Ali et al. *Journal of Inequalities and Applications* (2022) 2022:37
https://doi.org/10.1186/s13660-022-02770-8

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On new generalized θ_b-contractions and related fixed point theorems

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Abstract

The manuscript is an extension of the investigations concerning θ-contractions which have been newly proposed in (Jleli and Samet in J. Inequal. Appl. 2014:38, 2014). The authors generalize the notion of θ-contractions to the case of nonlinear θ_b-contraction mappings and prove multi-valued fixed point theorems based on the b-Bianchini–Grandolfi gauge function in b-metric spaces. The manuscript consists of a tangible example which displays the motivation for such investigations. The work is compiled by the application of the proposed nonlinear θ_b-contractions to Liouville–Caputo fractional differential equations.

Keywords: Fixed points; Multi-valued maps; θ_b-Contraction; b-Bianchini–Grandolfi gauge function; Liouville–Caputo fractional differential equations

1 Introduction

The Banach fixed point (or contraction) theorem is one of the most useful tools in functional analysis and its applications. It states conditions sufficient for the existence and uniqueness of a fixed point, and the theorem also provides an iterative system by which we can approximate to the fixed point and error bounds. This classical result of Banach has been described in different classes of distance spaces. In 1969, Nadler [19] generalized the concept of contraction theorem based on multi-valued mappings. He used the Hausdorff metric on it. Following the module of distance spaces, a number of authors have extended several results in this direction (see [1–30]). In 2007, Proinov [22] extended a contraction theorem with high order of iterative convergence of successive approximation by a new approach of contractive condition with respect to gauge function, high order of gauge function, and Bianchini–Grandolfi gauge function.

In 1989 Bakhtin [6] and in 1993 Czerwik [12] generalized for the first time the concept of metric space by reorganizing just the triangle inequality and called it a b-metric space. After that, Aydi et al. in [5] constituted common fixed point theorems for single-valued and set-valued contractions gratifying a weak structure of φ-contraction in b-metric spaces (see [8, 13–15, 20]).

In 2012 Wardowski [27] developed the concept of new contraction mappings named F-contraction and proved some fixed point theorems, which were a generalization of the contraction theorem. In the recent time, Jleli and Samet [17] provided the idea of a new class contraction called θ-contraction, which is a new aspect of the contraction theorem in the associated approach.

2 Preliminaries

Firstly, we recall the Hausdorff–Pompeiu b-metric and some basic manners of b-metric space for the main sequel.

Let $(\Gamma, \eth)$ be a b-metric space. For $\kappa \in \Gamma$, $A \subseteq \Gamma$ and let $\eth_b(\kappa, \Omega_1) = \inf\{\eth(\kappa, y) : y \in \Omega_2\}$. Define a mapping $\hat{H}_b : CB(\Gamma) \times CB(\Gamma) \to [0, +\infty)$ by

$$\hat{H}_b(\Omega_1, \Omega_2) = \max\Big\{\sup_{\kappa\in\Omega_1} \eth_b(\kappa, \Omega_2), \sup_{y\in\Omega_2} \eth_b(y, \Omega_1)\Big\}$$

for all $\Omega_1, \Omega_2 \in CB(\Gamma)$. Then $\hat{H}_b$ is known as Hausdorff–Pompeiu b-metric induced by $\eth$ on $CB(\Gamma)$, where $CB(\Gamma)$ is the class of all nonempty closed and bounded subsets of Γ. A point $\kappa \in \Gamma$ is called a fixed point of $\grave{I} : \Gamma \to CL(\Gamma)$ such that $\kappa \in \grave{I}\kappa$, where $CL(\Gamma)$ is the class of all nonempty closed subsets of Γ. If for $\kappa_0 \in \Gamma$ there is $\{\kappa_i\}$ in Γ such that $\kappa_i \in \grave{I}\kappa_{i-1}$, then $O(\grave{I}, \kappa_0) = \{\kappa_0, \kappa_1, \kappa_2, \dots\}$ is called an orbit of $\grave{I} : \Gamma \to CL(\Gamma)$. A map $f : \Gamma \to \mathbb{R}$ is called $\grave{I}$-orbitally lower semi-continuous if $\{\kappa_i\} \in O(\grave{I}, \kappa_0)$ and $\kappa_i \to \varrho$, which implies $f(\varrho) \leq \liminf_i f(\kappa_i)$.

Definition 2.1 ([12]) A b-metric space on a nonempty set M is a function $b : \Gamma \times \Gamma \to R^+ \cup \{0\}$ such that for each $\kappa_1, \kappa_2, \kappa_3 \in \Gamma$ with $s \geq 1$ being a given real number, if the following hold true:

(b_{i}) $\eth(\kappa_1, \kappa_2) = 0$ if and only if $\kappa_1 = \kappa_2$;

(b_{ii}) $\eth(\kappa_1, \kappa_2) = \eth(\kappa_2, \kappa_1)$;

(b_{iii}) $\eth(\kappa_1, \kappa_3) \leq s[\eth(\kappa_1, \kappa_2) + \eth(\kappa_2, \kappa_3)]$. The pair $(\Gamma, \eth)$ is known as a b-metric space.

Example 2.1 ([12]) Let $\Gamma = L_p[0, 1]$ be the space of all real-valued functions $\kappa(r)$, $0 \leq r \leq 1$ such that $\int_0^1 |\kappa(r)|^{\frac{1}{p}}\, dr < +\infty$. Define $b : \Gamma \times \Gamma \to R^+$ by $b(\kappa_1, \kappa_2) = (\int_0^1 |\kappa_1(r) - \kappa_2(r)|^p)^{\frac{1}{p}}$, then $(\Gamma, \eth)$ is known as a b-metric space with $s = 2^{\frac{1}{p}}$.

Lemma 2.1 *Let* $(\Gamma, \eth)$ *be a b-metric space and* $\Omega_1, \Omega_2 \in CB(\Gamma)$ *with* $\hat{H}_b(\Omega_1, \Omega_2) > 0$. *Then, for every* $h > 1$ *and* $\kappa \in \Omega_1$, *there is* $y = y(\kappa) \in \Omega_2$ *such that*

$$\eth(\kappa, y) < h\hat{H}_b(\Omega_1, \Omega_2).$$

Lemma 2.2 ([12]) *Let* $(\Gamma, \eth)$ *be a b-metric space. For any* $\Omega_1, \Omega_2 \in CB(\Gamma)$ *and any* $\kappa, y \in \Gamma$ *such that*

(i) $\eth_b(\kappa, \Omega_2) \leq \eth(\kappa, z)$ *for every* $z \in \Omega_2$;

(ii) $\eth_b(\kappa, \Omega_2) \leq \hat{H}_b(\Omega_1, \Omega_2)$;

(iii) $\eth_b(\kappa, \Omega_1) \leq s[\eth(\kappa, y) + \eth_b(\kappa, \Omega_2)]$.

Some other related recent developments are being recalled. Proinov [22] put forward the following concept.

Definition 2.2 Let $i \geq 1$. The function $\vartheta : J \to J$ is called a gauge function of order i on J, where $J = R^+ \cup \{0\}$, if the following conditions are satisfied:

(a) $\vartheta(\lambda\kappa) < \lambda^i \vartheta(\kappa)$ for each $\lambda \in (0,1)$ and $\kappa \in J$;
(b) $\vartheta(\kappa) < \kappa$ $\forall \kappa \in J - \{0\}$.

Easily, we conclude that the first case of Definition 2.2 is equivalent to (i) $\vartheta(0) = 0$ and (ii) $\vartheta(\kappa)/\kappa^i$ is nondecreasing on $J - \{0\}$.

Definition 2.3 ([22]) A gauge function $\vartheta : J \to J$ is called a Biachini–Grandolfi gauge function on J if

$$\sigma(\kappa) = \sum_{i=0}^{+\infty} \vartheta^i(\kappa) < \infty, \quad \forall \kappa \in J = R^{+} \cup \{0\}.$$

In view of the above observations, a Biachini–Grandolfi gauge function also satisfies the following functional equation:

$$\sigma(\kappa) = \sigma\left(\vartheta(\kappa)\right) + \kappa.$$

Lemma 2.3 ([22]) *For given $\kappa_0 \in \Lambda$ (Λ is a closed subset of Γ) such that $\eth(\kappa_0, \grave{I}\kappa_0) \in J$ and $\kappa_i \in \Lambda$ for some $i \geq 0$. Then $\eth(\kappa_i, \grave{I}\kappa_i) \in J$.*

Definition 2.4 Let $\kappa_0 \in \Lambda$ and $\eth(\kappa_0, \grave{I}\kappa_0) \in J$. Then, for each iterate $\kappa_i(i \geq 0) \in \Lambda$, define the closed ball $\overline{b}(\kappa_i, \rho)$ with center κ_i and center $\rho > 0$.

Lemma 2.4 *If an element $\kappa_0 \in \Lambda$ such that $\eth(\kappa_0, \grave{I}\kappa_0) \in J$ and $\overline{b}(\kappa_i, \rho) \subset \Lambda$ for some $i \geq 0$, then $\kappa_{i+1} \in \Lambda$ and $\overline{b}(\kappa_{i+1}, \rho) \subset \overline{b}(\kappa_i, \rho)$.*

In light of the above hypothesis the structure of a b-metric space in the class of Biachini–Grandolfi gauge function ϑ proceeds as follows.

Definition 2.5 A nondecreasing function $\vartheta : J \to J$ is called a b-Biachini–Grandolfi gauge function on J if

$$\sigma(\kappa) = \sum_{i=0}^{+\infty} s^i \vartheta^i(\kappa) < \infty, \quad \forall \kappa \in J, \tag{1}$$

where $J = R^{+}$ and $s \geq 1$. In light of the above observation, a b-Biachini–Grandolfi gauge function also satisfies the following functional equation:

$$\sigma(\kappa) = s\sigma\left(\vartheta(\kappa)\right) + \kappa. \tag{2}$$

Remark 1 Every b-Biachini–Grandolfi gauge function is also a Biachini–Grandolfi gauge function, but the converse may not be true in general. Furthermore, some definition of a gauge function in a b-metric space is the following:

$$\vartheta(\kappa) = \begin{cases} \frac{s\vartheta(\kappa)}{\kappa} & \text{if } \kappa \in J, \\ 0 & \text{if } \kappa = 0, \end{cases}$$

where s is the coefficient of a b-metric space.

Example 2.2 (i) $\vartheta(\kappa) = \frac{\lambda\kappa}{s}$ for each $\lambda \in (0,1)$ is a gauge function of order 1 on $\kappa \in J$;
(ii) $\vartheta(\kappa) = \frac{\lambda\kappa^k}{s}(\lambda > 0, k > 0)$ is a gauge function of order k on $J = (0,l)$ where $l = (\frac{1}{\lambda})^{\frac{1}{1-k}}$.

Definition 2.6 Suppose $\kappa_0 \in \Lambda$ and $\eth(\kappa_0, \grave{I}\kappa_0) \in J$. Then, for every iterate $\kappa_i(i \geq 0) \in \Lambda$, we define the closed ball $\overline{b}(\kappa_i, \rho_i)$ with center κ_i and radius $\rho_i = \sigma(\eth(\kappa_i, \grave{I}\kappa_i))$.

From now on, Jleli and Samet [17] examined the idea of a θ-contraction as follows.

Theorem 2.5 *Let* $(\Gamma, \eth)$ *be a complete metric space and* $\grave{I} : \Gamma \to \Gamma$ *be a mapping. Suppose that there are* $\theta \in \Xi$ *and* $k \in (0,1)$ *such that*

$$\kappa_1, \kappa_2 \in \Gamma, \quad \eth(\grave{I}\kappa_1, \grave{I}\kappa_2) > 0 \quad \textit{imply } \theta\big[\eth(\grave{I}\kappa_1, \grave{I}\kappa_2)\big] \leq \big[\theta\big(\eth(\kappa_1, \kappa_2)\big)\big]^k,$$

where Ξ *is the set of mappings* $\theta : (0,\infty) \to (1,\infty)$ *that satisfy* (θ_i)–(θ_{iii}):
(θ_i) θ *is nondecreasing and right-continuous*;
(θ_{ii}) *for all* $\{s_i\}$ *in* $(0,\infty)$, $\lim_{i\to\infty} \theta(s_i) = 1$ *if and only if* $\lim_{i\to\infty}(s_i) = 0$;
(θ_{iii}) *there are* $r \in (0,1)$ *and* $\kappa \in (0,+\infty]$ *such that* $\lim_{s\to 0^+} \frac{\theta(s)-1}{s^r} = \kappa$. *Then* $\grave{I}$ *has at least one fixed point.*

Example 2.3 The functions $\theta : (0,\infty) \to (1,\infty)$ defined by $\theta_1(r) = e^r$, $\theta_2(r) = e^{\sqrt{r}}, \theta_3(r) = e^{re^r}$, $\theta_4(r) = e^{\sqrt{re^r}}$, and $\theta_5(r) = 1 + \sqrt{r}$ are in Ξ.

The main purpose of this manuscript is to introduce a new concept of θ_b-contraction in a b-metric space, which is an extension of θ-contraction [17]. We prove multi-valued fixed point theorems via the b-Bianchini–Grandolfi gauge function in the class of b-metric spaces. As our generalized results are based on b-Bianchini–Grandolfi gauge function instead of the conventional operator, our newly proved works are the generalization of Ali et al. [2] and Proinov [22].

3 Main result

Further aspects of θ-contraction [17]. First, we give the following generalized definition.

We denote by Ξ_b the class of functions $\theta_b : (0,\infty) \to (1,\infty)$ satisfying the following statements:

(θ_i) θ_b is nondecreasing;
(θ_{ii}) for each $\{\kappa_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta_b(\kappa_n) = 1 \Leftrightarrow \lim_{n\to\infty}(\kappa_n) = 0$;
(θ_{iii}) there are $r \in (0,1)$ and $l \in (0,+\infty]$ such that $\lim_{\kappa\to 0^+} \frac{\theta_b(\kappa)-1}{\kappa^r} = l$;
(θ_{iv}) θ_b is right-continuous;
(θ_v) for each $\{\kappa_n\} \subset (0,\infty)$ such that $\theta_b(s\kappa_n) \leq [\theta_b(s\kappa_{n-1})]^k$ $\forall n \in N$, $s \geq 1$ and $k \in (0,1)$, we have

$$\theta_b\big(s^n\kappa_n\big) \leq \big[\theta_b\big(s^{n-1}\kappa_{n-1}\big)\big]^k \quad \forall n \in N.$$

Before going to our main exposition, first we prove an auxiliary lemma for our main sequel.

Lemma 3.1 *Let $(\Gamma, \eth, s)$ be a b-metric space and $\{\kappa_n\}$ be any sequence of Γ for which there are $k \in (0,1)$ and $\theta \in \Xi$ such that*

$$\theta_b\big[s\eth(\kappa_n, \kappa_{n+1})\big] \leq \big[\theta_b\big(\eth(\kappa_{n-1}, \kappa_n)\big)\big]^k, \quad n \in N.$$

Then $\{\kappa_n\}$ is a Cauchy sequence in Γ.

Proof In view of the given hypothesis, we have

$$\theta_b\big[s\eth(\kappa_n, \kappa_{n+1})\big] \leq \big[\theta_b\big(\eth(\kappa_{n-1}, \kappa_n)\big)\big]^k, \quad n \in N. \tag{3}$$

From (3), together with the equation (θ_v), we write

$$\theta_b\big[s^n\eth(\kappa_n, \kappa_{n+1})\big] \leq \big[\theta_b\big(s^{n-1}\eth(\kappa_{n-1}, \kappa_n)\big)\big]^k, \quad n \in N.$$

Consequently, we obtain

$$1 < \theta_b\big(s^n\eth(\kappa_n, \kappa_{n+1})\big) \leq \big[\theta_b\big(s^{n-1}\eth(\kappa_{n-1}, \kappa_n)\big)\big]^k \leq \cdots \leq \big[\theta_b\big(\eth(\kappa_0, \kappa_1)\big)\big]^{k^n}. \tag{4}$$

Taking the limit as $n \to \infty$ in (4), by appealing to $\theta_b \in \Xi_b$, we can write

$$\lim_{n\to\infty} \theta_b\big(s^n\eth(\kappa_n, \kappa_{n+1})\big) = 1.$$

In view of (θ_{ii}), we have

$$\lim_{n\to\infty} s^n\eth(\kappa_n, \kappa_{n+1}) = 0. \tag{5}$$

Now, we examine that $\{\kappa_n\}$ is a Cauchy sequence in Γ. Upon setting $\delta_n := \eth(\kappa_n, \kappa_{n+1})$ and in light of (θ_{iii}), there are $r \in (0,1)$ and $\kappa \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta_b(s^n\delta_n) - 1}{(s^n\delta_n)^r} = \kappa.$$

Taking $\lambda \in (0,\kappa)$ and from the definition of limit, there is $n_0 \in \mathbb{N}$ such that

$$\big[s^n\delta_n\big]^r \leq \lambda^{-1}\big[\theta_b\big(s^n\delta_n\big) - 1\big], \quad (\forall n > n_0).$$

Using (3) and in view of the above inequality, we obtain

$$n\big[s^n\delta_n\big]^r \leq \lambda^{-1}n\big(\big[\theta_b\big(s^n\delta_0\big)\big]^{k^n} - 1\big), \quad (\forall n > n_0),$$

which yields

$$\lim_{n\to\infty} n\big[s^n\delta_n\big]^r = \lim_{n\to\infty} n\big[s^n\delta_n\big]^r = 0.$$

Hence, there is $n_1 \in \mathbb{N}$ such that

$$\delta_n \leq \frac{1}{n^{1/r}} \quad (\forall n > n_1). \tag{6}$$

Let $p > n > n_1$. Then, using the triangular inequality and (6), we get

$$\eth(\kappa_n,\kappa_p) \le \sum_{j=n}^{p-1} s^n \eth(\kappa_j,\kappa_{j+1}) \le \sum_{j=n}^{\infty} s^n \eth(\kappa_j,\kappa_{j+1}) \le \sum_{j=n}^{\infty} \frac{1}{j^{1/r}} < \sum_{j=n}^{\infty} \frac{1}{j^{1/r}}.$$

Based on the convergence of the series $\sum_{j=n}^{\infty} \frac{1}{j^{1/r}}$, we get $\{\kappa_n\}$ is a Cauchy sequence in Γ. □

We start with the following.

Definition 3.1 Let $(\Gamma,\eth,s)$ be a b-metric space, Λ be a closed subset of Γ, and ϑ be a Bianchini–Grandolfi gauge function on J. A mapping $\grave{I} : \Lambda \to CB(\Gamma)$ is called multi-valued θ_b-contraction, if there is $\theta_b \in \Xi_b$ such that, for $\grave{I}\kappa \cap \Lambda \neq \emptyset$,

$$\frac{1}{2s}\min\{\eth_b(\kappa,\grave{I}\kappa \cap \Lambda), \eth_b(y,\grave{I}y \cap \Lambda)\} < \eth(\kappa,y)$$

implies that

$$\theta_b[sH_b(\grave{I}\kappa \cap \Lambda, \grave{I}y \cap \Lambda)] \le [\theta_b(\vartheta(\Omega(\kappa,y)))]^k, \tag{7}$$

where

$$\Omega(\kappa,y) = \max\left\{\eth(\kappa,y), \eth_b(\kappa,\grave{I}\kappa), \eth_b(y,\grave{I}y), \frac{\eth_b(\kappa,\grave{I}y) + \eth_b(y,\grave{I}\kappa)}{2s}\right\}$$

for each $\kappa \in \Lambda$, $y \in \grave{I}\kappa \cap \Lambda$ with $\eth(y,\grave{I}\kappa) \in J$, where $k \in (0,1)$.

Remark 2 Let $(\Gamma,\eth,s)$ be a b-metric space and $\grave{I} : \Lambda \to CB(\Gamma)$ be a multi-valued θ_b-contraction mapping satisfying (7) such that

$$\ln\theta_b(H_b(\grave{I}\kappa_1 \cap \Lambda, \grave{I}\kappa_2 \cap \Lambda)) \le k \ln\theta_b(\Omega(\kappa_1,\kappa_2)) < \ln\theta_b(\Omega(\kappa_1,\kappa_2)).$$

Owing to $\theta_b \in \Xi_b$, we have

$$H_b(\grave{I}\kappa_1 \cap \Lambda, \grave{I}\kappa_2 \cap \Lambda) < \Omega(\kappa_1,\kappa_2) \quad \text{for all } \kappa_1,\kappa_2 \in \kappa, \grave{I}\kappa_1 \cap \Lambda \neq \grave{I}\kappa_2 \cap \Lambda.$$

Theorem 3.2 *Let $(\Gamma,\eth,s)$ be a complete b-metric space and $\grave{I} : \Lambda \to CB(\Gamma)$ be a multi-valued θ_b-contraction. Suppose $\kappa_0 \in \Lambda$ such that $\eth(\kappa_0,c^*) \in J$ for some $c^* \in \grave{I}\kappa_0 \cap \Lambda$. Then there is an orbit $\{\kappa_i\}$ of $\grave{I}$ in Λ and $\vartheta^* \in \Lambda$ such that $\lim_{i\to\infty} \kappa_i = \vartheta^*$. In addition, ϑ^* is a fixed point of $\grave{I}$ if and only if the function $g(\kappa) := \eth(\kappa,\grave{I}\kappa \cap \Lambda)$ is $\grave{I}$-orbitally lower semi-continuous at ϑ^*.*

Proof Upon setting $\kappa_1 = c^* \in \grave{I}(\kappa_0) \cap \Lambda$. In the case that $\eth(\kappa_0,\kappa_1) = 0$, then our proof of Theorem 3.2 proceeds as follows. Thus, we assume that $\eth(\kappa_0,\kappa_1) \neq 0$. On the other hand, we obtain

$$\frac{1}{2s}\min\{\eth_b(\kappa_0,\grave{I}(\kappa_0) \cap \Lambda), \eth_b(\kappa_1,\grave{I}(\kappa_1) \cap \Lambda)\} < \eth(\kappa_0,\kappa_1). \tag{8}$$

Define $\rho = \sigma(\eth(\kappa_0,\kappa_1))$. From (2), we have $\sigma(r) \geq r$. Hence $\eth(\kappa_0,\kappa_1) \leq \rho$ and so $\kappa_1 \in \overline{b}(\kappa_0,\rho)$. Since $\eth(\kappa_0,\kappa_1) \in J$, from (7) and (8) it follows that

$$\theta_b\big[sH_d\big(\grave{I}(\kappa_0)\cap\Lambda,\grave{I}(\kappa_1)\cap\Lambda\big)\big] \leq \big[\theta_b\big(s\vartheta\big(\eth(\kappa_0,\kappa_1)\big)\big)\big]^k < \big[\theta_b\big(\Omega(\kappa_0,\kappa_1)\big)\big]^k. \tag{9}$$

By the right continuity of θ_b, there is a real number $h_1 > 1$ such that

$$\theta_b\big[sh_1H_d\big(\grave{I}(\kappa_0)\cap\Lambda,\grave{I}(\kappa_1)\cap\Lambda\big)\big] \leq \big[\theta_b\big(\Omega(\kappa_0,\kappa_1)\big)\big]^k. \tag{10}$$

From Lemma 2.2, we can write

$$s\eth_b\big(\kappa_1,\grave{I}(\kappa_1)\cap\Lambda\big) \leq sH_d\big(\grave{I}(\kappa_0)\cap\Lambda,\grave{I}(\kappa_1)\cap\Lambda\big) < sh_1H_d\big(\grave{I}(\kappa_0)\cap\Lambda,\grave{I}(\kappa_1)\cap\Lambda\big).$$

In the light of Lemma 2.1, there is $\kappa_2 \in \grave{I}(\kappa_1)\cap\Lambda$ such that $\eth(\kappa_1,\kappa_2) \leq h_1H_d(\grave{I}(\kappa_0)\cap\Lambda,\grave{I}(\kappa_1)\cap\Lambda)$. Due to (θ_i) and (10), this inequality gives that

$$\theta_b\big(s\eth(\kappa_1,\kappa_2)\big) \leq \theta_b\big[sh_1H_d\big(\grave{I}(\kappa_0)\cap\Lambda,\grave{I}(\kappa_1)\cap\Lambda\big)\big] \leq \big[\theta_b\big(\Omega(\kappa_0,\kappa_1)\big)\big]^k, \tag{11}$$

where

$$\begin{aligned}
\Omega(\kappa_0,\kappa_1) &= \max\left\{\eth(\kappa_0,\kappa_1),\eth_b\big(\kappa_0,\grave{I}(\kappa_0)\big),\eth_b\big(\kappa_1,\grave{I}(\kappa_1)\big),\frac{\eth_b(\kappa_0,\grave{I}(\kappa_1))+\eth_b(\kappa_1,\grave{I}(\kappa_0))}{2s}\right\}\\
&\leq \max\left\{\eth(\kappa_0,\kappa_1),\eth_b\big(\kappa_1,\grave{I}(\kappa_1)\big),\frac{\eth_b(\kappa_0,\grave{I}(\kappa_1))}{2s}\right\}\\
&\leq \max\big\{\eth(\kappa_0,\kappa_1),\eth_b\big(\kappa_1,\grave{I}(\kappa_1)\big)\big\}.
\end{aligned}$$

Now, we claim that

$$\theta_b\big(s\eth(\kappa_1,\kappa_2)\big) \leq \theta_b\big[sh_1H_d\big(\grave{I}(\kappa_0)\cap\Lambda,\grave{I}(\kappa_1)\cap\Lambda\big)\big] \leq \big[\theta_b\big(\eth(\kappa_0,\kappa_1)\big)\big]^k.$$

Let $\Phi = \max\{\eth(\kappa_0,\kappa_1),\eth_b(\kappa_1,\grave{I}(\kappa_1))\}$. If $\Phi = \eth_b(\kappa_1,\grave{I}(\kappa_1))$. Since $\kappa_2 \in \grave{I}(\kappa_1)\cap\Lambda$, we have

$$\theta_b\big(s\eth(\kappa_1,\kappa_2)\big) \leq \theta_b\big[sh_1H_d\big(\grave{I}(\kappa_0)\cap\Lambda,\grave{I}(\kappa_1)\cap\Lambda\big)\big] \leq \big[\theta_b\big(\eth(\kappa_1,\kappa_2)\big)\big]^k,$$

which is a contradiction. Thus, we get $\Phi = \eth(\kappa_0,\kappa_1)$. We consider $\eth(\kappa_1,\kappa_2) \neq 0$, otherwise our proof of Theorem 3.2 proceeds as follows. From Remark 2, we have $\eth(\kappa_1,\kappa_2) < \eth(\kappa_0,\kappa_1)$ and so $\eth(\kappa_1,\kappa_2) \in J$. Next, $\kappa_2 \in \overline{b}(\kappa_0,\rho)$ because

$$\left\{\begin{aligned}
\eth(\kappa_0,\kappa_2) &\leq s\eth(\kappa_0,\kappa_1) + s\eth(\kappa_1,\kappa_2)\\
&\leq s\eth(\kappa_0,\kappa_1) + s^2\eth(\kappa_1,\kappa_2)\\
&\leq s\eth(\kappa_0,\kappa_1) + s^2\vartheta(\eth(\kappa_0,\kappa_1))\\
&= s[\eth(\kappa_0,\kappa_1) + s\vartheta(\eth(\kappa_0,\kappa_1))]\\
&\leq s\sigma(\eth(\kappa_0,\kappa_1))\\
&\leq \eth(\kappa_0,\kappa_1) + s\sigma(\eth(\kappa_0,\kappa_1))\\
&= \sigma(\eth(\kappa_0,\kappa_1))\\
&= \rho.
\end{aligned}\right.$$

Also, by appealing to the above hypothesis, we have

$$\frac{1}{2s}\min\bigl\{\eth_b\bigl(\kappa_1,\grave{I}(\kappa_1)\cap\Lambda\bigr),\eth_b\bigl(\kappa_2,\grave{I}(\kappa_2)\cap\Lambda\bigr)\bigr\}<\eth(\kappa_1,\kappa_2),$$

using (7), we have

$$\theta_b\bigl[sH_d\bigl(\grave{I}(\kappa_1)\cap\Lambda,\grave{I}(\kappa_2)\cap\Lambda\bigr)\bigr]\leq\bigl[\theta_b\bigl(s\vartheta\bigl(\eth(\kappa_1,\kappa_2)\bigr)\bigr)\bigr]^k<\bigl[\theta_b\bigl(\Omega(\kappa_1,\kappa_2)\bigr)\bigr]^k. \tag{12}$$

Since θ_b is right continuous, there exists a real number $h_2>1$ such that

$$\theta_b\bigl[sh_2H_d\bigl(\grave{I}(\kappa_1)\cap\Lambda,\grave{I}(\kappa_2)\cap\Lambda\bigr)\bigr]\leq\bigl[\theta_b\bigl(\Omega(\kappa_1,\kappa_2)\bigr)\bigr]^k. \tag{13}$$

Further, from Lemma 2.2, we have

$$s\eth_b\bigl(\kappa_2,\grave{I}(\kappa_2)\cap\Lambda\bigr)\leq sH_d\bigl(\grave{I}(\kappa_1)\cap\Lambda,\grave{I}(\kappa_2)\cap\Lambda\bigr)<sh_2H_d\bigl(\grave{I}(\kappa_1)\cap\Lambda,\grave{I}(\kappa_2)\cap\Lambda\bigr),$$

and by Lemma 2.1, there is $\kappa_3\in\grave{I}(\kappa_2)\cap\Lambda$ such that $\eth(\kappa_2,\kappa_3)\leq h_2H_d(\grave{I}(\kappa_1)\cap\Lambda,\grave{I}(\kappa_2)\cap\Lambda)$. By (13), this inequality gives that

$$\begin{aligned}\theta_b\bigl(s\eth(\kappa_2,\kappa_3)\bigr)&\leq\theta_b\bigl[sh_2H_d\bigl(\grave{I}(\kappa_1)\cap\Lambda,\grave{I}(\kappa_2)\cap\Lambda\bigr)\bigr]\leq\bigl[s\theta_b\bigl(\Omega(\kappa_1,\kappa_2)\bigr)\bigr]^k\\&\leq\bigl[\theta_b\bigl(\Omega(\kappa_0,\kappa_1)\bigr)\bigr]^{k^2},\end{aligned}$$

where

$$\begin{aligned}\Omega(\kappa_1,\kappa_2)&=\max\left\{\eth(\kappa_1,\kappa_2),\eth_b\bigl(\kappa_1,\grave{I}(\kappa_1)\bigr),\eth_b\bigl(\kappa_2,\grave{I}(\kappa_2)\bigr),\frac{\eth_b(\kappa_1,\grave{I}(\kappa_2))+\eth_b(\kappa_2,\grave{I}(\kappa_1))}{2s}\right\}\\&\leq\max\left\{\eth(\kappa_1,\kappa_2),\eth_b\bigl(\kappa_2,\grave{I}(\kappa_2)\bigr),\frac{\eth_b(\kappa_1,\grave{I}(\kappa_2))}{2s}\right\}\\&\leq\max\bigl\{\eth(\kappa_1,\kappa_2),\eth_b\bigl(\kappa_2,\grave{I}(\kappa_2)\bigr)\bigr\}.\end{aligned}$$

Again, we claim that

$$\begin{aligned}\theta_b\bigl(s\eth(\kappa_2,\kappa_3)\bigr)&\leq\theta_b\bigl[sh_2H_d\bigl(\grave{I}(\kappa_1)\cap\Lambda,\grave{I}(\kappa_2)\cap\Lambda\bigr)\bigr]\leq\bigl[\theta_b\bigl(s\eth(\kappa_1,\kappa_2)\bigr)\bigr]^k\\&\leq\bigl[\theta_b\bigl(\eth(\kappa_0,\kappa_1)\bigr)\bigr]^{k^2}.\end{aligned}$$

Let $\Phi=\max\{\eth(\kappa_1,\kappa_2),\eth_b(\kappa_2,\grave{I}(\kappa_2))\}$. If $\Phi=\eth_b(\kappa_2,\grave{I}(\kappa_2))$, since $\kappa_3\in\grave{I}(\kappa_2)\cap\Lambda$, we have

$$\theta_b\bigl(s\eth(\kappa_2,\kappa_3)\bigr)\leq\theta_b\bigl[sh_2H_d\bigl(\grave{I}(\kappa_1)\cap\Lambda,\grave{I}(\kappa_2)\cap\Lambda\bigr)\bigr]\leq\bigl[\theta_b\bigl(\eth(\kappa_2,\kappa_3)\bigr)\bigr]^k,$$

which is a contradiction. Thus, we get $\Phi=\eth(\kappa_1,\kappa_2)$. We consider $\eth(\kappa_2,\kappa_3)\neq 0$, otherwise κ_2 is a fixed point of $\grave{I}$. From Remark 2 we have $\eth(\kappa_2,\kappa_3)<\eth(\kappa_1,\kappa_2)$, and so $\eth(\kappa_2,\kappa_3)\in J$.

Also, we have $\kappa_3 \in \overline{b}(\kappa_0, \rho)$ since

$$\begin{cases} \eth(\kappa_0, \kappa_3) \le s\eth(\kappa_0, \kappa_1) + s^2\eth(\kappa_1, \kappa_2) + s^3\eth(\kappa_2, \kappa_3) \\ \quad = s[\eth(\kappa_0, \kappa_1) + s\eth(\kappa_1, \kappa_2) + s^2\eth(\kappa_2, \kappa_3)] \\ \quad \le s[\eth(\kappa_0, \kappa_1) + \vartheta(\eth(\kappa_0, \kappa_1)) + \vartheta^2(\eth(\kappa_0, \kappa_1))] \\ \quad \le s\sigma\eth(\kappa_0, \kappa_1) \\ \quad \le \eth(\kappa_0, \kappa_1) + s\sigma(\eth(\kappa_0, \kappa_1)) \\ \quad = \sigma(\eth(\kappa_0, \kappa_1)) = \rho. \end{cases}$$

Continuing this set up, we have two sequences $\{\kappa_i\} \subset \overline{b}(\kappa_0, \rho)$ and $\{h_i\} \subset (1, \infty)$ such that $\kappa_{i+1} \in \grave{I}(\kappa_i) \cap \Lambda$, $\kappa_i \neq \kappa_{i+1}$ with $\eth(\kappa_i, \kappa_{i+1}) \in J$ and

$$\begin{aligned} 1 &< \theta_b\bigl(s\eth(\kappa_i, \kappa_{i+1})\bigr) \le \theta_b\bigl(sh_iH_d\bigl(\grave{I}(\kappa_{i-1}) \cap \Lambda, \grave{I}(\kappa_i) \cap \Lambda\bigr)\bigr) \\ &\le \bigl[\theta_b\bigl(\eth(\kappa_{i-1}, \kappa_i)\bigr)\bigr]^k < \theta_b\bigl(\eth(\kappa_{i-1}, \kappa_i)\bigr) \end{aligned}$$

for all $i \in \mathbb{N}$. It follows with (θ_v) that

$$\begin{aligned} 1 &< \theta_b\bigl(s^i\eth(\kappa_i, \kappa_{i+1})\bigr) \le \theta_b\bigl(s^ih_iH_d\bigl(\grave{I}(\kappa_{i-1}) \cap \Lambda, \grave{I}(\kappa_i) \cap \Lambda\bigr)\bigr) \\ &\le \bigl[\theta_b\bigl(s^{i-1}\eth(\kappa_{i-1}, \kappa_i)\bigr)\bigr]^k < \theta_b\bigl(s^{i-1}\eth(\kappa_{i-1}, \kappa_i)\bigr). \end{aligned}$$

Further, we obtain

$$\begin{aligned} 1 &< \theta_b\bigl(s^i\eth(\kappa_i, \kappa_{i+1})\bigr) \le \theta_b\bigl(s^ih_iH_d\bigl(\grave{I}(\kappa_{i-1}) \cap \Lambda, \grave{I}(\kappa_i) \cap \Lambda\bigr)\bigr) \\ &\le \bigl[\theta_b\bigl(s^{i-1}\eth(\kappa_{i-1}, \kappa_i)\bigr)\bigr]^k \\ &\le \bigl[\theta_b\bigl(s^{i-2}\eth(\kappa_{i-1}, \kappa_{i-2})\bigr)\bigr]^{k^2} \\ &\le \cdots \le \bigl[\theta_b\bigl(\eth(\kappa_0, \kappa_1)\bigr)\bigr]^{k^i}, \end{aligned}$$

which yields

$$1 < \theta_b\bigl(s^i\eth(\kappa_i, \kappa_{i+1})\bigr) \le \bigl[\theta_b\bigl(\eth(\kappa_0, \kappa_1)\bigr)\bigr]^{k^i} \quad (\forall i \in \mathbb{N}). \tag{14}$$

By Lemma 3.1, $\{\kappa_i\}$ is a Cauchy sequence in $\overline{b}(\kappa_0, \rho)$. Since $(\kappa, \eth, s)$ is complete and $\overline{b}(\kappa_0, \rho)$ is closed in Γ, there is $\vartheta^* \in \overline{b}(\kappa_0, \rho)$ such that $\kappa_i \to \vartheta^*$. Note that $\vartheta^* \in \Lambda$, because $\kappa_{i+1} \in \grave{I}\kappa_i \cap \Lambda$. Now, we claim that

$$\frac{1}{2s}\min\bigl\{\eth_b(\kappa_i, \grave{I}\kappa_i \cap \Lambda), \eth_b\bigl(\sigma^*, \grave{I}\sigma^* \cap \Lambda\bigr)\bigr\} < \eth\bigl(\kappa_i, \sigma^*\bigr) \tag{15}$$

or

$$\frac{1}{2s}\min\bigl\{\eth_b\bigl(\sigma^*, \grave{I}\sigma^* \cap \Lambda\bigr), \eth_b(\kappa_{i+1}, \grave{I}\kappa_{i+1} \cap \Lambda)\bigr\} < \eth\bigl(\kappa_{i+1}, \sigma^*\bigr) \tag{16}$$

for every $i \in \mathbb{N}$. Based on the contrary there is $i' \in \mathbb{N}$ such that

$$\frac{1}{2s}\min\{\eth_b(\kappa_{i'},\grave{I}\kappa_{i'}\cap\Lambda),\eth_b(\sigma^*,\grave{I}\sigma^*\cap\Lambda)\}\geq\eth(\kappa_{i'},\sigma^*) \tag{17}$$

and

$$\frac{1}{2s}\min\{\eth_b(\sigma^*,\grave{I}\sigma^*\cap\Lambda),\eth_b(\kappa_{i'+1},\grave{I}\kappa_{i'+1}\cap\Lambda)\}\geq\eth(\kappa_{i'+1},\sigma^*). \tag{18}$$

By (17), we have

$$\begin{aligned}2s\eth(\kappa_{i'},\sigma^*)&\leq\min\{\eth_b(\kappa_{i'},\grave{I}\kappa_{i'}\cap\Lambda),\eth_b(\sigma^*,\grave{I}\sigma^*\cap\Lambda)\}\\&\leq\min\{s[\eth(\kappa_{i'},\sigma^*)+\eth_b(\sigma^*,\grave{I}\kappa_{i'}\cap\Lambda)],\eth_b(\sigma^*,\grave{I}\sigma^*\cap\Lambda)\}\\&\leq s[\eth(\kappa_{i'},\sigma^*)+\eth_b(\sigma^*,\grave{I}\kappa_{i'}\cap\Lambda)]\\&< s[\eth(\kappa_{i'},\sigma^*)+\eth_b(\sigma^*,\grave{I}\kappa_{i'})]\\&\leq s[\eth(\kappa_{i'},\sigma^*)+\eth(\sigma^*,\kappa_{i'+1})],\end{aligned}$$

which implies

$$\eth(\kappa_{i'},\sigma^*)\leq\eth(\sigma^*,\kappa_{i'+1}).$$

From which together with (18), we have

$$\begin{aligned}\eth(\kappa_{i'},\sigma^*)&\leq\eth(\sigma^*,\kappa_{i'+1})\\&\leq\frac{1}{2s}\min\{\eth_b(\sigma^*,\grave{I}\sigma^*\cap\Lambda),\eth_b(\kappa_{i'+1},\grave{I}\kappa_{i'+1}\cap\Lambda)\}.\end{aligned} \tag{19}$$

Since

$$\frac{1}{2s}\min\{\eth_b(\kappa_{i'},\grave{I}\kappa_{i'}\cap\Lambda),\eth_b(\kappa_{i'+1},\grave{I}\kappa_{i'+1}\cap\Lambda)\}<\eth(\kappa_{i'},\kappa_{i'+1}),$$

by appealing to (7), we have

$$0<\theta_b[s\eth(\kappa_{i'+1},\kappa_{i'+2})]\leq\theta_b[sh_2H_b(\grave{I}\kappa_{i'}\cap\Lambda,\grave{I}\kappa_{i'+1}\cap\Lambda)]\leq\theta_b[\vartheta(\eth(\kappa_{i'},\kappa_{i'+1})))]^k, \tag{20}$$

where

$$\begin{aligned}\Omega(\kappa_{i'},\kappa_{i'+1})&=\max\left\{\begin{matrix}\eth(\kappa_{i'},\kappa_{i'+1}),\eth_b(\kappa_{i'},\grave{I}\kappa_{i'}),\eth_b(\kappa_{i'+1},\grave{I}\kappa_{i'+1}),\\\frac{\eth_b(\kappa_{i'},\grave{I}\kappa_{i'+1})+\eth_b(\kappa_{i'+1},\grave{I}\kappa_{i'})}{2s}\end{matrix}\right\}\\&\leq\max\left\{\begin{matrix}\eth(\kappa_{i'},\kappa_{i'+1}),\eth(\kappa_{i'+1},\kappa_{i'+2}),\\\frac{\eth(\kappa_{i'},\kappa_{i'+2})}{2s}\end{matrix}\right\}\\&\leq\max\{\eth(\kappa_{i'},\kappa_{i'+1}),\eth(\kappa_{i'+1},\kappa_{i'+2})\},\end{aligned}$$

which yields

$$\theta_b[s\eth(\kappa_{i'+1},\kappa_{i'+2})]\leq\theta_b[sh_2H_b(\grave{I}\kappa_{i'}\cap\Lambda,\grave{I}\kappa_{i'+1}\cap\Lambda)]\leq\theta_b[\vartheta(\eth(\kappa_{i'},\kappa_{i'+1})))]^k. \tag{21}$$

Let $\Delta = \max\{\eth(\kappa_{i'}, \kappa_{i'+1}), \eth(\kappa_{i'+1}, \kappa_{i'+2})\}$. If $\Delta = \eth(\kappa_{i'+1}, \kappa_{i'+2})$, since $\kappa_{i'+2} \in \grave{I}\kappa_{i'+1} \cap \Lambda$, we have

$$\theta_b\big[s\eth(\kappa_{i'+1}, \kappa_{i'+2})\big] \le \theta_b\big[sH_b(\grave{I}\kappa_{i'} \cap \Lambda, \grave{I}\kappa_{i'+1} \cap \Lambda)\big] \le \theta_b\big[\vartheta\left(\eth(\kappa_{i'}, \kappa_{i'+1})\right)\big]^k,$$

which is a contradiction. Thus, we conclude that $\Delta = \eth(\mu_{i'}, \mu_{i'+1})$. From Remark 2, we have

$$\eth(\kappa_{i'+1}, \kappa_{i'+2}) < \eth(\kappa_{i'}, \kappa_{i'+1}). \tag{22}$$

From (18), (19), and (22), we obtain

$$\begin{aligned}
\eth(\kappa_{i'+1}, \kappa_{i'+2}) &< \eth(\kappa_{i'}, \kappa_{i'+1}) \\
&\le s\big[\eth(\kappa_{i'}, \sigma^*) + \eth(\sigma^*, \kappa_{i'+1})\big] \\
&\le \begin{bmatrix} \frac{1}{2}\min\{\eth_b(\sigma^*, \grave{I}\sigma^* \cap \Lambda), \eth_b(\kappa_{i'+1}, \grave{I}\kappa_{i'+1} \cap \Lambda)\} \\ +\frac{1}{2}\min\{\eth_b(\sigma^*, \grave{I}\sigma^* \cap \Lambda), \eth_b(\kappa_{i'+1}, \grave{I}\kappa_{i'+1} \cap \Lambda)\} \end{bmatrix} \\
&\le \min\big\{\eth_b\big(\sigma^*, \grave{I}\sigma^* \cap \Lambda\big), \eth(\kappa_{i'+1}, \kappa_{i'+2})\big\} \\
&= \eth(\kappa_{i'+1}, \kappa_{i'+2}),
\end{aligned}$$

which is a contradiction. Hence (15) holds true, that is,

$$\frac{1}{2s}\min\big\{\eth_b(\kappa_i, \grave{I}\kappa_i \cap \Lambda), \eth_b\big(\sigma^*, \grave{I}\sigma^* \cap \Lambda\big)\big\} < \eth\big(\kappa_i, \sigma^*\big) \quad \forall i \ge 2. \tag{23}$$

In light of (23), we obtain

$$\frac{1}{2s}\min\big\{\eth_b(\kappa_i, \grave{I}\kappa_i \cap \Lambda), \eth_b(\kappa_{i+1}, \grave{I}\kappa_{i+1} \cap \Lambda)\big\} < \eth(\kappa_i, \kappa_{i+1}).$$

Moreover, $\eth(\kappa_i, \kappa_{i+1}) \in E\ \forall i$. Thus, from (7) we have

$$\begin{aligned}
\theta_b\big[s\eth_b(\kappa_{i+1}, \grave{I}\kappa_{i+1} \cap \Lambda)\big] &\le \theta_b\big[sH_b(\grave{I}\kappa_i \cap \Lambda, \grave{I}\kappa_{i+1} \cap \Lambda)\big] \\
&\le \big[\theta_b\big(\vartheta\left(\Omega(\kappa_i, \kappa_{i+1})\right)\big)\big]^k \\
&< \big[\theta_b\big(\Omega(\kappa_i, \kappa_{i+1})\big)\big]^k,
\end{aligned}$$

where

$$\begin{aligned}
\Omega(\kappa_i, \kappa_{i+1}) &= \max\left\{\begin{matrix} \eth(\kappa_i, \kappa_{i+1}), \eth_b(\kappa_i, \grave{I}\kappa_i), \eth_b(\kappa_{i+1}, \grave{I}\kappa_{i+1}), \\ \frac{\eth_b(\kappa_i, \grave{I}\kappa_{i+1}) + \eth_b(\kappa_{i+1}, \grave{I}\kappa_i)}{2s} \end{matrix}\right\} \\
&\le \max\left\{\begin{matrix} \eth(\kappa_i, \kappa_{i+1}), \eth(\kappa_{i+1}, \kappa_{i+2}), \\ \frac{\eth(\kappa_i, \kappa_{i+2})}{2s} \end{matrix}\right\} \\
&\le \max\big\{\eth(\kappa_i, \kappa_{i+1}), \eth(\kappa_{i+1}, \kappa_{i+2})\big\}
\end{aligned}$$

implies

$$\theta_b\big[s\eth(\kappa_{i+1}, \kappa_{i+2})\big] \le \theta_b\big[sH_b(\grave{I}\kappa_i \cap \Lambda, \grave{I}\kappa_{i+1} \cap \Lambda)\big] < \big[\theta_b\big(\eth(\kappa_i, \kappa_{i+1})\big)\big]^k. \tag{24}$$

Let $\Delta = \max\{\eth(\kappa_i,\kappa_{i+1}),\eth(\kappa_{i+1},\kappa_{i+2})\}$. If $\Delta = \eth(\kappa_{i+1},\kappa_{i+2})$, since $\kappa_{i+2} \in \grave{I}\kappa_{i+1} \cap \Lambda$, we have

$$\theta_b\big[s\eth(\kappa_{i+1},\kappa_{i+2})\big] \leq \theta_b\big[sH_b(\grave{I}\kappa_i \cap \Lambda, \grave{I}\kappa_{i+1} \cap \Lambda)\big] < \big[\theta_b\big(\eth(\kappa_{i+1},\kappa_{i+2})\big)\big]^k,$$

which is a contradiction. Based on Remark 2, we can write

$$\eth_b(\kappa_{i+1}, \grave{I}\kappa_{i+1} \cap \Lambda) < \eth(\kappa_i,\kappa_{i+1}). \tag{25}$$

Taking limit as $i \to +\infty$ in (25), we find

$$\lim_{i\to+\infty} \eth_b(\kappa_{i+1}, \grave{I}\kappa_{i+1} \cap \Lambda) = 0.$$

Since $g(\kappa) = \eth_b(\kappa, \grave{I}\kappa \cap \Lambda)$ is $\grave{I}$-orbitally lower semi-continuous at σ^*, then

$$\eth_b\big(\sigma^*, \grave{I}\sigma^* \cap \Lambda\big) = g\big(\sigma^*\big) \leq \liminf_i g(\kappa_{i+1}) = \liminf_i \eth_b(\kappa_{i+1}, \grave{I}\kappa_{i+1} \cap \Lambda) = 0.$$

Since $\grave{I}\sigma^*$ is closed, we have $\sigma^* \in \grave{I}\sigma^*$. Conversely, if σ^* is a fixed point of $\grave{I}$, then $g(\sigma^*) = 0 \leq \liminf_i g(\kappa_i)$ since $\sigma^* \in \Lambda$. □

Corollary 1 *Let $(\Gamma,\eth,s)$ be a complete b-metric space, ϑ be a b-Bianchini–Grandolfi gauge function on J, and let $\grave{I} : \Lambda \to CB(\Gamma)$ be a given set-valued mapping. If there exist $\theta_b \in \Xi_b$ and $k \in (0,1)$ such that*

$$\frac{1}{2s}\min\big\{\eth_b(\kappa, \grave{I}\kappa \cap \Lambda), \eth_b(y, \grave{I}y \cap \Lambda)\big\} < \eth(\kappa,y), \tag{26}$$
$$\Rightarrow \quad \theta_b\big[sH_b(\grave{I}\kappa, \grave{I}y)\big] \leq \big[\theta_b\big(\vartheta\big(\eth(\kappa,y)\big)\big)\big]^k$$

for every $\kappa \in \kappa$, $y \in \grave{I}\kappa$ with $\eth(\kappa,y) \in J$. Suppose that $\kappa_0 \in \kappa s.t \eth(\kappa_0,c^*) \in J$ for some $c^* \in \grave{I}\kappa_0$. Then there is an orbit $\{\kappa_i\}$ of $\grave{I}$ in Γ that converges to the fixed point $\vartheta^* \in \mathcal{F} = \{\kappa \in \Gamma : \eth(\kappa,\vartheta^*) \in J\}$ of $\grave{I}$.

Corollary 2 *Let $(\Gamma,\eth,s)$ be a complete b-metric space, ϑ be a b-Bianchini–Grandolfi gauge function on J, and let $\grave{I} : \Lambda \to CB(\Gamma)$ be a given set-valued mapping. If there are $\theta_b \in \Xi_b$ and $k \in (0,1)$ such that*

$$\frac{1}{2s}\eth_b(\kappa, \grave{I}\kappa \cap \Lambda) < \eth(\kappa,y), \tag{27}$$
$$\Rightarrow \quad \theta_b\big[sH_b(\grave{I}\kappa, \grave{I}y)\big] \leq \big[\theta_b\big(\vartheta\big(\eth(\kappa,y)\big)\big)\big]^k$$

for each $\kappa \in \kappa$, $y \in \grave{I}\kappa$ with $\eth(\kappa,y) \in J$. Suppose that $\kappa_0 \in \kappa s.t \eth(\kappa_0,c^*) \in J$ for some $c^* \in \grave{I}\kappa_0$. Then there is an orbit $\{\kappa_i\}$ of $\grave{I}$ in Γ that converges to the fixed point $\vartheta^* \in \mathcal{F} = \{\kappa \in \Gamma : \eth(\kappa,\vartheta^*) \in J\}$ of $\grave{I}$.

Corollary 3 *Let $(\Gamma,\eth,s)$ be a complete b-metric space, ϑ be a b-Bianchini–Grandolfi gauge function on an interval J, and let $\grave{I} : \Lambda \to CB(\Gamma)$ be a given set-valued mapping. If there are $\theta_b \in \Xi_b$ and $k \in (0,1)$ such that, for $\grave{I}\kappa \cap \Lambda \neq \emptyset$,*

$$\theta_b\big[sH_b(\grave{I}\kappa \cap \Lambda, \grave{I}y \cap \Lambda)\big] \leq \big[\theta_b\big(\vartheta\big(\eth(\kappa,y)\big)\big)\big]^k, \tag{28}$$

$\forall\kappa \in \Lambda$, $y \in \grave{I}\kappa \cap \Lambda$ with $\eth(\kappa,y) \in J$. Suppose $\kappa_0 \in \Lambda s.t \eth(\kappa_0,c^*) \in J$ for some $c^* \in \grave{I}\kappa_0 \cap \Lambda$. Then there exists an orbit $\{\kappa_i\}$ of $\grave{I}$ in Λ and $\vartheta^* \in \Lambda$ such that $\lim_{i\to\infty}\kappa_i = \vartheta^*$. Moreover, ϑ^* is a fixed point of $\grave{I}$ $\Leftrightarrow$, the function $g(\kappa) := \eth_b(\kappa, \grave{I}\kappa \cap \Lambda)$ is $\grave{I}$-orbitally lower semi-continuous at point ϑ^*.

Corollary 4 *Let $(\Gamma,\eth,s)$ be a complete b-m.s and ϑ be a b-Bianchini–Grandolfi gauge function on an interval J, and let $\grave{I}:\Lambda \to CB(\Gamma)$ be a multi-valued θ_b-contraction mapping. If there are $\theta_b \in \Xi_b$ and $k \in (0,1)$ such that, for $\grave{I}\kappa \cap \Lambda \neq \emptyset$,*

$$\frac{1}{2(1+\sigma)}\min\{\eth_b(\kappa,\grave{I}\kappa\cap\Lambda),\eth_b(y,\grave{I}y\cap\Lambda)\} < \eth(\kappa,y) \tag{29}$$

implies that

$$\theta_b\left[sH_b(\grave{I}\kappa\cap\Lambda,\grave{I}y\cap\Lambda)\right] \le \left[\theta_b\left(\vartheta\left(\Omega(\kappa,y)\right)\right)\right]^k,$$

where

$$\Omega(\kappa,y) = \max\left\{\eth(\kappa,y),\eth_b(\kappa,\grave{I}\kappa),\eth_b(y,\grave{I}y),\frac{\eth_b(\kappa,\grave{I}y)+\eth_b(y,\grave{I}\kappa)}{2s}\right\}$$

for all $\kappa \in \Lambda$, $y \in \grave{I}\kappa \cap \Lambda$, $\sigma > 0$ with $\eth(\kappa,y) \in J$. Moreover, suppose $\kappa_0 \in \Lambda$ such that $\eth(\kappa_0,c^*) \in J$ for some $c^* \in \grave{I}\kappa_0 \cap \Lambda$. Then there exists an orbit $\{\kappa_i\}$ of $\grave{I}$ in Λ, $\vartheta^* \in \Lambda$ such that $\lim_{i\to\infty}\kappa_i = \vartheta^*$ and ϑ^* is a fixed point of $\grave{I}$ $\Leftrightarrow$, the function $g(\kappa) := \eth_b(\kappa,\grave{I}\kappa \cap \Lambda)$ is $\grave{I}$-orbitally lower semi-continuous at ϑ^*.

Corollary 5 *Let $(\Gamma,\eth,s)$ be a complete b-metric space and ϑ be a b-Bianchini–Grandolfi gauge function on an interval J, and let $\grave{I}:\Lambda \to CB(\Gamma)$ be a multi-valued θ_b-contraction mapping. If there are $\theta_b \in \Xi_b$ and $k \in (0,1)$ such that, for $\grave{I}\kappa \cap \Lambda \neq \emptyset$,*

$$\frac{1}{2(1+\sigma)}\min\{\eth_b(\kappa,\grave{I}\kappa\cap\Lambda),\eth_b(y,\grave{I}y\cap\Lambda)\} < \eth(\kappa,y) \tag{30}$$

implies that

$$\theta_b\left[sH_b(\grave{I}\kappa\cap\Lambda,\grave{I}y\cap\Lambda)\right] \le \left[\theta_b\left(\vartheta\left(\eth(\kappa,y)\right)\right)\right]^k$$

for all $\kappa \in \Lambda$, $y \in \grave{I}\kappa \cap \Lambda$, $\sigma > 0$ with $\eth(\kappa,y) \in J$. In addition, suppose $\kappa_0 \in \Lambda$ such that $\eth(\kappa_0,c^*) \in J$ for some $c^* \in \grave{I}\kappa_0 \cap \Lambda$. Then $\exists$ an orbit $\{\kappa_i\}$ of $\grave{I}$ in Λ, $\vartheta^* \in \Lambda s.t \lim_{i\to\infty}\kappa_i = \vartheta^*$ and ϑ^* is a fixed point of $\grave{I}$ $\Leftrightarrow$ the function $g(\kappa) := \eth_b(\kappa,\grave{I}\kappa \cap \Lambda)$ is $\grave{I}$-orbitally lower semi-continuous at ϑ^*.

Example 3.1 Let $\Gamma = [-10,+\infty)$ be an b-metric $\eth$ defined by $\eth(\kappa,y) = |\kappa - y|^2$ for every $\kappa, y \in \Gamma$, and let $J = (0,\infty)$. Mapping $\grave{I}:\Lambda \to CB(\Gamma)$ is defined as

$$\grave{I}(\kappa) = \begin{cases} [0,\frac{\kappa}{8}], & \kappa \in [0,4], \\ \{0,\kappa\}, & \kappa \in [-10,0)\cup(4,\infty). \end{cases}$$

Clearly, $\frac{1}{2s}\min\{\eth_b(\kappa,\grave{I}\kappa\cap\Lambda),\eth_b(y,\grave{I}y\cap\Lambda)\}<\eth(\kappa,y)\Leftrightarrow\kappa,y\in[0,4]$. Let $\kappa_0=4$, then we have $c^*=\frac{1}{2}\in\grave{I}\kappa_0$ such that $\eth(\kappa_0,c^*)\in J$. Firstly, we claim that $\grave{I}$ satisfies inequality (26) with setting $\theta_b(r)=e^{\sqrt{re^r}}$, $\vartheta(r)=\frac{r}{2}$, and $k=\frac{1}{2}$. For $\kappa\in[0,4]$ and $y\in\grave{I}\kappa$, we obtain

$$\theta_b\big[H_b(\grave{I}\kappa,\grave{I}y)\big]=\theta_b\left(\frac{|\kappa-y|^2}{8}\right)\le e^{\frac{1}{2}\sqrt{\frac{|\kappa-y|^2}{2}e^{\frac{|\kappa-y|^2}{2}}}}=\big[\theta_b\big(\vartheta\big(\eth(\kappa,y)\big)\big)\big]^k.$$

Hence, the requirements of Corollary 1 are fulfilled and 0 is a fixed point of $\grave{I}$. For $\kappa=0$ and $y=5$,

$$\theta_b\big[H_b(\grave{I}\kappa,\grave{I}y)\big]=\theta_b\big[H_d(\grave{I}0,\grave{I}5)\big]=\theta_b(25)>\big[\theta_b(25)\big]^k=\big[\theta_b\big(\eth(\kappa,y)\big)\big]^k$$

$\forall\theta_b\in\Xi$ and $k\in(0,1)$. Therefore, Corollary 1 cannot satisfied.

Example 3.2 Let $\Gamma=\{\kappa_1,\kappa_2,\kappa_3\}$ be a b-metric $\eth$ with coefficient $s\ge\frac{D^2}{D^2-1}>1$ where $D\ge3$ is any positive integers defined by

$$\eth(\kappa_1,\kappa_2)=\frac{1}{D^2},\qquad\eth(\kappa_2,\kappa_3)=\frac{1}{D-1},\qquad\eth(\kappa_1,\kappa_3)=\frac{1}{D}.$$

The mapping $\grave{I}:\Lambda\to CB(\Gamma)$ is defined as

$$\grave{I}\kappa_i=\begin{cases}\{\kappa_1\} & \text{if } i=1;\\ \{\kappa_1\} & \text{if } i=2;\\ \{\kappa_2\} & \text{if } i=3.\end{cases}$$

Upon setting $\vartheta(r)=r^2$ and $\theta_b(r)=e^{\sqrt{re^r}}$, then ϑ is a b-Bianchini–Grandolfi gauge function on the interval $J=(0,\frac{1}{D-1}]$ with coefficient $\frac{D^2}{D^2-1}$ having order 2. Hence, in light of the above example it is easy to conclude that all conditions of Corollary 1 are satisfied.

4 An application

In this frame of study, we summarize by the application of the proposed nonlinear θ_b-contractions to Caputo fractional derivatives. Some new aspects of Liouville–Caputo fractional differential equations (L.C.F.D.E) in the module of complete b-metric space are presented. Define the L.C.F.D.E based on order $\kappa(\check{D}_{(c,\kappa)})$ by

$$\check{D}_{(c,\kappa)}\big(\alpha(g)\big)=\frac{1}{\Gamma(i-\kappa)}\int_0^g(g-t)^{i-\kappa-1}\alpha^{(i)}(t)\,dt,\tag{31}$$

where $i-1<\kappa<i$, $i=[\kappa]+1$, $\alpha\in C^i([0,+\infty])$, the collection $[\kappa]$ represents positive real number and Γ represents the gamma function. Let a b-metric space $\delta_\varsigma:C(I)\times C(I)\to R^+$ be given by

$$\delta_\varsigma(\varepsilon_{i-1},\varepsilon_i)=\big\|(\varepsilon_1-\varepsilon_2)^2\big\|_\infty=\sup_{a\in I}\big|\varepsilon_1(a)-\varepsilon_2(a)\big|^2\tag{32}$$

with setting $s = 2$. Now, consider the following fashion of F.D.E and its integral boundary valued problem:

$$\check{D}_{(c,\kappa)}\big(\beta(g)\big) = L\big(g,\beta(g)\big), \tag{33}$$

where $g \in (0,1)$, $\kappa \in (1,2]$ and

$$\begin{cases} \beta(0) = 0, \\ \beta(1) = \int_0^{\vartheta} \beta(g)\,dg, \quad \vartheta \in (0,1), \end{cases} \tag{34}$$

where $I = [0,1]$, $\beta \in C(I,R)$ and $L : I \times R \to R$ is a continuous function. Let $P : \Delta \to \Delta$ be defined by

$$Pv(r) = \begin{cases} \frac{1}{\Gamma(\kappa)} \int_0^g (g-t)^{\kappa-1} L(t,v(t))\,dt \\ \quad - \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^1 (1-t)^{\kappa-1} L(t,v(t))\,dt \\ \quad + \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^{\vartheta} (\int_0^{g_1} (g_1-t_1)^{\kappa-1} L(t_1,v(t_1))\,dt_1)\,dt \end{cases} \tag{35}$$

for $v \in \Delta$ and $g \in [0,1]$. Now, we start the main result.

Theorem 4.1 *Let $L : I \times R \to R$ be a continuous function and nondecreasing on the second variable. There is $\theta_b \in \Xi_b$ such that, for $\varepsilon_1, \varepsilon_2 \in \Delta$, $g \in [0,1]$, and $\alpha \in [0,1)$,*

$$\frac{1}{2s} \min\big\{\eth_b(\varepsilon_1, \grave{I}\varepsilon_1 \cap \Lambda), \eth_b(\varepsilon_2, \grave{I}\varepsilon_2 \cap \Lambda)\big\} < \eth(\varepsilon_1, \varepsilon_2)$$

implies that

$$\big|P\varepsilon_1(r) - P\varepsilon_2(r)\big| \le \Omega\Big(\Big[1 + \sqrt{\max_{g\in I} V(\varepsilon_1,\varepsilon_2)(r)}\Big]^{\alpha} - 1\Big)^2, \tag{36}$$

where $\Omega = \frac{(2\kappa-1)(\Gamma(\kappa+1))}{2(5\kappa+2)}$ and

$$V(\varepsilon_{i-1},\varepsilon_i)(r) = \max\left\{ \begin{array}{c} \big|\varepsilon_1(r) - \varepsilon_2(r)\big|^2, \big|\varepsilon_1(r) - \Upsilon\varepsilon_1(r)\big|^2, \big|\varepsilon_2(r) - \Upsilon\varepsilon_2(r)\big|^2, \\ \frac{|\varepsilon_1(r)-\Upsilon\varepsilon_2(r)|^2 + |\varepsilon_2(r)-\Upsilon\varepsilon_1(r)|^2}{2s} \end{array} \right\}.$$

Then equations (33) and (34) have precisely one solution, i.e., $\varepsilon^* \in \Delta$.

Proof For each $g \in I$ and owing to operator P, we write

$$\begin{aligned} &\big|P\varepsilon_1(r) - P\varepsilon_2(r)\big| \\ &\quad = \bigg|\bigg(\frac{1}{\Gamma(\kappa)} \int_0^g (g-t)^{\kappa-1} L\big(t,\varepsilon_1(t)\big)\,dt \\ &\qquad - \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^1 (1-t)^{\kappa-1} L\big(t,\varepsilon_{i-1}(t)\big)\,dt \\ &\qquad + \frac{2g}{(2-\vartheta^2)\Gamma(\kappa)} \int_0^{\vartheta} \bigg(\int_0^{g_1} (g_1-t_1)^{\kappa-1} L\big(t_1,\varepsilon_1(t_1)\big)\,dt_1\bigg)\,dt\bigg) \end{aligned}$$

$$\begin{aligned}&-\Big(\bigg(\frac{1}{\Gamma(\kappa)}\int_0^g (g-t)^{\kappa-1}L\big(t,\varepsilon_2(t)\big)\,dt\\&-\frac{2g}{(2-\vartheta^2)\Gamma(\kappa)}\int_0^1 (1-t)^{\kappa-1}L\big(t,\varepsilon_2(t)\big)\,dt\\&+\frac{2g}{(2-\vartheta^2)\Gamma(\kappa)}\int_0^\vartheta\bigg(\int_0^{g_1}(g_1-t_1)^{\kappa-1}L\big(t_1,\varepsilon_2(t_1)\big)\,dt_1\bigg)\,dt\bigg)\bigg|,\end{aligned}$$

which implies

$$\begin{aligned}&\big|P\varepsilon_1(r)-P\varepsilon_2(r)\big|\\&\quad\le\frac{1}{\Gamma(\kappa)}\int_0^g (g-t)^{\kappa-1}\big|L\big(t,\varepsilon_1(t)\big)-L\big(t,\varepsilon_2(t)\big)\big|\,dt\\&\qquad+\frac{2g}{(2-\vartheta^2)\Gamma(\kappa)}\int_0^1 (1-t)^{\kappa-1}\big|L\big(t,\varepsilon_1(t)\big)-L\big(t,\varepsilon_2(t)\big)\big|\,dt\\&\qquad+\frac{2g}{(2-\vartheta^2)\Gamma(\kappa)}\int_0^\vartheta\bigg|\int_0^{g_1}(g_1-t_1)^{\kappa-1}\big(L\big(t_1,\varepsilon_1(t_1)\big)-L\big(t,\varepsilon_2(t)\big)\big)\,dt_1\bigg|\,dt\\&\quad\le\frac{1}{\Gamma(\kappa)}\int_0^g (g-t)^{\kappa-1}\Omega\bigg(\bigg[1+\sqrt{\max_{g\in I}V(\varepsilon_1,\varepsilon_2)(r)}\bigg]^\alpha-1\bigg)^2\,dt\\&\qquad+\frac{2g}{(2-\vartheta^2)\Gamma(\kappa)}\int_0^1 (1-t)^{\kappa-1}\Omega\bigg(\bigg[1+\sqrt{\max_{g\in I}V(\varepsilon_1,\varepsilon_2)(r)}\bigg]^\alpha-1\bigg)^2\,dt\\&\qquad+\frac{2g}{(2-\vartheta^2)\Gamma(\kappa)}\int_0^\vartheta\int_0^{g_1}(g_1-t_1)^{\kappa-1}\Omega\bigg(\bigg[1+\sqrt{\max_{g\in I}V(\varepsilon_1,\varepsilon_2)(r)}\bigg]^\alpha-1\bigg)^2\,dt_1\,dt\\&\quad\le\frac{\Omega}{\Gamma(\kappa)}\bigg(\bigg[1+\sqrt{\max_{g\in I}V(\varepsilon_1,\varepsilon_2)(r)}\bigg]^\alpha-1\bigg)^2\left\{\begin{array}{c}\int_0^g (g-t)^{\kappa-1}\,dt\\+\frac{2g}{(2-\vartheta^2)}\int_0^1(1-t)^{\kappa-1}\,dt\\+\frac{2g}{(2-\vartheta^2)}\int_0^\vartheta\int_0^{g_1}(g_1-t_1)^{\kappa-1}\,dt_1\,dt\end{array}\right\},\end{aligned}$$

which yields

$$\begin{aligned}\big|P\varepsilon_1(r)-P\varepsilon_2(r)\big|&\le\frac{\Omega}{\Gamma(\kappa)}\bigg(\bigg[1+\sqrt{\max_{g\in I}V(\varepsilon_1,\varepsilon_2)(r)}\bigg]^\alpha-1\bigg)^2\\&\quad\times\bigg\{\frac{g^\kappa}{\kappa}+\frac{2g}{(2-\vartheta^2)}\frac{1}{\kappa}+\frac{2g}{(2-\vartheta^2)}\frac{\vartheta^{\kappa+1}}{\kappa(\kappa+1)}\bigg\}\\&\le\Omega\bigg(\bigg[1+\sqrt{\max_{g\in I}V(\varepsilon_1,\varepsilon_2)(r)}\bigg]^\alpha-1\bigg)^2\\&\quad\times\sup_{g\in(0,1)}\bigg\{g^\kappa+\frac{2g}{(2-\vartheta^2)}+\frac{2g}{(2-\vartheta^2)}\frac{\vartheta^{\kappa+1}}{(\kappa+1)}\bigg\}\\&=\frac{(2\kappa-1)}{2(5\kappa+2)}\bigg(\bigg[1+\sqrt{\max_{g\in I}V(\varepsilon_1,\varepsilon_2)(r)}\bigg]^\alpha-1\bigg)^2\\&\quad\times\sup_{g\in(0,1)}\bigg\{g^\kappa+\frac{2g}{(2-\vartheta^2)}+\frac{2g}{(2-\vartheta^2)}\frac{\vartheta^{\kappa+1}}{(\kappa+1)}\bigg\}\\&=\frac{(2\kappa-1)}{2(5\kappa+2)}\bigg(\bigg[1+\sqrt{\max_{g\in I}V(\varepsilon_1,\varepsilon_2)(r)}\bigg]^\alpha-1\bigg)^2.\end{aligned}$$

So, we have

$$\left|P\varepsilon_1(r) - P\varepsilon_2(r)\right| \leq \left(\left[1 + \sqrt{\max_{g\in I} V(\varepsilon_1, \varepsilon_2)(r)}\right]^{\alpha} - 1\right)^2. \tag{37}$$

Therefore,

$$\begin{aligned}\delta_{\varsigma}\left(P\varepsilon_1(r) - P\varepsilon_2(r)\right) &= \sup_{a\in I}\left|P\varepsilon_1(r) - P\varepsilon_2(r)\right|^2 \\ &\leq \left(\left[1 + \sqrt{\max_{g\in I} V(\varepsilon_1, \varepsilon_2)(r)}\right]^{\alpha} - 1\right)^2.\end{aligned} \tag{38}$$

By (38), we have

$$1 + \sqrt{\eth(\varpi_1, \varpi_2)} \leq \left[1 + \sqrt{\eth(\varpi_1, \varpi_2)}\right]^{\alpha}.$$

Now, by contractive condition (7) with setting of $\vartheta(\varepsilon) = 1$ for all $\varepsilon \in R$ and $\theta_b(\varepsilon) = 1 + \sqrt{\varepsilon}$, we get

$$\theta_b\left[sH_b(\grave{I}\varepsilon_1 \cap \Lambda, \grave{I}\varepsilon_2 \cap \Lambda)\right] \leq \left[\theta_b\left(\vartheta\left(\Omega(\varepsilon_1, \varepsilon_2)\right)\right)\right]^k$$

$\forall \varepsilon_1 \in \Lambda$, $\varepsilon_2 \in \grave{I}\kappa \cap \Lambda$ with $\eth(\varepsilon_2, \grave{I}\varepsilon_1) \in J$, where $k \in (0,1)$. Thus, all the required hypotheses of Theorem 3.2 are satisfied, and we approached that equations (33) and (34) have at least one solution of P. □

Example 4.1 Let L.C.F.D.E be based on order $\kappa(\check{D}_{(c,\kappa)})$ and its integral boundary valued problem

$$\check{D}_{(c,\frac{3}{2})}\left(\alpha(g)\right) = \frac{1}{(g+3)^2}\frac{|\alpha(g)|}{1 + |\alpha(g)|} \tag{39}$$

and

$$\begin{cases}\beta(0) = 0, \\ \beta(1) = \int_0^{\frac{3}{4}} \beta(g)\, dg, \quad \vartheta \in (0,1),\end{cases} \tag{40}$$

where $\kappa = \frac{3}{2}$, $\vartheta = \frac{3}{4}$ and $L(t, v(t)) = \frac{1}{(g+3)^2}\frac{|\alpha(g)|}{1+|\alpha(g)|}$. So, the above setting is an example of equations (3.3) and (3.4). Hence, clearly, equations (39) and (40) have at least one solution.

5 Concluding remarks

The paper deals with θ_b-contraction in b-metric spaces, which is an extension of θ-contraction. We prove multi-valued fixed point theorems of some generalized contractions which are defined on b-metric spaces that satisfy a θ_v-type condition. Our generalized results are based on b-Bianchini–Grandolfi gauge function instead of the conventional operator. Finally, we present an application dealing with the existence of solutions for Liouville–Caputo fractional differential equations.

Acknowledgements
The authors want to express their sincere thanks to the referee for his or her valuable remarks and suggestions, which made this paper more readable.

Funding
Not applicable.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AA was a major contributor in writing the manuscript. MA, AH, NH, and SMA performed the validation and formal analysis. All the authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 14 November 2021 Accepted: 8 March 2022 Published online: 04 April 2022

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