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Caccioppoli-type inequalities for Dirac operators

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Abstract

In this paper, we establish the Caccioppoli estimates for the nonlinear differential equation

$$-\overline{D}(|Dv|^{p-2}Dv) = \lambda|v|^{p-2}v, \quad 1 < p < \infty,$$

where D is the Dirac operator. Moreover, we obtain general weighted versions of the Caccioppoli-type inequalities for the Dirac operators.

MSC: 22E30; 43A80

Keywords: Caccioppoli inequality; Quaternions; Dirac operator; Picone identity

1 Introduction

In the Euclidean setting, we recall the Caccioppoli inequality

$$\int_\Omega \phi^p |\nabla v|^p \, dx \le p^p \int_\Omega v^p |\nabla \phi|^p \, dx \tag{1.1}$$

for all nonnegative functions $\phi \in C_0^\infty(\Omega)$, where a positive function v is a subsolution of the Dirichlet boundary value problem for p-Laplacian

$$\begin{cases} \nabla \cdot (|\nabla v|^{p-2}\nabla v) = \lambda |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

See, for example, [4] for the problem in the Euclidean setting. We also refer to [1, 5, 6, 8, 10] and references therein for discussions on the Caccioppoli-type estimates in different settings.

The main aim of this paper is to obtain the Caccioppoli-type inequality for the nonlinear equation

$$\begin{cases} -\overline{D}(|Dv|^{p-2}Dv) = \lambda |v|^{p-2} v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where v is the subsolution, D is the usual Dirac operator, and $\overline{D}$ is its conjugate. Also, we obtain weighted versions of the Caccioppoli-type inequality for the Dirac operator.

In what follows, we will work in $\mathbb{H}$, the skew-field of the quaternion. This means that each element $x' \in \mathbb{H}$ has the following representation:

$$x' = x_0 + \sum_{i=1}^{n} e_i x_i,$$

where $1, e_1, \dots, e_n$ are the basis elements of $\mathbb{H}$. For these elements, we have the multiplication rules

- $e_1^2 = \cdots = e_n^2 = -1,$
- $e_i e_j + e_j e_i = -2\delta + ij$ for all $i, j = 1, \dots, n$.

The conjugate element $\overline{x'}$ is given by $\overline{x'} = x_0 - \sum_{i=1}^{n} e_i x_i$, and we have the properties

$$\left|x'\right|^2 = x\overline{x'} = \overline{x'}x = x_0^2 + \sum_{i=1}^{n} x_i^2$$

and

$$|x|_q = \sqrt{\sum_{i=1}^{n} x_i^2} \tag{1.3}$$

for the norm on $\mathbb{H}$.

We recall the usual Dirac operator, which factorizes the n-dimensional Laplace operator,

$$Df = \sum_{i=1}^{n} e_i \frac{\partial f}{\partial x_i}$$

and its conjugate operator

$$\overline{D}f = -\sum_{i=1}^{n} e_i \frac{\partial f}{\partial x_i}.$$

The products of these operators

$$D\overline{D} = \overline{D}D = \Delta_n,$$

where Δ_n is the Laplacian for functions defined over domains in $\mathbb{R}^n$. For further discussions in this direction, we refer, for example, to [9] (see also [3] for theory of QDEs).

In Sect. 2, we discuss Picone's identity for the Dirac operator. The main results of this paper are presented in Sect. 3.

2 Picone's identity for the Dirac operator

Lemma 2.1 *Let u, v be a differentiable functions defined a.e. in $\Omega \subset \mathbb{H}$ such that $v > 0$ a.e. in Ω and $u \geq 0$. Define*

$$R(u, v) := |Du|^p - D\left(\frac{u^p}{v^{p-1}}\right)|Dv|^{p-2}Dv, \tag{2.1}$$

$$L(u,v) := |Du|^p - p\left(\frac{u}{v}\right)^{p-1}|Dv|^{p-2}DvDu + (p-1)\left(\frac{u}{v}\right)^p|Dv|^p,$$

where $p > 1$. *Then*

$$L(u,v) = R(u,v) \geq 0. \tag{2.2}$$

Also, $L(u,v) = 0$ *a.e. in* Ω *if and only if* $u = cv$ *a.e. in* Ω *with positive constant* c.

Proof of Lemma 2.1 A direct computation gives

$$\begin{aligned} R(u,v) &= |Du|^p - D\left(\frac{u^p}{v^{p-1}}\right)|Dv|^{p-2}Dv \\ &= |Du|^p - \frac{pu^{p-1}Duv^{p-1} - u^p(p-1)v^{p-2}Dv}{(v^{p-1})^2}|Dv|^{p-2}Dv \\ &= |Du|^p - p\frac{u^{p-1}}{v^{p-1}}|Dv|^{p-2}DvDu + (p-1)\frac{u^p}{v^p}|Dv|^p \\ &= L(u,v). \end{aligned}$$

This proves the equality in (2.2). Now we rewrite $L(u,v)$ to see that $L(u,v) \geq 0$:

$$\begin{aligned} L(u,v) &= |Du|^p - p\frac{u^{p-1}}{v^{p-1}}|Dv|^{p-1}|Du| + (p-1)\frac{u^p}{v^p}|Dv|^p \\ &\quad + p\frac{u^{p-1}}{v^{p-1}}|Dv|^{p-2}\big(|Dv||Du| - DvDu\big) \\ &= S_1 + S_2, \end{aligned}$$

where

$$\begin{aligned} S_1 := {} & p\left[\frac{1}{p}|Du|^p + \frac{p-1}{p}\left(\left(\frac{u}{v}|Dv|\right)^{p-1}\right)^{\frac{p}{p-1}}\right] \\ & - p\frac{u^{p-1}}{v^{p-1}}|Dv|^{p-1}|Du| \end{aligned}$$

and

$$S_2 := p\frac{u^{p-1}}{v^{p-1}}|Dv|^{p-2}\big(|Dv||Du| - DvDu\big).$$

We can see that $S_2 \geq 0$ due to $|Dv||Du| \geq DvDu$. To check that $S_1 \geq 0$, we need to use Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a \geq 0, b \geq 0, \tag{2.3}$$

where $p > 1$, $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. The equality holds if and only if $a^p = b^q$, that is, if $a = b^{\frac{1}{p-1}}$. Let us take $a = |Du|$ and $b = (\frac{u}{v}|Dv|)^{p-1}$ in (2.3) to get

$$p|Du|\left(\frac{u}{v}|Dv|\right)^{p-1} \leq p\left[\frac{1}{p}|Du|^p + \frac{p-1}{p}\left(\left(\frac{u}{v}|Dv|\right)^{p-1}\right)^{\frac{p}{p-1}}\right]. \tag{2.4}$$

From this we see that $S_1 \geq 0$, which proves that $L(u,v) = S_1 + S_2 \geq 0$. It is easy to see that $u = cv$ implies $R(u,v) = 0$. Now let us prove that $L(u,v) = 0$ implies $u = cv$. Due to $u(x) \geq 0$ and $L(u,v)(x_0) = 0$, $x_0 \in \Omega$, we consider the two cases $u(x_0) > 0$ and $u(x_0) = 0$.

(1) In the case $u(x_0) > 0$, from $L(u,v)(x_0) = 0$ it follows that $S_1 = 0$ and $S_2 = 0$. Then $S_1 = 0$ implies

$$|Du| = \frac{u}{v}|Dv|, \tag{2.5}$$

and $S_2 = 0$ implies

$$|Dv||Du| - DvDu = 0, \tag{2.6}$$

Combination of (2.5) and (2.6) gives

$$\frac{Du}{Dv} = \frac{u}{v} = c \quad \text{with } c \neq 0. \tag{2.7}$$

(2) Let us denote $\Omega^* := \{x \in \Omega | u(x) = 0\}$. If $\Omega^* \neq \Omega$, then suppose that $x_0 \in \partial\Omega^*$. Then there exists a sequence $x_k \notin \Omega^*$ such that $x_k \to x_0$. In particular, $u(x_k) \neq 0$, and hence by case (1) we have $u(x_k) = cv(x_k)$. Passing to the limit, we get $u(x_0) = cv(x_0)$. Since $u(x_0) = 0$ and $v(x_0) \neq 0$, we get that $c = 0$. Then by case (1) again, since $u = cv$ and $u \neq 0$ in $\Omega \backslash \Omega^*$, it is impossible that $c = 0$. This contradiction implies that $\Omega^* = \Omega$.

This completes the proof of Lemma 2.1. □

3 Caccioppoli-type inequalities

Let us consider the Dirichlet boundary problem for the Dirac operator

$$\begin{cases} -\overline{D}(|Dv|^{p-2}Dv) = \lambda|v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

We say that a weak solution of equation (3.1) means a function $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega |Dv|^{p-2}DvD\phi\,dx' - \lambda\int_\Omega |v|^{p-2}v\phi\,dx' = 0 \tag{3.2}$$

for all functions $\phi \in W^{1,p}_0(\Omega) \cap C(\Omega)$. The supsolution and subsolution of equation (3.1) mean a function $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega |Dv|^{p-2}Dv\overline{D}\phi\,dx' - \lambda\int_\Omega |v|^{p-2}v\phi\,dx' \geq 0 \tag{3.3}$$

and

$$\int_{\Omega} |Dv|^{p-2} Dv\overline{D}\phi \, dx' - \lambda \int_{\Omega} |v|^{p-2} v\phi \, dx' \leq 0 \tag{3.4}$$

for the test functions $\phi \in W_0^{1,p}(\Omega) \cap C(\Omega)$ with $\phi \geq 0$, respectively.

If we take the test function as $\phi = v$, then for the supsolution and subsolution, we have

$$\int_{\Omega} |Dv|^p \, dx' \geq \lambda \int_{\Omega} |v|^p \, dx' \tag{3.5}$$

and

$$\int_{\Omega} |Dv|^p \, dx' \leq \lambda \int_{\Omega} |v|^p \, dx'. \tag{3.6}$$

Now we are ready to establish a Caccioppoli-type inequality.

Theorem 3.1 *Let $\Omega \in \mathbb{H}$. Let v be a positive subsolution of equation (3.1) in Ω. For any fixed $q > p - 1$ and $q < p < \infty$, we have*

$$\int_{\Omega} v^{q-p} \phi^p |Dv|^p \, dx' \leq \left(\frac{p}{q-p+1} \right)^p \int_{\Omega} v^q |D\phi|^p \, dx' + \frac{\lambda p}{q-p+1} \int_{\Omega} v^q \phi^p \, dx' \tag{3.7}$$

for all nonnegative functions $\phi \in C_0^{\infty}(\Omega)$.

Remark 3.2 Note that Theorem 3.1 for the Finsler norm was obtained in [2].

- For the case $q = p$ and $\lambda = 0$ in Theorem 3.1, we have

$$\int_{\Omega} \phi^p |Dv|^p \, dx' \leq p^p \int_{\Omega} v^p |D\phi|^p \, dx'. \tag{3.8}$$

- For the case $q = 0$ in Theorem 3.1, we have

$$\int_{\Omega} \phi^p |D \log v|^p \, dx' \leq \left(\frac{p}{1-p} \right)^p \int_{\Omega} |D\phi|^p \, dx' + \frac{\lambda p}{1-p} \int_{\Omega} \phi^p \, dx'. \tag{3.9}$$

Proof of Theorem 3.1 Let us begin the proof by replacing $u = v^{\frac{q}{p}} \phi$ in $L(u, v)$, which gives

$$\begin{aligned}
\int_{\Omega} L\left(v^{q/p}\phi, v\right) dx' &= \int_{\Omega} \left|D\left(v^{q/p}\phi\right)\right|^p dx' + (p-1) \int_{\Omega} v^{q-p} \phi^p |Dv|^p \, dx' \\
&\quad - p \int_{\Omega} \left(v^{\frac{q-p}{p}} \phi\right)^{p-1} |Dv|^{p-2} D\left(v^{q/p}\phi\right) dx' \\
&= \int_{\Omega} \left|D\left(v^{q/p}\phi\right)\right|^p dx' + (p-1) \int_{\Omega} v^{q-p} \phi^p |Dv|^p \, dx' \\
&\quad - p \int_{\Omega} \left(v^{\frac{q-p}{p}} \phi\right)^{p-1} |Dv|^{p-2} \left(\frac{q}{p} v^{\frac{q-p}{p}} \phi Dv + v^{q/p} Dv D\phi \right) dx' \\
&= \int_{\Omega} \left|D\left(v^{q/p}\phi\right)\right|^p dx' - (q-p+1) \int_{\Omega} v^{q-p} \phi^p |Dv|^p \, dx'
\end{aligned}$$

$$+\,p\int_{\Omega}\big(\nu^{\frac{q-p}{p}}\phi\big)^{p-1}|D\nu|^{p-1}\nu^{q/p}|D\phi|\,dx'.$$

In the last line, we have used the Schwarz inequality. Now we apply the Young inequality of the form

$$ab^{p-1}\le\frac{a^p}{p\tau^{p-1}}+\frac{p-1}{p}\tau b^p,\quad a,b\ge 0,\tau>0.$$

By choosing $a=\nu^{q/p}|D\phi|$ and $b=\nu^{\frac{q-p}{p}}\phi|D\nu|$ and using inequality (3.6) we arrive at

$$\begin{aligned}0&\le\int_{\Omega}\big|D\big(\nu^{q/p}\phi\big)\big|^p\,dx'-(q-p+1)\int_{\Omega}\nu^{q-p}\phi^p|D\nu|^p\,dx'\\&\quad+\tau^{1-p}\int_{\Omega}\nu^q|D\phi|^p\,dx'+(p-1)\tau\int_{\Omega}\nu^{q-p}\phi^p|D\nu|^p\,dx'\\&\le\lambda\int_{\Omega}\nu^q\phi^p\,dx'-\big(q-p+1-\tau(p-1)\big)\int_{\Omega}\nu^{q-p}\phi^p|D\nu|^p\,dx'\\&\quad+\tau^{1-p}\int_{\Omega}\nu^q|D\phi|^p\,dx'.\end{aligned}$$

Thus we have the following inequality:

$$\begin{aligned}\int_{\Omega}\nu^{q-p}\phi^p|D\nu|^p\,dx'&\le\frac{\tau^{1-p}}{(q-p+1-\tau(p-1))}\int_{\Omega}\nu^q|D\phi|^p\,dx'\\&\quad+\frac{\lambda}{(q-p+1-\tau(p-1))}\int_{\Omega}\nu^q\phi^p\,dx'.\end{aligned}\tag{3.10}$$

Taking a suitable constant $\tau=\frac{q-p+1}{p}$ leads to

$$\int_{\Omega}\nu^{q-p}\phi^p|D\nu|^p\,dx'\le\left(\frac{p}{q-p+1}\right)^p\int_{\Omega}\nu^q|D\phi|^p\,dx'+\frac{\lambda p}{q-p+1}\int_{\Omega}\nu^q\phi^p\,dx'.$$

This proves the theorem. □

4 Weighted versions

Let us consider the following weighted operator:

$$\Delta_{p,w}f=\overline{D}\big(w(x)|Df|^{p-2}Df\big),\quad 1<p<\infty,\tag{4.1}$$

where $0\le w\in C^1(\mathbb{H})$.

Theorem 4.1 *Let $2\le p<\infty$. Let $0\le F\in C^{\infty}(\mathbb{H})$ and $0\le\eta\in L^1_{\mathrm{loc}}(\mathbb{H})$ be such that*

$$\eta F^{p-1}\le-\Delta_{p,w}F\quad a.e.\ in\ \mathbb{H}.\tag{4.2}$$

Then we have

$$\int_{\mathbb{H}}\eta(x)\big|f(x)\big|^p\,dx'+C_p\int_{\mathbb{H}}w(x)\big|F(x)\big|^p\big|D(f/F)\big|^p\,dx'\le\int_{\mathbb{H}}w(x)\big|Df(x)\big|^p\,dx'\tag{4.3}$$

for all real-valued functions $f\in C_0^{\infty}(\mathbb{H})$. Here C_p is a positive constant.

Note that the Carnot group version of Theorem 4.1 was obtained in [7].

Proof of Theorem 4.1 For all $a, b \in \mathbb{R}^n$, there exists a positive number C_p such that

$$|a|^p + C_p|b|^p + p|a|^{p-2}a \cdot b \le |a+b|^p, \quad 2 \le p < \infty. \tag{4.4}$$

Using this by taking $a = g(x)DF(x)$ and $b = F(x)Dg(x)$, we get

$$\begin{aligned} &\left|g(x)\right|^p \left|DF(x)\right|^p + C_p\left|F(x)\right|^p\left|Dg(x)\right|^p + F(x)\left|DF(x)\right|^{p-2}DF(x) \cdot D\left|g(x)\right|^p \\ &\quad \le \left|g(x)DF(x) + F(x)Dg(x)\right|^p = \left|Df(x)\right|^p, \end{aligned} \tag{4.5}$$

where $g = f/F$. It follows that

$$\begin{aligned} \int_{\mathbb{H}} w(x)\left|Df(x)\right|^p dx' \ge{}& \int_{\mathbb{H}} w(x)\left|DF(x)\right|^p\left|g(x)\right|^p dx' \\ &+ C_p \int_{\mathbb{H}} w(x)\left|Dg(x)\right|^p\left|F(x)\right|^p dx' \\ &- \int_{\mathbb{H}} \overline{D}\left(w(x)F(x)\left|DF(x)\right|^{p-2}DF(x)\right)\left|g(x)\right|^p dx' \\ \ge{}& C_p \int_{\mathbb{H}} w(x)\left|Dg(x)\right|^p\left|F(x)\right|^p dx' \\ &+ \int_{\mathbb{H}} -\overline{D}\left(w(x)\left|DF(x)\right|^{p-2}DF(x)\right)F(x)\left|g(x)\right|^p dx'. \end{aligned}$$

Using (4.2), this implies that

$$\begin{aligned} &\int_{\mathbb{H}} \eta(x)\left|g(x)\right|^p\left|F(x)\right|^p dx' + C_p\int_{\mathbb{H}} w(x)\left|Dg(x)\right|^p\left|F(x)\right|^p dx' \\ &\quad \le \int_{\mathbb{H}} w(x)\left|Df(x)\right|^p dx'. \end{aligned} \tag{4.6}$$

Since $g = f/F$, we arrive at

$$\int_{\mathbb{H}} \eta(x)\left|f(x)\right|^p dx' + C_p\int_{\mathbb{H}} w(x)\left|F(x)\right|^p\left|D(f/F)\right|^p dx' \le \int_{\Omega} w(x)\left|Df(x)\right|^p dx', \tag{4.7}$$

which proves (4.3). □

Remark 4.2 For $p = 2$, we have equality in inequality (4.4) with $C_2 = 1$, that is, the above proof gives the following remainder formula:

$$\int_{\mathbb{H}} w(x)\left|F(x)\right|^2\left|D(f/F)\right|^2 dx' = \int_{\mathbb{H}} w(x)\left|Df(x)\right|^2 dx' - \int_{\mathbb{H}} \eta(x)\left|f(x)\right|^2 dx'. \tag{4.8}$$

Remark 4.3 For $1 < p < 2$, inequality (4.4) can be stated as for all $a, b \in \mathbb{R}^n$, there exists a positive number C_p such that

$$|a|^p + C_p\frac{|b|^p}{(|a|+|b|)^{2-p}} + p|a|^{p-2}a \cdot b \le |a+b|^p, \quad 1 < p < 2. \tag{4.9}$$

In turn, from the proof it follows that

$$\begin{aligned}
&\int_{\mathbb{H}} w(x)\left|Df(x)\right|^p\,dx' \\
&\quad\geq \int_{\mathbb{H}} \eta(x)\left|f(x)\right|^p\,dx' \\
&\qquad + C_p\int_{\mathbb{H}} w(x)\left(\left|\frac{f(x)}{F(x)}DF(x)\right| + F\left|D\frac{f(x)}{F(x)}\right|\right)^{p-2}\left|F(x)\right|^2\left|D(f(x)/F(x)\right|^2\,dx'
\end{aligned}\tag{4.10}$$

for all real-valued functions $f \in C_0^\infty(\mathbb{H})$.

Proposition 4.4 *For $f \in C_0^\infty(\mathbb{H})$, we have*

$$\int_\Omega \frac{|f(x)|^p}{|x|_q^p}\,dx' \leq \left(\frac{p}{\gamma-p-2}\right)^p\int_\Omega \left|Df(x)\right|^p\,dx', \tag{4.11}$$

where $1 < p < \gamma - 2$ and $\gamma \leq 2 + n$ with $\gamma \in \mathbb{R}$.

Proof of Proposition 4.4 In Theorem 4.1, we take $w = 1$ and

$$F_\varepsilon = |x_\varepsilon|_q^{-\frac{\gamma-p-2}{p}} = \left((x_1+\varepsilon)^2 + \cdots + (x_n+\varepsilon)^2\right)^{-\frac{\gamma-p-2}{2p}}$$

for a given $\varepsilon > 0$. A direct computation gives

$$\begin{aligned}
&D|x_\varepsilon|_q^\alpha = \alpha|x_\varepsilon|_q^{\alpha-2}\sum_{i=1}^n e_i x_i, \\
&|D|x_\varepsilon|_q^\alpha|_q^{p-2} = |\alpha|^{p-2}|x_\varepsilon|_q^{(\alpha-2)(p-2)+(p-2)}, \\
&\overline{D}\left(|D|x_\varepsilon|_q^\alpha|_q^{p-2}D|x_\varepsilon|_q^\alpha\right) = \overline{D}\left(\alpha|\alpha|^{p-2}|x_\varepsilon|_q^{\alpha p-\alpha-p}\sum_{i=1}^n e_i x_i\right) \\
&\qquad\qquad = \alpha|\alpha|^{p-2}(\alpha p - \alpha - p + n)|x_\varepsilon|_q^{\alpha p-\alpha-p}.
\end{aligned}$$

By taking $\alpha = -\frac{\gamma-p-2}{2p}$ we compute

$$\begin{aligned}
-\Delta_{p,1}F_\varepsilon &= -\overline{D}\left(|DF_\varepsilon|^{p-2}DF_\varepsilon\right) \\
&= -\overline{D}\left(|D|x_\varepsilon|_q^{-\frac{\gamma-p-2}{p}}|_q^{p-2}D|x_\varepsilon|_q^{-\frac{\gamma-p-2}{p}}\right) \\
&= \frac{\gamma-p-2}{p}\left|\frac{\gamma-p-2}{p}\right|^{p-2}\left(\frac{\gamma-p-2}{p} - \gamma + 2 + n\right)|x_\varepsilon|_q^{-\frac{(\gamma-p-2)(p-1)}{p}-p} \\
&= \left(\left|\frac{\gamma-p-2}{p}\right|^p + \frac{\gamma-p-2}{p}\left|\frac{\gamma-p-2}{p}\right|^{p-2}(-\gamma+2+n)\right)|x_\varepsilon|_q^{-\frac{(\gamma-p-2)(p-1)}{p}-p}.
\end{aligned}$$

If $1 < p < \gamma - 2$ and $\gamma \leq 2 + n$, then the above expression gives

$$-\Delta_{p,1}F_\varepsilon \geq \left|\frac{\gamma-p-2}{p}\right|^p\frac{1}{|x_\varepsilon|_q^p}F_\varepsilon^{p-1}, \tag{4.12}$$

that is, according to the assumption in Theorem 4.1, we can put

$$\eta(x) = \left|\frac{\gamma - p - 2}{p}\right|^p \frac{1}{|x_\varepsilon|_q^p}, \tag{4.13}$$

which shows that (4.3) implies (4.11). □

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The authors declare that they have no competing interests.

Authors' contributions
GO wrote the initial draft after calculation of results, and AK originated the idea of this research. Both authors read and approved the final manuscript.

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