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On some Volterra–Fredholm and Hermite–Hadamard-type fractional integral inequalities

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Abstract

The main aim of this paper is establishing some new Volterra–Fredholm and Hermite–Hadamard-type fractional integral inequalities, which can be used as auxiliary tools in the study of solutions to fractional differential equations and fractional integral equations. Applications are also given to explicate the availability of our results.

Keywords: Volterra–Fredholm inequalities; Hermite–Hadamard inequalities; Fractional integral inequalities

1 Introduction

The subject of fractional calculus has gained considerable popularity and importance over the past few decades, mainly due to its validated applications in various fields of science and engineering [1–4]. Integral inequalities, especially fractional integral inequalities, have been paid more and more attention in recent years. These inequalities play important roles in the study of fractional differential equations and fractional integral equations. At present, many scholars are devoted to studying various integral inequalities, such as Volterra–Fredholm and Hermite–Hadamard-type inequalities. In [5–10] the authors generalized and analyzed the Volterra–Fredholm-type and delay integral inequalities. In addition, some applications in fractional differential equations were presented to illustrate the validity of their outcomes. Convex functions have found an important place in modern mathematics, as they can be seen in a large number of research papers and books today. In this context, the Hermite–Hadamard inequality can be regarded as the first fundamental result for convex functions, which is defined over an interval of real numbers with natural geometric interpretation and many applications. In [11–20] a number of Hermite–Hadamard-type inequalities are deduced involving the classical and Riemann–Liouville fractional integrals for different classes of convex functions such as (s,m)-convex, m-convex, log-convex, and prequasi-invex functions. In this paper, we consider Volterra–Fredholm and Hermite–Hadamard-type inequalities involving fractional integrals.

The structure of this paper is as follows. The first part gives some preliminary results about fractional integrals, derivatives, and convex functions. In the second part, we derive some new nonlinear Volterra–Fredholm-type fractional integral inequalities on time scales for one- and two-variable functions. In the third part, we establish Hermite–Hadamard-type inequalities and some other integral inequalities for the Riemann–Liouville fractional integral. Finally, we give some concluding remarks.

2 Preliminaries

In this section, we recall several definitions needed for the discussion.

Definition 2.1 ([21–23]) The Riemann–Liouville integral of a function $f(x)$ of order $\alpha > 0$ is defined as

$$ {}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} f(u)\,\mathrm{d}u, \quad t > a > 0, \tag{2.1} $$

where Γ is the gamma function.

Definition 2.2 ([21–23]) The Riemann–Liouville derivative of a function $f(x)$ of order α is defined as

$$ \begin{aligned} {}_{\mathrm{RL}}\mathrm{D}_{a,t}^{\alpha} f(t) &= \frac{\mathrm{d}^n}{\mathrm{d}t^n}\big({}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-(n-\alpha)} f(t)\big) \\ &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t (t-u)^{n-\alpha-1} f(u)\,\mathrm{d}u, \quad t > a > 0, \end{aligned} \tag{2.2} $$

where $n-1 < \alpha < n \in \mathbb{Z}^+$.

Definition 2.3 ([21–23]) The Caputo derivative of a function $f(x)$ of order α is defined as

$$ \begin{aligned} {}_{C}\mathrm{D}_{a,t}^{\alpha} f(t) &= {}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-(n-\alpha)} f^{(n)}(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-u)^{n-\alpha-1} f^{(n)}(u)\,\mathrm{d}u, \quad t > a, \end{aligned} \tag{2.3} $$

where $n-1 < \alpha < n \in \mathbb{Z}^+$.

Definition 2.4 ([11, 12]) A function $f : [a,b] \subset \mathbb{R} \mapsto \mathbb{R}$ is said to be convex on $[a,b]$ if

$$ f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y), \tag{2.4} $$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$.

Definition 2.5 ([11]) A function $f : [a,b] \subset \mathbb{R} \mapsto \mathbb{R}$ is said to be (s,m)-convex with modulus $\mu \ge 0$ (in the second sense) if

$$ f\big(\lambda x + m(1-\lambda)y\big) \le \lambda^s f(x) + m(1-\lambda)^s f(y) - \mu t(1-t)(x-y)^2 \tag{2.5} $$

for all $x, y \in [a,b]$, $\lambda \in [0,1]$, and $s, m \in [0,1]$.

Definition 2.6 ([18, 24]) A function $f : [a,b] \subset \mathbb{R} \mapsto \mathbb{R}$ is said to be s-convex (in the second sense) if

$$f\big(\lambda x + (1-\lambda)y\big) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{2.6}$$

for all $x, y \in [a,b]$, $\lambda \in [0,1]$, and $s \in [0,1]$.

Definition 2.7 ([25]) Let $f : [a,b] \subset \mathbb{R} \mapsto \mathbb{R}$ be a convex function. Then the Hermite–Hadamard inequality is given by

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \le \frac{f(a)+f(b)}{2}. \tag{2.7}$$

3 Nonlinear Volterra–Fredholm-type fractional integral inequalities

In this section, we show and prove certain Riemann–Liouville fractional integral inequalities of nonlinear Volterra–Fredholm-type by amplification, differentiation, integration, and inverse functions.

In the following discussion, we assume that

1. $x(t), f(t), c(t), g_1(t), g_2(t), r(t) \in C([a,b], \mathbb{R}_+)$ with $r(t) \le t$,
2. $h(t) \in C(\mathbb{R}_+, \mathbb{R}_+)$ is a nondecreasing function with $h(t) \ge 1$,
3. $Q(v) = \int_0^v \frac{\mathrm{d}r}{h(r)}$ for $v \ge 0$.

However, for fractional order α, we only consider the case $1 < \alpha < 2$.

Theorem 3.1 *If*

$$x(t) \le (t-a)c(t) + \int_a^t (t-u)^{\alpha-1} f(u) h\big(x(u)\big)\,\mathrm{d}u, \tag{3.1}$$

then

$$x(t) \le \left(\frac{t-a}{b-a}\right)^{\alpha-1} Q^{-1}\left[(b-a)^{\alpha-1}(t-a)^{2-\alpha}c(t) + (b-a)^{\alpha-1}\int_a^t f(s)\,\mathrm{d}s\right]. \tag{3.2}$$

Proof From (3.1) it follows that

$$\begin{aligned}(t-a)^{1-\alpha}x(t) &\le (t-a)^{2-\alpha}c(t) + (t-a)^{1-\alpha}\int_a^t (t-u)^{\alpha-1} f(u)h\big(x(u)\big)\,\mathrm{d}u \\ &\le (t-a)^{2-\alpha}c(t) + \int_a^t f(u)h\big(x(u)\big)\,\mathrm{d}u.\end{aligned} \tag{3.3}$$

For $t \in [a,b]$, defining the function $y(t)$ by the right-hand side of (3.3), we have

$$\begin{aligned}y'(t) &= \big[(t-a)^{2-\alpha}c(t)\big]' + f(t)h\big(x(t)\big) \\ &\le \big[(t-a)^{2-\alpha}c(t)\big]' + f(t)h\big((t-a)^{\alpha-1}y(t)\big) \\ &\le \big[(t-a)^{2-\alpha}c(t)\big]' + f(t)h\big((b-a)^{\alpha-1}y(t)\big),\end{aligned}$$

which implies

$$\begin{aligned}\frac{y'(t)}{h((b-a)^{\alpha-1}y(t))} &\le \frac{[(t-a)^{2-\alpha}c(t)]'}{h((b-a)^{\alpha-1}y(t))} + f(t)\\ &\le \big[(t-a)^{2-\alpha}c(t)\big]' + f(t).\end{aligned} \tag{3.4}$$

Multiplying both sides of (3.4) by $(b-a)^{\alpha-1}$, we get

$$\frac{(b-a)^{\alpha-1}y'(t)}{h((b-a)^{\alpha-1}y(t))} \le (b-a)^{\alpha-1}\big[(t-a)^{2-\alpha}c(t)\big]' + (b-a)^{\alpha-1}f(t). \tag{3.5}$$

Setting $t = s$ and integrating both sides of (3.5) over $[a,t]$, we find

$$Q\big((b-a)^{\alpha-1}y(t)\big) \le (b-a)^{\alpha-1}(t-a)^{2-\alpha}c(t) + (b-a)^{\alpha-1}\int_a^t f(s)\,\mathrm{d}s.$$

In fact, Q^{-1} is nondecreasing, and $y(a) = 0$. We can deduce that

$$y(t) \le (b-a)^{1-\alpha}Q^{-1}\left[(b-a)^{\alpha-1}(t-a)^{2-\alpha}c(t) + (b-a)^{\alpha-1}\int_a^t f(s)\,\mathrm{d}s\right]. \tag{3.6}$$

Using (3.3) and (3.6), we can derive the desired inequality (3.2). This ends the proof. □

Theorem 3.2 *If* $x(t)$ *satisfies*

$$\begin{aligned}x(t) \le{}& (t-a)c(t) + \int_a^t (t-u)^{\alpha-1}g_1(u)f(u)h\big(x(u)\big)\,\mathrm{d}u\\ &+ \int_a^t (t-u)^{\alpha-1}g_1(u)\int_a^u (u-v)^{\alpha-1}g_2(v)h\big(x(v)\big)\,\mathrm{d}v\,\mathrm{d}u,\end{aligned} \tag{3.7}$$

then

$$\begin{aligned}x(t) \le{}& \left(\frac{t-a}{b-a}\right)^{\alpha-1} Q^{-1}\bigg\{(b-a)^{\alpha-1}(t-a)^{2-\alpha}c(t) + (b-a)^{\alpha-1}\\ &\times \int_a^t g_1(s)\left[f(s) + 2^{2-\alpha}t^{\alpha-1}\int_a^s g_2(v)\,\mathrm{d}v - \int_a^s v^{\alpha-1}g_2(v)\,\mathrm{d}v\right]\mathrm{d}s\bigg\}.\end{aligned} \tag{3.8}$$

Proof Simplifying (3.7), we can easily get that

$$\begin{aligned}(t-a)^{1-\alpha}x(t) \le{}& (t-a)^{2-\alpha}c(t) + \int_a^t g_1(u)f(u)h\big(x(u)\big)\,\mathrm{d}u\\ &+ \int_a^t g_1(u)\int_a^u (u-v)^{\alpha-1}g_2(v)h\big(x(v)\big)\,\mathrm{d}v\,\mathrm{d}u.\end{aligned} \tag{3.9}$$

It is known that $(u-v)^{\alpha-1} \le 2^{2-\alpha}u^{\alpha-1} - v^{\alpha-1}$. Therefore

$$\begin{aligned}(t-a)^{1-\alpha}x(t) \le{}& (t-a)^{2-\alpha}c(t) + \int_a^t g_1(u)f(u)h\big(x(u)\big)\,\mathrm{d}u\\ &+ \int_a^t g_1(u)\int_a^u \big(2^{2-\alpha}u^{\alpha-1} - v^{\alpha-1}\big)g_2(v)h\big(x(v)\big)\,\mathrm{d}v\,\mathrm{d}u.\end{aligned} \tag{3.10}$$

For the convenience of calculation, denote the right-hand side of (3.10) as $y(t)$. Then

$$\begin{aligned} y'(t) &= \left[(t-a)^{2-\alpha}c(t)\right]' - g_1(t)\int_a^t v^{\alpha-1}g_2(v)h\bigl(x(v)\bigr)\,\mathrm{d}v \\ &\quad + g_1(t)\left[f(t)h\bigl(x(t)\bigr) + 2^{2-\alpha}t^{\alpha-1}\int_a^t g_2(v)h\bigl(x(v)\bigr)\,\mathrm{d}v\right] \\ &\leq \left[(t-a)^{2-\alpha}c(t)\right]' + g_1(t)h\bigl((t-a)^{\alpha-1}y(t)\bigr)\left[f(t) + 2^{2-\alpha}t^{\alpha-1}\int_a^t g_2(v)\,\mathrm{d}v\right] \\ &\quad - g_1(t)h\bigl((t-a)^{\alpha-1}y(t)\bigr)\int_a^t v^{\alpha-1}g_2(v)\,\mathrm{d}v. \end{aligned} \tag{3.11}$$

Hence

$$\begin{aligned} y'(t) &\leq \left[(t-a)^{2-\alpha}c(t)\right]' + g_1(t)h\bigl((b-a)^{\alpha-1}y(t)\bigr)\left[f(t) + 2^{2-\alpha}t^{\alpha-1}\int_a^t g_2(v)\,\mathrm{d}v\right] \\ &\quad - g_1(t)h\bigl((t-a)^{\alpha-1}y(t)\bigr)\int_a^t v^{\alpha-1}g_2(v)\,\mathrm{d}v. \end{aligned} \tag{3.12}$$

By the same steps from (3.4)–(3.6), as in the proof of Theorem 3.1, we have

$$\begin{aligned} y(t) &\leq (b-a)^{1-\alpha}Q^{-1}\biggl\{(t-a)^{2-\alpha}c(t) + (b-a)^{\alpha-1}\int_a^t g_1(s)f(s) + 2^{2-\alpha}t^{\alpha-1} \\ &\quad \times \left[\int_a^s g_2(v)\,\mathrm{d}v - \int_a^s v^{\alpha-1}g_2(v)\,\mathrm{d}v\right]\mathrm{d}s\biggr\}. \end{aligned} \tag{3.13}$$

Combining (3.10) and (3.13), we can easily find (3.8). The proof is completed. □

Theorem 3.3 *Let $S(t) = Q(2t + c(b) - 2c(a)) - Q(t)$ be a nondecreasing function. If there is a function $x(t)$ such that*

$$\begin{aligned} x(t) &\leq c(t) + \int_{r(a)}^{r(t)} \bigl(r(t)-u\bigr)^{\alpha-1}g_1(u)f(u)h\bigl(x(u)\bigr)\,\mathrm{d}u \\ &\quad + \int_{r(a)}^{r(b)} \bigl(r(b)-u\bigr)^{\alpha-1}g_1(u)f(u)h\bigl(x(u)\bigr)\,\mathrm{d}u \\ &\quad + \int_{r(a)}^{r(t)} \bigl(r(t)-u\bigr)^{\alpha-1}g_1(u)\int_{r(a)}^{u}(u-v)^{\alpha-1}g_2(v)h\bigl(x(v)\bigr)\,\mathrm{d}v\,\mathrm{d}u \\ &\quad + \int_{r(a)}^{r(b)} \bigl(r(b)-u\bigr)^{\alpha-1}g_1(u)\int_{r(a)}^{u}(u-v)^{\alpha-1}g_2(v)h\bigl(x(v)\bigr)\,\mathrm{d}v\,\mathrm{d}u, \end{aligned} \tag{3.14}$$

then

$$\begin{aligned} x(t) &\leq Q^{-1}\biggl\{c(t) - c(a) + Q\biggl[S^{-1}\biggl(c(b) - c(a) + \bigl(r(b)-r(a)\bigr)^{\alpha-1} \\ &\quad \times \int_{r(a)}^{r(b)} g_1(s)\biggl(f(s) + \int_{r(a)}^{s}\bigl(2^{2-\alpha}r^{\alpha-1}(s) - v^{\alpha-1}\bigr)g_2(v)\,\mathrm{d}v\biggr)\,\mathrm{d}s\biggr)\biggr] \end{aligned} \tag{3.15}$$

$$+ \big(r(b) - r(a)\big)^{\alpha-1} \int_{r(a)}^{r(t)} g_1(s) f(s)\,\mathrm{d}s + \big(r(b) - r(a)\big)^{\alpha-1} \\ \times \int_{r(a)}^{r(t)} g_1(s) \int_{r(a)}^{s} \big(2^{2-\alpha} r^{\alpha-1}(s) - \nu^{\alpha-1}\big) g_2(\nu)\,\mathrm{d}\nu\,\mathrm{d}s \Bigg\}.$$

Proof According to (3.14), we have

$$\begin{aligned} x(t) \le c(t) &+ \big(r(b) - r(a)\big)^{\alpha-1} \int_{r(a)}^{r(t)} g_1(u) f(u) h\big(x(u)\big)\,\mathrm{d}u \\ &+ \big(r(b) - r(a)\big)^{\alpha-1} \int_{r(a)}^{r(b)} g_1(u) f(u) h\big(x(u)\big)\,\mathrm{d}u \\ &+ \big(r(b) - r(a)\big)^{\alpha-1} \int_{r(a)}^{r(t)} g_1(u) \int_{r(a)}^{u} (u-\nu)^{\alpha-1} g_2(\nu) h(x(\nu)\,\mathrm{d}\nu\,\mathrm{d}u \\ &+ \big(r(b) - r(a)\big)^{\alpha-1} \int_{r(a)}^{r(b)} g_1(u) \int_{r(a)}^{u} (u-\nu)^{\alpha-1} g_2(\nu) h\big(x(\nu)\big)\,\mathrm{d}\nu\,\mathrm{d}u. \end{aligned} \tag{3.16}$$

Denote the right-hand side of (3.16) as $y(t)$. Inspired by (3.9)–(3.11), $y(t)$ has the following estimate:

$$\begin{aligned} y(t) \le Q^{-1}\Bigg\{ & Q\big(y(a)\big) + c(t) - c(a) + \big(r(b) - r(a)\big)^{\alpha-1} \\ & \times \int_{r(a)}^{r(t)} g_1(s) \bigg[f(s) + \int_{r(a)}^{s} \big(2^{2-\alpha} r^{\alpha-1}(s) - \nu^{\alpha-1}\big) g_2(\nu)\,\mathrm{d}\nu \bigg] \mathrm{d}s \Bigg\}. \end{aligned} \tag{3.17}$$

By the definition of $y(t)$ we get

$$\begin{aligned} y(b) &= c(b) + 2\big(r(b) - r(a)\big)^{\alpha-1} \int_{r(a)}^{r(b)} g_1(u) f(u) h\big(x(u)\big)\,\mathrm{d}u \\ &\quad + 2\big(r(b) - r(a)\big)^{\alpha-1} \int_{r(a)}^{r(b)} g_1(u) \int_{r(a)}^{u} (u-\nu)^{\alpha-1} g_2(\nu) h\big(x(\nu)\big)\,\mathrm{d}\nu\,\mathrm{d}u \\ &= c(b) + 2\big(y(a) - c(a)\big). \end{aligned} \tag{3.18}$$

According to (3.17), we have

$$\begin{aligned} Q\big(y(b)\big) \le Q\big(y(a)\big) &+ c(b) - c(a) + \big(r(b) - r(a)\big)^{\alpha-1} \\ & \times \int_{r(a)}^{r(b)} g_1(s) \bigg[f(s) + \int_{r(a)}^{s} \big(2^{2-\alpha} r^{\alpha-1}(s) - \nu^{\alpha-1}\big) g_2(\nu)\,\mathrm{d}\nu \bigg] \mathrm{d}s. \end{aligned} \tag{3.19}$$

Thus

$$\begin{aligned} S\big(y(a)\big) &= Q\big(2y(a) + c(b) - 2c(a)\big) - Q\big(y(a)\big) \\ &\le c(b) - c(a) + \big(r(b) - r(a)\big)^{\alpha-1} \\ &\quad \times \int_{r(a)}^{r(b)} g_1(s) \bigg[f(s) + \int_{r(a)}^{s} \big(2^{2-\alpha} r^{\alpha-1}(s) - \nu^{\alpha-1}\big) g_2(\nu)\,\mathrm{d}\nu \bigg] \mathrm{d}s. \end{aligned} \tag{3.20}$$

Since Q and S are nondecreasing, we have

$$\begin{aligned} &Q\big(y(a)\big) \\ &\quad\leq Q\bigg\{S^{-1}\bigg[c(b)-c(a)+\big(r(b)-r(a)\big)^{\alpha-1} \\ &\qquad\times\int_{r(a)}^{r(b)} g_1(s)\bigg(f(s)+\int_{r(a)}^{s}\big(2^{2-\alpha}r^{\alpha-1}(s)-v^{\alpha-1}\big)g_2(v)\,\mathrm{d}v\bigg)\,\mathrm{d}s\bigg]\bigg\}. \end{aligned} \tag{3.21}$$

Using (3.17) and (3.21), it follows that

$$\begin{aligned} y(t)\leq{}& Q^{-1}\bigg\{c(t)-c(a)+Q\bigg[S^{-1}\bigg(c(b)-c(a)+\big(r(b)-r(a)\big)^{\alpha-1} \\ &\times\int_{r(a)}^{r(b)} g_1(s)\bigg(f(s)+\int_{r(a)}^{s}\big(2^{2-\alpha}r^{\alpha-1}(s)-v^{\alpha-1}\big)g_2(v)\,\mathrm{d}v\bigg)\,\mathrm{d}s\bigg)\bigg] \\ &+\big(r(b)-r(a)\big)^{\alpha-1}\int_{r(a)}^{s}\big(2^{2-\alpha}r^{\alpha-1}(s)-v^{\alpha-1}\big)g_2(v)\,\mathrm{d}v\,\mathrm{d}s \\ &+\big(r(b)-r(a)\big)^{\alpha-1}\int_{r(a)}^{r(t)} g_1(s)f(s)\,\mathrm{d}s\bigg\}. \end{aligned} \tag{3.22}$$

From (3.16) and (3.22) the expected result follows. □

Similarly to the above case with single-variable functions, we will consider bivariate functions.

Let $I_1=[u_0,u_T]$ and $I_2=[v_0,v_T]$ with $u_0,v_0\geq 0$. Assume that:

1. $x(u,v), f(u,v), c(u,v), g_1(u,v), g_2(u,v)\in C(I_1\times I_2,\mathbb{R}_+)$,
2. $r_1(u)\in C(I_1,\mathbb{R}_+)$ and $r_2(v)\in C(I_2,\mathbb{R}_+)$ with $r_1(u)\leq u$, $r_2(v)\leq v$,
3. $h(t)\in C(\mathbb{R}_+,\mathbb{R}_+)$ is a nondecreasing function with $h(t)\geq 1$,
4. $Q(v)=\int_0^v\frac{\mathrm{d}r}{h(r)}$ for $v\geq 0$.

Under such conditions, we state the following theorem.

Theorem 3.4 *Suppose that $x(u,v)$ satisfies the following inequality*:

$$\begin{aligned} &x(u,v) \\ &\quad\leq\big(r_1(u)-r_1(u_0)\big)\big(r_2(v)-r_2(v_0)\big)c(u,v) \\ &\qquad+\int_{r_1(u_0)}^{r_1(u)}\int_{r_2(v_0)}^{r_2(v)}\big(r_1(u)-s\big)^{\alpha-1}\big(r_2(v)-\tau\big)^{\alpha-1}g_1(s,\tau)\bigg[f(s,\tau)h\big(x(s,\tau)\big) \\ &\qquad+\int_{r_1(u_0)}^{s}\int_{r_2(v_0)}^{\tau}(s-w)^{\alpha-1}(\tau-z)^{\alpha-1}g_2(w,z)h\big(x(w,z)\big)\,\mathrm{d}w\,\mathrm{d}z\bigg]\,\mathrm{d}s\,\mathrm{d}\tau \end{aligned} \tag{3.23}$$

for $u\in I_1$, $v\in I_2$. Then

$$\begin{aligned} x(u,v)\leq{}&\frac{1}{k}\big(u-r_1(u_0)\big)^{\alpha-1}\big(v-r_2(v_0)\big)^{\alpha-1} \\ &\times Q^{-1}\Big\{k\big(u-r_1(u_0)\big)^{2-\alpha}\big(v-r_2(v_0)\big)^{2-\alpha}c(u,v)-k_0kc(u_0,v) \end{aligned} \tag{3.24}$$

$$+ k\int_{r_1(u_0)}^{r_1(u)}\left[\int_{r_2(v_0)}^{r_2(v)} g_1(s,\tau)\left(f(s,\tau)+\int_{r_1(u_0)}^{s}\int_{r_2(v_0)}^{\tau} l(w,z,s,\tau)\right.\right.$$
$$\left.\left.\times g_2(w,z)\,\mathrm{d}w\,\mathrm{d}z\right)\mathrm{d}\tau\right]\mathrm{d}s\Bigg\},$$

where $l(w,z,s,\tau) = (2^{2-\alpha}s^{\alpha-1} - w^{\alpha-1})(2^{2-\alpha}\tau^{\alpha-1} - z^{\alpha-1})$. *The constants* k_0 *and* k *are defined by* $k_0 = (u - r_1(u_0))^{2-\alpha}(v - r_2(v_0))^{2-\alpha}$ *and* $k = (u_T - r_1(u_0))^{\alpha-1}(v_T - r_2(v_0))^{\alpha-1}$, *respectively.*

Proof From (3.23) we easily derive that

$$\bigl(u - r_1(u_0)\bigr)^{1-\alpha}\bigl(v - r_2(v_0)\bigr)^{1-\alpha}x(u,v) \le y(u,v), \tag{3.25}$$

where

$$\begin{aligned}
&y(u,v)\\
&\quad= \bigl(u - r_1(u_0)\bigr)^{2-\alpha}\bigl(v - r_2(v_0)\bigr)^{2-\alpha}c(u,v)\\
&\qquad+ \int_{r_1(u_0)}^{r_1(u)}\int_{r_2(v_0)}^{r_2(v)} g_1(s,\tau)f(s,\tau)h\bigl(x(s,\tau)\bigr)\,\mathrm{d}s\,\mathrm{d}\tau\\
&\qquad+ \int_{r_1(u_0)}^{r_1(u)}\int_{r_2(v_0)}^{r_2(v)} g_1(s,\tau)\int_{r_1(u_0)}^{s}\int_{r_2(v_0)}^{\tau} l(w,z,s,\tau)g_2(w,z)h\bigl(x(w,z)\bigr)\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}s\,\mathrm{d}\tau.
\end{aligned} \tag{3.26}$$

Taking the partial derivative of $y(u,v)$ with respect to u, we have

$$\begin{aligned}
\frac{\partial y(u,v)}{\partial u} &= \frac{\partial (u - r_1(u_0)^{2-\alpha}(v - r_2(v_0))^{2-\alpha}c(u,v))}{\partial u}\\
&\quad+ r_1'(u)\int_{r_2(v_0)}^{r_2(v)} g_1\bigl(r_1(u),\tau\bigr)f\bigl(r_1(u),\tau\bigr)h\bigl(x\bigl(r_1(u),\tau\bigr)\bigr)\,\mathrm{d}\tau\\
&\quad+ r_1'(u)\int_{r_2(v_0)}^{r_2(v)} g_1\bigl(r_1(u),\tau\bigr)\int_{r_1(u_0)}^{r_1(u)}\int_{r_2(v_0)}^{\tau} l(w,z,s,\tau)\\
&\quad\times g_2(w,z)h\bigl(x(w,z)\bigr)\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}\tau\\
&\le \frac{\partial (u - r_1(u_0))^{2-\alpha}(v - r_2(v_0))^{2-\alpha}c(u,v))}{\partial u}\\
&\quad+ r_1'(u)h\bigl(\bigl(u_T - r_1(u_0)\bigr)^{\alpha-1}\bigl(v_T - r_2(v_0)\bigr)^{\alpha-1}y(u,v)\bigr)\\
&\quad\times \int_{r_2(v_0)}^{r_2(v)} g_1\bigl(r_1(u),\tau\bigr)\left[f\bigl(r_1(u),\tau\bigr) + \int_{r_1(u_0)}^{r_1(u)}\int_{r_2(v_0)}^{\tau} l(w,z,s,\tau)g_2(w,z)\,\mathrm{d}w\,\mathrm{d}z\right]\mathrm{d}\tau.
\end{aligned} \tag{3.27}$$

Through a series of calculations, we get

$$\begin{aligned}
\frac{k\partial y(u,v)}{h(ky(u,v))\partial u} &\le \frac{k\partial((u - r_1(u_0))^{2-\alpha}(v - r_2(v_0))^{2-\alpha}c(u,v))}{\partial u}\\
&\quad+ kr_1'(u)\int_{r_2(v_0)}^{r_2(v)} g_1\bigl(r_1(u),\tau\bigr)f\bigl(r_1(u),\tau\bigr)\,\mathrm{d}\tau\\
&\quad+ kr_1'(u)\int_{r_1(u_0)}^{r_1(u)}\int_{r_2(v_0)}^{\tau} l(w,z,s,\tau)g_2(w,z)\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}\tau,
\end{aligned} \tag{3.28}$$

where

$$k = \left(u_T - r_1(u_0)\right)^{\alpha-1}\left(v_T - r_2(v_0)\right)^{\alpha-1}.$$

Integrating both sides of (3.28) with respect to t over $[u_0, u]$ yields the relation

$$\begin{aligned} &Q\left(ky(u,v)\right) \\ &\quad\le k\left(u - r_1(u_0)\right)^{2-\alpha}\left(v - r_2(v_0)\right)^{2-\alpha} c(u,v) \\ &\qquad - k_0 kc(u_0, v) + k\int_{r_1(u_0)}^{r_1(u)}\int_{r_2(v_0)}^{r_2(v)} g_1(s,\tau)f(s,\tau)\,\mathrm{d}\tau\,\mathrm{d}s \\ &\qquad + k\int_{r_1(u_0)}^{r_1(u)}\int_{r_2(v_0)}^{r_2(v)} g_1(s,\tau)\int_{r_1(u_0)}^{s}\int_{r_2(v_0)}^{\tau} l(w,z,s,\tau)g_2(w,z)\,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}\tau\,\mathrm{d}s, \end{aligned} \tag{3.29}$$

where

$$k_0 = \left(u - r_1(u_0)\right)^{2-\alpha}\left(v - r_2(v_0)\right)^{2-\alpha}.$$

Since Q^{-1} is an increasing function, in the light of (3.25) and (3.29), we observe that (3.24) holds. The theorem is proved. □

To illustrate our results, the following Volterra–Fredholm fractional integral equations for one and two variables are separately considered in Corollaries 3.1–3.3:

$$x(t) = A(t) + \int_a^t (t-u)^{\alpha-1} g_1(u)\left[f(u)x(u) + \int_a^u (u-v)^{\alpha-1} g_2(v)x(v)\,\mathrm{d}v\right]\mathrm{d}u, \tag{3.30}$$

$$\begin{aligned} x(t) &= A(t) + \int_{r(a)}^{r(t)} \left(r(t)-u\right)^{\alpha-1} g_1(u)f(u)\sum_{i=1}^{i=n}\gamma_i x^{\beta_i}(u)\,\mathrm{d}u \\ &\quad + \int_{r(a)}^{r(t)} \left(r(t)-u\right)^{\alpha-1} g_1(u)\int_{r(a)}^{u}(u-v)^{\alpha-1}g_2(v)\sum_{i=1}^{i=n}\gamma_i x^{\beta_i}(v)\,\mathrm{d}v\,\mathrm{d}u, \quad \beta_i > 0, \end{aligned} \tag{3.31}$$

and

$$x(u,v) = C(u,v) + \int_{u_0}^{u}\int_{v_0}^{v}(u-s)^{\alpha-1}(v-\tau)^{\alpha-1}F\left(s,\tau,x(s,\tau),G(s,\tau)\right)\mathrm{d}s\,\mathrm{d}\tau, \tag{3.32}$$

where $G(s,\tau) = \int_{u_0}^{s}\int_{v_0}^{\tau}(s-w)^{\alpha-1}(\tau-z)^{\alpha-1}x^p(w,z)\,\mathrm{d}w\,\mathrm{d}z$, $0 < p < 1$.

Corollary 3.1 *Suppose that $x(t)$ satisfies*

$$\begin{aligned} &x(t) \\ &\quad\le c(t) + \int_{r(a)}^{r(t)}\left(r(t)-u\right)^{\alpha-1}g_1(u)\left[f(u)x(u) + \int_{r(a)}^{u}(u-v)^{\alpha-1}g_2(v)x(v)\,\mathrm{d}v\right]\mathrm{d}u \\ &\qquad + \int_{r(a)}^{r(b)}\left(r(b)-u\right)^{\alpha-1}g_1(u)\left[f(u)x(u) + \int_{r(a)}^{u}(u-v)^{\alpha-1}g_2(v)x(v)\,\mathrm{d}v\right]\mathrm{d}u \end{aligned} \tag{3.33}$$

for $c(b) \le 2c(a) + 1$. *Then we can get an explicit estimation of* $x(t)$ *in* (3.30):

$$x(t) \le \frac{c(b) - (2c(a)+1)}{M-2}\left\{\exp\left[c(t) - c(a) + \left(r(b) - r(a)\right)^{\alpha-1} \times \int_{r(a)}^{r(t)} g_1(s)\left(f(s) + \int_{r(a)}^{s} \left(2^{2-\alpha} r^{\alpha-1}(s) - \nu^{\alpha-1}\right) g_2(\nu)\,\mathrm{d}\nu\right)\mathrm{d}s\right] - 1\right\}, \tag{3.34}$$

where

$$M = \exp\left\{c(b) - c(a) + \left(r(b) - r(a)\right)^{\alpha-1} \int_{r(a)}^{r(b)} g_1(s) \times \left[f(s) + \int_{r(a)}^{s} \left(2^{2-\alpha} r^{\alpha-1}(s) - \nu^{\alpha-1}\right) g_2(\nu)\,\mathrm{d}\nu\right]\mathrm{d}s\right\}.$$

Proof Since $x(t) \le x(t) + 1$, using $h(t) = t + 1$ in Theorem 3.3, we can get that $Q(\nu) = \log(\nu + 1)$, $Q^{-1}(t) = \exp(t) - 1$, $S(t) = \log(2t + c(b) - 2c(a) + 1) - \log(t+1)$, and $S^{-1}(t) = \frac{c(b)-(2c(a)+1)}{\exp(t)-2} - 1$. So (3.34) can be easily proved. □

Corollary 3.2 *If* $r(t)$, a, b, *and* $x(t)$ *in* (3.31) *meet the* $r(t) \le t$, $1 \le a, b \le \log(2) + 2c(a) - c(b)$, *and*

$$\begin{aligned} x(t) \le{}& c(t) + \int_{r(a)}^{r(t)} \left(r(t) - u\right)^{\alpha-1} g_1(u) f(u) \sum_{i=1}^{i=n} \gamma_i x^{\beta_i}(u)\,\mathrm{d}u \\ &+ \int_{r(a)}^{r(b)} \left(r(b) - u\right)^{\alpha-1} g_1(u) f(u) \sum_{i=1}^{i=n} \gamma_i x^{\beta_i}(u)\,\mathrm{d}u \\ &+ \int_{r(a)}^{r(t)} \left(r(t) - u\right)^{\alpha-1} g_1(u) \int_{r(a)}^{u} (u-\nu)^{\alpha-1} g_2(\nu) \sum_{i=1}^{i=n} \gamma_i x^{\beta_i}(\nu)\,\mathrm{d}\nu\,\mathrm{d}u \\ &+ \int_{r(a)}^{r(b)} \left(r(b) - u\right)^{\alpha-1} g_1(u) \int_{r(a)}^{u} (u-\nu)^{\alpha-1} g_2(\nu) \sum_{i=1}^{i=n} \gamma_i x^{\beta_i}(\nu)\,\mathrm{d}\nu\,\mathrm{d}u, \end{aligned} \tag{3.35}$$

then

$$x(t) \le \log\left(\frac{1}{1 - B - K(t)}\right), \tag{3.36}$$

where

$$K(t) = c(t) - c(a) + \left(\sum_{i=1}^{i=n} \gamma_i \beta_i^{\beta_i}\right)\left(r(b) - r(a)\right)^{\alpha-1} \int_{r(a)}^{r(t)} g_1(s) \times \left[f(s) + \int_{r(a)}^{s} \left(2^{2-\alpha} r^{\alpha-1}(s) - \nu^{\alpha-1}\right) g_2(\nu)\,\mathrm{d}\nu\right]\mathrm{d}s,$$

and

$$B = 1 - \frac{2\exp\left[\frac{c(b)K(b) - 2c(a)K(b)}{2}\right]}{\left[\exp(c(b) - 2c(a)) - 4K(b)\right]^{\frac{1}{2}} + 1}.$$

Proof Note that

$$\sum_{i=1}^{i=n} \gamma_i t^{\beta_i} \leq \sum_{i=1}^{i=n} \gamma_i (\beta_i + t)^{\beta_i} = \sum_{i=1}^{i=n} \gamma_i \beta_i^{\beta_i} \left(1 + \frac{t}{\beta_i}\right)^{\beta_i} \leq e^t \sum_{i=1}^{i=n} \gamma_i \beta_i^{\beta_i}. \tag{3.37}$$

We take $h(t) = e^t$. Then $Q(t) = 1 - e^{-t}$, $Q^{-1}(t) = \log(\frac{1}{1-t})$, $S(t) = \exp(-t) - \exp(-2t - c(b) + 2c(a))$, $S^{-1}(t) = \log[\exp(\frac{-c(b)+2c(a)}{2})(\exp(c(b) - 2c(a)) - 4t)^{\frac{1}{2}} + 1] - \log(2t)$, and $Q[S^{-1}(t)] = 1 - \frac{2\exp[\frac{c(b)t-2c(a)t}{2}]}{[\exp(c(b)-2c(a))-4t]^{\frac{1}{2}}+1}$. By applying Theorem 3.3 we deduce the corollary. □

Corollary 3.3 *If $F(y_1, y_2, y_3, y_4) \leq A(y_1, y_2)(y_3^p + y_4)$ and $C(u, v) \leq c(u, v)(u - u_0)(v - v_0)$ for $u \in [u_0, u_T]$, $v \in [v_0, v_T]$, then $x(u, v)$ defined by (3.32) satisfies*

$$\begin{aligned} x(u, v) \leq{}& \frac{1}{k}(u - u_0)^{\alpha-1}(v - v_0)^{\alpha-1}(1 - p)^{\frac{1}{1-p}} \biggl\{ k(u - u_0)^{2-\alpha}(v - v_0)^{2-\alpha} c(u, v) \\ & - k_0 k c(u_0, v) + k \int_{u_0}^{u} \int_{v_0}^{v} A(s, \tau)\bigl[1 + D(s, \tau)\bigr] \,\mathrm{d}\tau \,\mathrm{d}s \biggr\}^{\frac{1}{1-p}}, \end{aligned} \tag{3.38}$$

where

$$D(s, \tau) = \int_{u_0}^{s} \int_{v_0}^{\tau} \bigl(2^{2-\alpha} s^{\alpha-1} - w^{\alpha-1}\bigr)\bigl(2^{2-\alpha} \tau^{\alpha-1} - z^{\alpha-1}\bigr) \,\mathrm{d}w \,\mathrm{d}z, \tag{3.39}$$

and k_0 and k are as in Theorem 3.4.

Proof From the assumptions of the corollary we can deduce that

$$\begin{aligned} x(u, v) \leq{}& (u - u_0)(v - v_0) c(u, v) \\ & + \int_{u_0}^{u} \int_{v_0}^{v} (u - s)^{\alpha-1}(v - \tau)^{\alpha-1} A(s, \tau) \biggl[x^p(s, \tau) \\ & + \int_{u_0}^{s} \int_{v_0}^{\tau} (s - w)^{\alpha-1}(\tau - z)^{\alpha-1} x^p(w, z) \,\mathrm{d}w \,\mathrm{d}z \biggr] \,\mathrm{d}s \,\mathrm{d}\tau. \end{aligned} \tag{3.40}$$

Applying Theorem 3.4 to $h(t) = t^p$ completes the proof. □

4 Hermite–Hadamard-type fractional integral inequalities

In this section, we present some Hermite–Hadamard-type fractional integral inequalities by integration, differentiation, and convex functions.

Lemma 4.1 *Let $c, \alpha \in (0, 1)$, and let $f \in C^3([a, b])$. Then*

$${}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-\alpha} f(t)\big|_{t=b} = \frac{(b - a)^{\alpha}}{\Gamma(\alpha + 3)} T_{f,c}(a, b) + \frac{(b - a)^{\alpha+3}}{\Gamma(\alpha + 3)} \int_0^1 Q(t) f^{(3)}\bigl(at + (1 - t)b\bigr) \,\mathrm{d}t. \tag{4.1}$$

Furthermore, $Q(t)$ and $T_{f,c}(a, b)$ can be expressed as follows:

$$Q(t) = \begin{cases} Q_1(t) = t^{\alpha+2} - c^{\alpha} t^2, t \in [0, c], \\ Q_2(t) = t^{\alpha+2} + b_2 t^2 + b_1 t + b_0, t \in (c, 1], \end{cases} \tag{4.2}$$

and

$$T_{f,c}(a,b) = \alpha\frac{2c^{\alpha+1}-1}{c-1}f(a) + 2c^{\alpha}f(b) + 2\frac{1-c^{\alpha}}{(c-1)^2}f\bigl(ac+(1-c)b\bigr). \tag{4.3}$$

Here $b_0 = \frac{\alpha c}{2(c-1)}$, $b_1 = \frac{\alpha}{1-c}$, $b_2 = -\frac{\alpha(c-2)}{2(c-1)} - 1$, *and* $2(c^{\alpha+1}-1) = (\alpha+2)(c-1)$.

Proof We represent A as

$$\begin{aligned} A &= \int_0^1 Q(t)f^{(3)}\bigl(at+(1-t)b\bigr)\,\mathrm{d}t \\ &= \int_0^c Q_1(t)f^{(3)}\bigl(at+(1-t)b\bigr)\,\mathrm{d}t + \int_c^1 Q_2(t)f^{(3)}\bigl(at+(1-t)b\bigr)\,\mathrm{d}t \\ &= A_1 + A_2. \end{aligned} \tag{4.4}$$

First, we estimate A_1. It is clear that

$$\begin{aligned} A_1 &= \int_0^c Q_1(t)f^{(3)}\bigl(at+(1-t)b\bigr)\,\mathrm{d}t \\ &= \frac{1}{a-b}\int_0^c Q_1(t)\bigl[f^{(2)}\bigl(at+(1-t)b\bigr)\bigr]'\,\mathrm{d}t. \end{aligned} \tag{4.5}$$

Using integration by parts and the facts $Q_1(0) = Q_1(c) = 0$, $Q_1'(0) = 0$, and $Q_1'(c) = \alpha c^{\alpha+1}$, we have

$$\begin{aligned} A_1 &= \frac{1}{b-a}\int_0^c Q_1'(t)f^{(2)}\bigl(at+(1-t)b\bigr)\,\mathrm{d}t \\ &= -\frac{1}{(b-a)^2}\int_0^c Q_1'(t)\,\mathrm{d}\bigl[f'\bigl(at+(1-t)b\bigr)\bigr] \\ &= -\frac{\alpha c^{\alpha+1}}{(b-a)^2}f'\bigl(ac+(1-c)b\bigr) \\ &\quad - \frac{1}{(b-a)^3}Q_1''(t)f\bigl(at+(1-t)b\bigr)\Big|_0^c \\ &\quad + \frac{\alpha(\alpha+1)(\alpha+2)}{(b-a)^3}\int_0^c t^{\alpha-1}f\bigl(at+(1-t)b\bigr)\,\mathrm{d}t. \end{aligned} \tag{4.6}$$

In a similar manner, we find that

$$\begin{aligned} A_2 &= \frac{\alpha c^{\alpha+1}}{(b-a)^2}f'\bigl(ac+(1-c)b\bigr) \\ &\quad - \frac{1}{(b-a)^3}Q_2''(t)f\bigl(at+(1-t)b\bigr)\Big|_c^1 \\ &\quad + \frac{\alpha(\alpha+1)(\alpha+2)}{(b-a)^3}\int_c^1 t^{\alpha-1}f\bigl(at+(1-t)b\bigr)\,\mathrm{d}t. \end{aligned} \tag{4.7}$$

Thus A can be written as

$$\begin{aligned} A &= A_1 + A_2 \\ &= -\frac{1}{(b-a)^3}\bigl[-Q_1''(0)f(b) + Q_2''(1)f(a)\bigr] \\ &\quad - \frac{1}{(b-a)^3}\bigl[Q_1''(c) - Q_2''(c)\bigr]f\bigl(ac + (1-c)b\bigr) \\ &\quad + \frac{\alpha(\alpha+1)(\alpha+2)}{(b-a)^{\alpha+3}}\int_a^b (b-u)^{\alpha-1}f(u)\,\mathrm{d}u, \end{aligned} \tag{4.8}$$

which yields the desired result

$$ {}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-\alpha}f(t)\big|_{t=b} = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+3)}T_{f,c}(a,b) + \frac{(b-a)^{\alpha+3}}{\Gamma(\alpha+3)}\int_0^1 Q(t)f^{(3)}\bigl(at + (1-t)b\bigr)\,\mathrm{d}t \tag{4.9}$$

with

$$\begin{aligned} T_{f,c}(a,b) &= Q_2''(1)f(a) - Q_1''(0)f(b) + \bigl[Q_1''(c) - Q_2''(c)\bigr]f\bigl(ac + (1-c)b\bigr) \\ &= \alpha\frac{2c^{\alpha+1}-1}{c-1}f(a) + 2c^{\alpha}f(b) + 2\frac{1-c^{\alpha}}{(c-1)^2}f\bigl(ac + (1-c)b\bigr). \end{aligned} \tag{4.10}$$

The proof is completed. □

Corollary 4.1 *Let $c,\alpha \in (0,1)$, and let $f \in C^4([a,b])$. Under the assumptions of Lemma 4.1 for $1-\alpha$, we have*

$$ {}_{C}\mathrm{D}_{a,t}^{\alpha}f(t)\big|_{t=b} = \frac{(b-a)^{1-\alpha}}{\Gamma(4-\alpha)}T_{f',c}(a,b) + \frac{(b-a)^{4-\alpha}}{\Gamma(4-\alpha)}\int_0^1 Q(t)f^{(4)}\bigl(at + (1-t)b\bigr)\,\mathrm{d}t. \tag{4.11}$$

Theorem 4.1 *Let $f \in C^3([a,b])$. Denote by d a division of the interval $[a,b]$, i.e., $d: a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$, $h = t_{i+1} - t_i$, $i = 0,\dots,n-1$. Then for $c,\alpha \in (0,1)$,*

$$\begin{aligned} &{}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-\alpha}f(t)\big|_{t=t_n} \\ &\quad = \frac{h^{\alpha}}{\Gamma(\alpha+3)}\left[\sum_{i=0}^{n-2}T_{F_i,c}(t_i,t_{i+1}) + T_{f,c}(t_{n-1},t_n)\right] + \frac{h^{\alpha+3}}{\Gamma(\alpha+3)} \\ &\qquad \times \int_0^1 Q(t)\left[\sum_{i=0}^{n-2}F_i^{(3)}\bigl(t_it + (1-t)t_{i+1}\bigr) + f^{(3)}\bigl(t_{n-1}t + (1-t)t_n\bigr)\right]\mathrm{d}t, \end{aligned} \tag{4.12}$$

where $F_i(u) = (\frac{t_{i+1}-u}{t_n-u})^{1-\alpha}f(u)$, $i = 0,\dots,n-2$.

Proof From Lemma 4.1 we have

$$
\begin{aligned}
&{}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-\alpha}f(t)\big|_{t=t_n}\\
&\quad=\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}(t_n-u)^{\alpha-1}f(u)\,\mathrm{d}u\\
&\quad=\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}}(t_n-u)^{\alpha-1}f(u)\,\mathrm{d}u+\frac{1}{\Gamma(\alpha)}\int_{t_{n-1}}^{t_n}(t_n-u)^{\alpha-1}f(u)\,\mathrm{d}u\\
&\quad=\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}}(t_{i+1}-u)^{\alpha-1}\left(\frac{t_{i+1}-u}{t_n-u}\right)^{1-\alpha}f(u)\,\mathrm{d}u\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_{t_{n-1}}^{t_n}(t_n-u)^{\alpha-1}f(u)\,\mathrm{d}u\\
&\quad=\frac{1}{\Gamma(\alpha)}\sum_{i=0}^{n-2}\int_{t_i}^{t_{i+1}}(t_{i+1}-u)^{\alpha-1}F_i(u)\,\mathrm{d}u+\frac{1}{\Gamma(\alpha)}\int_{t_{n-1}}^{t_n}(t_n-u)^{\alpha-1}f(u)\,\mathrm{d}u\\
&\quad=\sum_{i=0}^{n-2}\frac{(t_{i+1}-t_i)^{\alpha}}{\Gamma(\alpha+3)}T_{F_i,c}(t_i,t_{i+1})+\sum_{i=0}^{n-2}\frac{(t_{i+1}-t_i)^{\alpha+3}}{\Gamma(\alpha+3)}\\
&\qquad\times\int_0^1 Q(t)F_i^{(3)}\big(t_it+(1-t)t_{i+1}\big)\,\mathrm{d}t+\frac{(t_n-t_{n-1})^{\alpha}}{\Gamma(\alpha+3)}T_{f,c}(t_{n-1},t_n)\\
&\qquad+\frac{(t_n-t_{n-1})^{\alpha+3}}{\Gamma(\alpha+3)}\int_0^1 Q(t)f^{(3)}\big(t_{n-1}t+(1-t)t_n\big)\,\mathrm{d}t\\
&\quad=\frac{h^{\alpha}}{\Gamma(\alpha+3)}\sum_{i=0}^{n-2}T_{F_i,c}(t_i,t_{i+1})+\frac{h^{\alpha}}{\Gamma(\alpha+3)}T_{f,c}(t_{n-1},t_n)\\
&\qquad+\frac{h^{\alpha+3}}{\Gamma(\alpha+3)}\int_0^1 Q(t)\left[\sum_{i=0}^{n-2}F_i^{(3)}\big(t_it+(1-t)t_{i+1}\big)\right]\mathrm{d}t\\
&\qquad+\frac{h^{\alpha+3}}{\Gamma(\alpha+3)}\int_0^1 Q(t)f^{(3)}\big(t_{n-1}t+(1-t)t_n\big)\,\mathrm{d}t\\
&\quad=\frac{h^{\alpha}}{\Gamma(\alpha+3)}\left[\sum_{i=0}^{n-2}T_{F_i,c}(t_i,t_{i+1})+T_{f,c}(t_{n-1},t_n)\right]+\frac{h^{\alpha+3}}{\Gamma(\alpha+3)}\\
&\qquad\times\int_0^1 Q(t)\left[\sum_{i=0}^{n-2}F_i^{(3)}\big(t_it+(1-t)t_{i+1}\big)+f^{(3)}\big(t_{n-1}t+(1-t)t_n\big)\right]\mathrm{d}t.
\end{aligned}
$$

This proves the theorem. □

Theorem 4.2 *Let $f\in C^4([a,b])$ and suppose that $|f^{(3)}|$ is an (s,m)-convex function on $[a,b]$. Under the assumptions of Lemma 4.1, we have the following inequality for $c,\alpha\in(0,1)$:*

$$
\begin{aligned}
&\left|{}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-\alpha}f(t)\big|_{t=b}-\frac{(b-a)^{\alpha}}{\Gamma(\alpha+3)}T_{f,c}(a,b)\right|\\
&\quad\le\frac{(b-a)^{\alpha+3}}{\Gamma(\alpha+3)}\left\{(1+b_1)\left[\frac{\alpha c^{\alpha+s+3}-\alpha(1-c)^{\alpha+s+3}}{(s+3)(\alpha+s+3)}+3\right]\big(\big|f^{(3)}(a)\big|+\big|f^{(3)}(b)\big|\big)\right.
\end{aligned}
\tag{4.13}
$$

$$
\begin{aligned}
&+ 2^{2-s} m(1+b_1)\left(\frac{c^{\alpha+s+3}}{\alpha+s+3}+\frac{c^{s+1}}{s+1}\right)\left(\left|f^{(3)}\left(\frac{a}{m}\right)\right|+\left|f^{(3)}\left(\frac{b}{m}\right)\right|\right) \\
&- \mu\left[\left(a-\frac{b}{m}\right)^2+\left(b-\frac{a}{m}\right)^2\right]\int_0^1 t^3\left(c^\alpha-t^\alpha\right)(1-t)\,\mathrm{d}t\bigg\}.
\end{aligned}
$$

Proof Equation (4.1) yields the inequality

$$
\begin{aligned}
&\left|{}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-\alpha} f(t)\big|_{t=b}-\frac{(b-a)^\alpha}{\Gamma(\alpha+3)}T_{f,c}(a,b)\right| \\
&\quad\le \frac{(b-a)^{\alpha+3}}{\Gamma(\alpha+3)}\int_0^1 \left|Q(t)\right|\left|f^{(3)}\left(at+(1-t)b\right)\right|\mathrm{d}t \\
&\quad= \frac{(b-a)^{\alpha+3}}{\Gamma(\alpha+3)}\left[\int_0^c \left|Q_1(t)\right|\left|f^{(3)}\left(at+(1-t)b\right)\right|\mathrm{d}t\right. \\
&\qquad\left.+\int_c^1 \left|Q_2(t)\right|\left|f^{(3)}\left(at+(1-t)b\right)\right|\mathrm{d}t\right] \\
&\quad= \frac{(b-a)^{\alpha+3}}{\Gamma(\alpha+3)}(I_1+I_2).
\end{aligned}
\tag{4.14}
$$

Since $|f^{(3)}|$ is (s,m)-convex, I_1 can be calculated as

$$
\begin{aligned}
I_1 &= \int_0^c \left|Q_1(t)\right|\left|f^{(3)}\left(at+(1-t)b\right)\right|\mathrm{d}t \\
&= \int_0^c t^2\left(c^\alpha-t^\alpha\right)\left|f^{(3)}\left(at+(1-t)b\right)\right|\mathrm{d}t \\
&\le \int_0^c t^2\left(c^\alpha-t^\alpha\right)\left\{t^s\left|f^{(3)}(a)\right|+m(1-t)^s\left|f^{(3)}\left(\frac{b}{m}\right)\right|\right. \\
&\qquad\left.-\mu t(1-t)\left(a-\frac{b}{m}\right)^2\right\}\mathrm{d}t \\
&= \left|f^{(3)}(a)\right|\int_0^c t^{s+2}\left(c^\alpha-t^\alpha\right)\mathrm{d}t+m\left|f^{(3)}\left(\frac{b}{m}\right)\right|\int_0^c t^2\left(c^\alpha-t^\alpha\right)(1-t)^s\,\mathrm{d}t \\
&\qquad-\mu\left(a-\frac{b}{m}\right)^2\int_0^c t^3\left(c^\alpha-t^\alpha\right)(1-t)\,\mathrm{d}t \\
&= \frac{\alpha c^{s+\alpha+3}|f^{(3)}(a)|}{(s+3)(s+\alpha+3)}+m\left|f^{(3)}\left(\frac{b}{m}\right)\right|\int_0^c t^2\left(c^\alpha-t^\alpha\right)(1-t)^s\,\mathrm{d}t \\
&\qquad-\mu\left(a-\frac{b}{m}\right)^2\int_0^c t^3\left(c^\alpha-t^\alpha\right)(1-t)\,\mathrm{d}t.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\int_0^c t^2\left(c^\alpha-t^\alpha\right)(1-t)^s\,\mathrm{d}t \\
&\quad\le 2^{1-s}\int_0^c t^2\left(1-t^\alpha\right)\left(1-t^s\right)\mathrm{d}t \\
&\quad= 2^{1-s}\left(\frac{c^{\alpha+s+3}}{\alpha+s+3}-\frac{c^{s+3}}{s+3}+\frac{c^{s+2}}{s+2}\right).
\end{aligned}
$$

Hence it is easily shown that

$$\begin{aligned} I_1 &\le \frac{\alpha c^{s+\alpha+3}|f^{(3)}(a)|}{(s+3)(s+\alpha+3)} + 2^{1-s}m\left(\frac{c^{\alpha+s+3}}{\alpha+s+3} - \frac{c^{s+3}}{s+3} + \frac{c^{s+2}}{s+2}\right)\left|f^{(3)}\left(\frac{b}{m}\right)\right| \\ &\quad - \mu\left(a-\frac{b}{m}\right)^2 \int_0^c t^3\left(c^\alpha - t^\alpha\right)(1-t)\,\mathrm{d}t \\ &\le \frac{\alpha c^{s+\alpha+3}|f^{(3)}(a)|}{(s+3)(s+\alpha+3)} + 2^{1-s}m\left(\frac{c^{\alpha+s+3}}{\alpha+s+3} - \frac{c^{s+3}}{s+3} + \frac{c^{s+1}}{s+1}\right)\left|f^{(3)}\left(\frac{b}{m}\right)\right| \\ &\quad - \mu\left(a-\frac{b}{m}\right)^2 \int_0^c t^3\left(c^\alpha - t^\alpha\right)(1-t)\,\mathrm{d}t. \end{aligned} \tag{4.15}$$

Next, using $c^\alpha - t^\alpha \le \alpha(c-t)t^{\alpha-1}$, we have

$$\begin{aligned} I_2 &= \int_c^1 \left|Q_2(t)\right|\left|f^{(3)}\left(at+(1-t)b\right)\right|\mathrm{d}t \\ &\le \left|f^{(3)}(b)\right| \int_0^{1-c} t^s\left|Q_2(1-t)\right|\mathrm{d}t + m\left|f^{(3)}\left(\frac{a}{m}\right)\right| \int_0^{1-c} (1-t)^s\left|Q_2(1-t)\right|\mathrm{d}t \\ &\quad - \mu\left(b-\frac{a}{m}\right)^2 \int_0^{1-c} t(1-t)\left|Q_2(1-t)\right|\mathrm{d}t. \end{aligned} \tag{4.16}$$

As for $|Q_2(1-t)|$, we have the inequality

$$(1-b_0)\left[m(t)-t+2\right] \le \left|Q_2(1-t)\right| \le (b_1+1)\left[m(t)+1\right], \tag{4.17}$$

in which $m(t) = (1-t)^{\alpha+2} + (1-t)^2$, and $(1-t)^r + t^r \le 1$ for $r \ge 1$.

Thus

$$\begin{aligned} I_2 &\le (1+b_1)\left[\frac{(1-c)^{\alpha+s+3}}{\alpha+s+3} + \frac{3(1-c)^{s+1}}{s+1} - \frac{(1-c)^{s+3}}{s+3}\right]\left|f^{(3)}(b)\right| \\ &\quad + m(1+b_1)\left(\frac{c^{\alpha+s+3}}{\alpha+s+3} + \frac{c^{s+3}}{s+3} + \frac{c^{s+1}}{s+1}\right)\left|f^{(3)}\left(\frac{a}{m}\right)\right| \\ &\quad - \mu\left(b-\frac{a}{m}\right)^2 \int_0^{1-c} t(1-t)\left|Q_2(1-t)\right|\mathrm{d}t \\ &\le (1+b_1)\left[\frac{-\alpha(1-c)^{\alpha+s+3}}{(s+3)(\alpha+s+3)} + 3\right]\left|f^{(3)}(b)\right| \\ &\quad + m(1+b_1)\left(\frac{c^{\alpha+s+3}}{\alpha+s+3} + \frac{c^{s+3}}{s+3} + \frac{c^{s+1}}{s+1}\right)\left|f^{(3)}\left(\frac{a}{m}\right)\right| \\ &\quad - \mu\left(b-\frac{a}{m}\right)^2 \int_0^{1-c} t(1-t)\left|Q_2(1-t)\right|\mathrm{d}t, \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} &\int_0^{1-c} t(1-t)\left|Q_2(1-t)\right|\mathrm{d}t \\ &\quad = \int_c^1 t(1-t)\left|Q_2(t)\right|\mathrm{d}t \end{aligned} \tag{4.19}$$

$$
\begin{aligned}
&\geq \int_c^1 t(1-t)\left(t^{\alpha+2}-b_2t^2\right)\mathrm{d}t \\
&\geq \int_c^1 t^3(1-t)\left(c^\alpha-t^\alpha\right)\mathrm{d}t.
\end{aligned}
$$

By (4.15), (4.18), and (4.19) we get the inequality

$$
\begin{aligned}
I_1+I_2 \leq{}& (1+b_1)\left[\frac{\alpha c^{\alpha+s+3}-\alpha(1-c)^{\alpha+s+3}}{(s+3)(\alpha+s+3)}+3\right]\left[\left|f^{(3)}(a)\right|+\left|f^{(3)}(b)\right|\right] \\
&+2^{2-s}m(1+b_1)\left(\frac{c^{\alpha+s+3}}{\alpha+s+3}+\frac{c^{s+1}}{s+1}\right)\left[\left|f^{(3)}\left(\frac{a}{m}\right)\right|+\left|f^{(3)}\left(\frac{b}{m}\right)\right|\right] \\
&-\mu\left[\left(a-\frac{b}{m}\right)^2+\left(b-\frac{a}{m}\right)^2\right]\int_0^1 t^3\left(c^\alpha-t^\alpha\right)(1-t)\,\mathrm{d}t.
\end{aligned} \tag{4.20}
$$

Finally, the proof can be fulfilled by (4.14) and (4.20). □

Lemma 4.2 *Let $f \in C([a,b])$. Then for $c,\alpha \in (0,1)$,*

$$
\begin{aligned}
{}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-\alpha}f(t)\big|_{t=b} ={}& \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}f\left(ca+(1-c)b\right) \\
&+\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1 p(t)f'\left(ta+(1-t)b\right)\mathrm{d}t,
\end{aligned} \tag{4.21}
$$

where

$$
p(t)=\begin{cases} t^\alpha, & t\in[0,c], \\ t^\alpha-1, & t\in(c,1]. \end{cases} \tag{4.22}
$$

Proof Let $J=\int_0^1 p(t)f'(ta+(1-t)b)\,\mathrm{d}t$. We have

$$
\begin{aligned}
J &= \int_0^c t^\alpha f'\left(ta+(1-t)b\right)\mathrm{d}t+\int_c^1\left(t^\alpha-1\right)f'\left(ta+(1-t)b\right)\mathrm{d}t \\
&= \frac{t^\alpha}{a-b}f\left(ta+(1-t)b\right)\Big|_{t=0}^{t=c}-\frac{\alpha}{a-b}\int_0^c t^{\alpha-1}f\left(ta+(1-t)b\right)\mathrm{d}t \\
&\quad+\frac{t^\alpha-1}{a-b}f\left(ta+(1-t)b\right)\Big|_{t=c}^{t=1}-\frac{\alpha}{a-b}\int_c^1 t^{\alpha-1}f\left(ta+(1-t)b\right)\mathrm{d}t \\
&= \frac{f(ca+(1-c)b)}{a-b}+\frac{\alpha}{b-a}\int_0^1 t^{\alpha-1}f\left(ta+(1-t)b\right)\mathrm{d}t \\
&= \frac{f(ca+(1-c)b)}{a-b}+\frac{\alpha}{(b-a)^{\alpha+1}}\int_a^b(b-t)^{\alpha-1}f(t)\,\mathrm{d}t.
\end{aligned} \tag{4.23}
$$

Multiplying both sides of (4.23) by $\frac{1}{\Gamma(\alpha)}$, we obtain (4.21). The lemma is proved. □

Theorem 4.3 *Let $c,\alpha \in (0,1)$, let $f:[a,b]\mapsto \mathbb{R}$ be a differentiable function on $[a,b]$, and let $p \geq 1$. If $|f'|^{\frac{p}{p-1}}$ is convex, then*

$$\begin{aligned}&\left|{}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-\alpha}f(t)\big|_{t=b}-\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}f\big(ca+(1-c)b\big)\right|\\&\quad\leq\frac{2^{\frac{1}{p}}(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\big(\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}\big)^{\frac{p-1}{p}}.\end{aligned}\tag{4.24}$$

Proof By (4.21) we have

$$\begin{aligned}&\left|{}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-\alpha}f(t)\big|_{t=b}-\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}f\big(ca+(1-c)b\big)\right|\\&\quad\leq\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^c t^{\alpha}\left|f'\big(ta+(1-t)b\big)\right|\,\mathrm{d}t+\int_c^1\big(1-t^{\alpha}\big)\left|f'\big(ta+(1-t)b\big)\right|\,\mathrm{d}t\\&\quad\leq\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\int_0^c t^{\alpha}\left|f'\big(ta+(1-t)b\big)\right|\,\mathrm{d}t+\int_c^1\big(1-t^{\alpha}\big)\left|f'\big(ta+(1-t)b\big)\right|\,\mathrm{d}t\right].\end{aligned}\tag{4.25}$$

Using the Hölder inequality for $q,p\geq 1$ such that $\frac{1}{p}+\frac{1}{q}=1$, we can prove that

$$\begin{aligned}&\left|{}_{\mathrm{RL}}\mathrm{D}_{a,t}^{-\alpha}f(t)\big|_{t=b}-\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}f\big(ca+(1-c)b\big)\right|\\&\quad\leq\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left\{\left(\int_0^c t^{\alpha p}\,\mathrm{d}t\right)^{\frac{1}{p}}\left[\int_0^c\left|f'\big(ta+(1-t)b\big)\right|^q\mathrm{d}t\right]^{\frac{1}{q}}\right.\\&\qquad\left.+\left[\int_c^1\big(1-t^{\alpha}\big)^p\,\mathrm{d}t\right]^{\frac{1}{p}}\left[\int_c^1\left|f'\big(ta+(1-t)b\big)\right|^q\mathrm{d}t\right]^{\frac{1}{q}}\right\}\\&\quad\leq\frac{(b-a)^{\alpha+1}}{2^{\frac{1}{q}}\Gamma(\alpha+1)}\left\{\left(\int_0^c t^{\alpha p}\,\mathrm{d}t\right)^{\frac{1}{p}}\left[c^2\left|f'(a)\right|^q+\big(1-(1-c)^2\big)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.\\&\qquad\left.+\left[\int_c^1\big(1-t^{\alpha}\big)^p\,\mathrm{d}t\right]^{\frac{1}{p}}\left[\big(1-c^2\big)\left|f'(a)\right|^q+(1-c)^2\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}\\&\quad\leq\frac{(b-a)^{\alpha+1}}{2^{\frac{1}{q}}\Gamma(\alpha+1)}\left\{\left(\int_0^c t^{\alpha p}\,\mathrm{d}t\right)^{\frac{1}{p}}+\left[\int_c^1\big(1-t^{\alpha}\big)^p\,\mathrm{d}t\right]^{\frac{1}{p}}\right\}\\&\qquad\times\left\{\left[c^2\left|f'(a)\right|^q+\big(1-(1-c)^2\big)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right.\\&\qquad\left.+\left[\big(1-c^2\big)\left|f'(a)\right|^q+(1-c)^2\left|f'(b)\right|^q\right]^{\frac{1}{q}}\right\}\\&\quad\leq\frac{(b-a)^{\alpha+1}}{2^{\frac{1}{q}}\Gamma(\alpha+1)}2^{1-1/p}\left(\int_0^c t^{\alpha p}\,\mathrm{d}t+\int_c^1\big(1-t^{\alpha}\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}}2^{1-1/q}\big(\left|f'(a)\right|^q+\left|f'(b)\right|^q\big)^{\frac{1}{q}}\\&\quad=\frac{2^{\frac{1}{p}}(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^c t^{\alpha p}\,\mathrm{d}t+\int_c^1\big(1-t^{\alpha}\big)^p\,\mathrm{d}t\right)^{\frac{1}{p}}\big(\left|f'(a)\right|^q+\left|f'(b)\right|^q\big)^{\frac{1}{q}}\\&\quad\leq\frac{2^{\frac{1}{p}}(b-a)^{\alpha+1}}{\Gamma(\alpha+1)}\big(\left|f'(a)\right|^q+\left|f'(b)\right|^q\big)^{\frac{1}{q}},\end{aligned}\tag{4.26}$$

where we used the inequalities $A^r+B^r\leq 2^{1-r}(A+B)^r$ for $A,B\geq 0$ and $0\leq r\leq 1$, and $(1-t^{\alpha})^p+t^{\alpha p}\leq 1$ for $0\leq t\leq 1$. The proof is completed. □

Proposition 4.1 *Let $p, \beta \geq 0, c, \alpha \in (0,1)$, and $0 \leq a \leq b$. Then*

$$\left|\mathrm{B}(\alpha,\beta)(b-a)^{\beta} - \left(ca + (1-c)b\right)^{\beta}\right| \leq \beta^{\frac{p}{p-1}} 2^{\frac{1}{p}} (b-a)\left(a^{\frac{p(\beta-1)}{p-1}} + b^{\frac{p(\beta-1)}{p-1}}\right), \tag{4.27}$$

where $\mathrm{B}(\alpha,\beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,\mathrm{d}u$.

Proof The proposition comes directly by applying Theorem 4.3 to $f(t) = t^{\beta}$. □

Proposition 4.2 *Let $\beta \geq 3$ and $c, \alpha \in (0,1)$ with $(\alpha+2)(c-1) = 2(c^{\alpha+1}-1)$. Then*

$$\left|\mathrm{B}(\alpha,\beta)(b-a)^{\beta} - \frac{T_{t^{\beta},c}(a,b)}{\alpha(\alpha+1)(\alpha+2)}\right| \leq \frac{\beta(\beta-1)(\beta-2)}{\alpha(\alpha+1)(\alpha+2)}(b-a)^{3}\int_0^1 \left|Q(u)\right|\,\mathrm{d}u. \tag{4.28}$$

Proof Applying Lemma 4.1 to $f(t) = t^{\beta}$ and $\beta \geq 3$, we arrive at (4.28). □

5 Conclusion

In this paper, we established some new Volterra–Fredholm and Hermite–Hadamard-type fractional integral inequalities. They extend some known inequalities and provide a handy tool for deriving bounds of solutions to fractional differential equations and fractional integral equations. In the meantime, we obtain new fractional integral inequalities for convex functions and show their applications. Finally, we present some estimates of the Riemann–Liouville fractional integral of functions whose absolute value is convex and the derivative is raised to a positive real power.

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Authors' contributions
MDB made the major analysis and the original draft preparation. JT contributed significantly in writing this paper by analyzing the results, reviewing and editing. Both authors read and approved the final manuscript.

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