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Some results of neutrosophic normed space VIA Tribonacci convergent sequence spaces

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Abstract

The concept of Tribonacci sequence spaces by the domain of a regular Tribonacci matrix was introduced by Yaying and Hazarika (Math. Slovaca 70(3):697–706, 2000). In this paper, by using the domain of regular Tribonacci matrix $T = (t_{ik})$ and the concept of neutrosophic convergence, we introduce some neutrosophic normed space in Tribonacci convergent spaces and prove some topological and algebraic properties based results with respect to these spaces.

Keywords: Tribonaaci matrix and sequence; Regular Tribonacci matrix; Convergence; t-norm; t-conorm; Neutrosophic normed space; Tribonacci convergence

1 Introduction

The theory of fuzzy sets was generalized from classical sets by Zadeh in 1965 [2], which was further generalized to intuitionistic fuzzy sets by Atanassov [3]. This theory deals with a situation that may be imprecise or vague or uncertain by attributing a degree of membership and a degree of non-membership to a certain object. Several literature works on their corresponding sequence spaces can be found in [4–6]. In 2004, Park laid the grounds of intuitionistic fuzzy metric space which was later redefined by Saadati [7] and Park [8] as fuzzy norm and intuitionistic fuzzy norm.

The idea of neutrosophic sets was introduced by Smarandache [9] as an extension of the intuitionistic fuzzy set. For the situation when the aggregate of the components is 1, in the wake of satisfying the condition by applying the neutrosophic set operators, different outcomes can be acquired by applying the intuitionistic fuzzy operators, since the operators disregard the indeterminacy, while the neutrosophic operators are taken into the cognizance of the indeterminacy at a similar level as truth-membership and falsehood-nonmembership. Using the idea of neutrosophic sets, the notion of neutrosophic bipolar vague soft set [25] and its application to decision making problems were defined. Further, Smarandache [10, 11] investigated neutroalgebra which is a generalization of partial algebra, neutroalgebraic structures, and antialgebraic structures. Neutrosophic set is a more adaptable and effective tool because it handles, aside from autonomous components, additionally partially independent and dependent information [12, 13]. Summability theory and matrix transformation have been necessary modes in developing the theory of nonconverging sequences. The motivation of it being able to transform the sequence or series which does not converge originally but approaches some number on applying the transformation. An infinite matrix is usually used for this approach, since it is the most natural operator between two sequence spaces. Some work on sequence spaces via matrix transformation can be found in [1].

Recently in [1], the authors defined the matrix corresponding to the Tribonacci sequence in [14, 15]. In this paper we aim to define novel neutrosophic sequence spaces with the help of neutrosophic norm and using the Tribonacci matrix as a mode. Also, we study Tribonacci convergent and Tribonacci Cauchy in neutrosophic normed space by using the Tribonacci matrix T. Prior to the introduction of new spaces of Tribonacci convergent sequence with respect to neutrosophic norm $(\mathsf{P},\mathsf{Q},\mathsf{R})$, we mention the following notions that will be used in the article.

2 Preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers respectively. By ω we denote a linear space of sequence of real or complex numbers. Any vector subspace of ω is called a sequence space.

$$\omega := \left\{\vartheta = (\vartheta_k), k \in \mathbb{N} | \vartheta = (\vartheta_k) \in \mathbb{R}, \text{ or } \mathbb{C}\right\}. \tag{2.1}$$

Let X_1 and X_2 be two sequence spaces and let $T = (t_{ik})$ be an infinite matrix of real entries. We write T_i to denote the sequence in the nth row of matrix T. Recalling that T defines a matrix mapping from sequence space X_1 to X_2 if for every sequence $\vartheta = (\vartheta_k)$, the W transform of ϑ is defined as $T\vartheta = \{T_i(\vartheta)\}_{i=1}^{\infty} \in X_2$, where

$$T_i(\vartheta) = \sum_{k=1} t_{ik}\vartheta_k, \quad i \in \mathbb{N}.$$

For any sequence space E, the sequence space E_T defined by

$$E_T = \left\{\vartheta = (\vartheta_k) \in w : T\vartheta \in E\right\}$$

is known as domain of the matrix T.

Definition 2.1 ([1, 16]) A matrix $T = (t_{ik})_{i,k\in\mathbb{N}}$ is said to be regular iff the following conditions hold:

(a) There exists $M > 0$ such that for every $i \in \mathbb{N}$, $\sum_k |t_{ik}| \le M$,
(b) $\lim_{i\to\infty} t_{ik} = 0$ for every $k \in \mathbb{N}$,
(c) $\lim_{i\to\infty} \sum_k t_{ik} = 1$.

First, we give some background about Tribonacci numbers. The studies on Tribonacci numbers were first initiated by a 14-year-old student Mark Feinberg in 1963. In 1963, Mark Feinberg [15, 19] defined the sequence $(t_n)_{n\in\mathbb{N}}$ of Tribonacci numbers given by third recurrence relation

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}, \quad n \ge 3 \text{ with } t_1 = t_2 = 1 \text{ and } t_3 = 2.$$

Thus, the first few numbers of Tribonacci sequence are $1, 1, 2, 4, 7, 13, 24, 44, 81, \ldots$ Some basic properties of Tribonacci sequence are:

$$\lim_{k\to\infty} \frac{t_k}{t_{k+1}} = 0.54368901\cdots$$

and

$$\sum_{k=1}^{i} t_k = \frac{t_{i+2} + t_i - 1}{2}$$

$$\lim_{k\to\infty} \frac{t_{k+1}}{t_k} = 1.83929 \quad \text{(approx)}.$$

Throughout this paper we use the lower triangular Tribonacci matrix $T = (t_{ik})$, defined in [1] as follows:

$$t_{ik} = \begin{cases} \frac{2t_k}{t_{i+2}+t_i-1} & \text{if } (1 \le k \le i), \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently,

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & \frac{2}{4} & 0 & 0 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{8} & \frac{2}{8} & \frac{4}{8} & 0 & 0 & \cdots \\ \frac{1}{15} & \frac{1}{15} & \frac{2}{15} & \frac{4}{15} & \frac{7}{15} & 0 & \cdots \\ \frac{1}{28} & \frac{1}{28} & \frac{2}{28} & \frac{4}{28} & \frac{7}{28} & \frac{13}{28} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.2}$$

It can be easily verified that T is a regular matrix (from Definition 2.1). By using the Tribonacci matrix (2.1), for any sequence $\vartheta = (\vartheta_k) \in \omega$, the $T-$ transformation of (ϑ_k) is defined as

$$T_i(\vartheta) = \sum_{k=1}^{i} \frac{2 J_k}{J_{i+2} + J_i - 1} \vartheta_k, \quad i \in \mathbb{N}. \tag{2.3}$$

Definition 2.2 ([17, 20, 23]) Given a binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-norm if

(a) $*$ is associative and commutative,
(b) $*$ is continuous,
(c) $\vartheta * 1 = \vartheta \ \forall \ \vartheta \in [0,1]$,
(d) $\vartheta * y \le w * z$ whenever $\vartheta \le w$ and $y \le z$ for each $\vartheta, y, w, z \in [0,1]$.

Example 2.1 For $\vartheta, y \in [0,1]$, define $\vartheta * y = \vartheta y$ or $\vartheta * y = \min\{\vartheta, y\}$, then $*$ is a continuous t-norm.

Definition 2.3 ([17, 20, 24]) Given a binary operation, $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be a continuous t-conorm if

(a) $\diamond$ is associative and commutative,
(b) $\diamond$ is continuous,
(c) $\vartheta \diamond 0 = \vartheta \ \forall \ \vartheta \in [0,1]$,
(d) $\vartheta \diamond y \leq w \diamond z$ whenever $\vartheta \leq w$ and $y \leq z$ for each $\vartheta, y, w, z \in [0,1]$.

Example 2.2 Let $\vartheta, y \in [0,1]$. Define $\vartheta \diamond y = \min\{\vartheta + y, 1\}$ or $\vartheta \diamond y = \max\{\vartheta, y\}$, then $\diamond$ is continuous t-conorm.

From the above definitions, we note that if we choose $0 < \epsilon_1, \epsilon_2 < 1$ for $\epsilon_1 > \epsilon_2$, then there exist $0 < \epsilon_3, \epsilon_4 < 0, 1$ such that $\epsilon_1 * \epsilon_3 \geq \epsilon_2, \epsilon_1 \geq \epsilon_4 \diamond \epsilon_2$.

Further, if we choose $\epsilon_5 \in (0,1)$, then there exist $\epsilon_6, \epsilon_7 \in (0,1)$ such that $\epsilon_6 * \epsilon_6 \geq \epsilon_5$ and $\epsilon_7 \diamond \epsilon_7 \leq \epsilon_5$.

Definition 2.4 ([18, 21, 22]) Assume $\star$ to be a continuous t-norm, $\diamond$ to be a continuous t-conorm, and Y to be a linear space over the neutrosophic field $\mathbb{R}$ or $\mathbb{C}$, and $\mathcal{Z} = \{< \vartheta, \mathsf{P}(\vartheta), \mathsf{Q}(\vartheta), \mathsf{R}(\vartheta) >: \vartheta \in Y\}$ to be a normed space such that $\mathcal{Z} : Y \times (0,\infty) \to [0,1]$. The four-tuple $(Y, \mathcal{Z}, \star, \diamond)$ is called a neutrosophic normed space (*NNS*) if the subsequent terms hold; for all $\vartheta, y, z \in Y$ and $\jmath, s > 0$,

(i) $0 \leq \mathsf{P}(\vartheta, \jmath) \leq 1, 0 \leq \mathsf{Q}(y, \jmath) \leq 1, 0 \leq \mathsf{R}(z, \jmath) \leq 1, \jmath \in R^+$,
(ii) $\mathsf{P}(\vartheta, \jmath) + \mathsf{Q}(\vartheta, \jmath) + \mathsf{R}(\vartheta, \jmath) \leq 3$ for $\jmath \in R^+$,
(iii) $\mathsf{P}(\vartheta, \jmath) = 1$ for $\jmath > 0$ iff $\vartheta = 0$,
(iv) $\mathsf{P}(\lambda\vartheta, \jmath) = \mathsf{P}(\vartheta, \frac{\jmath}{|\lambda|})$,
(v) $\mathsf{P}(\vartheta, \jmath) \star \mathsf{P}(y, s) \leq \mathsf{P}(\vartheta + y, \jmath + s)$,
(vi) $\mathsf{P}(\vartheta, \star)$ is a continuous nondecreasing function,
(vii) $\lim_{\jmath\to\infty} \mathsf{P}(\vartheta, \jmath) = 1$,
(viii) $\mathsf{Q}(y, \jmath) = 0$ for $\jmath > 0$ iff $\vartheta = 0$,
(ix) $\mathsf{Q}(\lambda y, \rho) = \mathsf{Q}(y, \frac{\jmath}{|\lambda|})$,
(x) $\mathsf{Q}(y, \jmath) \diamond \mathsf{Q}(z, \jmath) \geq \mathsf{Q}(y + z, \jmath + s)$,
(xi) $\mathsf{Q}(y, \diamond)$ is a continuous nonincreasing function,
(xii) $\lim_{\jmath\to\infty} \mathsf{Q}(\vartheta, \jmath) = 0$,
(xiii) $\mathsf{R}(\vartheta, \jmath) = 0$ for $\jmath > 0$ iff $\vartheta = 0$,
(xiv) $\mathsf{R}(\lambda\vartheta, \jmath) = \mathsf{R}(\vartheta, \frac{\jmath}{|\lambda|})$,
(xv) $\mathsf{R}(z, \jmath) \diamond \mathsf{R}(\vartheta, s) \geq \mathsf{R}(z + \vartheta, \jmath + s)$,
(xvi) $\mathsf{R}(z, \diamond)$ is a continuous nonincreasing function,
(xvii) $\lim_{\jmath\to\infty} \mathsf{R}(z, \jmath) = 0$,
(xviii) If $\jmath \leq 0$, then $\mathsf{P}(\vartheta, \jmath) = 0$, $\mathsf{Q}(y, \jmath) = 1$, $\mathsf{R}(z, \jmath) = 1$.

In such a case, $\mathcal{Z} = (\mathsf{P}, \mathsf{Q}, \mathsf{R})$ is called a neutrosophic norm (*NN*).

Example 2.3 ([18]) Let $(Y, \|\cdot\|)$ be an NNS. Given the operation $*$ and $\diamond$ as t-norm $\vartheta * y = \vartheta.y$ and t-conorm $\vartheta \diamond y = \vartheta + y - \vartheta y$ for $\jmath > \|y\|$ and $\jmath > 0$

$$\mathsf{P}(\vartheta, \jmath) = \frac{\jmath}{\jmath + \|\vartheta\|}, \qquad \mathsf{Q}(\vartheta, \jmath) = \frac{\|\vartheta\|}{\jmath + \|\vartheta\|} \quad \text{and} \quad \mathsf{R}(\vartheta, \jmath) = \frac{\|\vartheta\|}{\jmath} \tag{2.4}$$

for all $\vartheta, y \in Y$. If we take $\jmath \leq \|\vartheta\|$, then $\mathsf{P}(\vartheta, \jmath) = 0, \mathsf{Q}(\vartheta, \jmath) = 1$, and $\mathsf{R}(\vartheta, \jmath) = 1$. Then $(Y, \mathcal{Z}, *, \diamond)$ is an NNS where $\mathcal{Z} : Y \times \mathbb{R}^+ \to [0,1]$.

Example 2.4 Let $(Y = \mathbb{R}, \|\cdot\|)$ be an NNS where $\|c\| = |c|\ \forall c \in \mathbb{R}$. Given the operation $*$ and $\diamond$ as t-norm $\vartheta * y = \min\{\vartheta, y\}$ and t-conorm $\vartheta \diamond y = \max\{\vartheta, y\}$. $\forall \vartheta, y \in [0,1]$ and define

$$\mathsf{P}(\vartheta, \jmath) = \frac{\jmath}{\jmath + m\|\vartheta\|}, \qquad \mathsf{Q}(\vartheta, \jmath) = \frac{m\|\vartheta\|}{\jmath + m\|\vartheta\|}, \quad \text{and} \quad \mathsf{R}(\vartheta, \jmath) = \frac{m\|\vartheta\|}{\jmath}, \tag{2.5}$$

where $m > 0$. Then $\mathcal{Z} = \{(\vartheta, \jmath), \mathsf{P}(\vartheta, t), \mathsf{Q}(\vartheta, t), \mathsf{R}(\vartheta, \jmath) : (\vartheta, \jmath) \in Y \times \mathbb{R}^+\}$ is an NNS on Y.

Definition 2.5 Suppose that X is an NNS, the sequence $b = (b_i)$ in X is called convergent to $\xi \in X \iff \exists\, N \in \mathbb{N}$, with respect to NN-$\mathcal{Z} = (\mathsf{P}, \mathsf{Q}, \mathsf{R})$ if for every $\epsilon > 0$, $\jmath > 0$

$$\mathsf{P}\big((b_i) - \xi, \jmath\big) > 1 - \epsilon, \qquad \mathsf{Q}\big((b_i) - \xi, \jmath\big) < \epsilon \quad \text{and} \quad \mathsf{R}\big((b_i) - \xi, \jmath\big) < \epsilon \tag{2.6}$$

for all $i \geq N$, i.e.,

$$\lim_{i\to\infty} \mathsf{P}\big((b_i) - \xi, \jmath\big) = 1, \qquad \lim_{i\to\infty} \mathsf{Q}\big((b_i) - \xi, \jmath\big) = 0 \quad \text{and} \quad \lim_{i\to\infty} \mathsf{R}\big((b_i) - \xi, \jmath\big) = 0.$$

In such a case, we denote $\mathcal{Z} - \lim b_i = \xi$.

Definition 2.6 ([18]) Let $(Y, \mathcal{Z}, *, \diamond)$ be a neutrosophic normed space. A sequence $b = (b_i)$ is called a Cauchy sequence with respect to $\mathcal{Z}$ if for each $\epsilon > 0$ and $\jmath > 0$, $\exists\, \eta \in \mathbb{N}$ such that $\mathsf{P}(b_i - b_k, \jmath) > 1 - \epsilon$, $\mathsf{Q}(b_i - b_k, \jmath) < \epsilon$, and $\mathsf{R}(b_i - b_k, \jmath) < \epsilon$ for all $i, k \geq \eta$.

Definition 2.7 Consider $(Y, \mathcal{Z}, *, \diamond)$ to be an *NNS*. A subset $\mathring{H}$ of X is said to be neutrosophic bounded-(*NB*) if $\exists, \jmath > 0$ and $0 < \epsilon < 1$ such that $\mathsf{P}(\vartheta, \jmath) > 1 - \epsilon$ and $\mathsf{Q}(\vartheta, \jmath) < \epsilon$, $\mathsf{R}(\vartheta, \jmath) < \epsilon$ for each $\vartheta \in \mathring{H}$.

Definition 2.8 Let $(Y, \mathcal{Z}, *, \diamond)$ be an NNS. Then $(Y, \mathcal{Z}, *, \diamond)$ is said to be complete if every Cauchy sequence is convergent with respect to the norms $\mathcal{Z}$.

By using this T- transformation and notion of neutrosophic convergence, we define some sequence spaces, namely $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$, $\mathcal{L}_{0(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$, and $\mathcal{L}_{\infty(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$.

3 Main results

In this section, we introduce the following sequence spaces:

$$\begin{aligned}
&\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) \\
&\quad = \big\{\vartheta_k \in \omega : \big(T_i(\vartheta)\big) \text{ is convergent}\big\} \\
&\quad = \big\{\vartheta_k \in \omega : \text{ for some } \beta \in Y, \forall \epsilon > 0, \forall \jmath, \exists n \in \mathbb{N} : \forall i \geq n, \\
&\qquad \mathsf{P}\big(T_i(\vartheta) - \beta, \jmath\big) > 1 - \epsilon, \mathsf{Q}\big(T_i(\vartheta) - \beta, \jmath\big) < \epsilon \text{ and } \mathsf{R}\big(T_i(\vartheta) - \beta, \jmath\big) < \epsilon\big\} \\
&\quad = \big\{\vartheta = (\vartheta_k) \in \omega : \big\{i \in \mathbb{N} : \text{ for some } \beta \in Y, \forall \epsilon > 0, \mathsf{P}\big(T_i(\vartheta) - \beta, \jmath\big) \leq 1 - \epsilon \text{ or} \\
&\qquad \mathsf{Q}\big(T_i(\vartheta) - \beta, \jmath\big) \geq \epsilon \text{ or } \mathsf{R}\big(T_i(\vartheta) - \beta, \jmath\big) \geq \epsilon\big\} \text{ is finite}\big\}
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
&\mathcal{L}_{0(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)\\
&\quad= \{\vartheta_k \in \omega : (T_i(\vartheta)) \text{ is convergent}\}\\
&\quad= \{\vartheta_k \in \omega : \forall \epsilon > 0, \forall \jmath, \exists n \in \mathbb{N} : \forall i \geq n, \mathsf{P}(T_i(\vartheta), \jmath) > 1-\epsilon, \mathsf{Q}(T_i(\vartheta), \jmath) < \epsilon \text{ and}\\
&\qquad \mathsf{R}(T_i(\vartheta), \jmath) < \epsilon\} \qquad (3.2)\\
&\quad= \{\vartheta = (\vartheta_k) \in \omega : \{i \in \mathbb{N} : \forall \epsilon > 0, \mathsf{P}(T_i(\vartheta), \jmath) \leq 1-\epsilon \mathsf{Q}(T_i(\vartheta), \jmath) \geq \epsilon,\\
&\qquad \text{or } \mathsf{R}(T_i(\vartheta), \jmath) \geq \epsilon\} \text{ is finite}\}
\end{aligned}$$

$$\begin{aligned}
&\mathcal{L}_{\infty(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)\\
&\quad= \{\vartheta_k \in \omega : (T_i(\vartheta)) \text{ is convergent}\}\\
&\quad= \{\vartheta_k \in \omega : \forall \epsilon \in (0,1), \forall \jmath, \exists n \in \mathbb{N} : \forall i \geq n, \mathsf{P}(T_i(\vartheta), \jmath) > 1-\epsilon,\\
&\qquad \mathsf{Q}(T_i(\vartheta), \jmath) < \epsilon \text{ and } \mathsf{R}(T_i(\vartheta), \jmath) < \epsilon\} \qquad (3.3)\\
&\quad= \{\vartheta = (\vartheta_k) \in \omega : \{i \in \mathbb{N} : \forall \epsilon > 0, \mathsf{P}(T_i(\vartheta), \jmath) \leq 1-\epsilon, \mathsf{Q}(T_i(\vartheta), \jmath) \geq \epsilon,\\
&\qquad \text{or } \mathsf{R}(T_i(\vartheta), \jmath) \geq \epsilon\} \text{ is finite}\}.
\end{aligned}$$

We define the open ball and closed ball with the center at ϑ and the radius $r > 0$ with respect to the parameter of neutrosophic $\epsilon \in (0,1)$ as follows:

$$\begin{aligned}
&\mathscr{B}_{\vartheta}(r,\epsilon)(T)\\
&\quad= \{y = (y_k) \in \omega : \{i \in \mathbb{N} : \mathsf{P}(T_i(\vartheta) - T_i(y), r) \leq 1-\epsilon, \mathsf{Q}(T_i(\vartheta) - T_i(y), r) \geq \epsilon, \qquad (3.4)\\
&\qquad \text{or } \mathsf{R}(T_i(\vartheta) - T_i(y), r) \geq \epsilon\} \text{ is finite}\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathscr{B}_{\vartheta}[r,\epsilon](T)\\
&\quad= \{y = (y_k) \in \omega : \{j \in \mathbb{N} : \mathsf{P}(T_j\vartheta) - T_j(y), r) < 1-\epsilon \text{ or } \mathsf{Q}(T_j\vartheta) - T_j(y), r) > \epsilon, \qquad (3.5)\\
&\qquad \text{or } \mathsf{R}(T_j\vartheta) - T_j(y), r) > \epsilon,\} \text{ is finite}\}.
\end{aligned}$$

If $(\vartheta_n) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$, then (ϑ_n) converges to some $\beta \in Y$, denoted by $\vartheta_n \overset{(\mathsf{P},\mathsf{Q},\mathsf{R})(T)}{\longrightarrow} \beta$.

Lemma 3.1 *Consider the space $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Let $\vartheta = (\vartheta_k) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Then the following statements are equivalent*:

(a) $\mathcal{Z}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)\text{–}\lim(\vartheta) = \beta$;

(b) *For every $0 < \epsilon < 1$ and $\jmath > 0$, there exists $n \in \mathbb{N}$ such that for every $i \geq n$*

$$\mathsf{P}(T_i(\vartheta) - \beta, \jmath) \leq 1-\epsilon \quad \textit{and} \quad \mathsf{Q}(T_i(\vartheta) - \beta, \jmath) \geq \epsilon, \qquad \mathsf{R}(T_i(\vartheta) - \beta, \jmath) \geq \epsilon.$$

(c) *For every $0 < \epsilon < 1$ and $\jmath > 0$, the set*

$$\{i \in \mathbb{N} : \mathsf{P}(T_i(\vartheta) - \beta, \jmath) > 1-\epsilon \textit{ or } \mathsf{Q}(T_i(\vartheta) - \beta, \jmath) < \epsilon, \mathsf{R}(T_i(\vartheta) - \beta, \jmath) < \epsilon\}$$

is finite.

(*d*) *For every* $\jmath > 0$, $\lim_{i\to\infty} \mathsf{P}(T_i(\vartheta) - \beta, \jmath) = 1$, *and* $\lim_{i\to\infty} \mathsf{Q}(T_i(\vartheta) - \beta, \jmath) = 0$, $\lim_{i\to\infty} \mathsf{R}(T_i(\vartheta) - \beta, \jmath) = 0$.

Theorem 3.1 *The inclusion relation* $\mathcal{L}_{0(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) \subset \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) \subset \mathcal{L}_{\infty(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ *holds.*

Proof It can be easily seen that $\mathcal{L}_{0(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) \subset \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. We only show that $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) \subset \mathcal{L}_{\infty(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Let $\vartheta = (\vartheta_k) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Then there exists $\beta \in Y$ such that $\mathcal{Z}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$–$\lim(\vartheta_k) = \beta$. Thus, for every $0 < \epsilon < 1$ and $\jmath > 0$, the set A is finite

$$A = \left\{ i \in \mathbb{N} : \mathsf{P}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2} \right) > 1 - \epsilon \text{ or } \mathsf{Q}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2} \right) < \epsilon, \mathsf{R}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2} \right) < \epsilon \right\}.$$

Let $\mathsf{P}(\beta, \frac{\jmath}{2}) = p$, $\mathsf{Q}(\beta, \frac{\jmath}{2}) = q$, and $\mathsf{R}(\beta, \frac{\jmath}{2}) = r$ for all $\jmath > 0$. Since $p, q, r \in (0, 1)$ and $0 < \epsilon < 1$, there exist $s_1, s_2, s_3 \in (0, 1)$ such that $(1 - \epsilon) * p > 1 - s_1$, $\epsilon \diamond q < s_2$, and $\epsilon \diamond r < s_3$, and so for $\jmath \in Y$, we have

$$\begin{aligned}
\mathsf{P}\big(T_i(\vartheta), \jmath\big) &= \mathsf{P}\big(T_i(\vartheta) - \beta + \beta, \jmath\big) \\
&\geq \mathsf{P}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2} \right) * \mathsf{P}\left(\beta, \frac{\jmath}{2} \right) \\
&> (1 - \epsilon) * p \\
&> 1 - s_1, \\
\mathsf{Q}\big(T_i(\vartheta), \jmath\big) &= \mathsf{Q}\big(T_i(\vartheta) - \beta + \beta, \jmath\big) \\
&\leq \mathsf{Q}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2} \right) * \mathsf{Q}\left(\beta, \frac{\jmath}{2} \right) \\
&< \epsilon \diamond q \\
&< s_2
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{R}\big(T_i(\vartheta), \jmath\big) &= \mathsf{R}\big(T_i(\vartheta) - \beta + \beta, \jmath\big) \\
&\leq \mathsf{R}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2} \right) * \mathsf{R}\left(\beta, \frac{\jmath}{2} \right) \\
&< \epsilon \diamond r \\
&< s_3.
\end{aligned}$$

Taking $s = \max\{s_1, s_2, s_3\}$, we have the set

$$\begin{aligned}
&\big\{ i \in \mathbb{N}, \exists s \in (0,1) : \mathsf{P}\big(T_i(\vartheta), t\big) > 1 - s \text{ or } \mathsf{Q}\big(T_i(\vartheta) - \beta, \jmath\big) < s, \mathsf{R}\big(T_i(\vartheta) - \beta, t\big) < s \big\} \\
&\quad \Longrightarrow \quad \vartheta = (\vartheta_k) \in \mathcal{L}_{\infty(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) \quad \text{implies } \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) \subset \mathcal{L}_{\infty(\mathsf{P},\mathsf{Q},\mathsf{R})}(T).
\end{aligned}$$ □

The converse of the inclusion relation does not hold. We present the following examples in support of our claim.

Example 3.1 Let $(\mathbb{R}, \|\cdot\|)$ be an NNS space such that $\|\vartheta\| = \sup_k \mid \vartheta_k \mid$. Let $\vartheta * y = \min\{\vartheta, y\}$ and $\vartheta \diamond y = \max\{\vartheta, y\}$, $\forall \vartheta, y \in (0,1)$. Now define norms $(\mathsf{P}, \mathsf{Q}, \mathsf{R})$ on $Y^2 \times (0,\infty)$ as follows:

$$\mathsf{P}(\vartheta, \jmath) = \frac{\jmath}{\jmath + \|\vartheta\|}, \qquad \mathsf{Q}(\vartheta, \jmath) = \frac{\|\vartheta\|}{\jmath + \|\vartheta\|} \quad \text{and} \quad \mathsf{R}(\vartheta, \jmath) = \frac{\|\vartheta\|}{\jmath}.$$

Then $(\mathbb{R}, \mathcal{Z}, *, \diamond)$ is an NNS. Consider the sequence $(\vartheta_k) = \{1\}$. It can be easily seen that $(\vartheta_k) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ and $\vartheta_k \overset{\mathcal{Z}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)}{\longrightarrow} 1$, but $\vartheta_k \notin \mathcal{L}_{0(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$.

Example 3.2 Let $(\mathbb{R}, \|\cdot\|)$ be the NNS and $(\mathsf{P}, \mathsf{Q}, \mathsf{R})$ be the neutrosophic norms as defined in the above example. Consider the sequence $(\vartheta_k) = (-1)^k$. Then $(\vartheta_k) \in \mathcal{L}_{\infty(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$, but $(\vartheta_k) \notin \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$.

Theorem 3.2 *The spaces $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ and $\mathcal{L}_{0(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ are linear spaces.*

Proof We shall prove the result for $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. The proof of linearity of the space $\mathcal{L}_{0(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ follows similarly. Let $\vartheta = (\vartheta_k), y = (y_k) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Then there exist $\beta_1, \beta_2 \in Y$ such that (y_k) and (z_k) $\mathcal{Z}$—converge to β_1 and β_2 respectively. We shall show that for any scalars ζ_1 and ζ_2 the sequence $\zeta_1\vartheta_k + \zeta_2 y_k$ $\mathcal{Z}$—converges to $\zeta_1\beta_1 + \zeta_2\beta_2$. For $\jmath > 0$ and $0 < \epsilon < 1$, consider the following finite sets $\mathscr{C}_1$ and $\mathscr{C}_1$:

$$\begin{aligned}
\mathscr{C}_1 = \bigg\{ & i \in \mathbb{N} : \mathsf{P}\bigg(T_i(\vartheta) - \beta_1, \frac{\jmath}{2|\zeta_1|} \bigg) \leq 1 - \epsilon \text{ or } \mathsf{Q}\bigg(T_i(\vartheta) - \beta_1, \frac{\jmath}{2|\zeta_1|} \bigg) \geq \epsilon, \\
& \mathsf{R}\bigg(T_i(\vartheta) - \beta_1, \frac{\jmath}{2|\zeta_1|} \bigg) \geq \epsilon \bigg\}, \\
\mathscr{C}_1^c = \bigg\{ & i \in \mathbb{N} : \mathsf{P}\bigg(T_i(\vartheta) - \beta_1, \frac{\jmath}{2|\zeta_1|} \bigg) > 1 - \epsilon \text{ or } \mathsf{Q}\bigg(T_i(\vartheta) - \beta_1, \frac{\jmath}{2|\zeta_1|} \bigg) < \epsilon, \\
& \mathsf{R}\bigg(T_i(\vartheta) - \beta_1, \frac{\jmath}{2|\zeta_1|} \bigg) < \epsilon \bigg\}, \\
\mathscr{C}_2 = \bigg\{ & i \in \mathbb{N} : \mathsf{P}\bigg(T_i(y) - \beta_2, \frac{\jmath}{2|\zeta_2|} \bigg) \leq 1 - \epsilon \text{ or } \mathsf{Q}\bigg(T_i(y) - \beta_2, \frac{\jmath}{2|\zeta_2|} \bigg) \geq \epsilon, \\
& \mathsf{R}\bigg(T_i(y) - \beta_2, \frac{\jmath}{2|\zeta_2|} \bigg) \geq \epsilon \bigg\}, \\
\mathscr{C}_2^c = \bigg\{ & i \in \mathbb{N} : \mathsf{P}\bigg(T_i(y) - \beta_2, \frac{t}{2|\zeta_2|} \bigg) > 1 - \epsilon \text{ or } \mathsf{Q}\bigg(T_i(y) - \beta_2, \frac{\jmath}{2|\zeta_2|} \bigg) < \epsilon, \\
& \mathsf{R}\bigg(T_i(y) - \beta_2, \frac{\jmath}{2|\zeta_2|} \bigg) < \epsilon \bigg\}.
\end{aligned}$$

Define the set $\mathscr{C}_3 = \mathscr{C}_1 \cup \mathscr{C}_2$ so that $\mathscr{C}_3$ is finite. It follows that $\mathscr{C}_3^c \neq \phi$. We shall show that for each $(\vartheta), (y) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$,

$$\begin{aligned}
\mathscr{C}_3^c \subset \big\{ & i \in \mathbb{N} : \mathsf{P}\big((\zeta_1 T_i(\vartheta) - \zeta_2 T_i(y)) - (\zeta_1\beta_1 - \zeta_2\beta_2), \jmath \big) > 1 - \epsilon \text{ or} \\
& \mathsf{Q}\big((\zeta_1 T_i(\vartheta) - \zeta_2 T_i(y)) - (\zeta_1\beta_1 - \zeta_2\beta_2), \jmath \big) < \epsilon, \\
& \mathsf{R}\big((\zeta_1 T_i(\vartheta) - \zeta_2 T_i(y)) - (\zeta_1\beta_1 - \zeta_2\beta_2), \jmath \big) < \epsilon \big\}.
\end{aligned}$$

Let $q \in \mathscr{C}_3^c$. In this case,

$$\begin{aligned}
&\mathsf{P}\left(T_q(\vartheta)-\beta_1, \frac{J}{2|\zeta_1|}\right) > 1-\epsilon \quad \text{or} \quad \mathsf{Q}\left(T_q(\vartheta)-\beta_1, \frac{J}{2|\zeta_1|}\right) < \epsilon, \\
&\mathsf{R}\left(T_q(\vartheta)-\beta_1, \frac{J}{2|\zeta_1|}\right) < \epsilon
\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{P}\left(T_q(y)-\beta_2, \frac{J}{2|\zeta_2|}\right) > 1-\epsilon \quad \text{or} \quad \mathsf{Q}\left(T_q(y)-\beta_2, \frac{J}{2|\zeta_2|}\right) < \epsilon, \\
&\mathsf{R}\left(T_q(y)-\beta_2, \frac{J}{2|\zeta_2|}\right) < \epsilon, \\
&\mathsf{P}\big(\big(\zeta_1 T_q(\vartheta)-\zeta_2 T_q(y)\big)-(\zeta_1\beta_1-\zeta_2\beta_2), \jmath\big) \\
&\quad \geq \mathsf{P}\left(\zeta_1 T_q(\vartheta)-\zeta_1\beta_1, \frac{J}{2}\right) * \mathsf{P}\left(\zeta_2 T_q(y)-\zeta_2\beta_2, \frac{J}{2}\right) \\
&\quad = \mathsf{P}\left(T_q(\vartheta)-\beta_1, \frac{J}{2|\zeta_1|}\right) * \mathsf{P}\left(T_q(y)-\beta_2, \frac{J}{2|\zeta_2|}\right) \\
&\quad > (1-\epsilon) * (1-\epsilon) \\
&\quad = 1-\epsilon. \\
&\quad \Longrightarrow \quad \mathsf{P}\big(\big(\zeta_1 T_q(\vartheta)-\zeta_2 T_q(y)\big)-(\zeta_1\beta_1-\zeta_2\beta_2), \jmath\big) > 1-\epsilon.
\end{aligned}$$

In a similar way,

$$\begin{aligned}
&\mathsf{Q}\big(\big(\zeta_1 T_q(\vartheta)-\zeta_2 T_q(y)\big)-(\zeta_1\beta_1-\zeta_2\beta_2), \jmath\big) \\
&\quad \leq \mathsf{Q}\left(\zeta_1 T_q(\vartheta)-\zeta_1\beta_1, \frac{J}{2}\right) \diamond \mathsf{Q}\left(\zeta_2 T_q(y)-\zeta_2\beta_2, \frac{J}{2}\right) \\
&\quad = \mathsf{Q}\left(T_q(\vartheta)-\beta_1, \frac{J}{2|\zeta_1|}\right) \diamond \mathsf{Q}\left(T_q(y)-\beta_2, \frac{J}{2|\zeta_2|}\right) \\
&\quad < \epsilon \diamond \epsilon \\
&\quad = \epsilon, \\
&\quad \Longrightarrow \quad \mathsf{Q}\big(\big(\zeta_1 T_q(\vartheta)-\zeta_2 T_q(y)\big)-(\zeta_1\beta_1-\zeta_2\beta_2), \jmath\big) < \epsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{R}\big(\big(\zeta_1 T_q(\vartheta)-\zeta_2 T_q(y)\big)-(\zeta_1\beta_1-\zeta_2\beta_2), \jmath\big) \\
&\quad \leq \mathsf{R}\left(\zeta_1 T_q(\vartheta)-\zeta_1\beta_1, \frac{J}{2}\right) \diamond \mathsf{R}\left(\zeta_2 T_q(y)-\zeta_2\beta_2, \frac{J}{2}\right) \\
&\quad = \mathsf{R}\left(T_q(\vartheta)-\beta_1, \frac{J}{2|\zeta_1|}\right) \diamond \mathsf{R}\left(T_q(y)-\beta_2, \frac{J}{2|\zeta_2|}\right) \\
&\quad < \epsilon \diamond \epsilon \\
&\quad = \epsilon,
\end{aligned}$$

$$\implies \quad \mathsf{R}\big(\big(\zeta_1 T_q(\vartheta) - \zeta_2 T_q(y)\big) - (\zeta_1\beta_1 - \zeta_2\beta_2), \jmath\big) < \epsilon.$$

Therefore, we have

$$\begin{aligned}\mathscr{C}_3^c \subset \big\{q \in \mathbb{N} : \mathsf{P}\big(\big(\zeta_1 T_q(\vartheta) - \zeta_2 T_q(y)\big) - (\zeta_1\beta_1 - \zeta_2\beta_2), \jmath\big) > 1 - \epsilon,\\ \mathsf{Q}\big(\big(\zeta_1 T_q(\vartheta) - \zeta_2 T_q(y)\big) - (\zeta_1\beta_1 - \zeta_2\beta_2), \jmath\big) < \epsilon \text{ or}\\ \mathsf{R}\big(\big(\zeta_1 T_q(\vartheta) - \zeta_2 T_q(y)\big) - (\zeta_1\beta_1 - \zeta_2\beta_2), \jmath\big) < \epsilon\big\}\end{aligned}$$

and

$$\begin{aligned}\big\{q \in \mathbb{N} : \mathsf{P}\big(\big(\zeta_1 T_q(\vartheta) - \zeta_2 T_q(y)\big) - (\zeta_1\beta_1 - \zeta_2\beta_2), \jmath\big) > 1 - \epsilon,\\ \mathsf{Q}\big(\big(\zeta_1 T_q(\vartheta) - \zeta_2 T_q(y)\big) - (\zeta_1\beta_1 - \zeta_2\beta_2), \jmath\big) < \epsilon \text{ or}\\ \mathsf{R}\big(\big(\zeta_1 T_q(\vartheta) - \zeta_2 T_q(y)\big) - (\zeta_1\beta_1 - \zeta_2\beta_2), \jmath\big) < \epsilon\big\},\end{aligned}$$

which implies that the sequence $(\zeta_1\vartheta_i + \zeta_2 y_i)$ $\mathcal{Z}$—converges to $\zeta_1\beta_1 + \zeta_2\beta_2$. Therefore, $(\zeta_1\vartheta_i + \zeta_2 y_i) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Hence $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ is a linear space. □

Theorem 3.3 *Every open ball with the center at* ϑ *and the radius* $r > 0$ *with respect to the parameter of fuzziness* $0 < \epsilon < 1$, *i.e.,* $\mathscr{B}_\vartheta(r,\epsilon)(T)$ *is an open set in* $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$.

Proof Consider the open ball with center at ϑ and radius $r > 0$ with the parameter of neutrosophic $0 < \epsilon < 1$,

$$\begin{aligned}\mathscr{B}_\vartheta(r,\epsilon)(T) = \big\{\vartheta = (\vartheta_i) \in \omega : \big\{i \in \mathbb{N} : \mathsf{P}\big(T_i(\vartheta) - T_i(y), r\big) \le 1 - \epsilon \text{ or}\\ \mathsf{Q}\big(T_i(\vartheta) - T_i(y), r\big) \ge \epsilon,\\ \mathsf{R}\big(T_i(\vartheta) - T_i(y), r\big) \ge \epsilon\big\} \text{ is finite}\big\}.\end{aligned}$$

Then

$$\begin{aligned}\mathscr{B}_\vartheta^c(r,\epsilon)(T) = \big\{\vartheta = (\vartheta_k) \in \omega : \big\{i \in \mathbb{N} : \mathsf{P}\big(T_i(\vartheta) - T_i(y), r\big) > 1 - \epsilon \text{ or}\\ \mathsf{Q}\big(T_i(\vartheta) - T_i(y), r\big) < \epsilon,\\ \mathsf{R}\big(T_i(\vartheta) - T_i(y), r\big) < \epsilon\big\}\big\}.\end{aligned}$$

Let $y = (y_i) \in \mathscr{B}_\vartheta^c(r,\epsilon)(T)$. Then the set

$$\big\{i \in \mathbb{N} : \mathsf{P}\big(T_i(\vartheta) - T_i(y), r\big) > 1 - \epsilon \text{ or } \mathsf{Q}\big(T_i(\vartheta) - T_j(y), r\big) < \epsilon, \mathsf{R}\big(T_i(y) - T_i(z), r\big) < \epsilon\big\},$$
$$\mathsf{P}\big(T_i(\vartheta) - T_i(y), r\big) > 1 - \epsilon, \qquad \mathsf{Q}\big(T_i(\vartheta) - T_i(z), r\big) < \epsilon \quad \text{and} \quad \mathsf{R}\big(T_i(\vartheta) - T_i(y), r\big) < \epsilon,$$

there exists $r_0 \in (0, r)$ such that

$$\begin{aligned}&\mathsf{P}\big(T_i(\vartheta) - T_i(y), r_0\big) > 1 - \epsilon, \qquad \mathsf{Q}\big(T_i(\vartheta) - T_i(y), r_0\big) < \epsilon \quad \text{and}\\ &\mathsf{R}\big(T_i(\vartheta) - T_i(y), r_0\big) < \epsilon.\end{aligned}$$

Put $\epsilon_0 = \mathsf{P}(T_i(\vartheta) - T_i(y), r_0) \implies \epsilon_0 > 1 - \epsilon$. Then $\exists s \in (0,1)$ such that $\epsilon_0 > 1 - s > 1 - \epsilon$. For $\epsilon_0 > 1 - s$, we can have $\epsilon_1, \epsilon_2, \epsilon_3 \in (0,1)$ such that $\epsilon_0 * \epsilon_1 > 1 - s$, $(1 - \epsilon_0) \diamond (1 - \epsilon_2) < s$, and

$$(1 - \epsilon_0) \diamond (1 - \epsilon_3) < s.$$

Let $\epsilon_4 = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}$

Now consider the open ball $\mathscr{B}^c_\vartheta(r - r_0, 1 - \epsilon_4)(T)$.

We shall show that $\mathscr{B}^c_\vartheta(r - r_0, 1 - \epsilon_4)(T) \subset \mathscr{B}^c_\vartheta(r, \epsilon)(T)$.

Let $z = (z_i) \in \mathscr{B}^c_z(r - r_0, 1 - \epsilon_4)(T)$, then

$$\begin{aligned}&\bigl\{i \in \mathbb{N} : \mathsf{P}\bigl(T_i(\vartheta) - T_i(z), r - r_0\bigr) > \epsilon_4 \text{ or} \\ &\quad \mathsf{Q}\bigl(T_i(\vartheta) - T_i(z), r - r_0\bigr) < 1 - \epsilon_4, \mathsf{R}\bigl(T_i(\vartheta) - T_i(z), r - r_0\bigr) < 1 - \epsilon_4\bigr\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathsf{P}\bigl(T_i(\vartheta) - T_i(z), r\bigr) \\ &\quad \geq \mathsf{P}\bigl(T_i(\vartheta) - T_i(z), r_0\bigr) * \mathsf{P}\bigl(T_i(\vartheta) - T_i(z), r - r_0\bigr) \\ &\quad \geq \epsilon_0 * \epsilon_4 \geq \epsilon_0 * \epsilon_1 \\ &\quad > (1 - s) > (1 - \epsilon) \\ &\quad \implies \quad \bigl\{i \in \mathbb{N} : \mathsf{P}\bigl(T_i(\vartheta) - T_i(z), r\bigr) > 1 - \epsilon\bigr\} \\ &\mathsf{Q}\bigl(T_i(\vartheta) - T_i(z), r\bigr) \\ &\quad \leq \mathsf{Q}\bigl(T_i(\vartheta) - T_i(z), r_0\bigr) \diamond \mathsf{Q}\bigl(T_i(\vartheta) - T_i(z), r - r_0\bigr) \\ &\quad \leq (1 - \epsilon_0) \diamond (1 - \epsilon_4) \leq (1 - \epsilon_0) \diamond (1 - \epsilon_2) \\ &\quad \leq (s) < \epsilon \\ &\quad \implies \quad \bigl\{i \in \mathbb{N} : \mathsf{Q}\bigl(T_i(\vartheta) - T_i(z), r\bigr) < \epsilon\bigr\}\end{aligned}$$

and

$$\begin{aligned}&\mathsf{R}\bigl(T_i(\vartheta) - T_i(z), r\bigr) \\ &\quad \leq \mathsf{R}\bigl(T_i(\vartheta) - T_i(z), r_0\bigr) \diamond \mathsf{R}\bigl(T_i(\vartheta) - T_i(z), r - r_0\bigr) \\ &\quad \leq (1 - \epsilon_0) \diamond (1 - \epsilon_4) \leq (1 - \epsilon_0) \diamond (1 - \epsilon_3) \\ &\quad \leq (s) < \epsilon \\ &\quad \implies \quad \bigl\{i \in \mathbb{N} : \mathsf{R}\bigl(T_i(\vartheta) - T_i(z), r\bigr) < \epsilon\bigr\}.\end{aligned}$$

Therefore the set

$$\begin{aligned}&\bigl\{i \in \mathbb{N} : \mathsf{P}\bigl(T_i(\vartheta) - T_i(z), r - r_0\bigr) > 1 - \epsilon \text{ or } \mathsf{Q}\bigl(T_i(\vartheta) - T_i(z), r - r_0\bigr) < \epsilon, \\ &\quad \mathsf{R}\bigl(T_i(\vartheta) - T_i(z), r - r_0\bigr) < \epsilon\bigr\},\end{aligned}$$

$$\implies \quad \vartheta = (\vartheta_i) \in \mathscr{B}^c_\vartheta(r,\epsilon)(T)$$
$$\implies \quad \mathscr{B}^c_\vartheta(r-r_0, 1-\epsilon_4)(T) \subset \mathscr{B}^c_y(r,\epsilon)(T). \qquad \square$$

Remark 3.1 The spaces $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ and $\mathcal{L}_{0(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ are NNS with respect to neutrosophic norms $(\mathsf{P},\mathsf{Q},\mathsf{R})$.

Now define

$$\tau_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) = \bigl\{U \subset \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) : \text{ for each } \vartheta = (\vartheta_k) \in U \text{ there exist } r > 0 \text{ and } \epsilon \in (0,1) \text{ such that } \mathscr{B}_\vartheta(r,\epsilon) \subset U\bigr\}.$$

Then $\tau_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ defines a topology on the sequence space $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. The collection defined by $\mathcal{B} = \{\mathscr{B}_\vartheta(r,\epsilon) : \vartheta \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T), r > 0 \text{ and } \epsilon \in (0,1)\}$ is a base for the topology $\tau_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ on the space $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$.

Theorem 3.4 *The topology $\tau_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ on the space $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ is first countable.*

Proof For each $\vartheta = (\vartheta_i) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$, consider the set $\mathcal{B} = \{\mathscr{B}_\vartheta(\frac{1}{n}, \frac{1}{n}) : n = 1,2,3,4,\ldots\}$, which is a countable local base at ϑ. Therefore the topology $\tau_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ on the space $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ is first countable. $\square$

Theorem 3.5 *The spaces $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ and $\mathcal{L}_{0(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ are Hausdorff spaces.*

Proof We shall prove the result only for $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$, and the other one follows similarly. Let $\vartheta = (\vartheta_i)$ and $y = (y_i) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ such that $\vartheta \neq y$. Then, for each $i \in \mathbb{N}$ and $r > 0$, this implies

$$\begin{aligned} &0 < \mathsf{P}\bigl(T_i(\vartheta) - T_i(y), r\bigr) < 1, \qquad 0 < \mathsf{Q}\bigl(T_i(\vartheta) - T_i(y), r\bigr) < 1 \quad \text{and} \\ &0 < \mathsf{Q}\bigl(T_i(\vartheta) - T_i(y), r\bigr) < 1. \end{aligned} \tag{3.6}$$

Put

$$\begin{aligned} &\epsilon_1 = \mathsf{P}\bigl(T_i(\vartheta) - T_i(y), r\bigr), \qquad \epsilon_2 = \mathsf{Q}\bigl(T_i(\vartheta) - T_i(y), r\bigr), \\ &\epsilon_3 = \mathsf{R}\bigl(T_i(\vartheta) - T_i(y), r\bigr) \quad \text{and} \quad \epsilon = \max\{\epsilon_1, 1-\epsilon_2, 1-\epsilon_3\}. \end{aligned} \tag{3.7}$$

Then for each $\epsilon_0 > \epsilon$ there exist $\epsilon_4, \epsilon_5, \epsilon_6 \in (0,1)$ such that

$$\epsilon_4 * \epsilon_4 \geq \epsilon_0, \qquad (1-\epsilon_5) \diamond (1-\epsilon_5) \leq (1-\epsilon_0) \quad \text{and} \quad (1-\epsilon_6) \diamond (1-\epsilon_6) \leq (1-\epsilon_0).$$

Again putting $\epsilon_7 = \max\{\epsilon_4, \epsilon_5, \epsilon_6\}$, consider the open balls $\mathscr{B}_\vartheta(1-\epsilon_7, \frac{r}{2})(T)$ and $\mathscr{B}_y(1-\epsilon_7, \frac{r}{2})(T)$ centered at ϑ and y respectively. We show that $\mathscr{B}_\vartheta(1-\epsilon_7, \frac{r}{2})(T) \cap \mathscr{B}_y(1-\epsilon_7, \frac{r}{2})(T) = \phi$.

If possible, let $\vartheta = (\vartheta_i) \in \mathscr{B}_\vartheta(1-\epsilon_7, \frac{r}{2})(T) \cap \mathscr{B}_y(1-\epsilon_7, \frac{r}{2})(T)$.

Then, for the set $\{k \in \mathbb{N}\}$, we have

$$\begin{aligned}\epsilon_1 &= \mathsf{P}\big(T_k(\vartheta) - T_k(y), r\big)\\ &\geq \mathsf{P}\left(T_k(\vartheta) - T_k(z), \frac{r}{2}\right) * \mathsf{P}\left(T_k(z) - T_k(y), \frac{r}{2}\right)\\ &> \epsilon_7 * \epsilon_7 \geq \epsilon_4 * \epsilon_4 \geq \epsilon_0 > \epsilon_1,\end{aligned} \tag{3.8}$$

$$\begin{aligned}\epsilon_2 &= \mathsf{Q}\big(T_k(\vartheta) - T_k(y), r\big)\\ &\leq \mathsf{Q}\left(T_k(\vartheta) - T_k(z), \frac{r}{2}\right) \diamond \mathsf{Q}\left(T_k(z) - T_k(y), \frac{r}{2}\right)\\ &< (1-\epsilon_7) \diamond (1-\epsilon_7) \leq (1-\epsilon_5) \diamond (1-\epsilon_5)\\ &< (1-\epsilon_0) < \epsilon_2\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\epsilon_3 &= \mathsf{R}\big(T_k(\vartheta) - T_k(y), r\big)\\ &\leq \mathsf{R}\left(T_k(\vartheta) - T_j(z), \frac{r}{2}\right) \diamond \mathsf{R}\left(T_k(z) - T_k(y), \frac{r}{2}\right)\\ &< (1-\epsilon_7) \diamond (1-\epsilon_7) \leq (1-\epsilon_6) \diamond (1-\epsilon_6)\\ &< (1-\epsilon_0) < \epsilon_3.\end{aligned} \tag{3.10}$$

From equations (3.8), (3.9), and (3.10) we have a contradiction.

Therefore, $\mathscr{B}_\vartheta(1-\epsilon_7, \frac{r}{2})(T) \cap \mathscr{B}_y(1-\epsilon_7, \frac{r}{2})(T) = \phi$. Hence the space $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ is a Hausdorff space. □

4 On the Tribonacci sequence T_n

Definition 4.1 A sequence $\vartheta = (\vartheta_n) \in \omega$ is said to be Tribonacci convergent to $\beta \in Y$ if for every $\epsilon > 0$ the set B_1 is finite, where

$$B_1 = \big\{i \in \mathbb{N} : \big|T_i(\vartheta) - \beta\big| \geq \epsilon\big\}.$$

Definition 4.2 A sequence $\vartheta = (\vartheta_k) \in \omega$ is said to be neutrosophic Tribonacci convergent to $\beta \in Y$ with respect to neutrosophic norms- $(\mathsf{P},\mathsf{Q},\mathsf{R})$, denoted by $\vartheta_i \to \beta$, if for every $\epsilon \in (0,1)$ and $\jmath > 0$, the set T_1 is finite, where

$$T_1 = \big\{i \in \mathbb{N} : \mathsf{P}\big(T_i(\vartheta) - \beta, \jmath\big) \leq 1-\epsilon \text{ or } \mathsf{Q}\big(T_i(\vartheta) - \beta, \jmath\big) \geq \epsilon, \mathsf{R}\big(T_i(\vartheta) - \beta, \jmath\big) \geq \epsilon\big\},$$

and we write $\mathcal{Z}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)\text{–}\lim(\vartheta_i) = \beta$.

Definition 4.3 A sequence $\vartheta = (\vartheta_i) \in \omega$ is said to be Tribonacci Cauchy if for every $\epsilon > 0$ there exists $k = k(\epsilon) \in \mathbb{N}$ such that the set B_2 is finite, where

$$B_2 = \big\{i \in \mathbb{N} : \big|T_i(\vartheta) - T_k(\vartheta)\big| \geq \epsilon\big\}.$$

Definition 4.4 A sequence $\vartheta = (\vartheta_i) \in \omega$ is said to neutrosophic Tribonacci Cauchy with respect to neutrosophic norms-$(\mathsf{P},\mathsf{Q},\mathsf{R})$ if for every $\epsilon \in (0,1)$ and $\jmath > 0$ there exists $k \in \mathbb{N}$ such that the set T_2 is finite, where

$$\begin{aligned} T_2 = \bigl\{ i \in \mathbb{N} : \mathsf{P}\bigl(T_i(\vartheta) - T_k(\vartheta), \jmath\bigr) \leq 1 - \epsilon \text{ or } \mathsf{Q}\bigl(T_i(\vartheta) - T_k(\vartheta), \jmath\bigr) \geq \epsilon, \\ \mathsf{R}\bigl(T_i(\vartheta) - T_k(\vartheta), \jmath\bigr) \geq \epsilon \bigr\}. \end{aligned}$$

Definition 4.5 A sequence $\vartheta = (\vartheta_i) \in \omega$ is said to be Tribonacci bounded if there exists $M > 0$ such that the set

$$B_3 = \bigl\{ i \in \mathbb{N} : \bigl|T_i(\vartheta)\bigr| > M \bigr\}.$$

Definition 4.6 A subset D of ω is said to be neutrosophic Tribonacci bounded with respect to neutrosophic norms $(\mathsf{P},\mathsf{Q},\mathsf{R})$ if $\forall \vartheta \in D$ there exist $0 < \epsilon < 1$ and $\jmath > 0$ such that the set

$$\bigl\{ i \in \mathbb{N} : \mathsf{P}\bigl(T_i(\vartheta), \jmath\bigr) \leq 1 - \epsilon \text{ or } \mathsf{Q}\bigl(T_i(\vartheta), \jmath\bigr) \geq \epsilon, \mathsf{R}\bigl(T_i(\vartheta), \jmath\bigr) \geq \epsilon \bigr\}. \tag{4.1}$$

Theorem 4.1 *If a sequence $\vartheta = (\vartheta_i) \in \omega$ is neutrosophic Tribonacci convergent, then the $\mathcal{Z}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$-limit is unique.*

Proof Suppose $\vartheta = (\vartheta_i) \in \omega$ such that (ϑ_i) is neutrosophic Tribonacci convergent.

Let $\mathcal{Z}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) \lim(\vartheta_i) = \beta_1$ and $\mathcal{Z}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) - \lim(\vartheta_i) = \beta_2$. We show that $\beta_1 = \beta_2$. Now, for given $\epsilon \in (0,1)$, there exists $\epsilon_1 \in (0,1)$ such that $(1 - \epsilon_1) * (1 - \epsilon_1) > 1 - \epsilon$ and $\epsilon_1 \diamond \epsilon_1 < \epsilon$. Therefore the sets $\mathscr{C}_1$ and $\mathscr{C}_2$ are finite, where

$$\begin{aligned} \mathscr{C}_1 &= \biggl\{ i \in \mathbb{N} : \mathsf{P}\biggl(T_i(\vartheta) - \beta_1, \frac{\jmath}{2} \biggr) \leq 1 - \epsilon_1 \text{ or } \mathsf{Q}\biggl(T_i(\vartheta) - \beta_1, \frac{\jmath}{2} \biggr) \geq \epsilon_1, \\ &\quad \mathsf{R}\biggl(T_i(\vartheta) - \beta_1, \frac{\jmath}{2} \biggr) \geq \epsilon_1 \biggr\}, \\ \mathscr{C}_1 &= \biggl\{ i \in \mathbb{N} : \mathsf{P}\biggl(T_i(\vartheta) - \beta_2, \frac{\jmath}{2} \biggr) \leq 1 - \epsilon_1 \text{ or } \mathsf{Q}\biggl(T_i(\vartheta) - \beta_2, \frac{\jmath}{2} \biggr) \geq \epsilon_1, \\ &\quad \mathsf{R}\biggl(T_i(\vartheta) - \beta_2, \frac{\jmath}{2} \biggr) \geq \epsilon_1 \biggr\}, \\ &\quad \Longrightarrow \\ \mathscr{C}_1^c &= \biggl\{ i \in \mathbb{N} : \mathsf{P}\biggl(T_i(\vartheta) - \beta_1, \frac{\jmath}{2} \biggr) > 1 - \epsilon_1 \text{ or } \mathsf{Q}\biggl(T_i(\vartheta) - \beta_1, \frac{\jmath}{2} \biggr) < \epsilon_1, \\ &\quad \mathsf{R}\biggl(T_i(\vartheta) - \beta_1, \frac{\jmath}{2} \biggr) < \epsilon_1 \biggr\}, \\ \mathscr{C}_2^c &= \biggl\{ i \in \mathbb{N} : \mathsf{P}\biggl(T_i(\vartheta) - \beta_2, \frac{\jmath}{2} \biggr) > 1 - \epsilon_1 \text{ or } \mathsf{Q}\biggl(T_i(\vartheta) - \beta_2, \frac{\jmath}{2} \biggr) < \epsilon_1, \\ &\quad \mathsf{R}\biggl(T_i(\vartheta) - \beta_2, \frac{\jmath}{2} \biggr) < \epsilon_1 \biggr\}. \end{aligned}$$

Then $\mathscr{C}_1^c \cap \mathscr{C}_2^c \neq \phi$. Taking $i \in \mathscr{C}_1^c \cap \mathscr{C}_2^c$, we have

$$\begin{aligned}
\mathsf{P}(\beta_1 - \beta_2, \jmath) &\geq \mathsf{P}\left(T_i(\vartheta) - \beta_1, \frac{\jmath}{2}\right) * \mathsf{P}\left(T_i(\vartheta) - \beta_2, \frac{\jmath}{2}\right) \\
&> (1 - \epsilon_1) * (1 - \epsilon_1) \\
&> (1 - \epsilon), \\
\mathsf{Q}(\beta_1 - \beta_2, \jmath) &\leq \mathsf{Q}\left(T_i(\vartheta) - \beta_1, \frac{\jmath}{2}\right) \diamond \mathsf{Q}\left(T_i(\vartheta) - \beta_2, \frac{\jmath}{2}\right) \\
&< \epsilon_1 \diamond \epsilon_1 \\
&< \epsilon
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{R}(\beta_1 - \beta_2, \jmath) &\leq \mathsf{R}\left(T_i(\vartheta) - \beta_1, \frac{\jmath}{2}\right) \diamond \mathsf{R}\left(T_i(\vartheta) - \beta_2, \frac{\jmath}{2}\right) \\
&< \epsilon_1 \diamond \epsilon_1 \\
&< \epsilon.
\end{aligned}$$

Since $\epsilon \in (0,1)$ is arbitrary, therefore $\mathsf{P}(\beta_1 - \beta_2, \jmath) = 1$, $\mathsf{Q}(\beta_1 - \beta_2, \jmath) = 0$, and $\mathsf{R}(\beta_1 - \beta_2, \jmath) = 0$ for all $\jmath > 0$. Hence $\beta_1 - \beta_2 = 0$. Thus $\mathcal{Z}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$- limit is unique. □

Theorem 4.2 *A sequence $\vartheta = (\vartheta_i) \in \omega$ is neutrosophic Tribonacci convergent with respect to neutrosophic norms* $(\mathsf{P}, \mathsf{Q}, \mathsf{R})$ *iff it is neutrosophic Tribonacci Cauchy with respect to the same norms.*

Proof Suppose that $\vartheta = (\vartheta_i) \in \omega$ is neutrosophic Tribonacci convergent with respect to neutrosophic norms $(\mathsf{P}, \mathsf{Q}, \mathsf{R})$ such that $\mathcal{Z}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) - \lim(\vartheta_i) = \beta$. For given $\epsilon \in (0,1)$, there exists $\epsilon_1 \in (0,1)$ such that $(1-\epsilon_1) * (1-\epsilon_1) > 1-\epsilon$ and $\epsilon_1 \diamond \epsilon_1 < \epsilon$. Since $\mathcal{Z}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T) \lim(\vartheta_i) = \beta$, therefore for all $\jmath > 0$

$$\mathscr{C} = \left\{ i \in \mathbb{N} : \mathsf{P}\left(T_i(\vartheta) - \beta, \jmath\right) \leq 1 - \epsilon_1 \text{ or } \mathsf{Q}\left(T_i(\vartheta) - \beta, \jmath\right) \geq \epsilon_1, \mathsf{R}\left(T_i(\vartheta) - \beta, \jmath\right) \geq \epsilon_1 \right\}$$
$$\text{is finite,}$$

which implies

$$\begin{aligned}
\mathscr{C}^c = \big\{ i \in \mathbb{N} : {}& \mathsf{P}\left(T_i(\vartheta) - \beta, \jmath\right) > 1 - \epsilon_1 \quad \text{or} \quad \mathsf{Q}\left(T_i(\vartheta) - \beta, \jmath\right) < \epsilon_1, \\
& \mathsf{R}\left(T_i(\vartheta) - \beta, \jmath\right) < \epsilon_1 \big\}.
\end{aligned}$$

For $i \in \mathscr{C}^c$, we have

$$\mathsf{P}\left(T_i(\vartheta) - \beta, \jmath\right) > 1 - \epsilon_1 \quad \text{or} \quad \mathsf{Q}\left(T_i(\vartheta) - \beta, \jmath\right) < \epsilon_1, \qquad \mathscr{R}\left(T_j(y) - \beta, t\right) < \epsilon_1.$$

For fixed $k \in \mathscr{C}^c$, let

$$A = \left\{ i \in \mathbb{N} : \mathsf{P}\left(T_i(\vartheta) - T_k(\vartheta), \jmath\right) \leq 1 - \epsilon \text{ or } \mathsf{Q}\left(T_i(\vartheta) - T_k(\vartheta), \jmath\right) \geq \epsilon, \right.$$

$$\mathsf{R}\big(T_i(\vartheta) - T_k(\vartheta), \jmath\big) \geq \epsilon\big\} \quad \text{be finite}.$$

We show that $A \subset \mathscr{C}$. Let $i \in A$, we have

$$\mathsf{P}\big(T_i(\vartheta) - T_k(\vartheta), \jmath\big) \leq 1 - \epsilon \quad \text{or} \quad \mathsf{Q}\big(T_i(\vartheta) - T_k(\vartheta), \jmath\big) \geq \epsilon,$$
$$\mathsf{R}\big(T_i(\vartheta) - T_k(\vartheta), \jmath\big) \geq \epsilon.$$

We have two possible cases, firstly consider $\mathsf{P}(T_i(\vartheta) - T_k(\vartheta), \jmath) \leq 1 - \epsilon$. Then $\mathsf{P}(T_i(\vartheta) - \beta, \frac{\jmath}{2}) \leq 1 - \epsilon_1$. If possible, let $\mathsf{P}(T_i(\vartheta) - \beta, \frac{\jmath}{2}) > 1 - \epsilon_1$. Then

$$\begin{aligned} 1 - \epsilon &\geq \mathsf{P}\big(T_i(\vartheta) - T_k(\vartheta), \jmath\big) \\ &\geq \mathsf{P}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2}\right) * \mathsf{P}\left(T_k(\vartheta) - \beta, \frac{\jmath}{2}\right) \\ &> (1 - \epsilon_1) * (1 - \epsilon_1) \\ &> (1 - \epsilon), \end{aligned}$$

which is a contradiction. $\implies \mathsf{P}(T_i(\vartheta) - \beta, \frac{\jmath}{2}) \leq 1 - \epsilon_1$.

Similarly, consider $\mathsf{Q}(T_i(\vartheta) - T_k(\vartheta), \jmath) \geq \epsilon$, then $\mathsf{Q}(T_i(\vartheta) - \beta, \frac{\jmath}{2}) \geq \epsilon_1$.

If possible, suppose $\mathsf{Q}(T_i(\vartheta) - \beta, \frac{\jmath}{2}) < \epsilon_1$. Hence

$$\begin{aligned} \epsilon &\leq \mathsf{Q}\big(T_i(\vartheta) - T_k(\vartheta), \jmath\big) \\ &\leq \mathsf{Q}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2}\right) \diamond \mathsf{Q}\left(T_k(\vartheta) - \beta, \frac{\jmath}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 \\ &< \epsilon, \end{aligned}$$

which is again a contradiction. $\implies \mathsf{Q}(T_i(\vartheta) - \beta, \frac{\jmath}{2}) \geq \epsilon_1$.

Similarly,

$$\mathsf{R}\big(T_i(\vartheta) - T_k(\vartheta), \jmath\big) \geq \epsilon, \quad \text{then } \mathsf{R}\left(T_i(\vartheta) - \beta, \frac{t}{2}\right) \geq \epsilon_1.$$

If possible, suppose $\mathsf{R}(T_i(\vartheta) - \beta, \frac{\jmath}{2}) < \epsilon_1$. Hence

$$\begin{aligned} \epsilon &\leq \mathsf{R}\big(T_i(\vartheta) - T_k(\vartheta), \jmath\big) \\ &\leq \mathsf{R}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2}\right) \diamond \mathsf{R}\left(T_k(\vartheta) - \beta, \frac{\jmath}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 \\ &< \epsilon, \end{aligned}$$

which is again a contradiction

$$\implies \mathsf{R}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2}\right) \geq \epsilon_1.$$

Therefore, for $i \in A$, we have

$$\begin{aligned}
&\mathsf{P}\left(T_i(\vartheta)-\beta,\frac{J}{2}\right) \leq (1-\epsilon_1), \qquad \mathsf{Q}\left(T_i(\vartheta)-\beta,\frac{J}{2}\right) \geq \epsilon_1 \quad \text{and} \\
&\mathsf{R}\left(T_i(\vartheta)-\beta,\frac{J}{2}\right) \geq \epsilon_1 \\
&\quad \implies \quad i \in \mathscr{C}.
\end{aligned}$$

Hence, $A \subset \mathscr{C}$. Since $\mathscr{C}$ is finite, so the sequence $\vartheta = (\vartheta_i)$ is neutrosophic Tribonacci Cauchy with respect to the norms $(\mathsf{P},\mathsf{Q},\mathsf{R})$.

Conversely, suppose that the sequence $\vartheta = (\vartheta_i) \in \omega$ is neutrosophic Tribonacci Cauchy with respect to the norms $(\mathsf{P},\mathsf{Q},\mathsf{R})$. Let on the contrary the sequence $\vartheta = (\vartheta_i)$ be not neutrosophic Tribonacci convergent. Then there exists $i \in \mathbb{N}$ such that

$$\begin{aligned}
T_1 = \bigl\{ &i \in \mathbb{N} : \mathsf{P}\bigl(T_i(\vartheta)-T_k(\vartheta),J\bigr) \leq 1-\epsilon \text{ or } \mathsf{Q}\bigl(T_i(\vartheta)-T_k(\vartheta),J\bigr) \geq \epsilon, \\
&\mathsf{R}\bigl(T_i(\vartheta)-T_k(\vartheta),J\bigr) \geq \epsilon \bigr\}
\end{aligned}$$

but

$$\begin{aligned}
T_2 = \biggl\{ &i \in \mathbb{N} : \mathsf{P}\left(T_i(\vartheta)-\beta,\frac{J}{2}\right) > 1-\epsilon_1 \text{ or } \mathsf{Q}\left(T_i(\vartheta)-\beta,\frac{J}{2}\right) < \epsilon_1, \\
&\mathsf{R}\left(T_j(y)-\beta,\frac{J}{2}\right) < \epsilon_1 \biggr\}, \\
&\implies \\
1-\epsilon &\geq \mathsf{P}\bigl(T_i(\vartheta)-T_k(\vartheta),J\bigr) \\
&\geq \mathsf{P}\left(T_i(\vartheta)-\beta,\frac{J}{2}\right) * \mathsf{P}\left(T_k(\vartheta)-\beta,\frac{J}{2}\right) \\
&> (1-\epsilon_1) * (1-\epsilon_1) \\
&> 1-\epsilon,
\end{aligned}$$

which is a contradiction.

Now,

$$\begin{aligned}
\epsilon &\leq \mathsf{Q}\bigl(T_i(\vartheta)-T_k(\vartheta),J\bigr) \\
&\leq \mathsf{Q}\left(T_i(\vartheta)-\beta,\frac{J}{2}\right) \diamond \mathsf{Q}\left(T_k(\vartheta)-\beta,\frac{J}{2}\right) \\
&< \epsilon_1 \diamond \epsilon_1 \\
&< \epsilon,
\end{aligned}$$

which is again a contradiction; and

$$\epsilon \leq \mathsf{R}\bigl(T_i(\vartheta)-T_k(\vartheta),J\bigr)$$

$$
\begin{aligned}
&\leq \mathsf{R}\left(T_i(\vartheta)-\beta,\frac{J}{2}\right)\diamond \mathsf{R}\left(T_k(\vartheta)-\beta,\frac{J}{2}\right)\\
&< \epsilon_1 \diamond \epsilon_1\\
&< \epsilon,
\end{aligned}
$$

which is again a contradiction.

Therefore $T_2 \in \mathcal{Z}$ and hence $\vartheta = (\vartheta_i)$ is neutrosophic Tribonacci convergent. □

Theorem 4.3 *Every subset D of $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ is neutrosophic Tribonacci bounded.*

Proof The proof of this theorem follows from Theorem 3.1 and can be verified on similar grounds. □

Theorem 4.4 *Let NNS $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ and $\tau_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ be the topology on $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Let $(\vartheta^n) = (\vartheta^n_k)_{n=1}^{\infty}$ be a sequence of points in $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Then $\vartheta^n \longrightarrow \vartheta$ as $n \longrightarrow \infty$ iff*

$$
\begin{aligned}
&\mathsf{P}\bigl(T_i(\vartheta^n)-T_i(\vartheta),\jmath\bigr)\longrightarrow 1,\qquad \mathsf{Q}\bigl(T_i(\vartheta^n)-T_i(\vartheta),\jmath\bigr)\longrightarrow 0\quad \textit{and}\\
&\mathsf{R}\bigl(T_i(\vartheta^n)-T_i(\vartheta),\jmath\bigr)\longrightarrow 0\quad \textit{as } i\longrightarrow\infty.
\end{aligned}
$$

Proof Let $\vartheta^n \longrightarrow \vartheta$ as $n \longrightarrow \infty$. Fix $r > 0$. Then, for $0 < \epsilon < 1$, there exists $k \in \mathbb{N}$ such that $(\vartheta^n) \in \mathscr{B}_\vartheta(r,\epsilon)(T)$ for all $n \geq k$. Then the set A is finite, where

$$
\begin{aligned}
A = \bigl\{&i\in\mathbb{N} : \mathsf{P}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)\leq 1-\epsilon \text{ or } \mathsf{Q}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)\geq\epsilon,\\
&\mathsf{R}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)\geq\epsilon\bigr\},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
A^c = \bigl\{&i\in\mathbb{N} : \mathsf{P}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)> 1-\epsilon \text{ or } \mathsf{Q}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)<\epsilon,\\
&\mathsf{R}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)<\epsilon\bigr\}.
\end{aligned}
$$

For $\{i \in \mathbb{N}\} \subseteq A^c$,

$$
\begin{aligned}
&\mathsf{P}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)>1-\epsilon\\
&\quad\Longrightarrow\quad 1-\mathsf{P}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)>\epsilon,\qquad \mathsf{Q}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)<\epsilon\quad\text{and}\\
&\mathsf{R}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)<\epsilon.
\end{aligned}
$$

Therefore, for $n \longrightarrow \infty$,

$$
\begin{aligned}
&\Longrightarrow\quad 1-\mathsf{P}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)\longrightarrow 0,\qquad \mathsf{Q}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)\longrightarrow 0\quad\text{and}\\
&\mathsf{R}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)\longrightarrow 0,\\
&\Longrightarrow\quad \mathsf{P}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)\longrightarrow 1,\qquad \mathsf{Q}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)\longrightarrow 0\quad\text{and}\\
&\mathsf{R}\bigl(T_i(\vartheta^n)-T_i(\vartheta),r\bigr)\longrightarrow 0\quad\text{as } n\longrightarrow\infty.
\end{aligned}
$$

Conversely, suppose that for each $i > 0$, $\mathsf{P}(T_i(\vartheta^n) - T_i(\vartheta), r) \longrightarrow 1$, $\mathsf{Q}(T_i(\vartheta^n) - T_i(\vartheta), r) \longrightarrow 0$, and $\mathsf{R}(T_i(\vartheta^n) - T_i(\vartheta), r) \longrightarrow 0$ as $n \longrightarrow \infty$. Then for each $\epsilon \in (0,1)$ there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} &\Longrightarrow \quad 1 - \mathsf{P}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) \longrightarrow 0, \qquad \mathsf{Q}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) \longrightarrow 0 \quad \text{and} \\ &\mathsf{R}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) \longrightarrow 0 \end{aligned}$$

for all $n \geq k$,

$$\begin{aligned} &\mathsf{P}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) > 1 - \epsilon \quad \Longrightarrow \quad 1 - \mathsf{P}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) > \epsilon, \\ &\mathsf{Q}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) < \epsilon \quad \text{and} \\ &\mathsf{R}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) < \epsilon \quad \text{for all } n \geq k. \end{aligned}$$

Consider that the neutrosophic-$\mathcal{Z}$ generated by the set $\{t \in \mathbb{N} : t < k\}$ implies the collection of sets generated by the set $\{i \in \mathbb{N} : i \geq k\}$.

Thus

$$\begin{aligned} &\big\{i \in \mathbb{N} : \mathsf{P}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) > 1 - \epsilon \text{ or } \mathsf{Q}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) < \epsilon, \\ &\quad \mathsf{R}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) < \epsilon\big\} \\ &\quad \Longrightarrow \quad \big(\vartheta^n\big) \in \mathscr{B}_\vartheta(r, \epsilon)(T) \quad \text{for all } n \geq k. \end{aligned}$$

Hence $\vartheta^n \longrightarrow \vartheta$ as $n \longrightarrow \infty$. □

Theorem 4.5 *Every closed ball with the center at* ϑ *and the radius* $r > 0$ *with respect to the parameter of fuzziness* $0 < \epsilon < 1$, *i.e.,* $\mathscr{B}_\vartheta[r, \epsilon](T)$ *is a closed set in* $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$.

Proof Let $\vartheta = (\vartheta_k) \in \omega$ be such that $\vartheta \in \overline{\mathscr{B}_\vartheta[r, \epsilon](T)}$. Then there exists a sequence $(\vartheta^n) = (\vartheta^n_k) \in \mathscr{B}_\vartheta[r, \epsilon](T)$ such that $\vartheta^n \longrightarrow \vartheta$ as $n \longrightarrow \infty$. This implies the set

$$\begin{aligned} X = \big\{&i \in \mathbb{N} : \mathsf{P}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) \geq 1 - \epsilon \text{ or } \mathsf{Q}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) \leq \epsilon, \\ &\mathsf{P}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) \leq \epsilon\big\}. \end{aligned}$$

Since $\vartheta^n \longrightarrow \vartheta$ as $n \longrightarrow \infty$, by Theorem 4.4,

$$\begin{aligned} &\mathsf{P}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) \longrightarrow 1, \qquad \mathsf{Q}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) \longrightarrow 0 \quad \text{and} \\ &\mathsf{R}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) \longrightarrow 0 \quad \text{for all } t > 0 \text{ as } j \longrightarrow \infty. \end{aligned}$$

Hence for $i \in X$

$$\begin{aligned} \mathsf{P}\big(T_i(\vartheta) - T_i(y), \jmath + r\big) &\geq \lim_{i \to \infty} \mathsf{P}\big(T_i(\vartheta^n) - T_i(\vartheta), \jmath\big) * \mathsf{P}\big(T_i(\vartheta^n) - T_i(\vartheta), r\big) \\ &\geq 1 * (1 - \epsilon) \\ &= 1 - \epsilon, \end{aligned}$$

$$\begin{aligned}
\mathsf{Q}\big(T_i(\vartheta) - T_i(y), \jmath + r\big) &\leq \lim_{i\to\infty} \mathsf{Q}\big(T_i\big(\vartheta^n\big) - T_i(\vartheta), \jmath\big) \diamond \mathsf{Q}\big(T_i\big(\vartheta^n\big) - T_i(\vartheta), r\big) \\
&\leq 0 \diamond \epsilon \\
&= \epsilon
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{R}\big(T_i(\vartheta) - T_i(y), \jmath + r\big) &\leq \lim_{i\to\infty} \mathsf{R}\big(T_i\big(\vartheta^n\big) - T_i(\vartheta), \jmath\big) \diamond \mathsf{R}\big(T_i\big(\vartheta^n\big) - T_i(\vartheta), r\big) \\
&\leq 0 \diamond \epsilon \\
&= \epsilon.
\end{aligned}$$

In particular, for $k \in \mathbb{N}$, take $\jmath = \frac{1}{k}$. Then

$$\begin{aligned}
&\mathsf{P}\big(T_i(\vartheta) - T_i(y), r\big) = \lim_{k\to\infty} \mathsf{P}\left(T_i(\vartheta) - T_i(y), r + \frac{1}{k}\right) \geq 1 - \epsilon, \\
&\mathsf{Q}\big(T_i(\vartheta) - T_i(y), r\big) = \lim_{k\to\infty} \mathsf{Q}\left(T_i(\vartheta) - T_i(y), r + \frac{1}{k}\right) \leq \epsilon
\end{aligned}$$

and

$$\begin{aligned}
&\mathsf{R}\big(T_i(\vartheta) - T_i(y), r\big) = \lim_{k\to\infty} \mathsf{R}\left(T_i(\vartheta) - T_i(y), r + \frac{1}{k}\right) \leq \epsilon \\
\implies &\quad \big\{i \in \mathbb{N} : \mathsf{P}\big(T_i(\vartheta) - T_i(y), r\big) \geq 1 - \epsilon \text{ or } \mathsf{Q}\big(T_i(\vartheta) - T_i(y), r\big) \leq \epsilon, \\
&\quad \mathsf{R}\big(T_i(\vartheta) - T_i(y), r\big) \leq \epsilon\big\}. \\
\implies &\quad y \in \mathscr{B}_\vartheta[r, \epsilon](T).
\end{aligned}$$

Therefore $\mathscr{B}_\vartheta[r, \epsilon](T)$ is a closed set. □

Theorem 4.6 *Let $\vartheta = (\vartheta_k) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Then, for some $\beta \in Y$, $\vartheta_k \to \beta$ if and only if for every $\epsilon \in (0,1)$ and $\jmath > 0$ there exist positive integers $N = N(\vartheta, \epsilon, \jmath)$ such that*

$$\left\{N \in \mathbb{N} : \mathsf{P}\left(T_N(\vartheta) - \beta, \frac{\jmath}{2}\right) > 1 - \epsilon \textit{ or } \mathsf{Q}\left(T_N(\vartheta) - \beta, \frac{\jmath}{2}\right) < \epsilon, \mathsf{R}\left(T_N(\vartheta) - \beta, \frac{\jmath}{2}\right) < \epsilon\right\}.$$

Proof Suppose $\vartheta_k \to \beta$ for some $\beta \in Y$. For given $\epsilon \in (0,1)$, there exists $r \in (0,1)$ such that $(1-\epsilon) * (1-\epsilon) > 1 - r$ and $\epsilon \diamond \epsilon < r$. Since $\vartheta_k \to \beta$, for all $\jmath > 0$,

$$\begin{aligned}
\mathcal{S} = \Bigg\{&i \in \mathbb{N} : \mathsf{P}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2}\right) \leq 1 - \epsilon \text{ or } \mathsf{Q}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2}\right) \geq \epsilon, \\
&\mathsf{R}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2}\right) \geq \epsilon\Bigg\},
\end{aligned}$$

which implies that

$$\mathcal{S}^c = \left\{i \in \mathbb{N} : \mathsf{P}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2}\right) > 1 - \epsilon \text{ or } \mathsf{Q}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2}\right) < \epsilon, \mathsf{R}\left(T_i(\vartheta) - \beta, \frac{\jmath}{2}\right) < \epsilon\right\}.$$

Conversely, let us choose $N \in \mathcal{S}^c$. Then

$$\mathsf{P}\left(T_N(\vartheta) - \beta, \frac{J}{2}\right) > 1 - \epsilon \quad \text{or} \quad \mathsf{Q}\left(T_N(\vartheta) - \beta, \frac{J}{2}\right) < \epsilon, \qquad \mathsf{R}\left(T_N(\vartheta) - \beta, \frac{J}{2}\right) < \epsilon.$$

We show that there exists a positive integer $N = N(\vartheta, \epsilon, J)$ such that

$$\begin{aligned}&\big\{k \in \mathbb{N} : \mathsf{P}\big(T_k(\vartheta) - T_N(\vartheta), J\big) \le 1 - r \text{ or } \mathsf{Q}\big(T_k(\vartheta) - T_N(\vartheta), J\big) \ge r,\\ &\quad \mathsf{R}\big(T_k(\vartheta) - T_N(\vartheta), J\big) \ge r\big\}.\end{aligned}$$

So, for $\vartheta = (\vartheta_k) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$, define

$$\begin{aligned}\mathscr{C} = \big\{&k \in \mathbb{N} : \mathsf{P}\big(T_k(\vartheta) - T_N(\vartheta), J\big) \le 1 - r \text{ or } \mathsf{Q}\big(T_k(\vartheta) - T_N(\vartheta), J\big) \ge r,\\ &\mathsf{R}\big(T_k(\vartheta) - T_N(\vartheta), J\big) \ge r\big\}.\end{aligned}$$

We shall show that $\mathscr{C} \subseteq Y$. Let on the contrary $\mathscr{C} \nsubseteq Y$, i.e., there exists $m \in \mathscr{C}$ such that $m \notin Y$. Then

$$\mathsf{P}\big(T_m(\vartheta) - T_N(\vartheta), J\big) \le 1 - r \quad \text{or} \quad \mathsf{P}\left(T_m(\vartheta) - \beta, \frac{J}{2}\right) > 1 - \epsilon.$$

In particular,

$$\mathsf{P}\left(T_N(\vartheta) - \beta, \frac{J}{2}\right) > 1 - \epsilon.$$

Therefore we have

$$\begin{aligned}1 - r &\ge \mathsf{P}\big(T_m(\vartheta) - T_N(\vartheta), J\big)\\ &\ge \mathsf{P}\left(T_m(\vartheta) - \beta, \frac{J}{2}\right) * \mathsf{P}\left(T_N(\vartheta) - \beta, \frac{J}{2}\right)\\ &\ge (1 - \epsilon) * (1 - \epsilon)\\ &> 1 - r,\end{aligned}$$

which is a contradiction. Similarly,

$$\mathsf{Q}\big(T_m(\vartheta) - T_N(\vartheta), J\big) \ge r \quad \text{or} \quad \mathsf{Q}\left(T_m(\vartheta) - \beta, \frac{J}{2}\right) < \epsilon.$$

In particular,

$$\mathsf{Q}\left(T_N(\vartheta) - \beta, \frac{J}{2}\right) < \epsilon.$$

Therefore we have

$$r \le \mathsf{Q}\big(T_m(\vartheta) - T_N(\vartheta), J\big)$$

$$\begin{aligned}
&\le \mathsf{Q}\left(T_m(\vartheta)-\beta,\frac{J}{2}\right)\diamond \mathsf{Q}\left(T_N(\vartheta)-\beta,\frac{J}{2}\right)\\
&\le \epsilon\diamond\epsilon\\
&< r.
\end{aligned}$$

Similarly, in the other way,

$$\mathsf{R}\big(T_m(\vartheta)-T_N(\vartheta),J\big)\ge r \quad\text{or}\quad \mathsf{R}\left(T_m(\vartheta)-\beta,\frac{J}{2}\right)<\epsilon.$$

In particular,

$$\begin{aligned}
&\mathsf{R}\left(T_N(\vartheta)-\beta,\frac{J}{2}\right)<\epsilon,\\
r&\le \mathsf{R}\big(T_m(\vartheta)-T_N(\vartheta),J\big)\\
&\le \mathsf{R}\left(T_m(\vartheta)-\beta,\frac{J}{2}\right)\diamond \mathsf{R}\left(T_N(\vartheta)-\beta,\frac{J}{2}\right)\\
&\le \epsilon\diamond\epsilon\\
&< r,
\end{aligned}$$

which is again a contradiction.

Hence $\mathscr{C}\subseteq Y$ since $Y\in\mathcal{Z}$ implies $\mathscr{C}\in\mathcal{Z}$. □

Definition 4.7 Consider $D\subseteq\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Then D is compact if every open cover of D by the open set of $\tau_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ has a finite subcover.

Theorem 4.7 *Every finite subset D of $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ is compact.*

Proof Let $D=\{\vartheta_1,\vartheta_2,\vartheta_3,\dots,\vartheta_n\}$ be the finite subset of $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. For $r>0$ and $0<\epsilon<1$, let us assume that $\{\mathscr{B}_\vartheta(r,\epsilon)(T):\vartheta\in D\}$ is an open cover of D. Then $D\subseteq\bigcup_{\vartheta\in D}\mathscr{B}_\vartheta(r,\epsilon)(T)$.

Now, for all $\vartheta_i\in D$, $i=1,2,3,\dots n$, we have $\vartheta_i\in\bigcup_{\vartheta_i\in D}\mathscr{B}_{\vartheta_i}(r,\epsilon)(T)$. That implies $\vartheta_i\in\mathscr{B}_{\vartheta_i}(r,\epsilon)(T)$ for some $i\in\{1,2,3,\dots,n\}$. Then $\{\mathscr{B}_{\vartheta_i}(r,\epsilon)(T):i=1,2,3,\dots,n\}$ is a finite subcover of D.

Therefore D is compact. □

Theorem 4.8 *Let $D\subseteq\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Then D is compact iff every sequence in D has a convergent subsequence.*

Proof Suppose that D is a compact subset of $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Let $(\vartheta_k^n)=(\vartheta^n)_{n=1}^\infty$ be a sequence in D. For given $0<\epsilon<1$ and $r>0$, let $\{\mathscr{B}_\vartheta(\frac{r}{3},\epsilon)(T):\vartheta=(\vartheta_k)\in D\}$ be an open cover of S.

This implies $(\vartheta^n)\in\bigcup_{\vartheta\in D}\mathscr{B}_\vartheta(\frac{r}{3},\epsilon)(T)$. Then there exists some $\vartheta=(\vartheta_k)\in D$ such that $(\vartheta^n)\in\mathscr{B}_\vartheta(\frac{r}{3},\epsilon)(T)$. Therefore the set

$$Y_1=\left\{i\in\mathbb{N}:\mathsf{P}\left(T_i(\vartheta^n)-T_i(\vartheta),\frac{r}{3}\right)>1-\epsilon \text{ or } \mathsf{Q}\left(T_i(\vartheta^n)-T_i(\vartheta),\frac{r}{3}\right)<\epsilon,\right.$$

$$\mathsf{R}\left(T_i(\vartheta^n) - T_i(\vartheta), \frac{r}{3}\right) < \epsilon\Bigg\}.$$

Since D is compact, there exists a finite subcover $\{\mathscr{B}_{\vartheta_i}(\frac{r}{3}, \epsilon)(T) : \vartheta_i \in D \text{ and } i = 1, 2, 3, \ldots .m\}$ of D such that $D \subseteq \bigcup_{i=1}^{m} \mathscr{B}_{\vartheta_i}(\frac{r}{3}, \epsilon)(T)$.

Let (ϑ^{n_p}) be a subsequence of (ϑ^n). Then $(\vartheta^{n_p}) \in \bigcup_{i=1}^{m} \mathscr{B}_{\vartheta_i}(\frac{r}{3}, \epsilon)(T)$ implies $(\vartheta^{n_p}) \in \mathscr{B}_{\vartheta_i}(\frac{r}{3}, \epsilon)(T)$ for some $\vartheta_i \in D$. Therefore the set

$$\begin{aligned} Y_2 = \Bigg\{ & i \in \mathbb{N} : \mathsf{P}\left(T_i(\vartheta^{n_p}) - T_i(\vartheta_i), \frac{r}{3}\right) > 1 - \epsilon \text{ or } \mathsf{Q}\left(T_i(\vartheta^{n_p}) - T_i(\vartheta_i), \frac{r}{3}\right) < \epsilon, \\ & \mathsf{R}\left(T_i(\vartheta^{n_p}) - T_i(\vartheta_i), \frac{r}{3}\right) < \epsilon\Bigg\}. \end{aligned}$$

Now, for $k \in Y_1 \cap Y_2$,

$$\begin{aligned} & \mathsf{P}\big(T_i(\vartheta^{n_p}) - T_i(\vartheta), r\big) \\ & \quad \geq \mathsf{P}\left(T_i(\vartheta^{n_p}) - T_i(\vartheta_i), \frac{r}{3}\right) \\ & \qquad * \mathsf{P}\left(T_i(\vartheta^{n}) - T_i(\vartheta_i), \frac{r}{3}\right) \\ & \qquad * \mathsf{P}\left(T_i(\vartheta^{n}) - T_i(\vartheta), \frac{r}{3}\right) \\ & \quad > (1-\epsilon) * (1-\epsilon) * (1-\epsilon) \\ & \quad = (1-\epsilon), \\ & \quad \Longrightarrow \quad \mathsf{P}\big(T_i(\vartheta^{n_p}) - T_i(y), r\big) > 1 - \epsilon, \\ & \mathsf{Q}\big(T_i(\vartheta^{n_p}) - T_i(\vartheta), r\big) \\ & \quad \leq \mathsf{Q}\left(T_i(\vartheta^{n_p}) - T_i(\vartheta_i), \frac{r}{3}\right) \\ & \qquad \diamond \mathsf{Q}\left(T_i(\vartheta^{n}) - T_i(\vartheta_i), \frac{r}{3}\right) \\ & \qquad \diamond \mathsf{Q}\left(T_i(\vartheta^{n}) - T_i(\vartheta), \frac{r}{3}\right) \\ & \quad < \epsilon \diamond \epsilon \diamond \epsilon \\ & \quad = \epsilon, \end{aligned}$$

and

$$\begin{aligned} & \mathsf{R}\big(T_i(\vartheta^{n_p}) - T_i(\vartheta), r\big) \\ & \quad \leq \mathsf{R}\left(T_i(\vartheta^{n_p}) - T_i(\vartheta_i), \frac{r}{3}\right) \\ & \qquad \diamond \mathsf{R}\left(T_i(\vartheta^{n}) - T_i(\vartheta_i), \frac{r}{3}\right) \\ & \qquad \diamond \mathsf{R}\left(T_i(\vartheta^{n}) - T_i(\vartheta), \frac{r}{3}\right) \end{aligned}$$

$$\begin{aligned}
&< \epsilon \diamond \epsilon \diamond \epsilon \\
&= \epsilon.
\end{aligned}$$

Take $\epsilon = \frac{1}{p}$. Then

$$\begin{aligned}
&\lim_{p\to\infty} \mathsf{P}\big(T_i\big(\vartheta^{n_p}\big) - T_i(\vartheta), r\big) = \lim_{p\to\infty} 1 - \frac{1}{p} = 1, \\
&\lim_{p\to\infty} \mathsf{Q}\big(T_k\big(\vartheta^{n_p}\big) - T_k(\vartheta), r\big) = \lim_{p\to\infty} \frac{1}{p} = 0 \quad \text{and} \\
&\lim_{p\to\infty} \mathsf{R}\big(T_k\big(\vartheta^{n_p}\big) - T_k(\vartheta), r\big) = \lim_{p\to\infty} \frac{1}{p} = 0.
\end{aligned}$$

Hence, by Theorem (4.4), $\vartheta^{n_p} \to \vartheta$, as $p \to \infty$.

Conversely, suppose that (ϑ^{n_p}) is a subsequence of the sequence (ϑ^n) in D such that $(\vartheta^{n_p}) \to \vartheta$ in D. Let on the contrary D be not a compact subset of $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$.

Suppose that $\{\mathscr{B}_\vartheta(r,\epsilon)(T) : \vartheta \in D\}$ is an open cover of $D \implies D \subseteq \bigcup_{\vartheta\in d} \mathscr{B}_\vartheta(r,\epsilon)(T)$.

Therefore the set

$$\begin{aligned}
&\big\{i \in \mathbb{N} : \mathsf{P}\big(T_i\big(\vartheta^n\big) - T_i(\vartheta), r\big) > 1 - \epsilon \text{ or } \mathsf{Q}\big(T_i\big(\vartheta^n\big) - T_i(\vartheta), r\big) < \epsilon, \\
&\quad \mathsf{R}\big(T_i\big(\vartheta^n\big) - T_i(\vartheta), r\big) < \epsilon\big\}.
\end{aligned}$$

Since D is not compact, there exists a finite subcover $\{\mathscr{B}_{\vartheta_i}(r,\epsilon)(T) : \vartheta_i \in D, i = 1,2,3,\dots, m\}$ such that $D \nsubseteq \bigcup_{\vartheta_i\in D} \mathscr{B}_{\vartheta_i}(r,\epsilon)(T)$, which implies that the set

$$\begin{aligned}
&\big\{i \in \mathbb{N} : \mathsf{P}\big(T_i\big(\vartheta^{n_p}\big) - T_i(\vartheta_i), r\big) > 1 - \epsilon \text{ or } \mathsf{Q}\big(T_i\big(\vartheta^{n_p}\big) - T_i(\vartheta_i), r\big) < \epsilon, \\
&\qquad \mathsf{R}\big(T_i\big(\vartheta^{n_p}\big) - T_i(\vartheta_i), r\big) < \epsilon\big\} \\
&\implies \quad \big\{i \in \mathbb{N} : \mathsf{P}\big(T_i\big(\vartheta^{n_p}\big) - T_i(\vartheta), r\big) > 1 - \epsilon \text{ or } \mathsf{Q}\big(T_i\big(\vartheta^{n_p}\big) - T_i(\vartheta), r\big) < \epsilon, \\
&\qquad \mathsf{R}\big(T_i\big(\vartheta^{n_p}\big) - T_i(\vartheta_i), r\big) < \epsilon\big\} \\
&\implies \quad \text{For any } 0 < \epsilon < 1 \text{ and } r > 0, \big(\vartheta^{n_p}\big) \notin \mathscr{B}_\vartheta(r,\epsilon) \\
&\implies \quad \vartheta^{n_p} \nrightarrow \vartheta, \quad \text{which is a contradiction.}
\end{aligned}$$

Hence D is compact. □

Theorem 4.9 *Let D be the compact subset of $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ such that $\vartheta = (\vartheta_k) \notin D$. Then there exist two open sets U and V in $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ such that $D \subseteq V, \vartheta \in U$, and $U \cap V = \phi$.*

Proof Let D be a compact subset of $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ and $\vartheta \notin D$. Then, for any $s \in D$, we have $\vartheta \neq s$. Since $\mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ is a Hausdorff space, then for some $r > 0$ and $0 < \epsilon < 1$ there exist two open balls $U = \mathscr{B}_\vartheta(r,\epsilon)(T)$ and $V = \mathscr{B}_s(r,\epsilon)(T)$ such that $\vartheta \in U, s \in V$ and $U \cap V = \phi$.

Consider the open cover $V_s = \{\mathscr{B}_s(r,\epsilon)(T) : s \in D\}$ of D and D is compact.

Therefore there exists a finite subcover $V_{s_i} = \{\mathscr{B}_{s_i}(r,\epsilon)(T) : s_i \in D \text{ and } i = 1,2,3,\dots,n\}$ such that $D \subseteq \bigcup_{i=1}^n V_{s_i}$. Taking $V = \bigcap_{i=1}^n V_{s_i}$, we have $\vartheta \notin D$.

Hence U and V are open sets such that $D \subseteq V$ and $U \cap V = \phi$. □

Theorem 4.10 *Consider the NNS $\mathscr{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. Let $r > 0$ and $\epsilon_1, \epsilon_2 \in (0,1)$ such that $(1-\epsilon_1) * (1-\epsilon_1) \geq (1-\epsilon_2)$ and $\epsilon_1 \diamond \epsilon_1 \leq \epsilon_2$. Then, for any $\vartheta = (\vartheta_k) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$, $\overline{\mathscr{B}_\vartheta(\frac{r}{2}, \epsilon_1)(T)} \subseteq \mathscr{B}_\vartheta(r, \epsilon_2)(T)$.*

Proof Let $\ell = (\ell_k) \in \overline{\mathscr{B}_\vartheta(\frac{r}{2}, \epsilon_1)(T)}$ and $\mathscr{B}_\ell(\frac{r}{2}, \epsilon_1)(T)$ be an open ball with the center at ℓ and the radius ϵ_1. Hence $\mathscr{B}_\ell(\frac{r}{2}, \epsilon_1)(T) \cap \mathscr{B}_\vartheta(\frac{r}{2}, \epsilon_1)(T) \neq \phi$.

Suppose $z = (z_k) \in \mathscr{B}_\ell(\frac{r}{2}, \epsilon_1)(T) \cap \mathscr{B}_\vartheta(\frac{r}{2}, \epsilon_1)(T)$. Then the sets

$$\begin{aligned} Y_1 = \Bigg\{ & i \in \mathbb{N} : \mathsf{P}\left(T_i(\ell) - T_i(z), \frac{r}{2} \right) > 1 - \epsilon_1 \text{ or } \mathsf{Q}\left(T_i(\ell) - T_i(z), \frac{r}{2} \right) < \epsilon_1, \\ & \mathsf{R}\left(T_i(\ell) - T_i(z), \frac{r}{2} \right) < \epsilon_1 \Bigg\} \\ Y_2 = \Bigg\{ & i \in \mathbb{N} : \mathsf{P}\left(T_i(\vartheta) - T_i(z), \frac{r}{2} \right) > 1 - \epsilon_1 \text{ or } \mathsf{Q}\left(T_i(\vartheta) - T_i(z), \frac{r}{2} \right) < \epsilon_1, \\ & \mathsf{R}\left(T_i(\vartheta) - T_i(z), \frac{r}{2} \right) < \epsilon_1 \Bigg\} \end{aligned}$$

Consider $k \in Y_1 \cap Y_2$. Then

$$\begin{aligned} \mathsf{P}\big(T_k(\vartheta) - T_k(\ell), r\big) &\geq \mathsf{P}\left(T_k(\vartheta) - T_k(z), \frac{r}{2} \right) * \mathsf{P}\left(T_k(\ell) - T_k(z), \frac{r}{2} \right) \\ &> (1 - \epsilon_1) * (1 - \epsilon_1) \\ &\geq (1 - \epsilon_2) \\ \mathsf{Q}\big(T_k(\vartheta) - T_k(\ell), r\big) &\leq \mathsf{Q}\left(T_k(\vartheta) - T_k(z), \frac{r}{2} \right) \diamond \mathsf{Q}\left(T_k(\ell) - T_k(z), \frac{r}{2} \right) \\ &< \epsilon_1 \diamond \epsilon_1 \\ &\leq \epsilon_2 \end{aligned}$$

and

$$\begin{aligned} \mathsf{R}\big(T_k(\vartheta) - T_k(\ell), r\big) &\leq \mathsf{R}\left(T_k(\vartheta) - T_k(z), \frac{r}{2} \right) \diamond \mathsf{R}\left(T_k(\ell) - T_k(z), \frac{r}{2} \right) \\ &< \epsilon_1 \diamond \epsilon_1 \\ &\leq \epsilon_2. \end{aligned}$$

Therefore the set

$$\begin{aligned} & \big\{ i \in \mathbb{N} : \mathsf{P}\big(T_i(\vartheta) - T_i(\ell), r\big) > 1 - \epsilon_2 \text{ or } \mathsf{Q}\big(T_i(\vartheta) - T_i(\ell), r\big) < \epsilon_2, \mathsf{R}\big(T_i(\vartheta) - T_i(\ell), r\big) < \epsilon_2 \big\}. \\ & \quad \Longrightarrow \quad \ell = (\ell_k) \in \mathscr{B}_\vartheta(r, \epsilon_2)(T). \end{aligned}$$

Hence $\overline{\mathscr{B}_\vartheta(\frac{r}{2}, \epsilon_1)(T)} \subseteq \mathscr{B}_\vartheta(r, \epsilon_2)(T)$. □

Theorem 4.11 *Let $\vartheta = (\vartheta_k) \in \omega$. If there exists a sequence $y = (y_k) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$ such that $T_i(\vartheta) = T_i(y)$ for almost all i relative to neutrosophic-$\mathcal{Z}$, then $\vartheta \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$.*

Proof Suppose $T_i(\vartheta) = T_i(y)$ for almost all i relative to $\mathcal{Z}$. Then $\{i \in \mathbb{N} : T_i(\vartheta) \neq T_i(y)\} \in \mathcal{Z} \implies \{i \in \mathbb{N} : T_i(\vartheta) = T_i(y)\}$. Therefore, for all $\jmath > 0$,

$$\begin{aligned}
&\mathsf{P}\left(T_i(\vartheta) - T_i(y), \frac{\jmath}{2}\right) = 1 \quad \text{and} \quad \mathsf{Q}\left(T_i(\vartheta) - T(y), \frac{\jmath}{2}\right) = 0, \\
&\mathsf{R}\left(T_i(\vartheta) - T_i(y), \frac{\jmath}{2}\right) = 0.
\end{aligned}$$

Since $(y_k) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$, let $\lim(z_k) = \beta$. Then, for every $\epsilon \in (0,1)$ and $\jmath > 0$,

$$Y_1 = \left\{i \in \mathbb{N} : \mathsf{P}\left(T_i(y) - \beta, \frac{\jmath}{2}\right) > 1 - \epsilon \text{ or } \mathsf{Q}\left(T_i(y) - \beta, \frac{\jmath}{2}\right) < \epsilon, \mathsf{R}\left(T_i(y) - \beta, \frac{\jmath}{2}\right) < \epsilon\right\}.$$

Consider the set

$$Y_2 = \left\{i \in \mathbb{N} : \mathsf{P}\left(T_i(\vartheta) - \beta, \jmath\right) > 1 - \epsilon \text{ or } \mathsf{Q}\left(T_i(\vartheta) - \beta, \jmath\right) < \epsilon, \mathsf{R}\left(T_i(\vartheta) - \beta, \jmath\right) < \epsilon\right\}.$$

We show that $Y_1 \subset Y_2$. So for $j \in Y_1$ we have

$$\begin{aligned}
\mathsf{P}\left(T_i(\vartheta) - \beta, \jmath\right) &\geq \mathsf{P}\left(T_i(\vartheta) - T_i(y), \frac{\jmath}{2}\right) * \mathsf{P}\left(T_i(y) - \beta, \frac{\jmath}{2}\right) \\
&> 1 * (1 - \epsilon) \\
&= 1 - \epsilon, \\
\mathsf{Q}\left(T_i(\vartheta) - \beta, \jmath\right) &\leq \mathsf{P}\left(T_i(\vartheta) - T_i(y), \frac{\jmath}{2}\right) \diamond \mathsf{Q}\left(T_i(y) - \beta, \frac{\jmath}{2}\right) \\
&< 0 \diamond \epsilon \\
&= \epsilon,
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{R}\left(T_i(\vartheta) - \beta, \jmath\right) &\leq \mathsf{R}\left(T_i(\vartheta) - T_i(y), \frac{\jmath}{2}\right) \diamond \mathsf{R}\left(T_i(y) - \beta, \frac{\jmath}{2}\right) \\
&< 0 \diamond \epsilon \\
&= \epsilon
\end{aligned}$$

$\implies i \in Y_2$ and hence $Y_1 \subset Y_2$.

Hence $\vartheta = (\vartheta_k) \in \mathcal{L}_{(\mathsf{P},\mathsf{Q},\mathsf{R})}(T)$. □

5 Conclusion

Tribonacci numbers have been studied by several authors in the past who investigated Tribonacci identities, recurrence relations, and generalized Tribonacci numbers. However, in this paper, we focus on different directions by introducing a Tribonacci sequence space with the aid of a neutrosophic sequence space. We expect that our results might be a reference for further studies in this field. We have defined the Tribonacci matrix from neutrosophic convergence of sequence spaces and examine some topological and algebraic properties.

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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
VAK carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. MA and MDK participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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