Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

# Extremal properties of the beta-normal distribution

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**Abstract**

Asymptotic behaviors of the extremes of the beta-normal distribution are derived. The higher-order asymptotic expansions of the probability density and cumulative distribution functions for the maximum are given under an optimal normalizing constants. In particular, the associated rates of convergence are explicitly calculated.

**Keywords:** Rate of convergence; Beta-normal distribution; Extreme value distribution; Expansion

## 1 Introduction

Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed (i.i.d.) random variables and write $M_n$ for the partial maximum, i.e.,

$$M_n = \max\{X_1, X_2, \dots, X_n\}.$$

If there exist suitable normalizing constants $a_n > 0$ and $b_n \in R$, and a distribution $G(x)$ which is nondegenerate such that

$$\lim_{n\to\infty} P(M_n \le a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x) \tag{1.1}$$

for all continuity points of $G$. Then $G$ has one of the following three parametric forms:

$$\textit{Type I (Gumbel)}: \quad \Lambda(x) = \exp\{-e^{-x}\}, \quad x \in R,$$

$$\textit{Type II (Fréchet)}: \quad \Phi_\alpha(x) = \begin{cases} 0 & \text{if } x < 0, \\ \exp\{-x^{-\alpha}\} & \text{if } x \ge 0, \end{cases}$$

$$\textit{Type III (Weibull)}: \quad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & \text{if } x < 0, \\ 1 & \text{if } x \ge 0, \end{cases}$$

where $\alpha$ is a positive constant. If (1.1) holds for some sequences $\{a_n > 0\}$, $\{b_n\}$, we say that distribution $F$ belongs to the domain of attraction of $G$ and write $F \in D(G)$. Necessary and sufficient criteria for $F \in D(G)$ can be found in Leadbetter [15] and Resnick [23].

One meaningful issue in extreme value theory is to study the convergence rate and extremal properties related to the normalized maximum of a sample. Hall [10] studied optimal rates of uniform convergence for standard normal distribution. Nair [19] derived asymptotic expansions for the distribution and moments of extremes of normal samples with the same normalizing constants. Liao et al. [17] studied optimal convergence rates for the skew-normal distribution SN($\lambda$) with shape parameter $\lambda \in R$. Peng et al. [21] obtained similar results for the skew-$t$ distribution. Beranger et al. [2] derived Mills inequalities and ratio, and then the convergence rate of the univariate extended skew-normal ESN($\lambda$, $\tau$), where the parameters $\lambda \in R$ and $\tau \in R$ are known as the slant and extension parameters, respectively. For more efforts about asymptotic expansions and rates of convergence, see Lin et al. [18], Liao et al. [16], Jia et al. [13], Du and Chen [5], and Huang and Wang [11].

Our interest in this article is to study the extremal properties and convergence rate of the beta-normal distribution. The beta-normal distribution (BND) was first introduced by Eugene et al. [6]. It is a generalization of both the normal distribution and the normal order statistics. A random variable $X$ is said to have a standardized BND with shape parameters $\alpha > 0$ and $\beta > 0$ if its probability density function (p.d.f.) is given by

$$g_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\big[\Phi(x)\big]^{\alpha-1}\big[1-\Phi(x)\big]^{\beta-1}\phi(x), \tag{1.2}$$

where $-\infty < x < \infty$, $0 < \alpha, \beta < \infty$, $\Gamma(\cdot)$ denotes the gamma function, $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard normal cumulative distribution function (c.d.f.) and the standard normal probability density function (p.d.f.), respectively.

Note that $\alpha = 2, \beta = 1$ and $\alpha = 1, \beta = 2$ respectively stand for the skew-normal distribution with shape parameter $\lambda = 1$ and $\lambda = -1$ (Azzalini [1]). In addition, the normal distribution is a special case when $\alpha = 1$ and $\beta = 1$. Several properties of the beta-normal distribution have been studied in the literature: $n$th moment (Gupta and Nadarajah [9]); bimodality properties (Famoye and Lee [7]); bimodality region, hazard rate function, moments, quantile measures, generating function, mean deviations, and Shannon entropy (Rêgo, Cintra and Cordeiro [22]).

Throughout the paper, let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with the c.d.f. $G_{\alpha,\beta}$ which obey the beta-normal distribution. Let $M_n = \max\{X_k, 1 \leq k \leq n\}$ denote the partial maximum of $\{X_n, n \geq 1\}$.

In order to derive the asymptotic expansions of normalized maximum from BND, we introduce some preliminary but important results from Jiang and Li [14]. First the Mills type ratio of BND is stated as follows:

for fixed $\alpha, \beta > 0$,

$$\frac{1 - G_{\alpha,\beta}(x)}{g_{\alpha,\beta}(x)} \sim \frac{1}{\beta x} \quad \text{as } x \to \infty. \tag{1.3}$$

Jiang and Li [14] also showed that

$$1 - G_{\alpha,\beta}(x) = c(x)\exp\left(-\int_1^x \frac{h(t)}{f(t)}\,dt\right) \tag{1.4}$$

for large $x$, where

$$\begin{aligned} c(x) &\to \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}(2\pi e)^{-\frac{\beta}{2}} \quad \text{as } x \to \infty, \\ h(x) &= 1 + \frac{1}{x^2}, \qquad f(x) = \frac{1}{\beta x}. \end{aligned} \tag{1.5}$$

Since $\lim_{x\to\infty} h'(x) \to 1, f(x) > 0$ on $[1,\infty)$ and $\lim_{x\to\infty} f'(x) \to 0$, $G_{\alpha,\beta}(x) \in D(\Lambda)$ by Corollary 1.7 of Resnick [23]. The norming constants $a_n$ and $b_n$ can be given by

$$1 - G_{\alpha,\beta}(b_n) = n^{-1}, \qquad a_n = f(b_n) = \beta^{-1} b_n^{-1} \tag{1.6}$$

such that

$$\lim_{n\to\infty} G_{\alpha,\beta}^n(a_n x + b_n) = \Lambda(x) = \exp\bigl(-\exp(-x)\bigr). \tag{1.7}$$

By using Mills ratio of BND and Khintchine theorem in Leadbetter et al. [15], Jiang and Li [14] obtained another pair of normalized constants such that (1.7) holds:

$$\begin{aligned} \bar{a}_n &= (2\beta \log n)^{-\frac{1}{2}}, \\ \bar{b}_n &= \left(\frac{2}{\beta}\log n\right)^{\frac{1}{2}} + \frac{1}{(2\beta\log n)^{\frac{1}{2}}} \log \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)} \\ &\quad - \frac{1}{2}\left(\frac{\beta}{2\log n}\right)^{\frac{1}{2}} [\log 4\pi - \log\beta + \log\log n]. \end{aligned} \tag{1.8}$$

The remainder of this paper is organized as follows. Section 2 derives the main result on pointwise convergence rate of the maximum of BND and asymptotic expansions for distributions and densities of maximum from the BND sample. Some auxiliary lemmas and related proofs are given in Sect. 3.

## 2 Main results

In this section, we establish first the pointwise convergence rate of the distribution of $M_n$ for the norming constants $\bar{a}_n$ and $\bar{b}_n$ given by (1.8).

**Theorem 2.1** *Let $G_{\alpha,\beta}(x)$ represent the c.d.f. of BND. For normalizing constants $\bar{a}_n$ and $\bar{b}_n$ given by* (1.8), *we have*

$$G_{\alpha,\beta}^n(\bar{a}_n x + \bar{b}_n) - \Lambda(x) \sim \frac{\beta^2 \Lambda(x) e^{-x}}{16} \frac{(\log\log n)^2}{\log n}$$

*as $n \to \infty$.*

*Remark* 2.1 Beranger et al. [2] deduced the pointwise convergence rate of the extended skew-normal ESN$(\lambda, \tau)$ of $M_n$:

$$\frac{\Lambda(x)e^{-x}}{c} \frac{(\log\log n)^2}{\log n},$$

where $c = 16$ when $\lambda \geq 0$ and $c = 4$ when $\lambda < 0$. When $\tau = 0$ the extended skew-normal distribution reduces to the skew-normal $\mathrm{SN}(\lambda)$. The result is exactly the same as that of Liao et al. [17], regardless of the slant parameter $\lambda$ and extension parameter $\tau$. But in Theorem 2.1, the pointwise convergence rate of the BND is affected by the shape parameter $\beta$.

In the following, we shall derive asymptotic expansions for the c.d.f. and the p.d.f. of $M_n$ under the norming constants $a_n$ and $b_n$ given by (1.6).

**Theorem 2.2** *Let $G_{\alpha,\beta}(x)$ be the c.d.f. of BND. For normalizing constants $a_n$ and $b_n$ given by* (1.6), *we have*

$$b_n^2\bigl[b_n^2\bigl(G_{\alpha,\beta}^n(a_nx+b_n)-\Lambda(x)\bigr)-\kappa(x)\Lambda(x)\bigr]\longrightarrow\left(w(x)+\frac{\kappa(x)^2}{2}\right)\Lambda(x)$$

*as $n\to\infty$, where*

$$\kappa(x)=\frac{1}{2}\beta^{-1}\bigl(x^2+2\beta x\bigr)e^{-x}$$

*and*

$$\omega(x)=-e^{-x}\left[\frac{1}{8}\beta^{-2}x^4+\frac{1}{2}\beta^{-1}x^3+\frac{1}{2}\bigl(1+\beta^{-1}\bigr)x^2+2x\right].$$

*Remark* 2.2 According to the definition of $b_n$, one can check that $1/b_n^2 = O(1/\log n)$. Hence, the convergence rate of $G_{\alpha,\beta}^n(a_nx+b_n)$ to its limit c.d.f. $\Lambda(x)$ is proportional to $1/\log n$ by Theorem 2.2.

In the end of the section, we establish the high-order expansion of density of maxima from the BND.

Let

$$r_n(x)=\bigl(G_{\alpha,\beta}^n(a_nx+b_n)\bigr)'=na_nG_{\alpha,\beta}^{n-1}(a_nx+b_n)g_{\alpha,\beta}(a_nx+b_n)$$

denote the density of $(M_n-b_n)/a_n$, and

$$\Delta_n\bigl(r_n,\Lambda';x\bigr)=r_n(x)-\Lambda'(x).$$

By Proposition 2.5 in Resnick [23], we have $\Delta_n(r_n,\Lambda';x)\to 0$ as $n\to\infty$.

**Theorem 2.3** *Let $G_{\alpha,\beta}(x)$ denote the c.d.f. of BND, then for normalizing constants $a_n$ and $b_n$ given by* (1.6), *we have*

$$b_n^2\bigl[b_n^2\bigl(G_{\alpha,\beta}^n(a_nx+b_n)-\Lambda(x)\bigr)'-S(x)\Lambda'(x)\bigr]\longrightarrow R(x)\Lambda'(x)$$

*as $n\to\infty$, where*

$$S(x)=e^{-x}\left(\frac{1}{2}\beta^{-1}x^2+x\right)-\frac{1}{2}\beta^{-1}x^2+\bigl(\beta^{-1}-1\bigr)x+1 \tag{2.1}$$

*and*

$$
\begin{aligned}
R(x) = {} & e^{-2x}\left(\frac{1}{8}\beta^{-2}x^4 + \frac{1}{2}\beta^{-1}x^3 + \frac{1}{2}x^2\right) - e^{-x}\left(\frac{3}{8}\beta^{-2}x^4 + \frac{1}{2}\beta^{-2}(3\beta-1)x^3\right. \\
& \left. + \frac{1}{2}\beta^{-1}(3\beta-2)x^2 + x\right) + \frac{1}{8}\beta^{-2}x^4 + \frac{1}{2}\beta^{-2}(\beta-1)x^3 \\
& + \frac{1}{2}\beta^{-1}(\beta-2)x^2 + \left(1-\beta^{-1}\right)x - 2.
\end{aligned} \tag{2.2}
$$

*Remark* 2.3 Since $1/b_n^2 = O(1/\log n)$, by Theorem 2.3, we could derive the speed of $(G_{\alpha,\beta}^n(a_n x + b_n))'$ converging to its appropriate limit is proportional to $1/\log n$.

In addition to the theory of univariate maxima, multivariate cases have found an increasing interest in literature since the articles by Hüsler and Reiss [12], Nikoloulopoulos et al. [20], Fung and Seneta [8], Beranger et al. [3], and others. Sarabia et al. [24] introduced a bivariate BND which also is a beta-generated distribution. Further research on extremal properties of the bivariate beta-normal distribution is meaningful. It is obvious that our results will stimulate further multidimensional research work.

## 3 Proofs

In order to obtain expansions of a distribution and density for the maximum of the BND, we provide the following distributional tail decomposition of BND.

**Lemma 3.1** *Let $G_{\alpha,\beta}(x)$ represent the c.d.f. of BND. For large $x$, we have*

$$
\begin{aligned}
1 - G_{\alpha,\beta}(x) = {} & (2\pi e)^{-\frac{\beta}{2}} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)} \Phi(x)^{\alpha-1}\left(1 - \beta x^{-2} + 2^{-1}\beta(\beta+5)x^{-4}\right. \\
& \left. + O\left(x^{-6}\right)\right) \exp\left(-\int_1^x \frac{g(t)}{f(t)}\,dt\right)
\end{aligned} \tag{3.1}
$$

*with $g(t)$ and $f(t)$ given by* (1.5).

*Proof* By integration by parts, we have

$$
\begin{aligned}
1 - G_{\alpha,\beta}(x) = {} & \frac{1}{\beta}\frac{\Phi(-x)}{\phi(x)} g_{\alpha,\beta}(x)\left[1 + \frac{\alpha-1}{\beta+1}\left(\Phi^{-1}(x) - 1\right) + \frac{(\alpha-1)(\alpha-2)}{(\beta+1)(\beta+2)}\left(\Phi^{-1}(x)-1\right)^2\right. \\
& + \cdots + \frac{(\alpha-1)(\alpha-2)\dots(\alpha-n)}{(\beta+1)(\beta+2)\dots(\beta+n)}\left(\Phi^{-1}(x)-1\right)^n \\
& \left. + \frac{(\alpha-1)(\alpha-2)\dots(\alpha-n-1)}{(\beta+1)(\beta+2)\dots(\beta+n)} \frac{\int_x^\infty [1-\Phi(t)]^{\beta+n}\Phi(t)^{\alpha-n-2}\phi(t)\,dt}{\Phi(x)^{\alpha-1}[1-\Phi(x)]^\beta}\right].
\end{aligned} \tag{3.2}
$$

It is easy to check by L'Hospital's rule that

$$
\lim_{x\to\infty} \frac{\int_x^\infty [1-\Phi(t)]^{\beta+n}\Phi(t)^{\alpha-n-2}\phi(t)\,dt}{\Phi(x)^{\alpha-1-n}[1-\Phi(x)]^{\beta+n}} = 0 \tag{3.3}
$$

and

$$
x^r\left(\Phi(x)^{-1} - 1\right) \to 0 \quad \text{for all } r. \tag{3.4}
$$

Notice that

$$1 - \Phi(x) = \frac{\phi(x)}{x}\big(1 - x^{-2} + 3x^{-4} + O\big(x^{-6}\big)\big) \tag{3.5}$$

for large $x$ (Castro [4]). Thus, by (3.2), (3.3), (3.4), and (3.5), we have

$$\begin{aligned}
1 - G_{\alpha,\beta}(x) &= \frac{1}{\beta}\frac{\Phi(-x)}{\phi(x)}g_{\alpha,\beta}(x)\biggl[1 + \frac{\alpha-1}{\beta+1}\big(\Phi^{-1}(x) - 1\big) + \frac{(\alpha-1)(\alpha-2)}{(\beta+1)(\beta+2)}\big(\Phi^{-1}(x) - 1\big)^2 \\
&\quad + \cdots + \frac{(\alpha-1)(\alpha-2)\ldots(\alpha-n)}{(\beta+1)(\beta+2)\ldots(\beta+n)}\big(\Phi^{-1}(x) - 1\big)^n\big(1 + o(1)\big)\biggr] \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}\Phi(-x)^{\beta}\Phi(x)^{\alpha-1}\biggl[1 + \frac{\alpha-1}{\beta+1}\big(\Phi^{-1}(x) - 1\big) \\
&\quad + \frac{(\alpha-1)(\alpha-2)}{(\beta+1)(\beta+2)}\big(\Phi^{-1}(x) - 1\big)^2 \\
&\quad + \cdots + \frac{(\alpha-1)(\alpha-2)\ldots(\alpha-n)}{(\beta+1)(\beta+2)\ldots(\beta+n)}\big(\Phi^{-1}(x) - 1\big)^n\big(1 + o(1)\big)\biggr] \\
&= (2\pi e)^{-\frac{\beta}{2}}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}\big(1 - x^{-2} + 3x^{-4} + O\big(x^{-6}\big)\big)^{\beta}\Phi(x)^{\alpha-1} \\
&\quad \times \exp\biggl(-\int_1^x \frac{g(t)}{f(t)}\,dt\biggr) \\
&= (2\pi e)^{-\frac{\beta}{2}}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}\big(1 - \beta x^{-2} + 2^{-1}\beta(\beta+5)x^{-4} + O\big(x^{-6}\big)\big) \\
&\quad \times \Phi(x)^{\alpha-1}\exp\biggl(-\int_1^x \frac{g(t)}{f(t)}\,dt\biggr)
\end{aligned} \tag{3.6}$$

for large $x$, where $g(t)$ and $f(t)$ are given by (1.5). The proof is complete. □

In order to prove Theorem 2.2, we need the following auxiliary result.

**Lemma 3.2** *Let $H_{\alpha,\beta}(b_n;x) = G_{\alpha,\beta}(a_n x + b_n)$ and $h_{\alpha,\beta}(b_n;x) = n\log H_{\alpha,\beta}(b_n;x) + e^{-x}$ with constants $a_n$ and $b_n$ given by (1.6). Then*

$$\lim_{n\to\infty} b_n^2\big(b_n^2 h_{\alpha,\beta}(b_n;x) - \kappa(x)\big) = \omega(x),$$

*where $\kappa(x)$ and $\omega(x)$ are given by Theorem 2.2.*

*Proof* Since $1 - G_{\alpha,\beta}(b_n) = n^{-1}$, $b_n \to \infty$ if and only if $n \to \infty$. The following facts can be obtained:

$$\lim_{n\to\infty}\frac{1 - G_{\alpha,\beta}(a_n x + b_n)}{n^{-1}} = e^{-x} \tag{3.7}$$

and

$$\lim_{n\to\infty}\frac{1 - G_{\alpha,\beta}(a_n x + b_n)}{b_n^{-4}} = 0. \tag{3.8}$$

Set

$$A_{\alpha,\beta}(b_n) = \frac{\Phi(b_n)^{\alpha-1}(1-\beta b_n^{-2}+2^{-1}\beta(\beta+5)b_n^{-4}+O(b_n^{-6}))}{\Phi(b_n+\beta^{-1}b_n^{-1}x)^{\alpha-1}} \times \left(1-\beta\left(b_n+\beta^{-1}b_n^{-1}x\right)^{-2}+2^{-1}\beta(\beta+5)\left(b_n+\beta^{-1}b_n^{-1}x\right)^{-4} + O\left(\left(b_n+\beta^{-1}b_n^{-1}x\right)^{-6}\right)\right)^{-1}.$$

Then $\lim_{n\to\infty} A_{\alpha,\beta}(b_n) = 1$ and

$$A_{\alpha,\beta}(x) - 1 = \left(\Phi(b_n)^{\alpha-1} - \Phi\left(b_n+\beta^{-1}b_n^{-1}x\right)^{\alpha-1} - \beta b_n^{-2}\Phi(b_n)^{\alpha-1} + \beta\left(b_n+\beta^{-1}b_n^{-1}x\right)^{-2}\Phi\left(b_n+\beta^{-1}b_n^{-1}x\right)^{\alpha-1} + 2^{-1}\beta(\beta+5) \times b_n^{-4}\Phi(b_n)^{\alpha-1} - 2^{-1}\beta(\beta+5)\left(b_n+\beta^{-1}b_n^{-1}x\right)^{-4}\Phi\left(b_n+\beta^{-1}b_n^{-1}x\right)^{\alpha-1} + O\left(b_n^{-6}\right)\right)\left(1+o(1)\right).$$

Then

$$\lim_{n\to\infty} \frac{A_{\alpha,\beta}(b_n)-1}{b_n^{-2}} = 0 \tag{3.9}$$

and

$$\lim_{n\to\infty} \frac{A_{\alpha,\beta}(b_n)-1}{b_n^{-4}} = -2x. \tag{3.10}$$

By (3.1), we have

$$\begin{aligned}
&\frac{1-G_{\alpha,\beta}(b_n)}{1-G_{\alpha,\beta}(b_n+\beta^{-1}b_n^{-1}x)}e^{-x} \\
&\quad = A_{\alpha,\beta}(b_n)\exp\left(\int_0^x\left(b_n^{-2}\beta^{-1}t+\frac{b_n^{-2}}{1+b_n^{-2}\beta^{-1}t}\right)dt\right) \\
&\quad = A_{\alpha,\beta}(b_n)\left(1+\int_0^x\left(b_n^{-2}\beta^{-1}t+\frac{b_n^{-2}}{1+b_n^{-2}\beta^{-1}t}\right)dt+\frac{1}{2}\left(\int_0^x\left(b_n^{-2}\beta^{-1}t\right.\right.\right. \\
&\qquad \left.\left.\left.+\frac{b_n^{-2}}{1+b_n^{-2}\beta^{-1}t}\right)dt\right)^2\left(1+o(1)\right)\right).
\end{aligned} \tag{3.11}$$

Combining (1.3), (3.7), (3.8), (3.9), (3.10) with (3.11), we thus have

$$\begin{aligned}
&\lim_{n\to\infty} b_n^2 h_{\alpha,\beta}(b_n;x) \\
&\quad = \lim_{n\to\infty} \frac{\log H_{\alpha,\beta}(b_n;x)+n^{-1}e^{-x}}{n^{-1}b_n^{-2}} \\
&\quad = \lim_{n\to\infty} \frac{\log G_{\alpha,\beta}(b_n+\beta^{-1}b_n^{-1}x)+(1-G_{\alpha,\beta}(b_n))e^{-x}}{\beta^{-1}g_{\alpha,\beta}(b_n)b_n^{-3}} \\
&\quad = \lim_{n\to\infty}\left(-\left[1-G_{\alpha,\beta}\left(b_n+\beta^{-1}b_n^{-1}x\right)\right]-\frac{1}{2}\left[1-G_{\alpha,\beta}\left(b_n+\beta^{-1}b_n^{-1}x\right)\right]^2\left(1+o(1)\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left[1 - G_{\alpha,\beta}(b_n)\right]e^{-x}\Big) \\
&\Big/ \beta^{-1} g_{\alpha,\beta}(b_n) b_n^{-3} \\
&= \lim_{n\to\infty} \frac{b_n[1 - G_{\alpha,\beta}(b_n + \beta^{-1}b_n^{-1}x)]}{\beta^{-1}g_{\alpha,\beta}(b_n)} \frac{\frac{1-G_{\alpha,\beta}(b_n)}{1-G_{\alpha,\beta}(b_n+\beta^{-1}b_n^{-1}x)}e^{-x} - 1}{b_n^{-2}} \\
&= e^{-x} \lim_{n\to\infty} \frac{A_{\alpha,\beta}(b_n)(\int_0^x (b_n^{-2}\beta^{-1}t + \frac{b_n^{-2}}{1+b_n^{-2}\beta^{-1}t})\,dt)(1 + o(1)) + A_{\alpha,\beta}(b_n) - 1}{b_n^{-2}} \\
&= e^{-x} \lim_{n\to\infty} \int_0^x b_n^2 \left(b_n^{-2}\beta^{-1}t + \frac{b_n^{-2}}{1 + b_n^{-2}\beta^{-1}t}\right) dt \\
&= 2^{-1}\beta^{-1}\left(x^2 + 2\beta x\right)e^{-x} := \kappa(x),
\end{aligned}
\tag{3.12}
$$

where the last step is based on the dominated convergence theorem. By similar calculation we have

$$
\begin{aligned}
&\lim_{n\to\infty} b_n^2\left[b_n^2 h_{\alpha,\beta}(b_n;x) - \kappa(x)\right] \\
&= \lim_{n\to\infty} \frac{\log G_{\alpha,\beta}(b_n + \beta^{-1}b_n^{-1}x) + n^{-1}e^{-x} - n^{-1}b_n^{-2}\kappa(x)}{n^{-1}b_n^{-4}} \\
&= \lim_{n\to\infty} \frac{1 - G_{\alpha,\beta}(b_n + \beta^{-1}b_n^{-1}x)}{n^{-1}} \frac{\frac{1-G_{\alpha,\beta}(b_n)}{1-G_{\alpha,\beta}(b_n+\beta^{-1}b_n^{-1}x)}e^{-x}(1 - b_n^{-2}e^x\kappa(x)) - 1}{b_n^{-4}} \\
&= e^{-x} \lim_{n\to\infty} \left(A_{\alpha,\beta}(b_n)b_n^4\left(\int_0^x \left(b_n^{-2}\beta^{-1}t + \frac{b_n^{-2}}{1 + b_n^{-2}\beta^{-1}t}\right) dt - \kappa(x)e^x b_n^{-2}\right) dt\right. \\
&\quad + \frac{1}{2}A_{\alpha,\beta}(b_n)b_n^4\left(\int_0^x \left(b_n^{-2}\beta^{-1}t + \frac{b_n^{-2}}{1 + b_n^{-2}\beta^{-1}t}\right) dt\right)^2 - A_{\alpha,\beta}(b_n)b_n^2 e^x \\
&\quad \left. \times \kappa(x) \int_0^x \left(b_n^{-2}\beta^{-1}t + \frac{b_n^{-2}}{1 + b_n^{-2}\beta^{-1}t}\right) dt + b_n^4\left(A_{\alpha,\beta}(b_n) - 1\right)\right) \\
&= -e^{-x}\left(\frac{1}{8}\beta^{-2}x^4 + \frac{1}{2}\beta^{-1}x^3 + \frac{1}{2}\left(1 + \beta^{-1}\right)x^2 + 2x\right) \\
&:= \omega(x).
\end{aligned}
$$

The proof is complete. □

**Lemma 3.3** *Let $a_n$ and $b_n$ be defined by (1.6). For large $n$, we have*

$$
G_{\alpha,\beta}^{n-1}(a_n x + b_n) = D_n(x)\Lambda(x),
\tag{3.13}
$$

*where $D_n(x) = 1 + b_n^{-2}\kappa(x) + b_n^{-4}(\omega(x) + \frac{1}{2}\kappa(x)^2)(1 + o(1))$, $\kappa(x)$ and $\omega(x)$ are given by Theorem 2.2.*

*Proof* Obviously, by Theorem 2.2, we have

$$
G_{\alpha,\beta}^{n}(a_n x + b_n) = \left[1 + b_n^{-2}\kappa(x) + b_n^{-4}\left(\omega(x) + \frac{1}{2}\kappa(x)^2\right)\left(1 + o(1)\right)\right]\Lambda(x).
\tag{3.14}
$$

Noting that

$$G^n_{\alpha,\beta}(a_n x + b_n) \to \exp\bigl(-e^{-x}\bigr) \quad \text{as } n \to \infty,$$

we have

$$n\bigl(1 - G^n_{\alpha,\beta}(a_n x + b_n)\bigr) \to e^{-x}$$

and

$$1 - G^n_{\alpha,\beta}(a_n x + b_n) = O\bigl(n^{-1}\bigr)$$

as $n \to \infty$, which implies

$$\begin{aligned} G^{n-1}_{\alpha,\beta}(a_n x + b_n) &= \frac{G^n_{\alpha,\beta}(a_n x + b_n)}{1 - (1 - G_{\alpha,\beta}(a_n x + b_n))} \\ &= G^n_{\alpha,\beta}(a_n x + b_n)\bigl(1 + O\bigl(n^{-1}\bigr)\bigr). \end{aligned} \tag{3.15}$$

It is easy to check that

$$n^{-1} = o\bigl(b_n^{-4}\bigr) \tag{3.16}$$

holds for large $n$. □

Combining (3.14), (3.15), and (3.16), we obtain the desired result.

**Lemma 3.4** *Let $g_{\alpha,\beta}(x)$ denote the p.d.f. of BND, then*

$$g_{\alpha,\beta}(x) = \beta x\bigl(1 + x^{-2} - 2x^{-4} + O\bigl(x^{-6}\bigr)\bigr)\bigl(1 - G_{\alpha,\beta}(x)\bigr) \tag{3.17}$$

*for large $x$, and with normalizing constants $a_n$ and $b_n$ from (1.6), we have*

$$\frac{\beta^{-1} b_n^{-1} g_{\alpha,\beta}(\beta^{-1} b_n^{-1} x + b_n)}{1 - G_{\alpha,\beta}(\beta^{-1} b_n^{-1} x + b_n)} = 1 + A_1(x) b_n^{-2} + A_2(x) b_n^{-4} + O\bigl(b_n^{-6}\bigr)$$

*for large $n$, where*

$$A_1(x) = \beta^{-1} x + 1, \qquad A_2(x) = -\bigl(\beta^{-1} x + 2\bigr). \tag{3.18}$$

*Proof* According to Lemma 3.1 and

$$x^r\bigl(\Phi(x)^{-1} - 1\bigr) \to 0 \quad \text{for all } r,$$

we have

$$1 - G_{\alpha,\beta}(x)$$

$$
\begin{aligned}
&= \frac{1}{\beta}\frac{\Phi(-x)}{\phi(x)} g_{\alpha,\beta}(x)\biggl[1 + \frac{\alpha-1}{\beta+1}\bigl(\Phi^{-1}(x)-1\bigr) + \frac{(\alpha-1)(\alpha-2)}{(\beta+1)(\beta+2)}\bigl(\Phi^{-1}(x)-1\bigr)^2 \\
&\quad + \cdots + \frac{(\alpha-1)(\alpha-2)\ldots(\alpha-n)}{(\beta+1)(\beta+2)\ldots(\beta+n)}\bigl(\Phi^{-1}(x)-1\bigr)^n\bigl(1+o(1)\bigr)\biggr] \\
&= \beta^{-1}x^{-1}\bigl(1 - x^{-2} + 3x^{-4} + O\bigl(x^{-6}\bigr)\bigr) g_{\alpha,\beta}(x).
\end{aligned}
$$

Then

$$
\begin{aligned}
g_{\alpha,\beta}(x) &= \beta x\bigl(1 - x^{-2} + 3x^{-4} + O\bigl(x^{-6}\bigr)\bigr)^{-1}\bigl(1 - G_{\alpha,\beta}(x)\bigr) \\
&= \beta x\bigl(1 + x^{-2} - 2x^{-4} + O\bigl(x^{-6}\bigr)\bigr)\bigl(1 - G_{\alpha,\beta}(x)\bigr) \\
&:= C_n(x)\bigl(1 - G_{\alpha,\beta}(x)\bigr),
\end{aligned}
\tag{3.19}
$$

where

$$
C_n(x) = \beta x\bigl(1 + x^{-2} - 2x^{-4} + O\bigl(x^{-6}\bigr)\bigr).
$$

Therefore, for large $n$, we have

$$
\begin{aligned}
&\frac{\beta^{-1}b_n^{-1} g_{\alpha,\beta}(\beta^{-1}b_n^{-1}x + b_n)}{1 - G_{\alpha,\beta}(\beta^{-1}b_n^{-1}x + b_n)} \\
&\quad = \beta^{-1}b_n^{-1}C_n\bigl(\beta^{-1}b_n^{-1}x + b_n\bigr) \\
&\quad = \bigl(\beta^{-1}b_n^{-2}x + 1\bigr)\bigl(1 + \bigl(\beta^{-1}b_n^{-1}x + b_n\bigr)^{-2} - 2\bigl(\beta^{-1}b_n^{-1}x + b_n\bigr)^{-4} + O\bigl(\bigl(\beta^{-1}b_n^{-1}x + b_n\bigr)^{-6}\bigr)\bigr) \\
&\quad = \bigl(\beta^{-1}b_n^{-2}x + 1\bigr)\bigl(1 + b_n^{-2} - 2\beta^{-1}b_n^{-4}x - 2b_n^{-4} + O\bigl(b_n^{-6}\bigr)\bigr) \\
&\quad = 1 + \bigl(\beta^{-1}x + 1\bigr)b_n^{-2} - \bigl(\beta^{-1}x + 2\bigr)b_n^{-4} + O\bigl(b_n^{-6}\bigr) \\
&\quad := 1 + A_1(x)b_n^{-2} + A_2(x)b_n^{-4} + O\bigl(b_n^{-6}\bigr).
\end{aligned}
\tag{3.20}
$$

The proof is complete. □

*Proof of Theorem* 2.1 Let $u_n = \bar{a}_n x + \bar{b}_n$ and $\tau_n = n(1 - G_{\alpha,\beta}(u_n))$, where $\bar{a}_n$ and $\bar{b}_n$ are given by (1.8).

$$
\begin{aligned}
u_n &= (2\beta\log n)^{-\frac{1}{2}}x + \left(\frac{2}{\beta}\log n\right)^{\frac{1}{2}} + \frac{1}{(2\beta\log n)^{\frac{1}{2}}}\log\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)} \\
&\quad - \frac{1}{2}\left(\frac{\beta}{2\log n}\right)^{\frac{1}{2}}[\log 4\pi - \log\beta + \log\log n]
\end{aligned}
$$

implies

$$
\begin{aligned}
u_n^2 &= \frac{2}{\beta}\log n + \frac{2x}{\beta} + \frac{2}{\beta}\log\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)} - \log 4\pi + \log\beta - \log\log n \\
&\quad + \frac{\beta(\log\log n)^2}{8\log n}\bigl(1 + o(1)\bigr),
\end{aligned}
\tag{3.21}
$$

$$
u_n^{-1} = \left(\frac{2}{\beta}\log n\right)^{-\frac{1}{2}}\left(1 + O\left(\frac{\log\log n}{\log n}\right)\right),
\tag{3.22}
$$

and

$$O(u_n^{-2}) = O\left(\frac{\beta}{2\log n}\right). \tag{3.23}$$

Since

$$\log\Phi(u_n) = o\big((\log\log n)^2/\log n\big) \tag{3.24}$$

for large $n$, by using (3.6), (3.21), (3.22), and (3.23), we have

$$\begin{aligned}\tau_n &= n\big(1 - G_{\alpha,\beta}(u_n)\big)\\ &= \left(\frac{1}{2\pi}\right)^{\frac{\beta}{2}} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}\left(\frac{n}{u_n^{\beta}}\right)\exp\left(-\frac{\beta}{2}u_n^2 + (\alpha-1)\log\Phi(u_n)\right)\big(1+O(u_n^{-2})\big)\\ &= \left(\frac{2}{\beta}\log n\right)^{\frac{\beta}{2}}\left(\frac{1}{u_n^{\beta}}\right)\exp\left(-x - \frac{\beta^2}{16}\frac{(\log\log n)^2}{\log n}\big(1+o(1)\big)\right)\big(1+O(u_n^{-2})\big)\\ &= e^{-x}\left(1 - \frac{\beta^2}{16}\frac{(\log\log n)^2}{\log n}\big(1+o(1)\big)\right).\end{aligned}$$

Obviously, for $\tau(x) = e^{-x}$,

$$\tau(x) - \tau_n(x) = e^{-x}\frac{\beta^2}{16}\frac{(\log\log n)^2}{\log n}\big(1+o(1)\big) \sim e^{-x}\frac{\beta^2}{16}\frac{(\log\log n)^2}{\log n}$$

for large $n$. By Theorem 2.4.2 of Leadbetter et al. [15], the result follows. □

*Proof of Theorem* 2.2 It is followed by Lemma 3.2 that $h_{\alpha,\beta}(b_n;x) \to 0$ and

$$\left|\sum_{i=3}^{\infty}\frac{h_{\alpha,\beta}^{i-3}(b_n;x)}{i!}\right| < \exp\big(h_{\alpha,\beta}(b_n;x)\big) \to 1$$

as $n \to \infty$. By Lemma 3.2 once again, we have

$$\begin{aligned}&b_n^2\big[b_n^2\big(G_{\alpha,\beta}^n(a_nx+b_n) - \Lambda(x)\big) - \kappa(x)\Lambda(x)\big]\\ &\quad= b_n^2\big[b_n^2\big(\exp\big(h_{\alpha,\beta}(b_n;x)\big) - 1\big) - \kappa(x)\big]\Lambda(x)\\ &\quad= \left[b_n^2\big(b_n^2 h_{\alpha,\beta}(b_n;x)\big) - \kappa(x)\big) + b_n^4 h_{\alpha,\beta}^2(b_n;x)\left(\frac{1}{2} + h_{\alpha,\beta}(b_n;x)\sum_{i=3}^{\infty}\frac{h_{\alpha,\beta}^{i-3}(b_n;x)}{i!}\right)\right]\Lambda(x)\\ &\quad\longrightarrow \left(w(x) + \frac{\kappa(x)^2}{2}\right)\Lambda(x)\end{aligned}$$

as $n \to \infty$. The result follows. □

*Proof of Theorem* 2.3 Set $E_{\alpha,\beta}(b_n) = 1/A_{\alpha,\beta}(b_n)$, by (3.9) and (3.10), we have

$$\lim_{n\to\infty} b_n^2\big(E_{\alpha,\beta}(b_n) - 1\big) = 0 \tag{3.25}$$

and

$$\lim_{n\to\infty} b_n^4\big(E_{\alpha,\beta}(b_n)-1\big)=2x. \tag{3.26}$$

By (3.11), we have

$$\begin{aligned}
&\frac{1-G_{\alpha,\beta}(b_n+\beta^{-1}b_n^{-1}x)}{1-G_{\alpha,\beta}(b_n)}e^x\\
&\quad=E_{\alpha,\beta}(b_n)\left[1-\int_0^x \rho_{\alpha,\beta}(t)\,dt+\frac{1}{2}\left(\int_0^x \rho_{\alpha,\beta}(t)\,dt\right)^2\big(1+o(1)\big)\right]\\
&\quad=E_{\alpha,\beta}(b_n)\left[1-\int_0^x \rho_{\alpha,\beta}(t)\,dt+\frac{1}{2}\left(\int_0^x \rho_{\alpha,\beta}(t)\,dt\right)^2+o\big(b_n^{-4}\big)\right],
\end{aligned}\tag{3.27}$$

where

$$\rho_{\alpha,\beta}(x)=b_n^{-2}\beta^{-1}x+\frac{b_n^{-2}}{1+b_n^{-2}\beta^{-1}x}.$$

By (3.13), (3.19), and (3.20), we have

$$\begin{aligned}
&\beta^{-1}b_n^{-1}C_n\big(\beta^{-1}b_n^{-1}x+b_n\big)D_n(x)\\
&\quad=\big(1+A_1(x)b_n^{-2}+A_2(x)b_n^{-4}+O\big(b_n^{-6}\big)\big)\times\left(1+b_n^{-2}\kappa(x)+b_n^{-4}\left(\omega(x)\right.\right.\\
&\qquad\left.\left.+\frac{1}{2}\kappa(x)^2\right)\big(1+o(1)\big)\right)\\
&\quad=1+\big(A_1(x)+\kappa(x)\big)b_n^{-2}+\left(A_2(x)+\kappa(x)A_1(x)+\omega(x)+\frac{1}{2}\kappa^2(x)\right)b_n^{-4}\\
&\qquad+o\big(b_n^{-4}\big).
\end{aligned}\tag{3.28}$$

By Lemmas 3.3, 3.4 and combining (3.25)–(3.28), we have

$$\begin{aligned}
&\Delta_n\big(r_n,\Lambda';x\big)\\
&\quad=r_n(x)-\Lambda'(x)\\
&\quad=\big(1-G_{\alpha,\beta}(b_n)\big)^{-1}\beta^{-1}b_n^{-1}G_{\alpha,\beta}^{n-1}\big(\beta^{-1}b_n^{-1}x+b_n\big)g_{\alpha,\beta}\big(\beta^{-1}b_n^{-1}x+b_n\big)-\Lambda'(x)\\
&\quad=\big(1-G_{\alpha,\beta}(b_n)\big)^{-1}\beta^{-1}b_n^{-1}C_n\big(\beta^{-1}b_n^{-1}x+b_n\big)\Lambda(x)\big(1-G_{\alpha,\beta}\big(\beta^{-1}b_n^{-1}x+b_n\big)\big)\\
&\qquad\times D_n(x)-\Lambda'(x)\\
&\quad=\left[E_{\alpha,\beta}(b_n)\big(A_1(x)+\kappa(x)\big)b_n^{-2}+E_{\alpha,\beta}(b_n)\left(\omega(x)+\frac{1}{2}\kappa^2(x)+\kappa(x)A_1(x)+A_2(x)\right)b_n^{-4}\right.\\
&\qquad-E_{\alpha,\beta}(b_n)\left(1+\big(A_1(x)+\kappa(x)\big)b_n^{-2}+\left(\omega(x)+\frac{1}{2}\kappa^2(x)+\kappa(x)A_1(x)+A_2(x)\right)b_n^{-4}\right)\\
&\qquad\times\int_0^x \rho_{\alpha,\beta}(t)\,dt+\frac{1}{2}E_{\alpha,\beta}(b_n)\left(1+\big(A_1(x)+\kappa(x)\big)b_n^{-2}+\left(\omega(x)+\frac{1}{2}\kappa^2(x)+\kappa(x)A_1(x)\right.\right.\\
&\qquad\left.\left.\left.+A_2(x)\right)b_n^{-4}\right)\big)\left(\int_0^x \rho_{\alpha,\beta}(t)\,dt\right)^2+E_{\alpha,\beta}(b_n)-1+o\big(b_n^{-4}\big)\right]\Lambda'(x).
\end{aligned}\tag{3.29}$$

Hence

$$\begin{aligned}
&\lim_{n\to\infty} b_n^2 \Delta_n\big(r_n, \Lambda'; x\big) \\
&= \lim_{n\to\infty}\left(A_1(x) + \kappa(x) - b_n^2 \int_0^x \rho_{\alpha,\beta}(t)\,dt\right)\Lambda'(x) \\
&= 2^{-1}\beta^{-1}\big[\big(x^2 + 2\beta x\big)e^{-x} - x^2 + 2x(1-\beta) + 2\beta\big]\Lambda'(x) \\
&:= S(x)\Lambda'(x). \qquad (3.30)
\end{aligned}$$

Combining (3.29) and (3.30) together, we have

$$\begin{aligned}
&\lim_{n\to\infty} b_n^2\big(b_n^2 \Delta_n\big(r_n, \Lambda'; x\big) - S(x)\Lambda'(x)\big) \\
&= \lim_{n\to\infty} b_n^2\bigg[\big(E_{\alpha,\beta}(b_n) - 1\big)\big(\kappa(x) + A_1(x)\big) + E_{\alpha,\beta}(b_n)\bigg(\frac{1}{2}\kappa^2(x) + \kappa(x)A_1(x) \\
&\quad \times A_2(x) + \omega(x)\bigg)b_n^{-2} - b_n^2\int_0^x\big(\rho_{\alpha,\beta}(t) - \big(\beta^{-1}t + 1\big)b_n^{-2}\big)\,dt - E_{\alpha,\beta}(b_n)\bigg(\kappa(x) \\
&\quad + A_1(x) + \bigg(\frac{1}{2}\kappa^2(x) + \kappa(x)A_1(x) + A_2(x) + \omega(x)\bigg)b_n^{-2}\bigg)\int_0^x \rho_{\alpha,\beta}(t)\,dt \\
&\quad + \frac{1}{2}E_{\alpha,\beta}(b_n)b_n^2\bigg(1 + \big(\kappa(x) + A_1(x)\big)b_n^{-2} + \bigg(\frac{1}{2}\kappa^2(x) + \kappa(x)A_1(x) + A_2(x) \\
&\quad + \omega(x)\bigg)b_n^{-4}\bigg)\bigg(\int_0^x \rho_{\alpha,\beta}(t)\,dt\bigg)^2 + b_n^2\big(E_{\alpha,\beta}(b_n) - 1\big) + o\big(b_n^{-2}\big)\bigg]\Lambda'(x) \\
&= \lim_{n\to\infty}\bigg[b_n^4\big(E_{\alpha,\beta}(b_n) - 1\big) + \frac{1}{2}\kappa^2(x) + \kappa(x)A_1(x) + A_2(x) + \omega(x) \\
&\quad - b_n^4\int_0^x\big(\rho_{\alpha,\beta}(t) - \big(\beta^{-1}t + 1\big)b_n^{-2}\big)\,dt - \big(\kappa(x) + A_1(x)\big)b_n^2\int_0^x \rho_{\alpha,\beta}(t)\,dt \\
&\quad + \frac{1}{2}b_n^4\bigg(\int_0^x \rho_{\alpha,\beta}(t)\,dt\bigg)^2\bigg]\Lambda'(x) \\
&= \bigg[e^{-2x}\bigg(\frac{1}{8}\beta^{-2}x^4 + \frac{1}{2}\beta^{-1}x^3 + \frac{1}{2}x^2\bigg) - e^{-x}\bigg(\frac{3}{8}\beta^{-2}x^4 + \frac{1}{2}\beta^{-2}(3\beta - 1)x^3 \\
&\quad + \frac{1}{2}\beta^{-1}(3\beta - 2)x^2 + x\bigg) + \frac{1}{8}\beta^{-2}x^4 + \frac{1}{2}\beta^{-2}(\beta - 1)x^3 + \frac{1}{2}\beta^{-1}(\beta - 2)x^2 \\
&\quad + \big(1 - \beta^{-1}\big)x - 2\bigg]\Lambda'(x) \\
&:= R(x)\Lambda'(x).
\end{aligned}$$

The proof is complete. □

**Acknowledgements**
We thank the editor in chief and referees for a careful reading of the manuscript and a number of perceptive and useful suggestions which improved it greatly.

**Funding**
The research was supported by the Scientific Research Fund of Sichuan University of Science & Engineering under Grant 2019RC10 and the Opening Project of Sichuan Province University Key Laboratory of Bridge Non-destruction Detecting and Engineering Computing (2018QZJ01).

**Availability of data and materials**
Not applicable.

## Declarations

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
YJ: conceptualization, computation, funding acquisition, writing-original draft, writing-review and editing. BL: problem statement, supervision, writing-review, and provision of study resources. All authors read and approved the final manuscript.

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## Publisher's Note



Received: 12 July 2021 Accepted: 22 March 2022 Published online: 13 April 2022

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