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Generalized variational inequalities for linguistic interpretations using intuitionistic fuzzy relations and projected dynamical systems

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Abstract

The fuzzy set theory enables us to represent our knowledge under multiple interpretations and axiomatic foundations from linguistic to computational representations. While the intuitionistic fuzzy set, as a generalization of the fuzzy set, cannot only represent the tolerance levels, but also the intolerance levels which a decision maker can tolerate and cannot tolerate in the accomplishment of a linguistic interpretation, in this paper, we introduce the generalized variational inequalities for linguistic interpretations using intuitionistic fuzzy relations. It is shown that such problems can be transformed into the classical (nonfuzzy) generalized variational inequalities by means of level sets of the intuitionistic fuzzy relation. Furthermore, the equivalence between the generalized variational inequalities with intuitionistic fuzzy relations and the fuzzy fixed point problems is established. Finally, based on the projection method, we propose an iterative algorithm and a projected neural network model for the generalized variational inequalities with intuitionistic fuzzy relations, and the stability of the proposed projected dynamical system is also investigated.

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1 Introduction

A wide class of problems arising in diverse applied fields ranging from physics, economics, optimization to engineering can be formulated as variational inequalities. Variational inequality theory, where the function is a vector-valued mapping, was introduced by Hartman and Stampacchia [16] in 1965. It is well known that the theory of set-valued mappings, beside being an important mathematical theory, has become a significant tool in many practical areas, especially in economic analysis [22]. Subsequently, the variational inequality was generalized to the generalized variational inequality by Fang [9], where the function is a set-valued mapping. The generalized variational inequality is to find $x \in M$

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and $y \in f(x)$ such that

$$y^T(x' - x) \geq 0, \quad \forall x' \in M, \quad (1.1)$$

where $M \subseteq R^n$, and $f : M \rightarrow 2^{R^n}$ is a set-valued function. There is an equivalence between set-valued mappings and binary relations, thus the more convenient discussion framework system can be chosen between the two according to the actual needs. Indeed, for a given set-valued mapping $f : M \rightarrow R^n$, we can induce a relation by

$$\Gamma = \{(x, y) \in R^n \times R^n : x \in M \text{ and } y \in f(x)\}.$$

Conversely, for a given relation $\Gamma \in R^n \times R^n$, its domain is

$$\text{dom}(\Gamma) = \{x \in R^n : (x, y) \in \Gamma, \text{ for some } y \in R^n\},$$

and for every $x \in \text{dom}(\Gamma)$, its image set is

$$f(x) = \{y \in R^n : (x, y) \in \Gamma\}.$$

That is, f is a set-valued mapping from $\text{dom}(\Gamma)$ to the family of subsets of R^n . Thus, for a given set-valued mapping f on M , the generalized variational inequality (1.1) can also be represented as follows: the generalized variational inequality, denoted by $\text{GVI}(M, \Gamma)$, is to find all solutions (x, y) such that

$$\begin{aligned} x &\in M, \\ \langle y, x' - x \rangle &\geq 0, \quad \forall x' \in M, \\ (x, y) &\in \Gamma, \end{aligned} \quad (1.2)$$

where $M \subseteq R^n$, $\Gamma \subseteq R^n \times R^n$. The most basic result on the existence of solutions to the variational inequality $\text{VI}(M, F)$ requires the set M to be compact and convex, and the mapping f to be continuous [7], which is given by Brouwer's fixed point theorem. To proceed from this result, extended conclusions are derived by replacing the compactness of the set M (closed which is possibly unbounded) with additional conditions on F (e.g., pseudo-monotone, strongly monotone, coercive with respect to M) [7, 8, 15]. Similarly, the most fundamental existence theorem for $\text{GVI}(M, F)$ can also be proved by Kakutani fixed point theorem which is for a set-valued function.

Lemma 1.1 (Brouwer fixed point theorem) *Let $C \subseteq R^m$ be a nonempty compact convex set. Every continuous function $\Phi : C \rightarrow C$ has a fixed point in C .*

Lemma 1.2 (Kakutani fixed point theorem) *Let $C \subseteq R^m$ be a nonempty compact convex set. Let $\Phi : C \rightarrow 2^C$ be a set-valued map such that, for each $x \in C$, $\Phi(x)$ is a nonempty closed convex subset of C . If Φ is closed on C , then Φ has a fixed point.*

In the classical set, the nature of the element is required to be explicit, that is, it can be explicitly indicated that any element has or does not have this property. However, in

the objective world, many phenomena, which are based on numerous fuzzy phenomena and multi-valued logic, have fuzziness. For example, the linguistic interpretations such as “young” and “old”, “long” and “short” are fuzzy concepts in people’s concepts. Such vague concepts with unclear denotations cannot be expressed by the usual binary logic and therefore cannot be described by the classical set. In 1965, Zadeh [26] introduced the fuzzy set theory, which offers a wide variety of techniques for analyzing imprecise data and enables us to represent our knowledge under varied interpretations and axiomatic foundations from linguistic to computational representations. A fuzzy set u on R is a mapping $u: R \rightarrow [0, 1]$, and $u(x)$ is the degree of membership of the element x in the fuzzy set u . The fuzzy set is a generalization of the classical set whose characteristic function is valued in $\{0, 1\}$. By fuzziness, we mean a type of imprecision which is associated with fuzzy sets, that is, classes in which there is no sharp transition from membership to nonmembership. In fact, in sharp contrast to the notion of a class or a set in mathematics, most of the classes in the real world do not have crisp boundaries which separate those objects which belong to a class from those which do not. For notational purposes, it is convenient to have a device for indicating that a fuzzy set is obtained from a nonfuzzy set by fuzzifying the boundaries of the latter set. In 1970, Bellman and Zadeh [3] employed a wavy bar under a symbol which defines the nonfuzzy set. On the other hand, the classical binary relations were also extended to the fuzzy binary relations on two ordinary sets [20]. For two given ordinary sets A and B , a fuzzy relation is a fuzzy subset of the set $A \times B$. The uncertainty environment for a variational inequality leads to certain degrees of fuzziness in the classical relation. In 2001, Hu [17] introduced the fuzzy variational inequality over a compact set by using the tolerance approach. Subsequently, Hu [18] investigated the generalized variational inequality with fuzzy relation and showed that such problems can be transformed into regular optimization problems. In 2009, Hu and Liu [19] discussed mathematical programs with fuzzy parametric variational inequalities. In 2019, Xie and Gong [25] investigated the generalized variational-like inequalities for fuzzy-vector-valued functions.

With the research of fuzzy sets, Atanassov [1] presented the intuitionistic fuzzy set (IFS, for short) which is more powerful and sensitive than the fuzzy set in dealing with imperfect information and imprecise information. Actually, the intuitionistic fuzzy set is a straightforward generalization of Zadeh’s fuzzy set: a fuzzy set gives a degree to which an element belongs to the set, while an intuitionistic fuzzy set gives both a membership degree and a nonmembership degree. The membership and nonmembership values induce the hesitancy degree, which models the hesitancy of deciding the degree to which an object satisfies a particular property. For instance, this situation can be found in group decision making problems. Consider a voting situation in which human voters can be divided into three groups: vote for, vote against, and abstain. If we take $\langle x_1, 0.6, 0.3 \rangle$ as an element of intuitionistic fuzzy set A of voting, then we can interpret it as “the voting for the candidate x_1 is 0.6 for 0.3 against and 0.1 abstentions”. IFS theory has been widely applied in various fields such as decision analysis, pattern recognition, machine learning, image processing, and so on [12–14]. In this paper, we further discuss the generalized variational inequalities with intuitionistic fuzzy relations. In addition, real-time solutions to these problems are always needed in engineering applications, and thus they have to be solved in real time to optimize the performance of dynamical systems. As parallel computational models, recurrent neural networks possess many desirable properties for real-time information processing. In 2003, M.A. Noor [24] investigated some implicit projected dynamical systems

associated with quasi-variational inequalities by using the techniques of the projection and the Wiener–Hopf equations. Indeed, by means of level sets of the intuitionistic fuzzy relations, the generalized variational inequalities for linguistic interpretations using intuitionistic fuzzy relations can be transformed into the classical (nonfuzzy) generalized variational inequalities, and we further propose a projection neural network for solving the generalized variational inequalities with intuitionistic fuzzy relations. The stability of the projected dynamical system is also discussed.

The aim of this paper is to investigate the generalized variational inequality for linguistic interpretations using intuitionistic fuzzy relations and the stability of the associated dynamical system. The rest of the paper is structured as follows. In Sect. 2, we recall some preliminaries with respect to intuitionistic fuzzy sets. In Sect. 3, we introduce the generalized variational inequality for linguistic interpretations using intuitionistic fuzzy relations. In Sect. 4, we prove the existence theorem of solutions to the generalized variational inequalities with intuitionistic fuzzy relations. Furthermore, we propose an iterative algorithm. In Sect. 5, based on the projection method, we propose a projection neural network for solving the proposed problems, which is a dynamical system, and the stability of the projected dynamical system associated with generalized variational inequalities with intuitionistic fuzzy relations is discussed. Section 6 concludes this paper.

2 Preliminaries

For convenience of the reader, the basic properties of intuitionistic fuzzy sets is provided in this section. Let U be a nonempty set called the universe of discourse.

Definition 2.1 ([1]) Let U be a given set. An intuitionistic fuzzy set in U is an expression \tilde{A} given by

$$\tilde{A} = \{ \langle x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x) \rangle | x \in U \},$$

where $\mu_{\tilde{A}} : U \rightarrow [0, 1]$, $\nu_{\tilde{A}} : U \rightarrow [0, 1]$ with the condition $0 \leq \mu_{\tilde{A}} + \nu_{\tilde{A}}(x) \leq 1$ for all $x \in U$. The numbers $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ denote, respectively, the degree of membership and the degree of nonmembership of the element x in the set \tilde{A} . We call $\pi(x) = 1 - \mu_{\tilde{A}}(x) - \nu_{\tilde{A}}(x)$ the intuitionistic index or the hesitancy degree of the element x in the set \tilde{A} . We will denote by $\text{IFSs}(U)$ the set of all the intuitionistic fuzzy sets in U .

Obviously, when $\nu_{\tilde{A}}(x) = 1 - \mu_{\tilde{A}}(x)$, i.e., $\pi(x) = 0$, the set $\tilde{A} = \{ \langle x, \mu_{\tilde{A}}(x) \rangle | x \in U \}$ is a fuzzy set.

The operations of *IFS* are defined as follows [1, 2]: for every $\tilde{A}, \tilde{B} \in \text{IFSs}(U)$,

$\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ and $\nu_{\tilde{A}}(x) \geq \nu_{\tilde{B}}(x)$ for all $x \in U$.

$\tilde{A} = \tilde{B}$ if and only if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$.

$\tilde{A} \cap \tilde{B} = \{ \langle x, \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \max(\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)) \rangle | x \in U \}$.

$\tilde{A} \cup \tilde{B} = \{ \langle x, \max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)), \min(\nu_{\tilde{A}}(x), \nu_{\tilde{B}}(x)) \rangle | x \in U \}$.

The complementary of an *IFS* \tilde{A} is $\tilde{A}_c = \{ \langle x, \nu_{\tilde{A}}(x), \mu_{\tilde{A}}(x) \rangle | x \in U \}$.

Let $\tilde{A} \in \text{IFSs}(U)$, $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$. The (α, β) -level set of \tilde{A} , denoted by \tilde{A}_α^β , is defined by

$$\tilde{A}_\alpha^\beta = \{ x \in U | \mu_{\tilde{A}}(x) \geq \alpha, \nu_{\tilde{A}}(x) \leq \beta \}.$$

Definition 2.2 ([11]) An intuitionistic fuzzy set $\tilde{A} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) | x \in U\} \in \text{IFSs}(U)$ is called an intuitionistic fuzzy number if $\mu_{\tilde{A}}(x), \nu_{\tilde{A}}^c(x)$ are fuzzy numbers, where $\nu_{\tilde{A}}^c(x) = 1 - \nu_{\tilde{A}}(x)$.

Note that a fuzzy set u is called a fuzzy number if u is normal, convex, and upper semi-continuous, and $[u]^0 = \text{cl}(\text{supp } u) = \text{cl}(\bigcup_{r \in (0,1]} [u]^r)$ is compact.

Definition 2.3 ([5]) An intuitionistic fuzzy relation is an intuitionistic fuzzy subset of $U \times V$, that is, is an expression \tilde{R} given by

$$\tilde{R} = \{(x, y), \mu_{\tilde{R}}(x, y), \nu_{\tilde{R}}(x, y) | x \in U, y \in V\},$$

where $\mu_{\tilde{R}} : U \times V \rightarrow [0, 1], \nu_{\tilde{R}} : U \times V \rightarrow [0, 1]$ with the condition $0 \leq \mu_{\tilde{R}}(x, y) + \nu_{\tilde{R}}(x, y) \leq 1$ for all $x \in U$. We will denote by $\text{IFR}(U \times V)$ the set of all the intuitionistic fuzzy subsets in $U \times V$. If $U = V$, then we say that R is an intuitionistic fuzzy relation on U .

The complementary relation of R is

$$\tilde{R}_c = \{(x, y), \nu_{\tilde{R}}(x, y), \mu_{\tilde{R}}(x, y) | (x, y) \in U \times V\}.$$

For all $x \in M$, the image of \tilde{R} , denoted by $\tilde{R}(x)$, is

$$\tilde{R}(x) = \{y \in V | (x, y) \in \tilde{R}\}.$$

The (α, β) -level set of an intuitionistic fuzzy relation \tilde{R} is defined as

$$\tilde{R}_\alpha^\beta = \{(x, y) \in U \times V | \mu_{\tilde{R}}(x, y) \geq \alpha, \nu_{\tilde{R}}(x, y) \leq \beta\}.$$

For all $x \in M$, the (α, β) -level set of the image of \tilde{R} , denoted by $[\tilde{R}(x)]_\alpha^\beta$, is

$$[\tilde{R}(x)]_\alpha^\beta = \{y \in V | (x, y) \in \tilde{R}_\alpha^\beta\}.$$

Example 2.1 Assume that Mr. X wants to buy a car. Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a set of five candidate cars. Suppose that the set of candidate cars U can be characterized by a set of parameters $V = \{v_1, v_2, v_3, v_4\}$, where $v_j (j = 1, 2, 3, 4)$ stands for “being cheap”, “being beautiful”, “being safe”, and “being comfortable”, respectively. Mr. X thinks u_1 is very expensive and this fuzzy information cannot be expressed only by the two crisp numbers 0 and 1, a membership degree can be used instead, which is associated with each element and represented by a real number valued in the interval $[0, 1]$. Furthermore, Mr. X thinks u_1 is 0.8 in being cheap, 0.1 against, and the hesitancy degree is 0.1. In that case, the characteristics of five candidate choices under four parameters are represented by an intuitionistic fuzzy relation matrix $\tilde{R}(u_i, v_j)_{5 \times 4}$, which describes the attractiveness of the cars which Mr. X is going to buy, as follows:

$$\tilde{R} = \begin{pmatrix} (0.8, 0.1) & (0.4, 0.6) & (0.4, 0.2) & (0.3, 0.6) \\ (0.6, 0.2) & (0.3, 0.5) & (0.5, 0.4) & (0.7, 0.1) \\ (0.3, 0.6) & (0.6, 0.3) & (0.7, 0.1) & (0.4, 0.5) \\ (0.7, 0.3) & (0.2, 0.7) & (0.3, 0.7) & (0.7, 0.3) \\ (0.5, 0.4) & (0.1, 0.7) & (0.6, 0.2) & (0.8, 0.1) \end{pmatrix}$$

3 The generalized variational inequality with intuitionistic fuzzy relation

Definition 3.1 Let $M \subseteq R^n$, $f : M \rightarrow 2^{R^n}$ be a set-valued mapping and $\tilde{\Gamma}$ is an intuitionistic fuzzy relation on $M \times R^n$ with its membership function $\mu_{\tilde{\Gamma}}$ and nonmembership function $\nu_{\tilde{\Gamma}}$. Then the generalized variational inequality with intuitionistic fuzzy relation, denoted by $\text{GVI}(M, \tilde{\Gamma})$, is defined as

$$\begin{aligned} \text{GVI}(M, \tilde{\Gamma}) \quad & \text{find } (x, y) \\ & \text{subject to } x \in M, \\ & \langle y, x' - x \rangle \geq 0, \quad \forall x' \in M, \\ & \langle (x, y), \mu_{\tilde{\Gamma}}(x, y), \nu_{\tilde{\Gamma}}(x, y) \rangle \in \tilde{\Gamma}, \end{aligned} \quad (3.1)$$

where $\tilde{\Gamma} = \{ \langle (x, y), \mu_{\tilde{\Gamma}}(x, y), \nu_{\tilde{\Gamma}}(x, y) \rangle | y \approx f(x) \} \subseteq \text{IFR}(R^n \times R^n)$, here the wavy bar under a symbol plays the role of a fuzzifier, that is, a transformation which takes a nonfuzzy set into a fuzzy set which is approximately equal to it. In other words, $y \approx f(x)$ is a fuzzy equality and “ \approx ” denotes the fuzzified version of “=” with the linguistic interpretation “approximately equal to”.

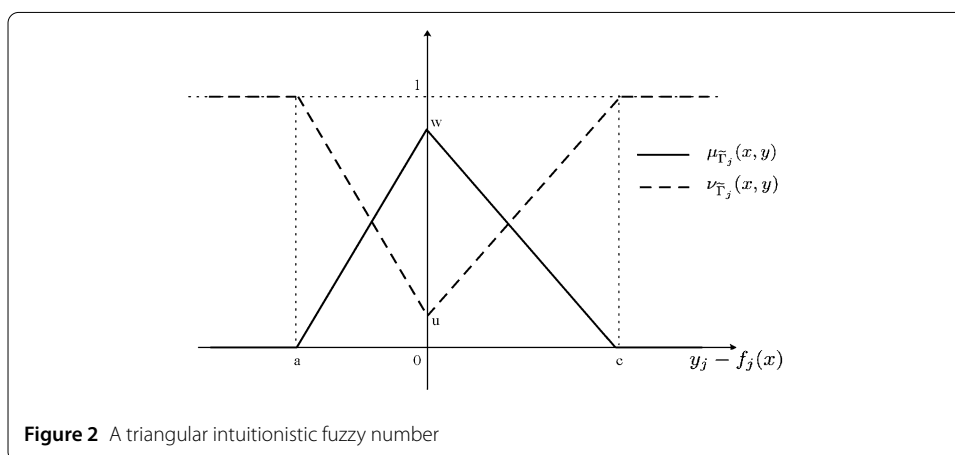
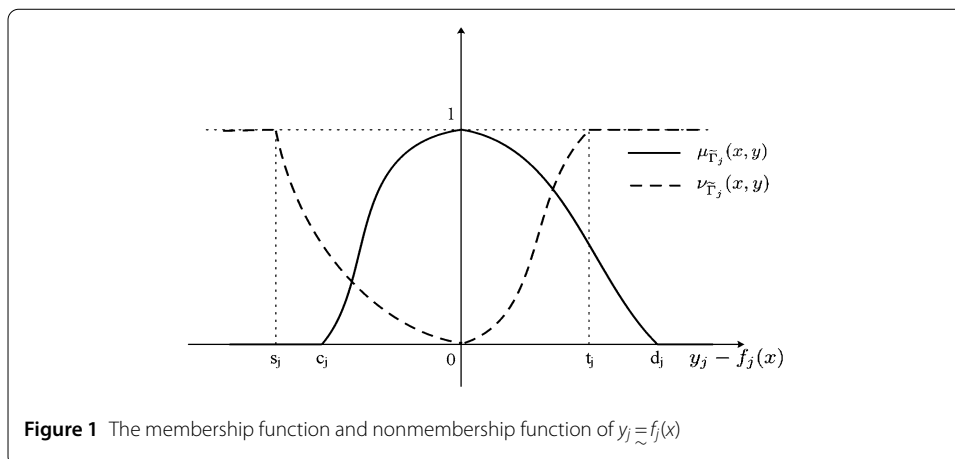
Remark 3.1 More specifically, for $y, f(x) \in R^n$, since $y \approx f(x)$, then $y_j \approx f_j(x)$, $j = 1, 2, \dots, n$, which actually determines an intuitionistic fuzzy set, whose membership function and nonmembership function are denoted by $\mu_{\tilde{\Gamma}_j}, \nu_{\tilde{\Gamma}_j}, j = 1, 2, \dots, n$, respectively. The membership grade $\mu_{\tilde{\Gamma}_j}(x, y)$ can be interpreted as the degree to which the regular equality $y_j = f_j(x)$, $j = 1, 2, \dots, n$, is satisfied. To specify the membership functions $\mu_{\tilde{\Gamma}_j}$, it is commonly assumed that $\mu_{\tilde{\Gamma}_j}(x, y)$ should be 0 if the regular equality $y_j = f_j(x)$ is strongly violated and 1 if it is satisfied, which is analogous to the nonmembership functions. In this sense, we can obtain a membership function and a nonmembership function in the following forms, respectively:

$$\mu_{\tilde{\Gamma}_j}(x, y) = \begin{cases} 1, & y_j - f_j(x) = 0, \\ \mu_{L_j}(y_j - f_j(x)), & -c_j \leq y_j - f_j(x) < 0, \\ \mu_{R_j}(y_j - f_j(x)), & 0 < y_j - f_j(x) \leq d_j, \\ 0, & \text{otherwise,} \end{cases}$$

$$\nu_{\tilde{\Gamma}_j}(x, y) = \begin{cases} 0, & y_j - f_j(x) = 0, \\ \nu_{L_j}(y_j - f_j(x)), & -s_j \leq y_j - f_j(x) < 0, \\ \nu_{R_j}(y_j - f_j(x)), & 0 < y_j - f_j(x) \leq t_j, \\ 1, & \text{otherwise,} \end{cases}$$

where $c_j, d_j \geq 0$, are the tolerance levels which a decision maker can tolerate in the accomplishment of the fuzzy equality $y_j \approx f_j(x)$, and $s_j, t_j \geq 0$ are the intolerance levels. We usually assume that $\mu_{L_j}, \nu_{R_j} \in [0, 1]$ are continuous and nondecreasing, respectively, on $[-d_j, 0]$, $[0, t_j]$, and $\mu_{R_j}, \nu_{L_j} \in [0, 1]$ are continuous and nonincreasing, respectively, on $[0, d_j], [-s_j, 0]$.

As shown in Fig. 1, $\mu_{\tilde{\Gamma}_j}(x, y)$ and $\nu_{\tilde{\Gamma}_j}(x, y)$ is the membership function and the nonmembership function of $y_j \approx f_j(x)$, respectively, where $\mu_{L_j}(-c_j) = 0$, $\mu_{R_j}(-d_j) = 0$, $\nu_{L_j}(s_j) = 0$, and



$\nu_{R_j}(t_j) = 0$. The membership function and the nonmembership function can be expressed by some special forms, such as triangular intuitionistic fuzzy numbers (TIFNs), trapezoidal intuitionistic fuzzy numbers (TrIFNs).

Example 3.1 Fig. 2 shows a triangular intuitionistic fuzzy number, denoted by $A = \langle (a, 0, c); w, u \rangle$, where w and u denote the maximum degree of membership and the minimum degree of nonmembership, satisfying the conditions $0 \leq w, u \leq 1$ and $0 \leq w + u \leq 1$, respectively.

Remark 3.2 Since all the components of $y = f(x)$ have to be satisfied, for the intuitionistic fuzzy relation Γ , we define its membership function and nonmembership function, respectively, as

$$\mu_{\tilde{\Gamma}}(x, y) = \min_{j=1,2,\dots,n} \mu_{\tilde{\Gamma}_j}(x, y),$$

$$\nu_{\tilde{\Gamma}}(x, y) = \max_{j=1,2,\dots,n} \nu_{\tilde{\Gamma}_j}(x, y).$$

Definition 3.2 We say (x, y) is a (α, β) -level solution to the problem $\text{GVI}(M, \tilde{\Gamma})$ if (x, y) solves the problem

$$\begin{aligned} \text{GVI}(M, \tilde{\Gamma}_\alpha^\beta) \quad & \text{find } (x, y) \\ & \text{subject to } x \in M, \\ & \langle y, x' - x \rangle \geq 0, \quad \forall x' \in M, \\ & \langle (x, y), \mu_{\tilde{\Gamma}}(x, y), \nu_{\tilde{\Gamma}}(x, y) \rangle \in \tilde{\Gamma}_\alpha^\beta, \end{aligned} \quad (3.2)$$

where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, and

$$\tilde{\Gamma}_\alpha^\beta = \{(x, y) \in M \times R^n \mid \mu_{\tilde{\Gamma}}(x, y) \geq \alpha, \nu_{\tilde{\Gamma}}(x, y) \leq \beta, \forall j = 1, 2, \dots, n\}.$$

4 The existence theorem and iterative algorithm of solutions to the generalized variational inequalities with intuitionistic fuzzy relations

Definition 4.1 Let $M \subseteq R^n$, \tilde{R} be an intuitionistic fuzzy relation on $M \times R^n$. For all $x_1, x_2 \in M$, the image of \tilde{R} is said to be

(1) strongly monotone, if there exists a constant $\delta \in (0, 1)$ such that

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geq \delta \|x_1 - x_2\| \quad (4.1)$$

for all $y_1 \in [\tilde{R}(x_1)]_\alpha^\beta$, $y_2 \in [\tilde{R}(x_2)]_\alpha^\beta$, where $\|\cdot\|$ and $\langle \cdot \rangle$ denote norm and inner product on R^n , respectively.

(2) Lipschitz continuous, if there exists a constant $L \in (0, 1)$ such that

$$D([\tilde{R}(x_1)]_\alpha^\beta, [\tilde{R}(x_2)]_\alpha^\beta) \leq L \|x_1 - x_2\|, \quad (4.2)$$

where D is the Hausdorff metric on R^n .

For simplicity, if the image of \tilde{R} is strongly monotone and Lipschitz continuous, we also say \tilde{R} is strongly monotone and Lipschitz continuous, respectively.

Definition 4.2 ([4]) The distance of a point $x_0 \in R^n$ to a closed set $C \subseteq R^n$, in the norm $\|\cdot\|$, is defined as

$$\text{dist}(x_0, C) = \inf\{\|x_0 - x\| : x \in C\}.$$

The infimum here is always achieved. We refer to any point $z \in C$ which is closest to x_0 , i.e., satisfies $\|z - x_0\| = \text{dist}(x_0, C)$, as a projection of x_0 on C , denoted by $P_C(x_0)$.

In other words, $P_C : R^n \rightarrow C$ and $P_C(x_0) = \text{argmin}\{\|x_0 - x\| : x \in C\}$, we refer to P_C as a projection on C .

Remark 4.1 In general, there can exist more than one projection of x_0 on C , that is, there are several points in C which are closest to x_0 . However, we can construct the projection of a point on a set which is unique. For instance, if C is closed and convex and the norm is strictly convex such as the Euclidean norm, then for any x_0 , there is always exactly one $z \in C$ which is closest to x_0 . Conversely, if for any x_0 there is a unique Euclidean projection of x_0 on C , then C is closed and convex.

In what follows, we suppose that the norm of $x \in R^n$ is the Euclidean norm, i.e.,

$$\|x\| = (x^T x)^{1/2} = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

Lemma 4.1 ([21]) *Let $M \subseteq R^n$ be a closed and convex set. Then*

$$(x - P_M(x))^T (y - P_M(x)) \leq 0, \quad \forall x \in R^n, \forall y \in M, \quad (4.3)$$

$$\|P_M(x) - P_M(y)\| \leq \|x - y\|, \quad \forall x, y \in R^n. \quad (4.4)$$

Theorem 4.1 *Let $M \subseteq R^n$, \tilde{R} be an intuitionistic fuzzy relation on $M \times R^n$. If \tilde{R} is Lipschitz continuous, then there exists a point $x \in M$ such that $x \in [\tilde{R}(x)]_\alpha^\beta$, where $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, that is, x is a fixed point of \tilde{R} .*

Proof Let $x_0 \in M$ and $x_1 \in [\tilde{R}(x_0)]_\alpha^\beta$. Then there exists $x_2 \in [\tilde{R}(x_1)]_\alpha^\beta$ and

$$\|x_2 - x_1\| \leq L\|x_1 - x_0\|,$$

where $L \in (0, 1)$. Since \tilde{R} and $x_2 \in [\tilde{R}(x_1)]_\alpha^\beta$, there is a point $x_3 \in [\tilde{R}(x_2)]_\alpha^\beta$ such that

$$\|x_3 - x_2\| \leq L\|x_2 - x_1\| \leq L^2\|x_1 - x_0\|.$$

Then we can obtain a sequence $\{x_n\}$ of points of M satisfying $x_{n+1} \in [\tilde{R}(x_n)]_\alpha^\beta$ and

$$\|x_{n+1} - x_n\| \leq L\|x_n - x_{n-1}\| \leq L^n\|x_1 - x_0\|$$

for all $n \geq 1$. Therefore, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq (L^{n+m-1} + \cdots + L^n)\|x_1 - x_0\| \\ &\leq \frac{L^n}{1-L}\|x_1 - x_0\| \end{aligned}$$

for all $n, m \geq 1$, thus, the sequence $\{x_{n+1}\}$ is a Cauchy sequence, which implies that $x_n \rightarrow x \in R^n$. Therefore, the sequence $[\tilde{R}(x_n)]_\alpha^\beta$ converges to $x_{n+1} \in [\tilde{R}(x)]_\alpha^\beta$ weakly, and since $x_{n+1} \in [\tilde{R}(x_n)]_\alpha^\beta$ for all n , then $x \in [\tilde{R}(x)]_\alpha^\beta$, therefore, x is a fixed point of \tilde{R} . \square

Theorem 4.2 *Let $M \subseteq R^n$ be a closed and convex set, $\tilde{\Gamma}$ be an intuitionistic fuzzy relation on $M \times R^n$. Then (x, y) is a solution of $\text{GVI}(M, \tilde{\Gamma})$ if and only if*

$$x = P_M[x - \rho y], \quad (4.5)$$

where $y \in [\tilde{\Gamma}(x)]_\alpha^\beta$ for $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$, $\rho > 0$ is a constant, and P_M is the projection of R^n on to M .

Proof If (x, y) is a solution to $\text{GVI}(M, \tilde{\Gamma})$, then $x \in M$, $y \in [\tilde{\Gamma}(x)]_\alpha^\beta$, and

$$\langle y, x' - x \rangle \geq 0, \quad \forall x' \in M.$$

Thus, for a constant $\rho > 0$, we have $\langle \rho y, x' - x \rangle \geq 0, \forall x' \in M$. Then, for all $v \in M$,

$$\begin{aligned}\|v - (x - \rho y)\|^2 &= \|v - x\|^2 + 2\langle v - x, \rho y \rangle + \|\rho y\|^2 \\ &\geq \|\rho y\|^2 \\ &= \|x - (x - \rho y)\|^2.\end{aligned}$$

Therefore, $x = \min_{x \in M} \frac{1}{2} \|v - (x - \rho y)\|^2$, that is, $x = P_M[x - \rho y]$, where $\rho > 0$.

Conversely, if $x = P_M[x - \rho y]$ and $y \in [\tilde{\Gamma}(x)]_\alpha^\beta$, where $\rho > 0$, then $x \in M$. By (4.3) of Lemma 4.1, we obtain

$$\langle P_M[x - \rho y] - (x - \rho y), v - P_M[x - \rho y] \rangle \geq 0, \quad \forall v \in M,$$

that is,

$$\langle x - (x - \rho y), v - x \rangle \geq 0, \quad \forall v \in M,$$

thus, we have $\langle \rho y, v - x \rangle \geq 0, \forall v \in M$. Since $\rho > 0$ is a constant, then $\langle y, v - x \rangle \geq 0, \forall v \in M$, where $y \in [\tilde{\Gamma}(x)]_\alpha^\beta$. Therefore, (x, y) is a solution of $\text{GVI}(M, \tilde{\Gamma})$. \square

Theorem 4.2 indicates that $\text{GVI}(M, \tilde{\Gamma})$ is equivalent to the following fuzzy fixed point problem:

$$H(x) = P_M[x - \rho y], \quad (4.6)$$

where $y \in [\tilde{\Gamma}(x)]_\alpha^\beta$. Accordingly, we can give the following iterative algorithm.

Algorithm 1 For given $x_0 \in M$ such that $y_0 \in [\tilde{\Gamma}(x_0)]_\alpha^\beta$.

Step 1. Let

$$x_1 = P_M[x_0 - \rho y_0],$$

where $\rho > 0$ is a constant.

Step 2. Since $y_0 \in [\tilde{\Gamma}(x_0)]_\alpha^\beta$, there exists $y_0 \in [\tilde{\Gamma}(x_0)]_\alpha^\beta$ such that $\|y_0 - y_1\| \leq D([\tilde{\Gamma}(x_0)]_\alpha^\beta, [\tilde{\Gamma}(x_1)]_\alpha^\beta)$. Let

$$x_2 = P_M[x_1 - \rho y_1].$$

Step 3. Find x_n and y_n by the following iterative methods:

$$\begin{aligned}\|y_{n+1} - y_n\| &\leq D([\tilde{\Gamma}(x_{n+1})]_\alpha^\beta, [\tilde{\Gamma}(x_n)]_\alpha^\beta), \\ x_{n+1} &= P_M[x_n - \rho y_n], \quad n = 1, 2, \dots\end{aligned} \quad (4.7)$$

Theorem 4.3 Let (x_n, y_n) and (x, y) be the solutions to (4.7) and (3.1), respectively. If $\tilde{\Gamma}$ is strongly monotone and Lipschitz continuous, then $x_n \rightarrow x$ strongly, and $y_n \rightarrow y$ strongly.

Proof According to (4.7) of Algorithm 1, we obtain $x_{n+1} - x_n = P_M[x_n - \rho y_n] - P_M[x_{n-1} - \rho y_{n-1}]$, where $\rho > 0$. Since $\tilde{\Gamma}$ is strongly monotone and Lipschitz continuous, then there exist constants $L, \delta \in (0, 1)$, and combining with (4.4) of Lemma 4.1, we have

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|P_M[x_n - \rho y_n] - P_M[x_{n-1} - \rho y_{n-1}]\|^2 \\ &\leq \|x_n - x_{n-1} - \rho(y_n - y_{n-1})\|^2 \\ &= \|x_n - x_{n-1}\|^2 + \rho^2 \|y_n - y_{n-1}\|^2 - 2\rho \langle y_n - y_{n-1}, x_n - x_{n-1} \rangle \\ &\leq (1 + \rho^2 L^2 - 2\rho\delta) \|x_n - x_{n-1}\|^2.\end{aligned}$$

Setting $\theta = \sqrt{1 - 2\rho\delta + \rho^2 L^2} < 1$ for $0 < \rho < \frac{2\delta}{L^2}$, then $\|x_{n+1} - x_n\| = \theta \|x_n - x_{n-1}\|$, thus, $\{x_n\}$ is a Cauchy sequence, that is, $x_n \rightarrow x$ strongly ($n \rightarrow \infty$).

On the other hand, since $\tilde{\Gamma}$ is Lipschitz continuous, then there exists a constant $L \in (0, 1)$ such that

$$\begin{aligned}\|y_{n+1} - y_n\| &\leq D([\tilde{\Gamma}(x_{n+1})]_\alpha^\beta, [\tilde{\Gamma}(x_n)]_\alpha^\beta) \\ &\leq L \|x_n - x_{n-1}\|,\end{aligned}$$

therefore, y_n is a Cauchy sequence, that is, $y_n \rightarrow y$ strongly ($n \rightarrow \infty$). Furthermore, $y \in [\tilde{\Gamma}(x)]_\alpha^\beta$. Indeed,

$$\begin{aligned}d(y, [\tilde{\Gamma}(x)]_\alpha^\beta) &\leq \|y - y_n\| + d(y_n, [\tilde{\Gamma}(x)]_\alpha^\beta) \\ &\leq \|y - y_n\| + D([\tilde{\Gamma}(x_n)]_\alpha^\beta, [\tilde{\Gamma}(x)]_\alpha^\beta) \\ &\leq \|y_n - y\| + L \|x_n - x\|,\end{aligned}$$

thus, when $n \rightarrow \infty$, $d(y, [\tilde{\Gamma}(x)]_\alpha^\beta) \rightarrow 0$, and since $d(y, [\tilde{\Gamma}(x)]_\alpha^\beta) = \inf\{\|y - t\| : t \in [\tilde{\Gamma}(x)]_\alpha^\beta\}$, then we obtain $d(y, [\tilde{\Gamma}(x)]_\alpha^\beta) = 0$, that is, $y \in [\tilde{\Gamma}(x)]_\alpha^\beta$.

Hence, according to Theorem 4.1, we have $x \in M$, $y \in R^n$ satisfying $y \in [\tilde{\Gamma}(x)]_\alpha^\beta$ are the solutions to (3.1), and $x_n \rightarrow x$ strongly, $y_n \rightarrow y$ strongly. \square

5 Stability of the dynamical system for generalized variational inequalities with intuitionistic fuzzy relations

Let $M \subseteq R^n$ be a closed and convex set, $\tilde{\Gamma}$ be an intuitionistic fuzzy relation on $M \times R^n$. Consider the following projected neural network associated with the generalized variational inequality with intuitionistic fuzzy relation (3.1):

$$\frac{dx(t)}{dt} = \lambda \{P_M[x - \rho y] - x\}, \quad x(t_0) = x_0, \quad (5.1)$$

where $\rho > 0$, λ are constants, $y \in [\tilde{\Gamma}(x)]_\alpha^\beta$. $x(t) = (x_1(t), x_2(t), \dots, x_m(t))^T$ denotes the state vector of neurons, m is the number of neurons, and the initial value x_0 is given randomly. It is a dynamical system.

Without loss of generality, consider the following nonlinear dynamical system [6]:

$$\begin{cases} \frac{dx}{dt} = f(t, x), \\ x(t_0) = x_0, \end{cases} \quad (5.2)$$

where $t \in R$, $x \in M \subseteq R^n$, x_0 is the initial state. If there exists a state x^* in the state space satisfying

$$f(t, x^*) = 0, \quad \forall t \geq t_0,$$

then we say x^* is an equilibrium state or an equilibrium point of system (5.2). The equilibrium point x^* is said to be stable in the sense of Lyapunov if, for any $\varepsilon > 0$, there exists $\delta > 0$, when $\|x(t_0) - x^*\| < \delta$, we have $\|x(t) - x^*\| < \varepsilon$ ($t \geq t_0$); x^* is said to be asymptotically stable if x^* is stable and satisfies $x(t) \rightarrow x^*$ ($t \rightarrow \infty$); x^* is said to be globally asymptotically stable if, for any initial point, x^* is asymptotically stable; x^* is said to be globally exponentially stable if, for any solution of the system $x(t)$, there exist $k > 0$, $\eta > 0$ such that

$$\|x(t) - x^*\| \leq k \|x(t_0) - x^*\| \exp(-\eta(t - t_0)), \quad \forall t \geq t_0.$$

System (5.2) is said to globally converge to the set $M' \subseteq R^n$ if, for any initial point, the solution of the system $x(t)$ satisfies

$$\lim_{t \rightarrow \infty} \text{dist}(x(t), M') = 0,$$

where $\text{dist}(x(t), M') = \inf_{y \in M'} \|x - y\|$.

Lemma 5.1 (LaSalle's invariance principle [23]) *Let $f(t, x)$ be continuous in system (5.2). If there exists a continuously differentiable function $V : R^n \rightarrow R^1$ satisfying the following conditions:*

- (i) *There exists a constant $r > 0$ such that the set $M_r = \{x \in R^n : V(x) \leq r\}$ is bounded;*
- (ii) *for all $x \in M_r$, $\frac{dV(x)}{dt} \leq 0$,*

then for all $x_0 \in M_r$, when $t \rightarrow \infty$, $x(t)$ converges to the largest invariant subset of the set $\{x \in R^n : \frac{dV(x)}{dt} \leq 0\}$.

Lemma 5.2 (Gronwall's inequality [10]) *Let $x(t)$, $y(t)$ be real-valued nonnegative continuous functions with domain $\{t : t \geq t_0\}$, and let $a(t) = a_0(|t - t_0|)$, where a_0 is a monotone increasing function. If, for $t \geq t_0$,*

$$x(t) \leq a(t) + \int_{t_0}^t x(s)y(s) ds,$$

then

$$x(t) \leq a(t) \exp\left(\int_{t_0}^t y(s) ds\right).$$

Theorem 5.1 *Let $M \subseteq R^n$ be a closed and convex set, $\tilde{\Gamma}$ be an intuitionistic fuzzy relation on $M \times R^n$. (x^*, y^*) is a solution of $\text{GVI}(M, \tilde{\Gamma})$ if and only if x^* is an equilibrium point of dynamical system (5.1).*

Proof According to Theorem 4.2, (x^*, y^*) is a solution of $\text{GVI}(M, \tilde{\Gamma})$ if and only if

$$x^* = P_M[x^* - \rho y^*],$$

where $y^* \in [\tilde{\Gamma}(x^*)]_\alpha^\beta$, $\rho > 0$ is a constant, that is,

$$P_M[x^* - \rho y^*] - x^* = 0,$$

namely, x^* is an equilibrium point of dynamical system (5.1). \square

Theorem 5.2 *Let $M \subseteq R^n$ be a closed and convex set, $\tilde{\Gamma}$ be an intuitionistic fuzzy relation on $M \times R^n$. If $\tilde{\Gamma}$ is Lipschitz continuous, then for any $x_0 \in R^n$ there exists a unique continuous solution $x(t)$ of dynamical system (5.1) with $x(t_0) = x_0$, where $t \in [t_0, \infty)$.*

Proof Let

$$G(x) = \lambda \{P_M[x - \rho y] - x\}, \quad y \in [\tilde{\Gamma}(x)]_\alpha^\beta, \alpha, \beta \in [0, 1].$$

Then, for any $x_1, x_2 \in R^n$, since $\tilde{\Gamma}$ is Lipschitz continuous, and by (4.4), we have

$$\begin{aligned} \|G(x_1) - G(x_2)\| &\leq \lambda \{ \|P_M[x_1 - \rho y_1] - P_M[x_2 - \rho y_2]\| + \|x_1 - x_2\| \} \\ &\leq \lambda \{ \|x_1 - x_2\| + \|(x_1 - \rho y_1) - (x_2 - \rho y_2)\| \} \\ &\leq \lambda \{2 + \rho L\} \|x_1 - x_2\|, \end{aligned}$$

where $y_1 \in [\tilde{\Gamma}(x_1)]_\alpha^\beta$, $y_2 \in [\tilde{\Gamma}(x_2)]_\alpha^\beta$, $\rho > 0$, $L > 0$. Thus, $G(x)$ is Lipschitz continuous. Then, by the existence and uniqueness theorem of solutions for an ordinary differential equation, for any $x_0 \in R^n$, there exists a unique continuous solution $x(t)$ of dynamical system (5.1) with $x(t_0) = x_0$ over $[t_0, T]$.

On the other hand, since for any $x \in R^n$

$$\begin{aligned} \|G(x)\| &= \lambda \{ \|P_M[x - \rho y] - x\| \} \\ &\leq \lambda \{ \|P_M[x - \rho y] - P_M[x]\| + \|P_M(x) - P_M[x^*]\| \} + \|P_M[x^*] - x\| \\ &\leq \lambda \rho \|y\| + \lambda \|x - x^*\| + \lambda \|P_M[x^*]\| + \lambda \|x\| \\ &\leq \lambda(2 + \rho L) \|x\| + \lambda \{ \|x^*\| + \|P_M[x^*]\| \}, \end{aligned}$$

then

$$\|x(t)\| \leq \|x_0\| + \int_{t_0}^t \|Tx(s)\| ds \leq (\|x_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^t \|x(s)\| ds,$$

where $k_1 = \lambda \{ \|x^*\| + \|P_M[x^*]\| \}$, $k_2 = \lambda(2 + \rho L)$. Therefore, by Lemma 5.2, we have

$$\|x(t)\| \leq \{ \|x_0\| + k_1(t - t_0) \exp(k_2(t - t_0)) \}, \quad t \in [t_0, T].$$

It implies that $x(t)$ is bounded on $[t_0, T)$, then by the extension theorem of solutions for an ordinary differential equation, we have $T = \infty$. \square

Theorem 5.3 *Let $M \subseteq R^n$ be a closed and convex set, $\tilde{\Gamma}$ be an intuitionistic fuzzy relation on $M \times R^n$. If $\tilde{\Gamma}$ is pseudo-monotone and Lipschitz continuous, then dynamical system (5.1) is stable in the sense of Lyapunov and globally converges to the solution set S of $\text{GVI}(M, \tilde{\Gamma})$.*

Proof Since $\tilde{\Gamma}$ is Lipschitz continuous, by Theorem 5.2, dynamical system (5.1) has a unique continuous solution $x(t)$. Suppose that $x^* \in M$ is an equilibrium point of dynamical system (5.1), then x^* is a solution of $\text{GVI}(M, \tilde{\Gamma})$, it follows that $(y^*)^T(x - x^*) \geq 0, \forall x \in M$, where $y^* \in [\tilde{\Gamma}(x^*)]_{\alpha}^{\beta}$, and since $\tilde{\Gamma}$ is pseudo-monotone, then we have $y^T(x - x^*) \geq 0, \forall x \in M$, where $y \in [\tilde{\Gamma}(x)]_{\alpha}^{\beta}$. Setting $x = P_M[x - \rho y]$, then

$$\langle y, P_M[x - \rho y] - x^* \rangle \geq 0.$$

On the other hand, for $x^* \in M$, by (4.3) of Lemma 4.1, we have

$$\langle P_M[x - \rho y] - (x - \rho y), x^* - P_M[x - \rho y] \rangle \geq 0,$$

that is,

$$\langle P_M[x - \rho y] - x, x^* - P_M[x - \rho y] \rangle + \langle \rho y, x^* - P_M[x - \rho y] \rangle \geq 0.$$

Therefore, we obtain

$$\langle P_M[x - \rho y] - x, x^* - x + (x - P_M[x - \rho y]) \rangle \geq 0.$$

Thus, we have

$$\langle x - x^*, x - P_M[x - \rho y] \rangle \geq \|x - P_M[x - \rho y]\|^2.$$

Hence, for the following Lyapunov function

$$V(x) = \lambda \|x - x^*\|^2, \quad x \in R^n,$$

we have

$$\frac{dV(x)}{dt} = \frac{dV}{dx} \frac{dx}{dt} = 2\lambda \langle x - x^*, P_M[x - \rho y] - x \rangle \leq 0,$$

where $x \in M_0 = \{x \in M : V(x) \leq V(x_0)\}$. Therefore, dynamical system (5.1) is stable in the sense of Lyapunov.

Furthermore, since $V(x)$ is continuously differentiable on the bounded set M_0 , by LaSalle's invariance principle, $x(t)$ converges to the largest invariant subset of the set $\{x \in M : \frac{dV}{dt} = 0\}$. Since $\frac{dV}{dt} = 0 \Leftrightarrow \frac{dx}{dt} = 0$, then $\{x \in M : \frac{dV}{dt} = 0\} = \{x \in M : \frac{dx}{dt} = 0\} = M_0 \cap S$, therefore, $\lim_{t \rightarrow \infty} \text{dist}(x(t), S) = 0$, that is, dynamical system (5.1) globally converges to the solution set S of $\text{GVI}(M, \tilde{\Gamma})$. \square

Theorem 5.4 *Let $M \subseteq R^n$ be a closed and convex set, $\tilde{\Gamma}$ be an intuitionistic fuzzy relation on $M \times R^n$. If $\tilde{\Gamma}$ is Lipschitz continuous, then for $\lambda < 0$, dynamical system (5.1) globally exponentially converges to the solution of $\text{GVI}(M, \tilde{\Gamma})$.*

Proof Since $\tilde{\Gamma}$ is Lipschitz continuous, by Theorem 5.2, dynamical system (5.1) has a unique continuous solution $x(t)$. Let $x^* \in M$ be an equilibrium point of dynamical system (5.1), and consider the following Lyapunov function:

$$V(x) = \lambda \|x - x^*\|^2, \quad x \in R^n,$$

we have

$$\begin{aligned}\frac{dV}{dt} &= 2\lambda \langle x(t) - x^*, P_M[x(t) - \rho y] - x(t) \rangle \\ &= -2\lambda \|x(t) - x^*\|^2 + 2\lambda \langle x(t) - x^*, P_M[x(t) - \rho y] - x^* \rangle.\end{aligned}$$

On the other hand, for the equilibrium point $x^* \in M$, by Theorem 5.1, we have x^* is a solution of $\text{GVI}(M, \tilde{\Gamma})$, that is, $x^* = P_M[x^* - \rho y^*]$, thus, by (4.3) of Lemma 4.1 and $\tilde{\Gamma}$ is Lipschitz continuous, we obtain

$$\begin{aligned}\|P_M[x(t) - \rho y] - x^*\| &= \|P_M[x(t) - \rho y] - P_M[x^* - \rho y^*]\| \\ &\leq \|x - x^* - \rho(y - y^*)\| \\ &\leq \|x - x^*\| + \rho L \|x - x^*\| \\ &\leq (1 + \rho L) \|x - x^*\|,\end{aligned}$$

where $\rho > 0$, $L > 0$, $y \in [\tilde{\Gamma}(x)]_\alpha^\beta$, $y^* \in [\tilde{\Gamma}(x^*)]_\alpha^\beta$, $\alpha, \beta \in [0, 1]$. Therefore, we have

$$\frac{dV}{dt} = \frac{d}{dt} (\lambda \|x(t) - x^*\|^2) \leq 2\alpha\lambda^2 \|x(t) - x^*\|^2,$$

where $\alpha = \rho L$. Setting $\lambda_1 = -\lambda$, then $\lambda_1 > 0$, and we have

$$\|x(t) - x^*\| \leq \|x(t_0) - x^*\| \exp(-\alpha\lambda_1(t - t_0)),$$

that is, dynamical system (5.1) globally exponentially converges to the solution of $\text{GVI}(M, \tilde{\Gamma})$. \square

6 Conclusions

In this paper, we have investigated the generalized variational inequalities with intuitionistic fuzzy relations. We have obtained the existence theorem of solutions to the generalized variational inequalities with intuitionistic fuzzy relations. Furthermore, we have analyzed an iterative algorithm and a projected neural network model for this type variational inequality by using the projection method, and the proposed projected dynamical system is shown to be stable in the sense of Lyapunov, globally convergent and globally exponentially convergent under various conditions.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

TX made the major analysis and the methodology, and was a major contributor in writing the manuscript. DL dealt with investigation and provision of study resources. All authors read and approved the final manuscript.

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