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Optimal bounds for Seiffert-like elliptic integral mean by harmonic, geometric, and arithmetic means

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Abstract
In this article, we present the optimal bounds for a special elliptic integral mean in terms of the harmonic combinations of harmonic, geometric, and arithmetic means. As consequences, several new bounds for the complete elliptic integral of the second kind are discovered, which are the improvements of many previously known results.

MSC: 26D15; 33E05

Keywords: Seiffert-like mean; Complete elliptic integral; Harmonic mean; Geometric mean; Arithmetic mean

1 Introduction

For $r \in (0,1)$, Legendre's complete elliptic integrals of the first kind $\mathcal{K}(r)$ and second kind $\mathcal{E}(r)$ [1–8] are defined by

$$\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2\theta\right)^{-1/2} d\theta$$

and

$$\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} \left(1 - r^2 \sin^2\theta\right)^{1/2} d\theta,$$

respectively.

It is well known that $\mathcal{K}(r)$ is strictly increasing from $(0,1)$ onto $(\pi/2, \infty)$ and $\mathcal{E}(r)$ is strictly decreasing from $(0,1)$ onto $(1, \pi/2)$, they satisfy the derivative formulas

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2},$$
$$\frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r}$$

and Landen identities

$$\mathcal{K}\left(\frac{2\sqrt{r}}{1+r}\right)=(1+r)\mathcal{K}(r),\qquad \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathcal{E}-r'^2\mathcal{K}}{1+r},$$

where and in what follows we denote $r'=\sqrt{1-r^2}$ for $r\in(0,1)$.

Let $a,b>0$ with $a\neq b$. Then the harmonic mean $H(a,b)$, geometric mean $G(a,b)$, arithmetic mean $A(a,b)$, arithmetic–geometric mean $AG(a,b)$ [9–11], and Toader mean $TD(a,b)$ [12–15] are given by

$$H(a,b)=\frac{2ab}{a+b},\qquad G(a,b)=\sqrt{ab},\qquad A(a,b)=\frac{a+b}{2}, \tag{1.1}$$

$$AG(a,b)=\frac{\pi}{2\int_0^{\pi/2}(a^2\cos^2\theta+b^2\sin^2\theta)^{-1/2}\,d\theta}=\begin{cases}\frac{\pi a}{2\mathcal{K}(\sqrt{1-(b/a)^2})}, & a>b,\\ \frac{\pi b}{2\mathcal{K}(\sqrt{1-(a/b)^2})}, & a<b,\end{cases}$$

and

$$TD(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\left(a^2\cos^2\theta+b^2\sin^2\theta\right)^{1/2}d\theta=\begin{cases}\frac{2a}{\pi}\mathcal{E}(\sqrt{1-(b/a)^2}), & a>b,\\ \frac{2b}{\pi}\mathcal{E}(\sqrt{1-(a/b)^2}), & a<b.\end{cases}$$

Recently, the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ of the first and second kinds have attracted the attention of many researchers [16–22] because they have wide applications in many branches of mathematics including the geometric function theory, differential equations, number theory, and mean value theory. For instance, the perimeter $\mathcal{L}(a,b)$ of an ellipse with semi-axes a, b and eccentricity $e=\sqrt{1-b^2/a^2}$ is given by

$$\mathcal{L}(a,b)=4\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta=4a\mathcal{E}(e). \tag{1.2}$$

Many remarkable inequalities and properties for the complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ can be found in the literature [23–31]. Barnard et al. [32] and Alzer and Qiu [33] proved that $\lambda=3/2$ and $\mu=\log 2/\log(\pi/2)$ are the best possible constants such that the double inequality

$$\frac{\pi}{2}\left(\frac{1+r'^{\lambda}}{2}\right)^{1/\lambda}<\mathcal{E}(r)<\frac{\pi}{2}\left(\frac{1+r'^{\mu}}{2}\right)^{1/\mu} \tag{1.3}$$

holds for all $r\in(0,1)$.

Later, Wang and Chu [34] improved the lower bound of (1.3) and proved that the double inequality

$$\begin{aligned}&\frac{\pi[(\alpha+(1-\alpha)r')^2+(1-\alpha+\alpha r')^2]^2}{(1+r')^3}<\mathcal{E}(r)\\ &\quad<\frac{\pi}{2}\left[\frac{(\beta+(1-\beta)r')^2+(1-\beta+\beta r')^2}{2}\right]^{1/2}\end{aligned}$$

holds for all $r\in(0,1)$ with the best possible constants $\alpha=\frac{4+\sqrt{2}}{8}$ and $\beta=\frac{1+\sqrt{(4/\pi)-1}}{2}$.

Very recently, Yang et al. [35] found the high accuracy asymptotic bounds for $\mathcal{E}(r)$ and proved that

$$\frac{\pi}{2}J(r') - \left(\frac{51\pi}{160} - 1\right)r^{16} < \mathcal{E}(r) < \frac{\pi}{2}J(r') - \frac{5\pi}{3\times 2^{31}}r^{16}$$

for all $r \in (0,1)$, where

$$J(r) = \frac{51r^2 + 20r\sqrt{r} + 50r + 20\sqrt{r} + 51}{16(5r + 2\sqrt{r} + 5)}.$$

The following Seiffert-like elliptic integral mean

$$\begin{aligned} V(a,b) &= \frac{\pi H(a,b)}{2\mathcal{E}(\frac{|a-b|}{a+b})} = \frac{\pi H(a,b)}{2\mathcal{E}(\sqrt{1-\frac{G^2(a,b)}{A^2(a,b)}})} \\ &= \frac{\pi G^2(a,b)}{2\int_0^{\pi/2}\sqrt{A^2(a,b)\cos^2\theta + G^2\sin^2\theta}\,d\theta} \end{aligned} \tag{1.4}$$

was introduced by Witkowski in [36], in which Witkowski investigated the so-called Seiffert-like means

$$M_f(a,b) = \begin{cases} \frac{|a-b|}{2f(\frac{|a-b|}{a+b})}, & a \neq b, \\ a, & a = b, \end{cases}$$

where the function $f : (0,1) \mapsto \mathbb{R}$ (called Seiffert function) satisfies the double inequality

$$\frac{x}{1+x} \le f(x) \le \frac{x}{1-x}.$$

From (1.3) we clearly see that

$$r' < \frac{2}{\pi}\mathcal{E}(r) < 1$$

for $r \in (0,1)$, which in conjunction with (1.4) gives

$$H(a,b) < V(a,b) < G(a,b) < A(a,b) \tag{1.5}$$

for all $a, b > 0$ with $a \neq b$.

Inspired by (1.5), the main purpose of the article is to find the optimal bounds for $V(a,b)$ in terms of the harmonic combinations of $H(a,b)$ and $G(a,b)$ (or $H(a,b)$ and $A(a,b)$). Our main results are the following Theorems 1.1 and 1.2.

Theorem 1.1 *The double inequality*

$$\frac{\alpha_1}{H(a,b)} + \frac{1-\alpha_1}{G(a,b)} < \frac{1}{V(a,b)} < \frac{\beta_1}{H(a,b)} + \frac{1-\beta_1}{G(a,b)} \tag{1.6}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \le 1/2$ and $\beta_1 \ge 2/\pi = 0.6366\ldots$.

Theorem 1.2 *The double inequality*

$$\frac{\alpha_2}{H(a,b)}+\frac{1-\alpha_2}{A(a,b)}<\frac{1}{V(a,b)}<\frac{\beta_2}{H(a,b)}+\frac{1-\beta_2}{A(a,b)} \tag{1.7}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_2\leq 2/\pi$ *and* $\beta_2\geq 3/4$.

To further improve and refine the lower bound in (1.6) and the upper bound in (1.7), we also establish the following Theorems 1.3 and 1.4.

Theorem 1.3 *The double inequality*

$$\begin{aligned}&\alpha_3\left[\frac{3}{4H(a,b)}+\frac{1}{4A(a,b)}\right]+(1-\alpha_3)\left[\frac{1}{2H(a,b)}+\frac{1}{2G(a,b)}\right]<\frac{1}{V(a,b)}\\&\quad<\beta_3\left[\frac{3}{4H(a,b)}+\frac{1}{4A(a,b)}\right]+(1-\beta_3)\left[\frac{1}{2H(a,b)}+\frac{1}{2G(a,b)}\right]\end{aligned}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_3\leq 1/4$ *and* $\beta_3\geq 2(4/\pi-1)=0.5464\ldots$.

Theorem 1.4 *The double inequality*

$$\begin{aligned}&\left[\frac{3}{4H(a,b)}+\frac{1}{4A(a,b)}\right]^{\alpha_4}\left[\frac{1}{2H(a,b)}+\frac{1}{2G(a,b)}\right]^{1-\alpha_4}\\&\quad<\frac{1}{V(a,b)}\\&\quad<\left[\frac{3}{4H(a,b)}+\frac{1}{4A(a,b)}\right]^{\beta_4}\left[\frac{1}{2H(a,b)}+\frac{1}{2G(a,b)}\right]^{1-\beta_4}\end{aligned}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_4\leq 1/4$ *and* $\beta_4\geq[\log(4/\pi)]/\log(3/2)=0.5957\ldots$.

2 Lemmas

In order to prove our main results, we need several lemmas which we present in this section.

Lemma 2.1 (See [1, Theorem 1.25]) *Let* $-\infty<a<b<\infty$, *and* $f,g:[a,b]\to\mathbb{R}$ *be continuous and differentiable on* (a,b) *such that* $f(a)=g(a)=0$ *or* $f(b)=g(b)=0$. *Assume that* $g'(x)\neq 0$ *for each* $x\in(a,b)$. *If* f'/g' *is (strictly) increasing (decreasing) on* (a,b), *then so is* f/g.

Lemma 2.2 *The functions*

(i) $r\mapsto(\mathcal{E}-r'^2\mathcal{K})/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,1)$;
(ii) $r\mapsto(\mathcal{K}-\mathcal{E})/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,\infty)$;
(iii) $r\mapsto(\mathcal{E}^2-r'^2\mathcal{K}^2)/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi^2/32,1)$;
(iv) $r\mapsto[(1+r'^2)\mathcal{K}-2\mathcal{E}]/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/16,\infty)$;
(v) $r\mapsto\varrho(r)=[(1-r')(3+r')]/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(2,3)$;
(vi) $r\mapsto\rho(r)=(1-r')\mathcal{E}/r^2$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/4,1)$.

Proof Parts (i)–(iv) can be found in [1, Theorem 3.21 (1) and Exercise 3.43 (11), (16), (29)].

For part (v), $\varrho(r)$ can be rewritten as

$$\varrho(r) = \frac{(1-r')(3+r')}{r^2} = 1 + \frac{2}{1+r'},$$

which gives the monotonicity of $\varrho(r)$. Note that $\varrho(0^+) = 2$ and $\varrho(1^-) = 3$.

For part (vi), differentiating $\rho(r)$ and making use of part (iii), we get

$$\rho'(r) = \frac{r(1-r')}{r'(\mathcal{E} + r'\mathcal{K})} \cdot \frac{\mathcal{E}^2 - r'^2\mathcal{K}^2}{r^4} > 0.$$

This in conjunction with $\rho(0^+) = \pi/4$ and $\rho(1^-) = 1$ gives the desired result. □

Lemma 2.3 *The function*

$$\varphi(r) = \frac{r'[(1+r'^2)\mathcal{E} - 2r'^2\mathcal{K}]}{r^4}$$

is strictly decreasing from $(0,1)$ *onto* $(0, 3\pi/16)$.

Proof Differentiating $\varphi(r)$ yields

$$\varphi'(r) = \frac{(3r^4 - 11r^2 + 8)\mathcal{K} + (7r^2 - 8)\mathcal{E}}{r'r^5} = \frac{\varphi_1(r)}{r'r^5}, \tag{2.1}$$

where

$$\varphi_1(r) = \left(3r^4 - 11r^2 + 8\right)\mathcal{K} + \left(7r^2 - 8\right)\mathcal{E}.$$

Simple computations lead to

$$\varphi_1(0) = 0, \tag{2.2}$$

$$\varphi_1'(r) = -9r^5\left[\frac{(1+r'^2)\mathcal{K} - 2\mathcal{E}}{r^4}\right]. \tag{2.3}$$

Therefore, Lemma 2.3 follows easily from (2.1)–(2.3) and Lemma 2.2(iv) together with $\varphi(0^+) = 3\pi/16$ and $\varphi(1^-) = 0$. □

Lemma 2.4 *The function*

$$\phi(r) = \frac{r'[(3+r'^2)\mathcal{K} - (4+r^2)\mathcal{E}]}{r^4}$$

is strictly decreasing from $(0,1)$ *onto* $(0, 3\pi/8)$.

Proof Let

$$\phi_1(r) = \left(3r^4 - 16r^2 + 16\right)\mathcal{K} - 8\left(2 - r^2\right)\mathcal{E},$$

$$\phi_2(r) = \left(8 - 7r^2\right)\mathcal{E} - \left(1 - r^2\right)\left(8 - 3r^2\right)\mathcal{K}.$$

Then simple computations lead to

$$\phi_1(0) = \phi_2(0) = 0, \tag{2.4}$$

$$\phi'(r) = -\frac{\phi_1(r)}{r' r^5}, \tag{2.5}$$

$$\phi_1'(r) = \frac{3r}{r'^2}\phi_2(r), \tag{2.6}$$

$$\phi_2'(r) = 9r^5\left[\frac{(1+r'^2)\mathcal{K} - 2\mathcal{E}}{r^4}\right]. \tag{2.7}$$

Therefore, Lemma 2.4 follows easily from (2.4)–(2.7) and Lemma 2.2(iv) together with $\phi(0^+) = 3\pi/8$ and $\phi(1^-) = 0$. □

3 Proofs of Theorems 1.1–1.4

In this section, we assume that $a > b > 0$ because all the bivariate means $H(a,b)$, $G(a,b)$, $A(a,b)$, and $V(a,b)$ are symmetric and homogeneous of degree one.

Proof of Theorem 1.1 Let $r = (a-b)/(a+b) \in (0,1)$. Then from (1.1) and (1.4) we obtain

$$\begin{aligned} &H(a,b) = A(a,b)\left(1-r^2\right), \quad G(a,b) = A(a,b)\sqrt{1-r^2}, \\ &V(a,b) = A(a,b)\frac{\pi(1-r^2)}{2\mathcal{E}}. \end{aligned} \tag{3.1}$$

From (3.1), inequality (1.6) can be rewritten as

$$\frac{\frac{1}{V(a,b)} - \frac{1}{G(a,b)}}{\frac{1}{H(a,b)} - \frac{1}{G(a,b)}} = \frac{\frac{2\mathcal{E}}{\pi r'^2} - \frac{1}{r'}}{\frac{1}{r'^2} - \frac{1}{r'}} = 1 - f(r), \tag{3.2}$$

where

$$f(r) = \frac{1 - 2\mathcal{E}/\pi}{1 - r'}.$$

Let $f_1(r) = 1 - 2\mathcal{E}/\pi$ and $f_2(r) = 1 - r'$. Then we clearly see that $f(r) = f_1(r)/f_2(r)$ and $f_1(0) = f_2(0) = 0$, and simple computations lead to

$$\frac{f_1'(r)}{f_2'(r)} = \frac{2}{\pi}\frac{r'(\mathcal{K} - \mathcal{E})}{r^2}, \tag{3.3}$$

$$\left[\frac{r'(\mathcal{K} - \mathcal{E})}{r^2}\right]' = -\frac{2[(1+r'^2)\mathcal{K} - 2\mathcal{E}]}{\pi r' r^3}. \tag{3.4}$$

Lemma 2.1 and Lemma 2.2(iv) together with (3.3) and (3.4) lead to the conclusion that $f(r)$ is strictly decreasing on $(0,1)$. Note that

$$f\left(0^+\right) = \frac{1}{2}, \qquad f\left(1^-\right) = 1 - \frac{2}{\pi}. \tag{3.5}$$

Therefore, Theorem 1.1 follows from (3.2) and (3.5) together with the monotonicity of $f(r)$. □

Proof of Theorem 1.2 Let $r = (a-b)/(a+b) \in (0,1)$. Then it follows from (3.1) that

$$\frac{\frac{1}{V(a,b)} - \frac{1}{A(a,b)}}{\frac{1}{H(a,b)} - \frac{1}{A(a,b)}} = \frac{\frac{2\mathcal{E}}{\pi r'^2} - 1}{\frac{1}{r'^2} - 1} = 1 - g(r), \tag{3.6}$$

where

$$g(r) = \frac{1 - 2\mathcal{E}/\pi}{r^2}.$$

Let $g_1(r) = 1 - 2\mathcal{E}/\pi$ and $g_2(r) = r^2$. Then elementary computations lead to

$$g(r) = \frac{g_1(r)}{g_2(r)}, \qquad g_1(0) = g_2(0) = 0, \tag{3.7}$$

$$\frac{g_1'(r)}{g_2'(r)} = \frac{1}{\pi}\frac{\mathcal{K} - \mathcal{E}}{r^2}. \tag{3.8}$$

Lemma 2.1 and Lemma 2.2(ii) together with (3.7) and (3.8) lead to the conclusion that $g(r)$ is strictly increasing on $(0,1)$. Note that

$$g(0^+) = \frac{1}{4}, \qquad g(1^-) = 1 - \frac{2}{\pi}. \tag{3.9}$$

Therefore, Theorem 1.2 follows easily from (3.6) and (3.9) together with the monotonicity of $g(r)$. □

Proof of Theorem 1.3 Let $r = (a-b)/(a+b) \in (0,1)$. Then from (3.1) we get

$$\begin{aligned} &\frac{\frac{1}{V(a,b)} - [\frac{1}{2H(a,b)} + \frac{1}{2G(a,b)}]}{[\frac{3}{4H(a,b)} + \frac{1}{4A(a,b)}] - [\frac{1}{2H(a,b)} + \frac{1}{2G(a,b)}]} \\ &\quad = \frac{\frac{2\mathcal{E}}{\pi r'^2} - (\frac{1}{2r'^2} + \frac{1}{2r'})}{(\frac{3}{4r'^2} + \frac{1}{4}) - (\frac{1}{2r'^2} + \frac{1}{2r'})} = 1 - h(r), \end{aligned} \tag{3.10}$$

where

$$h(r) = \frac{3 + r'^2 - 8\mathcal{E}/\pi}{(1-r')^2}.$$

Let $h_1(r) = 3 + r'^2 - 8\mathcal{E}/\pi$, $h_2(r) = (1-r')^2$, $h_3(r) = 4(\mathcal{K}-\mathcal{E})/(\pi r^2) - 1$, and $h_4(r) = 1/r' - 1$. Then we clearly see that $h_1(0) = h_2(0) = h_3(0) = h_4(0) = 0$. Simple computations lead to

$$h(r) = \frac{h_1(r)}{h_2(r)}, \qquad \frac{h_1'(r)}{h_2'(r)} = \frac{h_3(r)}{h_4(r)}, \tag{3.11}$$

$$\frac{h_3'(r)}{h_4'(r)} = \frac{4r'[(1+r'^2)\mathcal{E} - 2r'^2\mathcal{K}]}{\pi r^4} = \frac{4}{\pi}\varphi(r), \tag{3.12}$$

where $\varphi(r)$ is defined in Lemma 2.3.

Lemmas 2.1 and 2.3 together with (3.11) and (3.12) lead to the conclusion that $h(r)$ is strictly decreasing on $(0,1)$. Moreover, by Taylor's formula, one has

$$h(0^+) = \lim_{r\to 0^+} \frac{(1+r')^2[3r^4/16 + o(r^4)]}{r^4} = \frac{3}{4}, \qquad h(1^-) = 3 - \frac{8}{\pi}. \tag{3.13}$$

Therefore, Theorem 1.3 follows easily from (3.10) and (3.13) together with the monotonicity of $h(r)$. □

Proof of Theorem 1.4 Let $r = (a-b)/(a+b) \in (0,1)$. Then it follows from (3.1) that

$$\begin{aligned} &\frac{\log[\frac{1}{V(a,b)}] - \log[\frac{1}{2H(a,b)} + \frac{1}{2G(a,b)}]}{\log[\frac{3}{4H(a,b)} + \frac{1}{4A(a,b)}] - \log[\frac{1}{2H(a,b)} + \frac{1}{2G(a,b)}]} \\ &\quad = \frac{\log[(2\mathcal{E})/(\pi r'^2)] - \log[(1+r')/(2r'^2)]}{\log[(3+r'^2)/(4r'^2)] - \log[(1+r')/(2r'^2)]} := 1 - j(r), \end{aligned} \tag{3.14}$$

where

$$j(r) = \frac{\log[(3+r'^2)/4] - \log[(2\mathcal{E})/\pi]}{\log[(3+r'^2)/4] - \log[(1+r')/2]}.$$

Let $j_1(r) = \log[(3+r'^2)/4] - \log[(2\mathcal{E})/\pi]$ and $j_2(r) = \log[(3+r'^2)/4] - \log[(1+r')/2]$. Then elaborated computations lead to

$$j(r) = \frac{j_1(r)}{j_2(r)} = \frac{j_1(r) - j_1(0)}{j_2(r) - j_2(0)}, \tag{3.15}$$

$$\frac{j_1'(r)}{j_2'(r)} = \frac{r'[(4-r^2)\mathcal{K} - (4+r^2)\mathcal{E}]}{(1-r')^2(r'+3)\mathcal{E}} = \frac{\phi(r)}{\varrho(r)\rho(r)}, \tag{3.16}$$

where $\varrho(r)$, $\rho(r)$, and $\phi(r)$ are defined as in Lemma 2.2(v), (vi) and Lemma 2.4, respectively.

Lemma 2.1, Lemma 2.2(v), (vi), and Lemma 2.4 together with (3.15) and (3.16) lead to the conclusion that $j(r)$ is strictly decreasing on $(0,1)$. Moreover, by L'Hôpital's rule we get

$$j(0^+) = \lim_{r\to 0^+} \frac{j_1'(r)}{j_2'(r)} = \frac{3}{4}, \qquad j(1^-) = \frac{\log(3\pi) - 3\log 2}{\log 3 - \log 2}. \tag{3.17}$$

Therefore, Theorem 1.4 follows easily from (3.14) and (3.17) together with the monotonicity of $j(r)$. □

As a consequence of Theorems 1.1–1.4, we can derive the following Corollary 3.1 immediately.

Corollary 3.1 *Let $l(r) = (1+r)/2$ and $u(r) = (3+r^2)/4$. Then the double inequalities*

$$\frac{\pi}{2} l(r') < \mathcal{E}(r) < 1 + \left(\frac{\pi}{2} - 1\right) r',$$

$$1 + \left(\frac{\pi}{2} - 1\right) r'^2 < \mathcal{E}(r) < \frac{\pi}{2} u(r'),$$

$$\frac{\pi}{2}\left[\frac{u(r')}{4}+\frac{3l(r')}{4}\right]<\mathcal{E}(r)<\frac{\pi}{2}\left[\sigma u\left(r'\right)+(1-\sigma)l\left(r'\right)\right],$$
$$\frac{\pi}{2}u\left(r'\right)^{1/4}l\left(r'\right)^{3/4}<\mathcal{E}(r)<\frac{\pi}{2}u\left(r'\right)^{\tau}l\left(r'\right)^{1-\tau}$$

hold for all $r\in(0,1)$, where $\sigma=2(4/\pi-1)$ and $\tau=[\log(4/\pi)]/\log(3/2)$ are given in Theorems 1.3 and 1.4, respectively.

In order to compare the lower and upper bounds in Corollary 3.1, we provide Theorem 3.2 as follows.

Theorem 3.2 *The double inequality*

$$\max_{r\in(0,1)}\left\{1+\left(\frac{\pi}{2}-1\right)r'^2,\frac{\pi}{32}\left(3+r'\right)^2\right\}<\mathcal{E}(r)<\frac{\pi}{4}r'\left(3-r'\right)+\left(1-r'\right)^2$$

holds for all $r\in(0,1)$.

Proof We clearly see that the function

$$r\mapsto\frac{u(r)}{l(r)}=\frac{1}{2}\left(r+1+\frac{4}{r+1}-2\right)$$

is strictly decreasing on $(0,1)$. Therefore, $u(r)/l(r)\in(1,3/2)$ and

$$\frac{u(r')}{4}+\frac{3l(r')}{4}>l\left(r'\right),\qquad\sigma u\left(r'\right)+(1-\sigma)l\left(r'\right)<u\left(r'\right).\tag{3.18}$$

It is well known that

$$\frac{u(r')}{4}+\frac{3l(r')}{4}>u\left(r'\right)^{1/4}l\left(r'\right)^{3/4}.\tag{3.19}$$

It is not difficult to verify that the functions $1+(\frac{\pi}{2}-1)r'^2$ and $\frac{\pi}{2}[\frac{u(r')}{4}+\frac{3l(r')}{4}]$ are not comparable on $(0,1)$ due to

$$\frac{1+(\frac{\pi}{2}-1)r'^2-\frac{\pi}{2}[\frac{u(r')}{4}+\frac{3l(r')}{4}]}{r^2}\to\begin{cases}1-\frac{3\pi}{8}<0, & r\to0^+,\\ 1-\frac{9\pi}{32}>0, & r\to1^-.\end{cases}$$

This in conjunction with (3.18) and (3.19) implies that

$$\begin{aligned}&\max_{r\in(0,1)}\left\{\frac{\pi}{2}l\left(r'\right),1+\left(\frac{\pi}{2}-1\right)r'^2,\frac{\pi}{2}\left[\frac{u(r')}{4}+\frac{3l(r')}{4}\right],\frac{\pi}{2}u\left(r'\right)^{1/4}l\left(r'\right)^{3/4}\right\}\\&\quad=\max_{r\in(0,1)}\left\{1+\left(\frac{\pi}{2}-1\right)r'^2,\frac{\pi}{32}\left(3+r'\right)^2\right\}.\end{aligned}$$

We now claim that

$$s(x)=\sigma x^{1-\tau}+(1-\sigma)x^{-\tau}<1\tag{3.20}$$

for $x \in (1, 3/2)$. Indeed, differentiating $s(x)$ yields

$$s'(x) = \sigma(1-\tau)x^{-1-\tau}\left[x - \frac{\tau(1-\sigma)}{\sigma(1-\tau)}\right],$$

which together with $\tau(1-\sigma)/[\sigma(1-\tau)] = 1.223\ldots$ enables us to know that $s(x)$ is convex on $(1, 3/2)$. Therefore, inequality (3.20) follows from $s(1) = s(3/2) = 1$.

It follows from (3.20) and $1 < u(r)/l(r) < 3/2$ that

$$\begin{aligned}
&\sigma u(r) + (1-\sigma)l(r) - u(r)^\tau l(r)^{1-\tau} \\
&\quad = u(r)^\tau l(r)^{1-\tau}\left[\sigma\left(\frac{u(r)}{l(r)}\right)^{1-\tau} + (1-\sigma)\left(\frac{l(r)}{u(r)}\right)^{\tau} - 1\right] \\
&\quad = u(r)^\tau l(r)^{1-\tau}\left[s\big(u(r)/l(r)\big) - 1\right] < 0. \qquad (3.21)
\end{aligned}$$

Moreover, it is not difficult to verify that

$$\frac{\pi}{2}\left[\sigma u(r') + (1-\sigma)l(r')\right] - \left[1 + \left(\frac{\pi}{2} - 1\right)r'\right] = -\left(1 - \frac{\pi}{4}\right)r'(1-r') < 0.$$

This in conjunction with (3.18) and (3.21) implies that

$$\begin{aligned}
&\min_{r\in(0,1)}\left\{1 + \left(\frac{\pi}{2} - 1\right)r', \frac{\pi}{2}u(r'), \frac{\pi}{2}\left[\sigma u(r') + (1-\sigma)l(r')\right], \frac{\pi}{2}u(r')^\tau l(r')^{1-\tau}\right\} \\
&\quad = \frac{\pi}{4}r'(3-r') + (1-r')^2.
\end{aligned}$$

□

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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
FZ: conceptualization, computation, writing–original draft, writing–review and editing. WQ: problem statement, conceptualization, methodology, computation, writing–original draft, supervision, and funding acquisition. HZX: computation, writing–review and editing. All authors read and approved the final manuscript.

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