the power function as intermediate variables

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Abstract

In this paper, by virtue of the symmetry principle, applying the techniques of real analysis and Euler–Maclaurin summation formula, we construct proper weight coefficients and use them to establish a reverse Hardy–Hilbert inequality with the power function as intermediate variables. Then, we obtain the equivalent forms and some equivalent statements of the best possible constant factor related to several parameters. Finally, we illustrate how the obtained results can generate some particular reverse Hardy–Hilbert inequalities.

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Keywords: Weight coefficient; Hardy–Hilbert inequality; Equivalent statement; Parameter; Power function; Reverse

1 Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $0 < \sum_{m=1}^{\infty} a_m^p < \infty$, and $0 < \sum_{n=1}^{\infty} b_n^q < \infty$. We have the following well-known Hardy–Hilbert inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

In 2006, by introducing multi parameters $\lambda_i \in (0,2]$ $(i = 1,2)$, $\lambda_1 + \lambda_2 = \lambda \in (0,4]$,an extension of (1) was provided by Krnić et al. [2] as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty} m^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty} n^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}, \tag{2}$$

where the constant factor $B(\lambda_1, \lambda_2)$ is the best possible and

$$B(u,v) = \int_0^{\infty}\frac{t^{u-1}}{(1+t)^{u+v}}\,dt \quad (u,v > 0) \tag{3}$$

is the beta function. For $p = q = 2$, $\lambda_1 = \lambda_2 = \frac{\lambda}{2}$, inequality (2) reduces to Yang's inequality in [3] as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(\sum_{m=1}^{\infty} m^{1-\lambda}a_m^2 \sum_{n=1}^{\infty} n^{1-\lambda}b_n^2\right)^{\frac{1}{2}}. \tag{4}$$

Recently, by using inequality (2), Adiyasuren et al. [4] gave a new Hardy–Hilbert inequality with the best possible constant factor $\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ involving two partial sums as follows: For $\lambda_i \in (0,1] \cap (0,\lambda)$ $(i = 1,2)$, $\lambda_1 + \lambda_2 = \lambda \in (0,2]$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < \lambda_1\lambda_2 B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty} m^{-p\lambda_1-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-q\lambda_2-1}B_n^q\right)^{\frac{1}{q}}, \tag{5}$$

where, for $a_m, b_n \geq 0$, two partial sums $A_m = \sum_{i=1}^{m} a_i$ $B_n = \sum_{k=1}^{n} b_k$ are indicated, satisfying

$$0 < \sum_{m=1}^{\infty} m^{-p\lambda_1-1}A_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=1}^{\infty} n^{-q\lambda_2-1}B_n^q < \infty.$$

Inequalities (1) and (2) with their integral analogues and the reverses play an important role in the analysis and its applications (cf. [5–16]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [1], Theorem 351): If $K(t)$ $(t > 0)$ is a decreasing function, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi(s) = \int_0^{\infty} K(t)t^{s-1}\,dt < \infty$, $a_n \geq 0$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, then we have

$$\int_0^{\infty} x^{p-2}\left(\sum_{n=1}^{\infty} K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^{\infty} a_n^p. \tag{6}$$

Some new extensions of (6) with their reverses were provided by [17–22].

In 2016, Hong et al. [23] obtained some equivalent statements of the extensions of (1) with the best possible constant factor related to several parameters. The other similar works were given by [24–31]. In 2019–2020, Luo et al. [32] considered a new inequality of the extension of (2) with the general decreasing kernel as $k_{\lambda}(m^{\alpha}, n^{\beta})$ $(\lambda, \alpha, \beta > 0)$; Huang et al. [33] also gave a reverse of (2) by using the Euler–Maclaurin summation formula.

In this paper, following the way of [2, 23], by virtue of the symmetry principle, by means of the weight coefficients, the idea of introduced parameters, and the techniques of real analysis, we apply the Euler–Maclaurin summation formula to provide a reverse Hardy–Hilbert inequality with the kernel as follows:

$$\frac{1}{(m^{\alpha}+n^{\beta})^{\lambda}} \quad (\lambda \in (0,6], \alpha, \beta \in (0,1]),$$

which is an extension of [33]'s work. The equivalent forms, some equivalent statements of the best possible constant factor related to several parameters, and some particular inequalities are also obtained.

2 Some lemmas

In what follows, we suppose that $0 < p < 1$ $(q < 0)$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda \in (0,6]$, $\alpha, \beta \in (0,1]$, $\lambda_1 \in (0, \frac{2}{\alpha}] \cap (0,\lambda)$, $\lambda_2 \in (0, \frac{2}{\beta}] \cap (0,\lambda)$,

$$k_\lambda(\lambda_i) := B(\lambda_i, \lambda - \lambda_i) \quad (i = 1,2).$$

$a_m, b_n \geq 0$ $(m, n \in \mathrm{N} = \{1,2,\ldots\})$ such that

$$0 < \sum_{m=1}^{\infty} m^{p[1-\alpha(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})]-1} a_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=1}^{\infty} n^{q[1-\beta(\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})]-1} b_n^q < \infty. \tag{7}$$

Lemma 1 *For $\lambda_2 \in (0, \frac{2}{\beta}] \cap (0,\lambda)$ $(\lambda_1 \in (0,\lambda))$, define the following weight coefficient*:

$$\varpi(\lambda_2, m) := m^{\alpha(\lambda-\lambda_2)} \sum_{n=1}^{\infty} \frac{\beta n^{\beta\lambda_2 - 1}}{(m^\alpha + n^\beta)^\lambda} \quad (m \in \mathrm{N}). \tag{8}$$

We have the following inequalities:

$$0 < k_\lambda(\lambda_2)\left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right) < \varpi(\lambda_2, m) < k_\lambda(\lambda_2) \quad (m \in \mathrm{N}), \tag{9}$$

where $O(\frac{1}{m^{\alpha\lambda_2}}) := \frac{1}{k_\lambda(\lambda_2)} \int_0^{\frac{1}{m^\alpha}} \frac{u^{\lambda_2 - 1}}{(1+u)^\lambda}\, du > 0$.

Proof For fixed $m \in \mathrm{N}$, we set the real function $g(m,t)$ as follows:

$$g(m,t) := \frac{\beta t^{\beta\lambda_2 - 1}}{(m^\alpha + t^\beta)^\lambda} \quad (t > 0).$$

By means of the Euler–Maclaurin summation formula (cf. [2, 3]) and the Bernoulli function of 1-order $P_1(t) := t - [t] - \frac{1}{2}$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} g(m,n) &= \int_1^\infty g(m,t)\, dt + \frac{1}{2} g(m,1) + \int_1^\infty P_1(t) g'(m,t)\, dt \\
&= \int_0^\infty g(m,t)\, dt - h(m), \\
h(m) &:= \int_0^1 g(m,t)\, dt - \frac{1}{2} g(m,1) - \int_1^\infty P_1(t) g'(m,t)\, dt.
\end{aligned}$$

We obtain $-\frac{1}{2} g(m,1) = \frac{-\beta}{2(m^\alpha + 1)^\lambda}$ and

$$\begin{aligned}
-g'(m,t) &= -\frac{\beta(\beta\lambda_2 - 1) t^{\beta\lambda_2 - 2}}{(m^\alpha + t^\beta)^\lambda} + \frac{\beta^2 \lambda t^{\beta + \beta\lambda_2 - 2}}{(m^\alpha + t^\beta)^{\lambda+1}} \\
&= -\frac{\beta(\beta\lambda_2 - 1) t^{\beta\lambda_2 - 2}}{(m^\alpha + t^\beta)^\lambda} + \frac{\beta^2 \lambda (m^\alpha + t^\beta - m^\alpha) t^{\beta\lambda_2 - 2}}{(m^\alpha + t^\beta)^{\lambda+1}} \\
&= \frac{\beta(\beta\lambda - \beta\lambda_2 + 1) t^{\beta\lambda_2 - 2}}{(m^\alpha + t^\beta)^\lambda} - \frac{\beta^2 \lambda m^\alpha t^{\beta\lambda_2 - 2}}{(m^\alpha + t^\beta)^{\lambda+1}}.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
\int_0^1 g(m,t)\,dt &= \int_0^1 \frac{\beta t^{\beta\lambda_2-1}}{(m^\alpha+t^\beta)^\lambda}\,dt \overset{u=t^\beta}{=} \int_0^1 \frac{u^{\lambda_2-1}}{(m^\alpha+u)^\lambda}\,du \\
&= \frac{1}{\lambda_2}\int_0^1 \frac{du^{\lambda_2}}{(m^\alpha+u)^\lambda} \\
&= \frac{1}{\lambda_2}\frac{u^{\lambda_2}}{(m^\alpha+u)^\lambda}\Big|_0^1 + \frac{\lambda}{\lambda_2}\int_0^1 \frac{u^{\lambda_2}}{(m^\alpha+u)^{\lambda+1}}\,du \\
&= \frac{1}{\lambda_2}\frac{1}{(m^\alpha+1)^\lambda} + \frac{\lambda}{\lambda_2(\lambda_2+1)}\int_0^1 \frac{du^{\lambda_2+1}}{(m^\alpha+u)^{\lambda+1}} \\
&> \frac{1}{\lambda_2}\frac{1}{(m^\alpha+1)^\lambda} + \frac{\lambda}{\lambda_2(\lambda_2+1)}\left[\frac{u^{\lambda_2+1}}{(m^\alpha+u)^{\lambda+1}}\right]_0^1 \\
&\quad + \frac{\lambda(\lambda+1)}{\lambda_2(\lambda_2+1)(m^\alpha+1)^{\lambda+2}}\int_0^1 u^{\lambda_2+1}\,du \\
&= \frac{1}{\lambda_2}\frac{1}{(m^\alpha+1)^\lambda} + \frac{\lambda}{\lambda_2(\lambda_2+1)}\frac{1}{(m^\alpha+1)^{\lambda+1}} \\
&\quad + \frac{\lambda(\lambda+1)}{\lambda_2(\lambda_2+1)(\lambda_2+2)}\frac{1}{(m^\alpha+1)^{\lambda+2}}.
\end{aligned}$$

For $0<\lambda_2\leq\frac{2}{\beta}$, $0<\beta\leq 1$, $\lambda_2<\lambda\leq 6$, it follows that

$$(-1)^i\frac{d^i}{dt^i}\left[\frac{t^{\beta\lambda_2-2}}{(m^\alpha+t^\beta)^\lambda}\right]>0,\ (-1)^i\frac{d^i}{dt^i}\left[\frac{t^{\beta\lambda_2-2}}{(m^\alpha+t^\beta)^{\lambda+1}}\right]>0 \quad (i=0,1,2,3).$$

Still using the Euler–Maclaurin summation formula (cf. [2]), we obtain

$$\begin{aligned}
&\beta(\beta\lambda-\beta\lambda_2+1)\int_1^\infty P_1(t)\frac{t^{\beta\lambda_2-2}}{(m^\alpha+t^\beta)^\lambda}\,dt > -\frac{\beta(\beta\lambda-\beta\lambda_2+1)}{12(m^\alpha+1)^\lambda}, \\
&-\beta^2 m^\alpha\lambda\int_1^\infty P_1(t)\frac{t^{\beta\lambda_2-2}}{(m^\alpha+t^\beta)^{\lambda+1}}\,dt \\
&\quad > \frac{\beta^2 m^\alpha\lambda}{12(m^\alpha+1)^{\lambda+1}} - \frac{\beta^2 m^\alpha\lambda}{720}\left[\frac{t^{\beta\lambda_2-2}}{(m^\alpha+t^\beta)^{\lambda+1}}\right]''_{t=1} \\
&\quad > \frac{\beta^2(m^\alpha+1-1)\lambda}{12(m^\alpha+1)^{\lambda+1}} - \frac{\beta^2(m^\alpha+1)\lambda}{720} \\
&\qquad \times\left[\frac{(\lambda+1)(\lambda+2)\beta^2}{(m^\alpha+1)^{\lambda+3}} + \frac{\beta(\lambda+1)(5-\beta-2\beta\lambda_2)}{(m^\alpha+1)^{\lambda+2}} + \frac{(2-\beta\lambda_2)(3-\beta\lambda_2)}{(m^\alpha+1)^{\lambda+1}}\right] \\
&\quad = \frac{\beta^2\lambda}{12(m^\alpha+1)^\lambda} - \frac{\beta^2\lambda}{12(m^\alpha+1)^{\lambda+1}} - \frac{\beta^2\lambda}{720}\left[\frac{(\lambda+1)(\lambda+2)\beta^2}{(m^\alpha+1)^{\lambda+2}}\right. \\
&\qquad \left. + \frac{\beta(\lambda+1)(5-\beta-2\beta\lambda_2)}{(m^\alpha+1)^{\lambda+1}} + \frac{(2-\beta\lambda_2)(3-\beta\lambda_2)}{(m^\alpha+1)^\lambda}\right],
\end{aligned}$$

and then we have

$$h(m) > \frac{1}{(m^\alpha+1)^\lambda}h_1 + \frac{\lambda}{(m^\alpha+1)^{\lambda+1}}h_2 + \frac{\lambda(\lambda+1)}{(m^\alpha+1)^{\lambda+2}}h_3,$$

$$\begin{aligned}
h_1 &:= \frac{1}{\lambda_2} - \frac{\beta}{2} - \frac{\beta - \beta^2\lambda_2}{12} - \frac{\beta^2\lambda(2-\beta\lambda_2)(3-\beta\lambda_2)}{720}, \\
h_2 &:= \frac{1}{\lambda_2(\lambda_2+1)} - \frac{\beta^2}{12} - \frac{\beta^3(\lambda+1)(5-\beta-2\beta\lambda_2)}{720}, \quad \text{and} \\
h_3 &:= \frac{1}{\lambda_2(\lambda_2+1)(\lambda_2+2)} - \frac{\beta^4(\lambda+2)}{720}.
\end{aligned}$$

We find that

$$h_1 \geq \frac{1}{\lambda_2} - \frac{\beta}{2} - \frac{\beta - \beta^2\lambda_2}{12} - \frac{\lambda\beta^2(2-\beta\lambda_2)(3-\beta\lambda_2)}{720} = \frac{g(\lambda_2)}{720\lambda_2},$$

where we indicate a real function $g(\sigma)$ ($\sigma \in (0, \frac{2}{\beta}]$) as follows:

$$g(\sigma) := 720 - \left(420\beta + 6\lambda\beta^2\right)\sigma + \left(60\beta^2 + 5\lambda\beta^3\right)\sigma^2 - \lambda\beta^4\sigma^3.$$

We obtain that, for $\beta \in (0,1]$, $\lambda \in (0,6]$ and $\sigma \in (0, \frac{2}{\beta}]$,

$$\begin{aligned}
g'(\sigma) &= -\left(420\beta + 6\lambda\beta^2\right) + 2\left(60\beta^2 + 5\lambda\beta^3\right)\sigma - 3\beta^4\sigma^2 \\
&\leq -420\beta - 6\lambda\beta^2 + 2\left(60\beta^2 + 5\lambda\beta^3\right)\frac{2}{\beta} \\
&= (14\lambda\beta - 180)\beta < 0,
\end{aligned}$$

and then it follows that $h_1 \geq \frac{g(\lambda_2)}{720\lambda_2} \geq \frac{g(2/\beta)}{720\lambda_2} = \frac{1}{6\lambda_2} > 0$. We also obtain that, for $\lambda_2 \in (0, \frac{2}{\beta}]$,

$$\begin{aligned}
h_2 &> \frac{\beta^2}{6} - \frac{\beta^2}{12} - \frac{5(\lambda+1)\beta^2}{720} = \left(\frac{1}{12} - \frac{\lambda+1}{140}\right)\beta^2 > 0 \quad (0 < \lambda \leq 6), \quad \text{and} \\
h_3 &\geq \left(\frac{1}{24} - \frac{\lambda+2}{720}\right)\beta^3 > 0 (0 < \lambda \leq 6).
\end{aligned}$$

Hence, we have $h(m) > 0$. Setting $t = m^{\alpha/\beta}u^{1/\beta}$, it follows that

$$\begin{aligned}
\varpi(\lambda_2, m) &= m^{\alpha(\lambda-\lambda_2)} \sum_{n=1}^{\infty} g(m,n) < m^{\alpha(\lambda-\lambda_2)} \int_0^{\infty} g(m,t)\,dt \\
&= m^{\alpha(\lambda-\lambda_2)} \int_0^{\infty} \frac{\beta t^{\beta\lambda_2-1}}{(m^\alpha + t^\beta)^\lambda}\,dt = \int_0^{\infty} \frac{u^{\lambda_2-1}}{(1+u)^\lambda}\,du = B(\lambda_2, \lambda - \lambda_2).
\end{aligned}$$

On the other hand, by using the Euler–Maclaurin summation formula, we also have

$$\begin{aligned}
\sum_{n=1}^{\infty} g(m,n) &= \int_1^{\infty} g(m,t)\,dt + \frac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)\,dt \\
&= \int_1^{\infty} g(m,t)\,dt + H(m), \\
H(m) &:= \frac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)\,dt.
\end{aligned}$$

We have obtained that $\frac{1}{2}g(m,1) = \frac{\beta}{2(m^\alpha+1)^\lambda}$ and

$$g'(m,t) = -\frac{\beta(\beta\lambda - \beta\lambda_2 + 1)t^{\beta\lambda_2-2}}{(m^\alpha + t^\beta)^\lambda} + \frac{\beta^2\lambda m^\alpha t^{\beta\lambda_2-2}}{(m^\alpha + t^\beta)^{\lambda+1}}.$$

For $\lambda_2 \in (0, \frac{2}{\beta}] \cap (0,\lambda)$, $0 < \lambda \le 6$, by means of the Euler–Maclaurin summation formula, we obtain

$$-\beta(\beta\lambda - \beta\lambda_2 + 1)\int_1^\infty P_1(t)\frac{t^{\beta\lambda_2-2}}{(m^\alpha + t^\beta)^\lambda}\,dt > 0, \quad \text{and}$$
$$\beta^2 m^\alpha\lambda\int_1^\infty P_1(t)\frac{t^{\beta\lambda_2-2}}{(m^\alpha + t^\beta)^{\lambda+1}}\,dt > -\frac{\beta^2 m^\alpha\lambda}{12(m^\alpha+1)^{\lambda+1}} > -\frac{\beta^2\lambda}{12(m^\alpha+1)^\lambda}.$$

Hence, we have

$$H(m) > \frac{\beta}{2(m^\alpha+1)^\lambda} - \frac{\beta^2\lambda}{12(m^\alpha+1)^\lambda} \ge \frac{\beta}{2(m^\alpha+1)^\lambda} - \frac{6\beta}{12(m^\alpha+1)^\lambda} = 0,$$

and then we obtain

$$\begin{aligned}\varpi(\lambda_2,m) &= m^{\alpha(\lambda-\lambda_2)}\sum_{n=1}^\infty g(m,n) > m^{\alpha(\lambda-\lambda_2)}\int_1^\infty g(m,t)\,dt \\ &= m^{\alpha(\lambda-\lambda_2)}\int_0^\infty g(m,t)\,dt - m^{\alpha(\lambda-\lambda_2)}\int_0^1 g(m,t)\,dt \\ &= k_\lambda(\lambda_2)\left[1 - \frac{1}{k_\lambda(\lambda_2)}\int_0^{\frac{1}{m^\alpha}}\frac{u^{\lambda_2-1}}{(1+u)^\lambda}\,du\right] > 0,\end{aligned}$$

where we indicate $O(\frac{1}{m^{\alpha\lambda_2}}) = \frac{1}{k_\lambda(\lambda_2)}\int_0^{\frac{1}{m^\alpha}}\frac{u^{\lambda_2-1}}{(1+u)^\lambda}\,du > 0$,satisfying

$$0 < \int_0^{\frac{1}{m^\alpha}}\frac{u^{\lambda_2-1}}{(1+u)^\lambda}\,du < \int_0^{\frac{1}{m^\alpha}} u^{\lambda_2-1}\,du = \frac{1}{\lambda_2 m^{\alpha\lambda_2}}.$$

Therefore, we obtain inequalities (7).

The lemma is proved. □

Lemma 2 *We have the following reverse Hardy–Hilbert inequality with the intermediate variables*:

$$\begin{aligned}I = \sum_{n=1}^\infty\sum_{m=1}^\infty\frac{a_m b_n}{(m^\alpha + n^\beta)^\lambda} &> \left(\frac{1}{\beta}k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_\lambda(\lambda_1)\right)^{\frac{1}{q}} \\ &\times\left\{\sum_{m=1}^\infty\left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right)m^{p[1-\alpha(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})]-1}a_m^p\right\}^{\frac{1}{p}} \\ &\times\left\{\sum_{n=1}^\infty n^{q[1-\beta(\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p})]-1}b_n^q\right\}^{\frac{1}{q}}.\end{aligned} \tag{10}$$

Proof In the same way, for $\lambda_1 \in (0, \frac{2}{\alpha}] \cap (0,\lambda)$ $(\lambda_2 \in (0,\lambda))$, $n \in \mathbf{N}$, we obtain the following inequalities for the other weight coefficient:

$$0 < k_\lambda(\lambda_1)\left(1 - O\left(\frac{1}{n^{\beta\lambda_1}}\right)\right) < \omega(\lambda_1, n) := n^{\beta(\lambda-\lambda_1)} \sum_{m=1}^{\infty} \frac{\alpha m^{\alpha\lambda_1 - 1}}{(m^\alpha + n^\beta)^\lambda} < k_\lambda(\lambda_1), \tag{11}$$

where $O(\frac{1}{n^{\beta\lambda_1}}) := \frac{1}{k_\lambda(\lambda_1)} \int_0^{\frac{1}{n^\beta}} \frac{u^{\lambda_1 - 1}}{(1+u)^\lambda}\, du > 0$.

By the reverse Hölder inequality (cf. [34]), we obtain

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{(m^\alpha + n^\beta)^\lambda} \left[\frac{m^{\alpha(1-\lambda_1)/q}(\beta n^{\beta-1})^{1/p}}{n^{\beta(1-\lambda_2)/p}(\alpha m^{\alpha-1})^{1/q}} a_m\right]\left[\frac{n^{\beta(1-\lambda_2)/p}(\alpha m^{\alpha-1})^{1/q}}{m^{\alpha(1-\lambda_1)/q}(\beta n^{\beta-1})^{1/p}} b_n\right] \\ &\geq \left[\frac{1}{\beta}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\beta}{(m^\alpha + n^\beta)^\lambda} \frac{m^{\alpha(1-\lambda_1)(p-1)} n^{\beta-1} a_m^p}{n^{\beta(1-\lambda_2)}(\alpha m^{\alpha-1})^{p-1}}\right]^{\frac{1}{p}} \\ &\quad \times \left[\frac{1}{\alpha}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\alpha}{(m^\alpha + n^\beta)^\lambda} \frac{n^{\beta(1-\lambda_2)(q-1)} m^{\alpha-1} b_n^q}{m^{\alpha(1-\lambda_1)}(\beta n^{\beta-1})^{q-1}}\right]^{\frac{1}{q}} \\ &= \frac{1}{\alpha^{1/q}\beta^{1/p}} \left\{\sum_{m=1}^{\infty} \varpi(\lambda_2, m) m^{p[1-\alpha(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})]-1} a_m^p\right\}^{\frac{1}{p}} \\ &\quad \times \left\{\sum_{n=1}^{\infty} \omega(\lambda_1, n) n^{q[1-\beta(\frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p})]-1} b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then, by (9) and (11) (for $0 < p < 1$ $(q < 0)$), we have (10).

The lemma is proved. □

Remark 1 By (10), for $\lambda_1 + \lambda_2 = \lambda \in (0,6]$, $K_\lambda(\lambda_1) := \frac{1}{\alpha^{1/q}\beta^{1/p}} B(\lambda_1, \lambda_2)$, we find

$$\omega(\lambda_1, n) = n^{\beta\lambda_2} \sum_{m=1}^{\infty} \frac{\alpha m^{\alpha\lambda_1 - 1}}{(m^\alpha + n^\beta)^\lambda}, \tag{12}$$

$$0 < \sum_{m=1}^{\infty} m^{p(1-\alpha\lambda_1)-1} a_m^p < \infty, \qquad 0 < \sum_{n=1}^{\infty} n^{q(1-\beta\lambda_2)-1} b_n^q < \infty,$$

and the following inequality:

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m^\alpha + n^\beta)^\lambda} \\ &\quad > K_\lambda(\lambda_1)\left[\sum_{m=1}^{\infty}\left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right) m^{p(1-\alpha\lambda_1)-1} a_m^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} n^{q(1-\beta\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{13}$$

Lemma 3 *The constant factor $K_\lambda(\lambda_1)$ in (13) is the best possible.*

Proof For any $0 < \varepsilon < p\lambda_1$, we set

$$\tilde{a}_m := m^{\alpha(\lambda_1 - \frac{\varepsilon}{p})-1}, \qquad \tilde{b}_n := n^{\beta(\lambda_2 - \frac{\varepsilon}{q})-1} \quad (m, n \in \mathrm{N}).$$

If there exists a constant $M \geq K_\lambda(\lambda_1)$ such that (13) is valid when we replace $K_\lambda(\lambda_1)$ with M, then in particular, by substitution of $a_m = \tilde{a}_m$ and $b_n = \tilde{b}_n$ in (13), we have

$$\begin{aligned}\tilde{I} &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{(m^\alpha + n^\beta)^\lambda}\\ &> M\left[\sum_{m=1}^{\infty}\left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right)m^{p(1-\alpha\lambda_1)-1}\tilde{a}_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\lambda_2)-1}\tilde{b}_n^q\right]^{\frac{1}{q}}.\end{aligned}$$

In the following, we show that $M \leq K_\lambda(\lambda_1)$,from which it follows that $M = K_\lambda(\lambda_1)$ is the best possible constant factor of (13). By the decreasingness property of series, we obtain

$$\begin{aligned}\tilde{I} &> M\left[\sum_{m=1}^{\infty}\left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right)m^{p(1-\alpha\lambda_1)-1}m^{p\alpha\lambda_1-\alpha\varepsilon-p}\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\lambda_2)-1}n^{q\beta\lambda_2-\beta\varepsilon-q}\right]^{\frac{1}{q}}\\ &= M\left[\sum_{m=1}^{\infty}m^{-\alpha\varepsilon-1} - \sum_{m=1}^{\infty}O\left(m^{-\alpha(\lambda_2+\varepsilon)-1}\right)\right]^{\frac{1}{p}}\left(1 + \sum_{n=2}^{\infty}n^{-\beta\varepsilon-1}\right)^{\frac{1}{q}}\\ &> M\left(\int_1^{\infty}x^{-\alpha\varepsilon-1}\,dx - O(1)\right)^{\frac{1}{p}}\left(1 + \int_1^{\infty}y^{-\beta\varepsilon-1}\,dy\right)^{\frac{1}{q}}\\ &= \frac{M}{\varepsilon}\left(\frac{1}{\alpha} - \varepsilon O(1)\right)^{\frac{1}{p}}\left(\varepsilon + \frac{1}{\beta}\right)^{\frac{1}{q}}.\end{aligned}$$

By (12), setting $\hat{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} \in (0, \frac{2}{\alpha}) \cap (0, \lambda)$ $(0 < \hat{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p} = \lambda - \hat{\lambda}_1 < \lambda)$, we find

$$\begin{aligned}\tilde{I} &= \sum_{n=1}^{\infty}\left[n^{\beta\hat{\lambda}_2}\sum_{m=1}^{\infty}\frac{1}{(m^\alpha + n^\beta)^\lambda}m^{\alpha\hat{\lambda}_1-1}\right]n^{-\beta\varepsilon-1}\\ &= \frac{1}{\alpha}\sum_{n=1}^{\infty}\omega(\hat{\lambda}_1, n)n^{-\beta\varepsilon-1} < \frac{1}{\alpha}k_\lambda(\hat{\lambda}_1)\sum_{n=1}^{\infty}n^{-\beta\varepsilon-1}\\ &= \frac{1}{\alpha}k_\lambda(\hat{\lambda}_1)\left(1 + \sum_{n=2}^{\infty}n^{-\beta\varepsilon-1}\right) < \frac{1}{\alpha}k_\lambda(\hat{\lambda}_1)\left(1 + \int_1^{\infty}x^{-\beta\varepsilon-1}\,dx\right)\\ &= \frac{1}{\varepsilon\alpha\beta}k_\lambda(\hat{\lambda}_1)(\varepsilon\beta + 1).\end{aligned}$$

By virtue of the above results, we have

$$\frac{1}{\alpha\beta}B\left(\lambda_1 - \frac{\varepsilon}{p}, \lambda_2 + \frac{\varepsilon}{p}\right)(\varepsilon\beta + 1) > \varepsilon\tilde{I} > M\left(\frac{1}{\alpha} - \varepsilon O(1)\right)^{\frac{1}{p}}\left(\varepsilon + \frac{1}{\beta}\right)^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, in view of the continuity of the beta function, it follows that

$$K_\lambda(\lambda_1) = \frac{1}{\alpha^{1/q}\beta^{1/p}}B(\lambda_1, \lambda_2) \geq M.$$

Hence, $M = K_\lambda(\lambda_1)$ is the best possible constant factor of (13).

The lemma is proved. □

Remark 2 Setting $\tilde{\lambda}_1 := \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$, $\tilde{\lambda}_2 := \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p}$ in (10), we find

$$\tilde{\lambda}_1 + \tilde{\lambda}_2 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} + \frac{\lambda-\lambda_1}{q} + \frac{\lambda_2}{p} = \frac{\lambda}{p} + \frac{\lambda}{q} = \lambda, \quad \text{and}$$

$$\begin{aligned} I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\beta)^\lambda} &> \left(\frac{1}{\beta}k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_\lambda(\lambda_1)\right)^{\frac{1}{q}} \\ &\times \left[\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right)m^{p(1-\alpha\tilde{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\tilde{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{14}$$

(i) For $\lambda-\lambda_1-\lambda_2 \in (-\lambda_1 p, (\lambda-\lambda_1)p)$, we have

$$0 < \tilde{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q} < \lambda, \quad 0 < \tilde{\lambda}_2 = \lambda - \tilde{\lambda}_1 < \lambda;$$

(ii) For $\lambda-\lambda_1-\lambda_2 \in [(\lambda-\lambda_1-\frac{2}{\beta})p, (\frac{2}{\alpha}-\lambda_1)p]$ ($\lambda \le \min\{6, \frac{2}{\alpha}+\frac{2}{\beta}\}$), we have

$$\tilde{\lambda}_1 \le \frac{2}{\alpha}, \qquad \tilde{\lambda}_2 \le \frac{2}{\beta}.$$

In view of (i) and (ii), we can rewrite (10) as follows:

$$\begin{aligned} &\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m^\alpha+n^\beta)^\lambda} \\ &\quad > K_\lambda(\lambda_1)\left[\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\alpha\tilde{\lambda}_2}}\right)\right)m^{p(1-\alpha\tilde{\lambda}_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\beta\tilde{\lambda}_2)-1}b_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{15}$$

Lemma 4 *If the constant factor* $(\frac{1}{\beta}k_\lambda(\lambda_2))^{\frac{1}{p}}(\frac{1}{\alpha}k_\lambda(\lambda_1))^{\frac{1}{q}}$ *in* (10) (*or* (14)) *is the best possible, then for* $0 < \lambda \le \min\{6, \frac{2}{\alpha}+\frac{2}{\beta}\}$ *and*

$$\lambda-\lambda_1-\lambda_2 \in \left(-\lambda_1 p, (\lambda-\lambda_1)p\right) \cap \left[\left(\lambda-\lambda_1-\frac{2}{\beta}\right)p, \left(\frac{2}{\alpha}-\lambda_1\right)p\right] (\supset \{0\}), \tag{16}$$

we have $\lambda_1+\lambda_2=\lambda$.

Proof If the constant factor $(\frac{1}{\beta}k_\lambda(\lambda_2))^{\frac{1}{p}}(\frac{1}{\alpha}k_\lambda(\lambda_1))^{\frac{1}{q}}$ in (10) (or (14)) is the best possible, then in view of (16) and (15), we have the following inequality:

$$\left(\frac{1}{\beta}k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_\lambda(\lambda_1)\right)^{\frac{1}{q}} \ge K_\lambda(\tilde{\lambda}_1) = \frac{1}{\beta^{1/p}\alpha^{1/q}}k_\lambda(\tilde{\lambda}_1),$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) \ge k_\lambda(\tilde{\lambda}_1)$.

By the reverse Hölder inequality (cf. [34]), we obtain

$$k_\lambda(\tilde{\lambda}_1) = k_\lambda\left(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}\right)$$

$$
\begin{aligned}
&= \int_0^\infty \frac{1}{(1+u)^\lambda} u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}-1}\,du = \int_0^\infty \frac{1}{(1+u)^\lambda}\big(u^{\frac{\lambda-\lambda_2-1}{p}}\big)\big(u^{\frac{\lambda_1-1}{q}}\big)\,du \\
&\geq \left[\int_0^\infty \frac{1}{(1+u)^\lambda} u^{\lambda-\lambda_2-1}\,du\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{1}{(1+u)^\lambda} u^{\lambda_1-1}\,du\right]^{\frac{1}{q}} \\
&= \left[\int_0^\infty \frac{1}{(1+v)^\lambda} v^{\lambda_2-1}\,dv\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{1}{(1+u)^\lambda} u^{\lambda_1-1}\,du\right]^{\frac{1}{q}} \\
&= k_\lambda^{\frac{1}{p}}(\lambda_2) k_\lambda^{\frac{1}{q}}(\lambda_1).
\end{aligned}
\tag{17}
$$

Hence, we have $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1) = k_\lambda(\tilde{\lambda}_1)$, from which it follows that (17) keeps the form of equality.

We observe that (17) keeps the form of equality if and only if there exist constants A and B such that they are not both zero and (cf. [34])

$$
Au^{\lambda-\lambda_2-1} = Bu^{\lambda_1-1} \quad \text{a.e. in } \mathrm{R}_+.
$$

Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_2-\lambda_1} = \frac{B}{A}$ a.e. in R_+, and then $\lambda - \lambda_2 - \lambda_1 = 0$. Hence, we have $\lambda_1 + \lambda_2 = \lambda$.

The lemma is proved. □

3 Main results

Theorem 1 *Inequality* (14) *is equivalent to the following inequalities*:

$$
\begin{aligned}
J &:= \left\{\sum_{n=1}^\infty n^{p\beta\tilde{\lambda}_2-1}\left[\sum_{m=1}^\infty \frac{1}{(m^\alpha+n^\beta)^\lambda} a_m\right]^p\right\}^{\frac{1}{p}} \\
&> \left(\frac{1}{\beta}k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_\lambda(\lambda_1)\right)^{\frac{1}{q}}\left[\sum_{m=1}^\infty \left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right) m^{p(1-\alpha\tilde{\lambda}_1)-1} a_m^p\right]^{\frac{1}{p}},
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
J_1 &:= \left\{\sum_{m=1}^\infty \frac{m^{q\alpha\tilde{\lambda}_1-1}}{(1-O(\frac{1}{m^{\alpha\lambda_2}}))^{q-1}}\left[\sum_{n=1}^\infty \frac{1}{(m^\alpha+n^\beta)^\lambda} b_n\right]^q\right\}^{\frac{1}{q}} \\
&> \left(\frac{1}{\beta}k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_\lambda(\lambda_1)\right)^{\frac{1}{q}}\left[\sum_{n=1}^\infty n^{q(1-\beta\tilde{\lambda}_2)-1} b_n^q\right]^{\frac{1}{q}}.
\end{aligned}
\tag{19}
$$

If the constant factor in (14) *is the best possible, then so is the same constant factor in* (18) *and* (19).

Proof Suppose that (18) is valid. By the reverse Hölder inequality (cf. [34]), we have

$$
I = \sum_{n=1}^\infty \left[n^{\frac{-1}{p}+\beta\tilde{\lambda}_2}\sum_{m=1}^\infty \frac{1}{(m^\alpha+n^\beta)^\lambda} a_m\right]\big(n^{\frac{1}{p}-\beta\tilde{\lambda}_2} b_n\big) \geq J\left[\sum_{n=1}^\infty n^{q(1-\beta\tilde{\lambda}_2)-1} b_n^q\right]^{\frac{1}{q}}.
\tag{20}
$$

Then, by (18), we obtain (14). On the other hand, assuming that (14) is valid, we set

$$
b_n := n^{p\beta\tilde{\lambda}_2-1}\left[\sum_{m=1}^\infty \frac{1}{(m^\alpha+n^\beta)^\lambda} a_m\right]^{p-1}, \quad n \in \mathbf{N}.
$$

If $J = \infty$, then (18) is naturally valid; if $J = 0$, then it is impossible that makes (18) valid, namely, $J > 0$. Suppose that $0 < J < \infty$. By (14), we have

$$\begin{aligned} 0 &> \sum_{n=1}^{\infty} n^{q(1-\beta\tilde{\lambda}_2)-1} b_n^q = J^p = I > \left(\frac{1}{\beta}k_\lambda(\lambda_2)\right)^{\frac{1}{p}} \left(\frac{1}{\alpha}k_\lambda(\lambda_1)\right)^{\frac{1}{q}} \\ &\quad \times \left[\sum_{m=1}^{\infty}\left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right) m^{p(1-\alpha\tilde{\lambda}_1)-1} a_m^p\right]^{\frac{1}{p}} J^{p-1} > 0, \\ J &= \left[\sum_{n=1}^{\infty} n^{q(1-\beta\tilde{\lambda}_2)-1} b_n^q\right]^{\frac{1}{p}} > \left(\frac{1}{\beta}k_\lambda(\lambda_2)\right)^{\frac{1}{p}} \left(\frac{1}{\alpha}k_\lambda(\lambda_1)\right)^{\frac{1}{q}} \\ &\quad \times \left[\sum_{m=1}^{\infty}\left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right) m^{p(1-\alpha\tilde{\lambda}_1)-1} a_m^p\right]^{\frac{1}{p}}, \end{aligned}$$

namely, (18) follows, which is equivalent to (14).

Suppose that (19) is valid. By the reverse Hölder inequality (cf. [34]), we have

$$\begin{aligned} I &= \sum_{m=1}^{\infty}\left[\left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right)^{\frac{1}{p}} m^{\frac{1}{q}-\alpha\tilde{\lambda}_1} a_m\right]\left[\frac{m^{\frac{-1}{q}+\alpha\tilde{\lambda}_1}}{(1 - O(\frac{1}{m^{\alpha\lambda_2}}))^{1/p}} \sum_{n=1}^{\infty} \frac{1}{(m^\alpha + n^\beta)^\lambda} b_n\right] \\ &\geq \left[\sum_{m=1}^{\infty}\left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right) m^{p(1-\alpha\overset{\leftrightarrow}{\lambda}_1)-1} a_m^p\right]^{\frac{1}{p}} J_1. \end{aligned} \tag{21}$$

Then, by (19), we obtain (14). On the other hand, assuming that (14) is valid, we set

$$a_m := \frac{m^{q\alpha\tilde{\lambda}_1-1}}{(1 - O(\frac{1}{m^{\alpha\lambda_2}}))^{q-1}} \left[\sum_{n=1}^{\infty} \frac{1}{(m^\alpha + n^\beta)^\lambda} b_n\right]^{q-1}, \quad m \in \mathrm{N}.$$

If $J_1 = \infty$, then (19) is naturally valid; if $J_1 = 0$, then it is impossible that makes (19) valid, namely, $J_1 > 0$. Suppose that $0 < J_1 < \infty$. By (14), we have

$$\begin{aligned} \infty &> \sum_{m=1}^{\infty}\left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right) m^{p(1-\alpha\tilde{\lambda}_1)-1} a_m^p = J_1^q = I \\ &> \left(\frac{1}{\beta}k_\lambda(\lambda_2)\right)^{\frac{1}{p}} \left(\frac{1}{\alpha}k_\lambda(\lambda_1)\right)^{\frac{1}{q}} J_1^{q-1} \left[\sum_{n=1}^{\infty} n^{q(1-\beta\tilde{\lambda}_2)-1} b_n^q\right]^{\frac{1}{q}} > 0, \\ J_1 &= \left[\sum_{m=1}^{\infty}\left(1 - O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right) m^{p(1-\alpha\tilde{\lambda}_1)-1} a_m^p\right]^{\frac{1}{q}} \\ &> \left(\frac{1}{\beta}k_\lambda(\lambda_2)\right)^{\frac{1}{p}} \left(\frac{1}{\alpha}k_\lambda(\lambda_1)\right)^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} n^{q(1-\beta\tilde{\lambda}_2)-1} b_n^q\right]^{\frac{1}{q}}, \end{aligned}$$

namely, (19) follows, which is equivalent to (14).

Hence, inequalities (14), (18), and (19) are equivalent.

If the constant factor in (14) is the best possible, then so is the constant factor in (18) and (19). Otherwise, by (20) (or (21)), we would reach a contradiction that the constant factor in (14) is not the best possible.

The theorem is proved. □

Theorem 2 *The following statements* (i), (ii), (iii), *and* (iv) *are equivalent*:

(i) *Both* $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ *and* $k_\lambda(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q})$ *are independent of* p, q;

(ii) $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ *is expressible as the following single integral*:

$$k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)=k_\lambda(\tilde{\lambda}_1)=\int_0^\infty\frac{1}{(1+u)^\lambda}u^{\tilde{\lambda}_1-1}\,du;$$

(iii) $(\frac{1}{\beta}k_\lambda(\lambda_2))^{\frac{1}{p}}(\frac{1}{\alpha}k_\lambda(\lambda_1))^{\frac{1}{q}}$ *in* (14) *is the best possible constant factor*;

(iv) *If* $0<\lambda\leq\min\{6,\frac{2}{\alpha}+\frac{2}{\beta}\}$ *and*

$$\lambda-\lambda_1-\lambda_2\in(-\lambda_1 p,(\lambda-\lambda_1)p)\cap\left[\left(\lambda-\lambda_1-\frac{2}{\beta}\right)p,\left(\frac{2}{\alpha}-\lambda_1\right)p\right],$$

then we have $\lambda_1+\lambda_2=\lambda$.

If statement (iv) follows, namely, $\lambda_1+\lambda_2=\lambda$, then we have (13) and the following equivalent inequalities with the best possible constant factor $K_\lambda(\lambda_1)$:

$$\begin{aligned}&\left\{\sum_{n=1}^\infty n^{p\beta\lambda_2-1}\left[\sum_{m=1}^\infty\frac{1}{(m^\alpha+n^\beta)^\lambda}a_m\right]^p\right\}^{\frac{1}{p}}\\&\quad>K_\lambda(\lambda_1)\left[\sum_{m=1}^\infty\left(1-O\left(\frac{1}{m^{\alpha\lambda_2}}\right)\right)m^{p(1-\alpha\lambda_1)-1}a_m^p\right]^{\frac{1}{p}},\end{aligned}\tag{22}$$

$$\left\{\sum_{m=1}^\infty\frac{m^{q\alpha\lambda_1-1}}{(1-O(\frac{1}{m^{\alpha\lambda_2}}))^{q-1}}\left[\sum_{n=1}^\infty\frac{1}{(m^\alpha+n^\beta)^\lambda}b_n\right]^q\right\}^{\frac{1}{q}}>K_\lambda(\lambda_1)\left[\sum_{n=1}^\infty n^{q(1-\beta\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}.\tag{23}$$

Proof (i) ⇒ (ii). By (i), in view of the continuity of the beta function, we have

$$k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)=\lim_{q\to-\infty}\lim_{p\to1^-}k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)=k_\lambda(\lambda_2),$$

$$k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)=\lim_{q\to-\infty}\lim_{p\to1^-}k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)=k_\lambda(\lambda-\lambda_2)=k_\lambda(\lambda_2),$$

namely, $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ is expressible as the following single integral:

$$k_\lambda\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}\right)=k_\lambda(\tilde{\lambda}_1)=\int_0^\infty\frac{1}{(1+u)^\lambda}u^{\tilde{\lambda}_1-1}\,du.$$

(ii) ⇒ (iv). If $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)=k_\lambda(\tilde{\lambda}_1)$, then (17) keeps the form of equality. In view of the proof of Lemma 4, it follows that $\lambda_1+\lambda_2=\lambda$.

(iv) ⇒ (i). If $\lambda_1 + \lambda_2 = \lambda$, then both $k_\lambda^{\frac{1}{p}}(\lambda_2)k_\lambda^{\frac{1}{q}}(\lambda_1)$ and $k_\lambda(\frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q})$ are equal to $k_\lambda(\lambda_1)$, which is independent of p, q. Hence, it follows that (i) ⇔ (ii) ⇔ (iv).

(iii) ⇒ (iv). By the assumption and Lemma 4, we have $\lambda_1 + \lambda_2 = \lambda$.

(iv) ⇒ (iii). By Lemma 3, for $\lambda = \lambda_1 + \lambda_2$,

$$\left(\frac{1}{\beta}k_\lambda(\lambda_2)\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}k_\lambda(\lambda_1)\right)^{\frac{1}{q}}\left(=K_\lambda(\lambda_1)\right)$$

is the best possible constant factor of (14). Therefore, we have (iii) ⇔ (iv).

Hence, statements (i), (ii), (iii), and (iv) are equivalent.

The theorem is proved. □

Remark 3 (i) For $\alpha = \beta = 1$, $\lambda_1, \lambda_2 \in (0, 2]$ ($\lambda_1 + \lambda_2 = \lambda \in (0, 4]$) in (13), (22), and (23), we have the following equivalent inequalities with the best possible constant factor $B(\lambda_1, \lambda_2)$:

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^\lambda}\\&\quad > B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\lambda_2}}\right)\right)m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}},\end{aligned}\tag{24}$$

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty}n^{p\lambda_2-1}\left[\sum_{m=1}^{\infty}\frac{1}{(m+n)^\lambda}a_m\right]^p\right\}^{\frac{1}{p}}\\&\quad > B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\lambda_2}}\right)\right)m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}},\end{aligned}\tag{25}$$

$$\left\{\sum_{m=1}^{\infty}\frac{m^{q\lambda_1-1}}{(1-O(\frac{1}{m^{\lambda_2}}))^{q-1}}\left[\sum_{n=1}^{\infty}\frac{1}{(m+n)^\lambda}b_n\right]^q\right\}^{\frac{1}{q}} > K_\lambda(\lambda_1)\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}.\tag{26}$$

Inequality (24) is the reverse of (2) (cf. [33]).

(ii) For $\alpha = \beta = \frac{1}{2}$, $\lambda_1, \lambda_2 \in (0, 4]$ ($\lambda_1 + \lambda_2 = \lambda \in (0, 6]$) in (13), (22), and (23), we have the following equivalent inequalities with the best possible constant factor $2B(\lambda_1, \lambda_2)$:

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(\sqrt{m}+\sqrt{n})^\lambda}\\&\quad > 2B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\lambda_2/2}}\right)\right)m^{p(1-\frac{\lambda_1}{2})-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda_2}{2})-1}b_n^q\right]^{\frac{1}{q}},\end{aligned}\tag{27}$$

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty}n^{\frac{p\lambda_2}{2}-1}\left[\sum_{m=1}^{\infty}\frac{1}{(\sqrt{m}+\sqrt{n})^\lambda}a_m\right]^p\right\}^{\frac{1}{p}}\\&\quad > 2B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{\lambda_2/2}}\right)\right)m^{p(1-\frac{\lambda_1}{2})-1}a_m^p\right]^{\frac{1}{p}},\end{aligned}\tag{28}$$

$$\left\{\sum_{m=1}^{\infty}\frac{m^{q\lambda_1/2-1}}{(1-O(\frac{1}{m^{\lambda_2/2}}))^{q-1}}\left[\sum_{n=1}^{\infty}\frac{1}{(\sqrt{m}+\sqrt{n})^{\lambda}}b_n\right]^q\right\}^{\frac{1}{q}}$$
$$> 2B(\lambda_1,\lambda_2)\left[\sum_{n=1}^{\infty}n^{q(1-\frac{\lambda_2}{2})-1}b_n^q\right]^{\frac{1}{q}}. \tag{29}$$

(iii) For $\alpha=\beta=\frac{2}{3}$, $\lambda_1,\lambda_2\in(0,3]$ $(\lambda_1+\lambda_2=\lambda\in(0,6])$ in (13), (22), and (23), we have the following equivalent inequalities with the best possible constant factor $\frac{3}{2}B(\lambda_1,\lambda_2)$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(\sqrt[3]{m^2}+\sqrt[3]{n^2})^{\lambda}}$$
$$> \frac{3}{2}B(\lambda_1,\lambda_2)$$
$$\times\left[\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{2\lambda_2/3}}\right)\right)m^{p(1-\frac{2\lambda_1}{3})-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\frac{2\lambda_2}{3})-1}b_n^q\right]^{\frac{1}{q}}, \tag{30}$$

$$\left\{\sum_{n=1}^{\infty}n^{\frac{2p\lambda_2}{3}-1}\left[\sum_{m=1}^{\infty}\frac{1}{(\sqrt[3]{m^2}+\sqrt[3]{n^2})^{\lambda}}a_m\right]^p\right\}^{\frac{1}{p}}$$
$$> \frac{3}{2}B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}\left(1-O\left(\frac{1}{m^{2\lambda_2/3}}\right)\right)m^{p(1-\frac{2\lambda_1}{3})-1}a_m^p\right]^{\frac{1}{p}}, \tag{31}$$

$$\left\{\sum_{m=1}^{\infty}\frac{m^{2q\lambda_1/3-1}}{(1-O(\frac{1}{m^{2\lambda_2/3}}))^{q-1}}\left[\sum_{n=1}^{\infty}\frac{b_n}{(\sqrt[3]{m^2}+\sqrt[3]{n^2})^{\lambda}}\right]^q\right\}^{\frac{1}{q}}$$
$$> \frac{3}{2}B(\lambda_1,\lambda_2)\left[\sum_{n=1}^{\infty}n^{q(1-\frac{2\lambda_2}{3})-1}b_n^q\right]^{\frac{1}{q}}. \tag{32}$$

4 Conclusions

In this paper, by virtue of the symmetry principle, by means of the techniques of real analysis and Euler–Maclaurin summation formula, we construct proper weight coefficients and use them to establish a reverse Hardy–Hilbert inequality with the power function as intermediate variables and the equivalent forms in Lemma 2 and Theorem 1. Then, we obtain some equivalent statements of the best possible constant factor related to several parameters in Theorem 2. Finally, we illustrate how the obtained results can generate some particular reverse Hardy–Hilbert inequalities in Remark 3. The lemmas and theorems provide an extensive account of this type of inequalities.

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Availability of data and materials
The data used to support the findings of this study are included within the article.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. XH and RL participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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