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# Explicit iteration and unbounded solutions for fractional $q$ -difference equations with boundary conditions on an infinite interval

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## Abstract

In this work, a proposed system of fractional boundary value problems is investigated concerning its unbounded solutions' existence for a class of nonlinear fractional  $q$ -difference equations in the context of the Riemann–Liouville fractional  $q$ -derivative on an infinite interval. The system's solution is formulated with the help of Green's function. A compactness criterion is established in a special space. All the obtained results of uniqueness and existence are investigated with the help of fixed-point theorems. Some essential examples are illustrated to support our main outcomes.

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## 1 Introduction

Fractional differential equations are much better than integer ones with respect to their nature in the descriptions of phenomena and processes of several scientific and engineering phenomena. Various studies of fractional differential equations have been recently conducted in the context of fractional-order derivatives such as the Riemann–Liouville fractional derivative and the Caputo fractional derivative.

Fractional calculus and  $q$ -calculus in general, and fractional differential equations in particular, are well established in current research works. Fractional calculus belongs to a large portion of mathematical analysis. The notion of  $q$ -difference equations goes back to 1910, where it was introduced by Jackson [1]. In the past decades, the subject has attracted many authors, and  $q$ -difference equations have appeared as a promising research field, on both applied and theoretical levels (see [2–7]). For more details on this subject, refer to [8–20].

Recently, fractional boundary value problems on both bounded and unbounded domains have been extensively investigated (see [21–29]). On the one hand, a large number of research works on fractional calculus are dedicated to the fractional differential equations' solvability on a finite interval. On the other hand, fractional boundary value

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problems on an infinite interval have been rarely studied (see [30–38]). Arara *et al.* [30] investigated the bounded solutions' existence for fractional differential equations in the sense of the Caputo fractional derivative on the unbounded domain expressed as:

$$\begin{cases} {}^c\mathbb{D}_0^\varsigma \wp(t) = \mathfrak{w}(t, \wp(t)), & t \geq 0, \\ \wp(0) = \wp_0, \\ \wp \text{ is bounded on } J := [0, \infty), \end{cases}$$

where  $\varsigma \in (1, 2)$ ,  ${}^c\mathbb{D}_0^\varsigma$  is the Caputo fractional derivative of order  $\varsigma$ ,  $\wp_0 \in \mathbb{R}$ , and  $\mathfrak{w} : J \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous. From a combined Schauder fixed-point theorem with a diagonalization method, Su [31] discussed the solutions' existence for the following fractional boundary value problem:

$$\begin{cases} \mathbb{D}_{0+}^\varsigma \wp(t) = \mathfrak{w}(t, \wp(t)), & t \in J, \\ \wp(0) = 0, & \mathbb{D}_{0+}^{\varsigma-1} \wp(\infty) = \wp_\infty, \end{cases}$$

where  $1 < \varsigma \leq 2$ ,  $\mathfrak{w} \in \mathcal{C}(J \times \mathbb{E}, \mathbb{E})$ ,  $\wp_\infty \in \mathbb{E}$ ,  $\mathbb{D}_{0+}^\varsigma$  and  $\mathbb{D}_{0+}^{\varsigma-1}$  are Riemann–Liouville fractional derivatives. The applied technique relies on the Kuratowski measure properties of non-compactness and the Darbo fixed-point theorem on infinite intervals in Banach space. Zhao *et al.* [32] proved the positive solutions' existence for the following fractional differential equation:

$$\begin{cases} \mathbb{D}_{0+}^\varsigma \wp(t) + \mathfrak{w}(t, \wp(t)) = 0, & t \in J, \\ \wp(0) = 0, & \lim_{t \rightarrow +\infty} \mathbb{D}_{0+}^{\varsigma-1} \wp(t) = \beta \wp(v), \end{cases}$$

via the Leray–Schauder nonlinear alternative theorem, where  $1 < \varsigma \leq 2$ ,  $\mathfrak{w} \in \mathcal{C}(J \times \mathbb{R}, J)$ ,  $0 \leq v < \infty$  and  $\mathbb{D}_{0+}^\varsigma$  is the usual Riemann–Liouville fractional derivative. Zhao *et al.* [2] investigated the fractional multipoint boundary value problem:

$$\begin{cases} \mathbb{D}_{0+}^\varsigma \wp(t) + \mathfrak{w}(t, \wp(t)) = 0, & t \in J, \\ \wp(0) = 0, & \mathbb{D}_{0+}^{\varsigma-1} \wp(+\infty) = \sum_{i=1}^m \varsigma_i \wp(v_i), \end{cases}$$

where  $1 < \varsigma \leq 2$ ,  $0 < v_1 < v_2 < \dots < v_m < \infty$ ,  $\mathbb{D}_{0+}^\varsigma$  is a usual Riemann–Liouville fractional derivative. Three positive solutions' existence was proven via the Leggett–Williams fixed-point theorem. Rezapour *et al.* investigated the multisingular fractional integrodifferential pointwise equation

$$\begin{cases} \mathbb{D}_q^\varsigma \wp(t) = \mathfrak{w}(t, \wp(t), \wp'(t), \mathbb{D}^{\gamma_1} \wp(t), \mathbb{I}^{\gamma_2} \wp(t)), & (\forall t \in [0, 1]), \\ \wp'(0) = \wp(a), & \wp(1) = \int_0^b \wp(\xi) d\xi, \end{cases} \quad (1)$$

$\wp^{(j)}(0) = 0$ ,  $j = 2, \dots, [\sigma] - 1$ , where  $\wp \in C^1([0, 1])$ ,  $\gamma_1, \gamma_2 \in [2, \infty)$ ,  $a, b \in (0, 1)$ ,  $\mathbb{D}^\varsigma$  is the Caputo fractional  $q$ -derivative of ord  $\varsigma$ , and  $\mathfrak{w} : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$  is a function such that  $\mathfrak{w}(t, \cdot, \cdot, \cdot, \cdot)$  is singular at some points  $t \in [0, 1]$  [6]. The authors in [24] applied the Leray–Schauder nonlinear alternative theorem and Leggett–Williams fixed-point theorem to

prove the positive solutions' existence to the following fractional differential equation:

$$\begin{cases} (\mathbb{D}_q^\varsigma \wp)(t) + \mathfrak{w}(t, \wp(t)) = 0, & t \in J, \\ \wp(0) = 0, & \mathbb{D}_q^{\varsigma-1} \wp(+\infty) = \sum_{i=1}^m a_i \wp(v_i), \end{cases}$$

where  $0 < q < 1, 1 < \varsigma < 2$ ,

$$0 \leq \sum_{i=1}^m a_i v_i^{\varsigma-1} < \Gamma_q(\varsigma),$$

$\mathbb{D}_q^\varsigma$  is a Riemann–Liouville fractional  $q$ -derivative.  $\mathfrak{w} : (J, \mathbb{R}) \rightarrow J$  is a continuous function. Zhao *et al.* [32] applied the Altman fixed-point theorem to study the solutions' existence to the following fractional differential equation:

$$\begin{cases} \mathbb{D}^\varsigma \wp(t) + \mathfrak{w}(t, \wp(t), \mathcal{T}\wp(t)) = 0, & 3 < \varsigma \leq 4, \\ \wp(0) = \wp'(0) = \wp''(0) = 0, \\ \mathbb{D}^{\varsigma-1} \wp(\infty) = \nu \mathbb{I}^\zeta \wp(\eta), & \zeta > 0, \end{cases} \quad (2)$$

where  $t \in J, \mathfrak{w} \in \mathcal{C}[J \times \mathbb{R}^2, \mathbb{R}]$ ,  $\nu \in \mathbb{R}, \eta \in J$ ,  $\mathbb{D}^\varsigma$  is the Riemann–Liouville fractional derivative of order  $\varsigma$ ,  $\mathbb{I}^\zeta$  is the Riemann–Liouville fractional integral of order  $\zeta$  and

$$\mathcal{T}\wp(t) = \int_0^t \mathbb{G}(t, \xi) \wp(\xi) d\xi,$$

with  $\mathbb{G}(t, s) \in \mathcal{C}[\mathcal{E}, \mathbb{R}]$ ,  $\mathcal{E} = \{(t, s) \in \mathbb{R}^2 : 0 \leq s \leq t\}$ . In [38], Matar *et al.* considered the FDE

$$\begin{cases} \frac{d}{dt}(\varphi_p({}^C \mathbb{D}^{\varsigma_1, \rho} \wp(t))) = \mathfrak{w}(t, \wp(t), {}^C \mathbb{D}^{\varsigma_2, \rho} \wp(t)), & (\forall t \in [0, 1]), \\ \wp(0) + \mu \wp(1) = \mathfrak{v}_1(\wp(0), \wp(1)), & (\mu \neq 1), \\ \wp'(1) = \mathfrak{v}_2(\wp(0), \wp(1)), \end{cases} \quad (3)$$

here  ${}^C \mathbb{D}^{\varsigma_1, \rho}$  and  ${}^C \mathbb{D}^{\varsigma_2, \rho} \wp(t)$  are the generalized Caputo fractional derivative of order  $1 < \sigma_1 < 2$  and  $0 < \sigma_2 < 1$ , respectively,  $\rho > 1$ ,  $\varphi_p(p > 1)$  is a  $p$ -Laplacian operator and the nonlinear functions  $\mathfrak{w} : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $\mathfrak{v}_1, \mathfrak{v}_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  are given continuous functions.

The solutions' existence for the following boundary value problem of nonlinear fractional  $q$ -difference equations on an infinite interval are investigated in this work:

$$\begin{cases} \mathbb{D}_q^\varsigma \wp(t) + \mathfrak{w}(t, \wp(t)) = 0, & 3 < \varsigma \leq 4, \\ \wp(0) = \wp'(0) = \wp''(0) = 0, \\ \mathbb{D}_q^{\varsigma-1} \wp(\infty) = \sum_{i=1}^m {}_1 a_i \wp({}_1 \eta_i) + \sum_{i=1}^m {}_2 a_i \mathbb{D}_q^\nu \wp({}_2 \eta_i), & \nu > 0, \end{cases} \quad (4)$$

where  $t \in J, \mathfrak{w} \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ ,  $0 < {}_1 \eta_i, {}_2 \eta_i < +\infty$ ,  $i = 1, \dots, m$ ,  $\mathbb{D}_q^{(\cdot)}$  is the Riemann–Liouville fractional  $q$ -derivative with  $0 < q < 1$ ,  ${}_1 a_i, {}_2 a_i > 0$  such that

$$\left\{ \sum_{i=1}^m {}_1 a_i \eta_i^{\varsigma-1} + \frac{\Gamma_q(\varsigma)}{\Gamma_q(\varsigma-\nu)} \sum_{i=1}^m {}_2 a_i {}_2 \eta_i^{\varsigma-\nu-1} \right\} \leq \Gamma_q(\varsigma),$$

and  $\mathfrak{w} : (J, \mathbb{R}) \rightarrow \mathbb{R}$  is a continuous function.

A variable's range is studied in this research work on the unbounded domain. A modified compactness criterion is utilized to investigate the studied compact operator  $\mathfrak{H}$  due to the failure of the Arzelà–Ascoli theorem in space  $\mathcal{C}_\infty$ . With the help of the Banach fixed-point theorem, the Leray–Schauder nonlinear alternative theorem, and the Schauder fixed-point theorem, the investigated system's uniqueness and existence are proven.

The remainder of this article is organized as follows: the main definitions and lemmas are discussed in Sect. 2 to offer a guide for proving our results by providing a necessary background with some properties to formulate a Green's function that is suitable for the investigated problem on an unbounded domain. In Sect. 3, the solution's existence and uniqueness are investigated for the boundary value problem (4). In Sect. 4, two examples are presented to apply our outcomes.

## 2 Preliminaries

Essential  $q$ -derivative and  $q$ -integral notions are presented in this section. For more background information, refer to [4, 11, 12, 33, 39, 40]. Consider the real number  $v$  and put  $[v]_q = (1 - q^v)(1 - q)^{-1}$ . The  $q$ -analog of the power  $(v - \hat{v})^l$  is  $(v - \hat{v})^{(0)} = 1$ ,

$$(v - \hat{v})^{(l)} = \prod_{\kappa=0}^{l-1} (v - \hat{v}q^\kappa), \quad (v, \hat{v} \in \mathbb{R}, l \in \mathbb{N}).$$

Generally,  $\forall v, \hat{v}$  and  $\varsigma \in \mathbb{R}$ ,

$$(v - \hat{v})^{(\varsigma)} = v^\varsigma \prod_{\kappa=0}^{\infty} \left( \frac{v - \hat{v}q^\kappa}{v - \hat{v}q^{\kappa+\varsigma}} \right).$$

The  $q$ -gamma function is expressed as  $\Gamma_q(\varsigma) = (1 - q)^{(\varsigma-1)}(1 - q)^{1-\varsigma}$ , for  $\varsigma \in \mathbb{R} - \{\dots, -2, -1, 0\}$  [39]. The  $q$ -gamma function satisfies  $(\Gamma_q(1 + \varsigma) = [\varsigma]_q \Gamma_q(\varsigma))$ . For any  $\varsigma, \nu > 0$ , the  $q$ -beta function is expressed as

$$\mathbb{B}_q(\varsigma, \nu) = \int_0^1 \xi^{(\varsigma-1)}(1 - q\xi)^{(\nu-1)} d_q \xi, \quad q \in (0, 1),$$

where the  $q$ -beta function's expression in terms of the  $q$ -gamma function is ([39])

$$\mathbb{B}_q(\varsigma, \nu) = \frac{\Gamma_q(\varsigma)\Gamma_q(\nu)}{\Gamma_q(\varsigma + \nu)}.$$

The  $q$ -derivative of order  $l \in \mathbb{N}$ , of a function  $\wp$  is expressed as  $\mathbb{D}_q^0 \wp(t) = \wp(t)$ ,

$$\mathbb{D}_q \wp(t) := \mathbb{D}_q^1 \wp(t) = \frac{\wp(t) - \wp(qt)}{(1 - q)t}, \quad (t \neq 0),$$

$\mathbb{D}_q \wp(0) = \lim_{t \rightarrow 0} \mathbb{D}_q \wp(t)$ , and  $\mathbb{D}_q^l \wp(t) = \mathbb{D}_q \mathbb{D}_q^{l-1} \wp(t)$ ,  $\forall t \in J$  [39]. Set

$$J_t := \{tq^l : l \in \mathbb{N}\} \cup \{0\}.$$

The  $q$ -integral of a function  $\wp : J_t \rightarrow \mathbb{R}$  is written as

$$\mathbb{I}_q \wp(t) = \int_0^t \wp(\xi) d_q \xi = \sum_{\kappa=0}^{\infty} tq^\kappa (1 - q) \wp(tq^\kappa),$$

iff the series converges [39]. Obviously,  $\mathbb{D}_q \mathbb{I}_q \wp(t) = \wp(t)$ . Suppose that  $\wp$  is continuous at 0, then  $\mathbb{I}_q \mathbb{D}_q \wp(t) = \wp(t) - \wp(0)$  [39]. The fractional  $q$ -integral of the Riemann–Liouville type of order  $\alpha \geq 0$ , of a function  $\wp : \mathbb{I} \rightarrow \mathbb{R}$  is expressed as  $\mathbb{I}_q^\alpha \wp(t) = \wp(t)$ , and  $\forall t \in \mathbb{I}$ , [9]

$$\mathbb{I}_q^\varsigma \wp(t) = \int_0^t \frac{(t - q\xi)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} \wp(\xi) d_q \xi.$$

**Lemma 2.1** ([4]) *Assume that  $\varsigma, \nu \geq 0$ , and  $\wp$  is a function defined on  $[0, 1]$ . Then, the following hold  $\mathbb{I}_q^\varsigma \mathbb{I}_q^\nu \wp(t) = \mathbb{I}_q^{\varsigma+\nu} \wp(t) = \mathbb{I}_q^\nu \mathbb{I}_q^\varsigma \wp(t)$ , and*

$$\mathbb{I}_q^\varsigma t^\nu = \frac{\Gamma_q(\nu+1)}{\Gamma_q(\varsigma+\nu+1)} t^{\varsigma+\nu}, \quad \forall \nu \in (-1, \infty), \varsigma \in J, t > 0.$$

*In particular, if  $\wp \equiv 1$ , then  $\mathbb{I}_q^\varsigma 1(t) = \frac{1}{\Gamma_q(1+\varsigma)} t^{(\varsigma)}, \forall t > 0$ .*

**Definition 2.2** ([40]) *The fractional  $q$ -derivative of the Riemann–Liouville type of order  $\varsigma \geq 0$ , of a function  $\wp : J \rightarrow \mathbb{R}$  is defined by  $\mathbb{D}_q^0 \wp(t) = \wp(t)$ , and*

$$\mathbb{D}_q^\varsigma \wp(t) = \mathbb{D}_q^{[\varsigma]} \mathbb{I}_q^{[\varsigma]-\varsigma} \wp(t) = \frac{1}{\Gamma_q(\iota - \varsigma)} \int_0^t \frac{\wp(\xi)}{(t - q\xi)^{\varsigma-\iota+1}} d_q \xi,$$

where  $\iota = [\varsigma]$  is the integer part of  $\varsigma$ .

**Lemma 2.3** ([12]) *Assume that  $\varsigma > 0$  and  $\iota \in \mathbb{N}$ . Then, an equality holds as follows:*

$$\mathbb{I}_q^\varsigma \mathbb{D}_q^\iota \wp(t) = \mathbb{D}_q^\iota \mathbb{I}_q^\varsigma \wp(t) - \sum_{\kappa=0}^{\varsigma-1} \frac{t^{\varsigma-\iota+\kappa}}{\Gamma_q(\varsigma+\kappa-\iota+1)} (\mathbb{D}_q^\kappa \wp)(0).$$

*Obviously, we have:*

$$\mathbb{D}_q^\varsigma t^{v-1} = \frac{\Gamma_q(v)}{\Gamma_q(v-\varsigma)} t^{v-\varsigma-1}. \quad (5)$$

**Lemma 2.4** ([11]) *Assume that  $J_0$  and  $J$  are intervals involving  $0 \in J_0 \subseteq J$ . Assume that  $\wp_l, (l \in \mathbb{N})$  and  $\wp$  are functions defined in  $J$   $\lim_{l \rightarrow \infty} \wp_l(t) = \wp(t), (\forall t \in J)$ , and  $\wp_l$  tends uniformly to  $\wp$  on  $J_0$ . Then,*

$$\lim_{l \rightarrow \infty} \int_0^v \wp_l(\xi) d_q \xi = \int_0^v \wp(\xi) d_q \xi \quad (\forall v \in J).$$

The following results are important in the remainder of the paper, where we give the exact expression of the Green's function associated with the system (4).

**Lemma 2.5** *A given function:  $\hbar \in C(\mathbb{R}^+)$  is assumed. Then, the boundary value problem:*

$$\mathbb{D}_q^\varsigma \wp(t) + \hbar(t) = 0, \quad t \in J, \quad (6)$$

via boundary conditions

$$\begin{cases} \wp(0) = \wp'(0) = \wp''(0) = 0, \\ \mathbb{D}_q^{\zeta-1} \wp(+\infty) = \sum_{i=1}^m {}_1a_i \wp({}_1\eta_i) + \sum_{i=1}^m {}_2a_i \mathbb{D}_q^v \wp({}_2\eta_i) \end{cases} \quad (7)$$

has a unique solution

$$\begin{aligned} \wp(t) = & \int_0^\infty {}_1\mathbb{G}_q(t, \xi) h(\xi) d_q \xi \\ & + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^\infty {}_2\mathbb{G}_q({}_1\eta_i, \xi) h(\xi) d_q \xi \\ & + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^\infty {}_2\mathbb{G}_q({}_2\eta_i, \xi) h(s) d_q \xi, \end{aligned} \quad (8)$$

with  $\Lambda$  defined by

$$\Lambda = \left\{ \Gamma_q(\zeta) - \left( \sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1} + \frac{\Gamma_q(\zeta)}{\Gamma_q(\zeta-\nu)} \sum_{i=1}^m {}_2a_i {}_2\eta_i^{\zeta-\nu-1} \right) \right\}, \quad (9)$$

where

$${}_1\mathbb{G}_q(t, s) = \begin{cases} \frac{t^{\zeta-1} - (t-q s)^{(\zeta-1)}}{\Gamma_q(\zeta)}, & s \leq t, \\ \frac{t^{\zeta-1}}{\Gamma_q(\zeta)}, & t \leq s, \end{cases}$$

and

$${}_2\mathbb{G}_q(t, s) = \begin{cases} \frac{t^{\zeta-\nu-1} - (t-q s)^{(\zeta-\nu-1)}}{\Gamma_q(\zeta-\nu)}, & s \leq t, \\ \frac{t^{\zeta-\nu-1}}{\Gamma_q(\zeta-\nu)}, & t \leq s, \end{cases}$$

$\forall t, s \in J$  is the Green's function of boundary value problem (6) and (7).

*Proof* By integrating both sides of (6)

$$\mathbb{I}_q^\zeta \mathbb{D}_q^\zeta \wp(t) = -\mathbb{I}_q^\zeta h(t).$$

The Definition 2.2 and Lemma 2.3, imply that

$$\wp(t) = -\frac{1}{\Gamma_q(\zeta)} \int_0^t (t-q\xi)^{(\zeta-1)} h(\xi) d_q \xi + c_1 t^{\zeta-1} + c_2 t^{\zeta-2} + c_3 t^{\zeta-3} + c_4 t^{\zeta-4}.$$

The boundary conditions implies that  $\wp(0) = \wp'(0) = \wp''(0) = 0$ , thus,

$$c_2 = c_3 = c_4 = 0,$$

hence,

$$\wp(t) = -\frac{1}{\Gamma_q(\zeta)} \int_0^t (t-q\xi)^{(\zeta-1)} h(\xi) d_q \xi + c_1 t^{\zeta-1}. \quad (10)$$

We obtain:

$$\begin{aligned} \sum_{i=1}^m {}_1a_i \wp({}_1\eta_i) &= c_1 \sum_{i=1}^m {}_1a_i \eta_i^{\varsigma-1} \\ &\quad - \frac{1}{\Gamma_q(\varsigma)} \sum_{i=1}^m {}_1a_i \int_0^{1\eta_i} ({}_1\eta_i - q\xi)^{(\varsigma-1)} \hbar(\xi) d_q\xi. \end{aligned}$$

By differentiating both sides of (10) along with (5), we obtain

$$\begin{aligned} \sum_{i=1}^m {}_2a_i \mathbb{D}_q^\nu \wp({}_2\eta_i) &= c_1 \sum_{i=1}^m {}_2a_i \frac{\Gamma(\varsigma)}{\Gamma(\varsigma-\nu)^2} \eta_i^{\varsigma-\nu-1} \\ &\quad - \frac{1}{\Gamma_q(\varsigma-\nu)} \sum_{i=1}^m {}_2a_i \int_0^{2\eta_i} ({}_2\eta_i - q\xi)^{(\varsigma-1)} \hbar(\xi) d_q\xi, \end{aligned}$$

and  $\mathbb{D}_q^{\varsigma-1} \wp(t) = -\mathbb{I}_q \hbar(t) + c_1 \Gamma_q(\varsigma)$ . Therefore,

$$\mathbb{D}_q^{\varsigma-1} \wp(\infty) = - \int_0^{+\infty} \hbar(\xi) d_q\xi + c_1 \Gamma_q(\varsigma).$$

By the second condition of (7)

$$\begin{aligned} - \int_0^{+\infty} \hbar(\xi) d_q\xi + c_1 \Gamma_q(\varsigma) &= c_1 \sum_{i=1}^m {}_1a_i \eta_i^{\varsigma-1} \\ &\quad - \frac{1}{\Gamma_q(\varsigma)} \sum_{i=1}^m {}_1a_i \int_0^{1\eta_i} ({}_1\eta_i - q\xi)^{(\varsigma-1)} \hbar(\xi) d_q\xi \\ &\quad + c_1 \sum_{i=1}^m {}_2a_i \frac{\Gamma(\varsigma)}{\Gamma(\varsigma-\nu)^2} \eta_i^{\varsigma-\nu-1} \\ &\quad - \frac{1}{\Gamma_q(\varsigma-\nu)} \sum_{i=1}^m {}_2a_i \int_0^{2\eta_i} ({}_2\eta_i - q\xi)^{(\varsigma-1)} \hbar(\xi) d_q\xi. \quad (11) \end{aligned}$$

Moreover, equation (11) implies that

$$\begin{aligned} c_1 &= \frac{1}{\Lambda} \int_0^{+\infty} \hbar(\xi) d_q\xi - \frac{1}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^{1\eta_i} \frac{({}_1\eta_i - q\xi)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} \hbar(\xi) d_q\xi \\ &\quad - \frac{1}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^{2\eta_i} \frac{({}_2\eta_i - q\xi)^{(\varsigma-\nu-1)}}{\Gamma_q(\varsigma-\nu)} \hbar(\xi) d_q\xi, \quad (12) \end{aligned}$$

where  $\Lambda$  is given by (9). Consequently, the unique solution of problem (6) and (7) is given by the following formula

$$\begin{aligned} \wp(t) &= \frac{t^{\varsigma-1}}{\Lambda} \int_0^{+\infty} \hbar(\xi) d_q\xi \\ &\quad - \frac{t^{\varsigma-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^{1\eta_i} \frac{({}_1\eta_i - q\xi)^{(\varsigma-1)}}{\Gamma_q(\varsigma)} \hbar(\xi) d_q\xi \end{aligned}$$

$$\begin{aligned}
& -\frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^{\eta_i} \frac{({}_2\eta_i - q\xi)^{(\zeta-\nu-1)}}{\Gamma_q(\zeta-\nu)} h(\xi) d_q\xi \\
& -\frac{1}{\Gamma_q(\zeta)} \int_0^t (t-q\xi)^{(\zeta-1)} h(\xi) d_q\xi, \\
& = \frac{t^{\zeta-1} \Gamma_q(\zeta)}{\Lambda} \int_0^{+\infty} \frac{h(\xi)}{\Gamma_q(\zeta)} d_q\xi \\
& -\frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^{{}_1\eta_i} \frac{({}_1\eta_i - q\xi)^{(\zeta-1)}}{\Gamma_q(\zeta)} h(\xi) d_q\xi \\
& -\frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^{{}_2\eta_i} \frac{({}_2\eta_i - q\xi)^{(\zeta-\nu-1)}}{\Gamma_q(\zeta-\nu)} h(\xi) d_q\xi \\
& -\frac{1}{\Gamma_q(\zeta)} \int_0^t (t-q\xi)^{(\zeta-1)} h(\xi) d_q\xi, \\
& = \frac{1}{\Gamma_q(\zeta) - \sum_{i=1}^m {}_1a_i {}_1\eta_i - \sum_{i=1}^m {}_2a_i \frac{\Gamma(\zeta)}{\Gamma(\zeta-\nu)} {}_2\eta_i^{\zeta-\nu-1}} \left[ \left( \Gamma_q(\zeta) \right. \right. \\
& - \sum_{i=1}^m {}_1a_i {}_1\eta_i + \sum_{i=1}^m {}_1a_i {}_1\eta_i - \sum_{i=1}^m {}_2a_i \frac{\Gamma(\zeta)}{\Gamma(\zeta-\nu)} {}_2\eta_i^{\zeta-\nu-1} \\
& \left. \left. + \sum_{i=1}^m {}_2a_i \frac{\Gamma(\zeta)}{\Gamma(\zeta-\nu)} {}_2\eta_i^{\zeta-\nu-1} \right) \right] \frac{t^{\zeta-1}}{\Gamma_q(\zeta)} \int_0^{+\infty} h(\xi) d_q\xi \\
& -\frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^{{}_1\eta_i} \frac{({}_1\eta_i - q\xi)^{(\zeta-1)}}{\Gamma_q(\zeta)} h(\xi) d_q\xi \\
& -\frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^{{}_2\eta_i} \frac{({}_2\eta_i - q\xi)^{(\zeta-\nu-1)}}{\Gamma_q(\zeta-\nu)} h(\xi) d_q\xi \\
& -\frac{1}{\Gamma_q(\zeta)} \int_0^t (t-q\xi)^{(\zeta-1)} h(\xi) d_q\xi, \\
& = \frac{1}{\Gamma_q(\zeta)} \int_0^t (t^{\zeta-1} - (t-q\xi)^{(\zeta-1)}) h(\xi) d_q\xi \\
& + \frac{1}{\Gamma_q(\zeta)} \int_t^{+\infty} t^{\zeta-1} h(\xi) d_q\xi \\
& -\frac{t^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} \sum_{i=1}^m {}_1a_i \int_0^{{}_1\eta_i} ({}_1\eta_i^{\zeta-1} - ({}_1\eta_i - q\xi)^{(\zeta-1)}) h(\xi) d_q\xi \\
& + \frac{t^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} \sum_{i=1}^m {}_1a_i \int_{{}_1\eta_i}^{\infty} {}_1\eta_i^{\zeta-1} h(\xi) d_q\xi \\
& -\frac{t^{\zeta-1}}{\Lambda \Gamma_q(\zeta-\nu)} \sum_{i=1}^m {}_2a_i \int_0^{{}_2\eta_i} ({}_2\eta_i^{\zeta-\nu-1} - ({}_2\eta_i - q\xi)^{(\zeta-\nu-1)}) h(\xi) d_q\xi \\
& + \frac{t^{\zeta-1}}{\Lambda \Gamma_q(\zeta-\nu)} \sum_{i=1}^m {}_2a_i \int_{{}_2\eta_i}^{\infty} {}_2\eta_i^{\zeta-\nu-1} h(\xi) d_q\xi, \\
& = \int_0^{\infty} {}_1\mathbb{G}_q(t, \xi) h(\xi) d_q\xi - \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^{\infty} {}_2\mathbb{G}_q({}_1\eta_i, \xi) h(\xi) d_q\xi
\end{aligned}$$

$$\begin{aligned}
& - \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^\infty {}_2G_q({}_2\eta_i, \xi) h(\xi) d_q \xi, \\
& = \int_0^\infty {}_1G_q(t, \xi) h(\xi) d_q \xi.
\end{aligned}$$

The proof is completed.  $\square$

Let us now mention some Green's function's properties that are essential to our study.

**Lemma 2.6** *Suppose that  ${}_1G_q(t, s)$  and  ${}_2G_q(t, s)$  are the Green's functions of the linear system (6) and (7) provided in Lemma 2.5. Then, we obtain:*

- (A1)  ${}_1G_q(t, s)$  is a continuous function for  $(t, s) \in J^2$ ;
- (A2)  ${}_1G_q(t, s) \geq 0$  and  ${}_1G_q(t, s) \leq {}_1G_q(qs, s)$ ,  $\forall t, s \in J$ .
- (A3) for  $(t, s) \in J^2$ ,

$$\frac{{}_1G_q(t, s)}{1+t^{\zeta-1}} \leq \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta-\nu)}.$$

*Proof* Clearly, (A1) and (A2) hold. Obviously,  ${}_1G_q(t, s) \geq 0$ , for  $t, s \in J$ . The monotonicity of  ${}_1G_q(t, s)$  indicates

$$\sup_{t \in J} {}_1G_q(t, s) = {}_1G_q(qs, s) = \frac{(qs)^{\zeta-1}}{\Gamma_q(\zeta)}, \quad \forall s \in J. \quad (13)$$

To prove (A3), for  $(tq, t) \in J^2$ , we obtain

$$\begin{aligned}
\frac{{}_1G_q(t, s)}{1+t^{\zeta-1}} &= \frac{{}_1G_q(t, s)}{1+t^{\zeta-1}} + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \frac{{}_1G_q({}_1\eta_i, s)}{1+t^{\zeta-1}} \\
&\quad + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \frac{{}_1G_q({}_2\eta_i, s)}{1+t^{\zeta-1}} \\
&\leq \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta-\nu)} \right) =: \nabla.
\end{aligned}$$

The proof of Lemma 2.6 is completed.  $\square$

### 3 Main results

First, we define the norm space

$$\mathcal{C}_\infty(J, \mathbb{R}) = \left\{ \wp \in \mathcal{C}(J, \mathbb{R}) : \lim_{t \rightarrow +\infty} \frac{\wp(t)}{1+t^{\zeta-1}} \text{ exists} \right\},$$

equipped with the norm

$$\|\wp\| = \sup_{t \in J} \frac{|\wp(t)|}{1+t^{\zeta-1}}.$$

The space  $\mathcal{C}_\infty$  is a Banach space [7]. A map  $\wp(t) \in \mathcal{C}(J, \mathbb{R})$  with its  $q$ -derivative of Riemann–Liouville of order  $\zeta$  that exists on  $J$  is named as a solution of (6) if (7) is satisfied. For

$\wp \in \mathcal{C}_\infty$ , let an operator  $\mathfrak{H}$  be expressed as:

$$\begin{aligned}\mathfrak{H}\wp(t) &= \int_0^\infty {}_1\mathbb{G}_q(t, \xi)h(\xi) d_q\xi \\ &\quad - \frac{t^{\varsigma-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^\infty {}_2\mathbb{G}_q({}_1\eta_i, \xi)h(\xi) d_q\xi \\ &\quad - \frac{t^{\varsigma-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^\infty {}_2\mathbb{G}_q({}_2\eta_i, \xi)h(\xi) d_q\xi.\end{aligned}\quad (14)$$

Clearly, we have a continuous,  $\wp \in \mathcal{C}_\infty(J, \mathbb{R})$ , then  $\wp$  is the problem (4) solution if  $\wp(t) = \mathfrak{H}\wp(t)$  for any  $t \in \mathbb{R}$ . The operator  $\mathfrak{H}$  compactness is shown by proving that  $\mathfrak{H}$  has a fixed point on  $\mathcal{C}_\infty(J, \mathbb{R})$ .

**Remark 3.1** Note that to apply the Arzelà–Ascoli theorem in basic space  $\mathcal{C}_\infty$ , we need to establish the following modified compactness criterion to show that  $\mathfrak{H}$  is compact.

Let us introduce the following hypotheses for convenience:

(A4) for each  $q \in (0, 1)$ ,

$$0 \leq \left\{ \sum_{i=1}^m {}_1a_i \eta_i^{\varsigma-1} + \sum_{i=1}^m {}_2a_i \frac{\Gamma(\varsigma)}{\Gamma(\varsigma-\nu)} {}_2\eta_i^{\varsigma-\nu-1} \right\} < \Gamma_q(\varsigma).$$

(A5) Let

$$\mathfrak{w}_*(t, \wp) = \mathfrak{w}(t, (1 + t^{\varsigma-1})\wp),$$

and  $|\mathfrak{w}_*(t, \wp)| \leq \psi(t)p(|\wp|)$ , on  $J$  with nondecreasing function  $p \in \mathcal{C}(J, \mathbb{R})$  and  $\psi \in \mathcal{C}(J)$ ,

$$\int_0^\infty \psi(\xi) d_q\xi < +\infty.$$

(A6)  $\exists$  is a constant  $\gamma$ , for any  $\xi \in J$ ,  $\wp, \wp' \in \mathbb{R}$

$$|\mathfrak{w}_*(\xi, \wp'(\xi)) - \mathfrak{w}_*(\xi, \wp(\xi))| \leq \gamma |\wp' - \wp|.$$

**Lemma 3.1** ([7]) Consider the two sets  $\Upsilon = \{\wp \in \mathcal{C}_\infty : \|\wp\| < \mu\} (\mu > 0)$ , and

$$\hat{\Upsilon} = \left\{ \frac{\wp(t)}{1 + t^{\varsigma-1}} : \wp \in \Upsilon \right\}.$$

If  $\hat{\Upsilon}$  is equicontinuous on any compact intervals of  $J$  and equiconvergent at  $\infty$ , then  $\Upsilon$  is relatively compact on  $\mathcal{C}_\infty$

**Remark 3.2**  $\hat{\Upsilon}$  is termed equiconvergent at  $\infty$  iff, for any given positive number  $\epsilon$ ,  $\exists N = N_\epsilon > 0$ ,

$$\left| \frac{\wp(t)}{1 + t^{\varsigma-1}} - \frac{\wp(\acute{t})}{1 + \acute{t}^{\varsigma-1}} \right| < \epsilon \quad \forall \wp \in \hat{\Upsilon}, t, \acute{t} \geq N.$$

**Lemma 3.2** A cone  $\mathbb{P} \subset \mathcal{C}_\infty$  is defined by

$$\mathbb{P} = \{\wp \in \mathcal{C}_\infty : \wp(t) \geq 0, t \in J\}.$$

If (A4) and (A5) hold, then  $\mathfrak{H} : \mathbb{P} \rightarrow \mathbb{P}$  is completely continuous.

*Proof* This proof consists of 3 essential steps.

*STEP A:* Let us prove that  $\mathfrak{H} : \mathbb{P} \rightarrow \mathbb{P}$  is continuous. From the continuity and nonnegativity of  ${}_1\mathcal{G}_q$  and  $\mathfrak{w}$ , we obtain  $\mathfrak{H}\wp(t) \geq 0$  for  $t \in J$ . For each  $\wp \in \mathbb{P}$ , by (A4), we obtain:

$$\int_0^\infty \mathfrak{w}(\xi, \wp(\xi)) \, d_q \xi \leq p(\mu) \int_0^\infty \Psi(\xi) \, d_q \xi < \infty,$$

and

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{(\mathfrak{H}\wp)(t)}{1+t^{\zeta-1}} &= \int_0^\infty {}_1\mathcal{G}_q(t, \xi) \mathfrak{w}(\xi, \wp(\xi)) \, d_q \xi \\ &\quad - \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^\infty {}_2\mathcal{G}_q({}_1\eta_i, \xi) \mathfrak{w}(\xi, \wp(\xi)) \, d_q \xi \\ &\quad - \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^\infty {}_2\mathcal{G}_q({}_2\eta_i, \xi) \mathfrak{w}(\xi, \wp(\xi)) \, d_q \xi. \end{aligned}$$

Thus,

$$\lim_{t \rightarrow +\infty} \frac{(\mathfrak{H}\wp)(t)}{1+t^{\zeta-1}},$$

exists. Thus,  $\mathfrak{H}(\mathbb{P}) \subset \mathbb{P}$ . Let  $\wp_1(t) = \frac{\wp(t)}{1+t^{\zeta-1}}$  and

$$\wp_{n1}(t) = \frac{\wp_n(t)}{1+t^{\zeta-1}}.$$

Then by (A4),

$$\mathfrak{w}(\xi, \wp(\xi)) = \mathfrak{w}_*\left(\xi, \frac{\wp_n(\xi)}{1+\xi^{\zeta-1}}\right) = \mathfrak{w}_*(\xi, \wp_1(\xi)).$$

Taking  $\wp_n \rightarrow \wp$  as  $n \rightarrow +\infty$  in  $\mathcal{C}_\infty$ , by (A5), we obtain:

$$\begin{aligned} \left| \mathfrak{w}(t, \wp_n(t)) - \mathfrak{w}(t, \wp(t)) \right| &= \left| \mathfrak{w}_*\left(t, \frac{\wp_n(t)}{1+t^{\zeta-1}}\right) - \mathfrak{w}_*\left(t, \frac{\wp(t)}{1+t^{\zeta-1}}\right) \right| \\ &= \left| \mathfrak{w}_*(t, \wp_{n1}(t)) - \mathfrak{w}_*(t, \wp_1(t)) \right| \\ &\leq \gamma |\wp_{n1}(t) - \wp_1(t)| \rightarrow 0 \quad \text{uniformly on } J. \end{aligned}$$

So  $\mathfrak{w}(t, \wp_n(t)) \rightarrow \mathfrak{w}(t, \wp(t))$  uniformly on  $J$ . By Lemma 2.4, we obtain

$$\lim_{n \rightarrow \infty} \int_0^\infty \mathfrak{w}(\xi, \wp_n(\xi)) \, d_q \xi = \int_0^\infty \mathfrak{w}(\xi, \wp(\xi)) \, d_q \xi. \quad (15)$$

Hence, combining (15), we obtain

$$\begin{aligned}
 & \left| \frac{\mathfrak{H}\wp_n(t)}{1+t^{\zeta-1}} - \frac{\mathfrak{H}\wp(t)}{1+t^{\zeta-1}} \right| \\
 &= \left| \int_0^\infty \frac{{}_1\mathbb{G}_q(t, \xi)}{1+t^{\zeta-1}} (\mathfrak{w}(\xi, \wp_n(\xi)) - \mathfrak{w}(\xi, \wp(\xi))) d_q \xi \right. \\
 &\quad + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_1\eta_i, \xi)}{1+t^{\zeta-1}} (\mathfrak{w}(\xi, \wp_n(\xi)) - \mathfrak{w}(\xi, \wp(\xi))) d_q \xi \\
 &\quad \left. + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_2\eta_i, \xi)}{1+t^{\zeta-1}} (\mathfrak{w}(\xi, \wp_n(\xi)) - \mathfrak{w}(\xi, \wp(\xi))) d_q \xi \right| \\
 &\leq \left| \int_0^\infty \frac{{}_1\mathbb{G}_q(t, \xi)}{1+t^{\zeta-1}} (\mathfrak{w}(\xi, \wp_n(\xi)) - \mathfrak{w}(\xi, \wp(\xi))) d_q \xi \right| \\
 &\quad + \left| \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_1\eta_i, \xi)}{1+t^{\zeta-1}} (\mathfrak{w}(\xi, \wp_n(\xi)) - \mathfrak{w}(\xi, \wp(\xi))) d_q \xi \right| \\
 &\quad + \left| \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_2\eta_i, \xi)}{1+t^{\zeta-1}} (\mathfrak{w}(\xi, \wp_n(\xi)) - \mathfrak{w}(\xi, \wp(\xi))) d_q \xi \right| \\
 &\leq \frac{1}{\Gamma_q(\zeta)} \left| \int_0^\infty (\mathfrak{w}(\xi, \wp_n(\xi)) - \mathfrak{w}(\xi, \wp(\xi))) d_q \xi \right| \\
 &\quad + \left| \frac{\sum_{i=1}^m {}_1a_i {}_1\eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} \int_0^\infty (\mathfrak{w}(\xi, \wp_n(\xi)) - \mathfrak{w}(\xi, \wp(\xi))) d_q \xi \right| \\
 &\quad + \left| \frac{\sum_{i=1}^m {}_2a_i {}_2\eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta-\nu)} \int_0^\infty (\mathfrak{w}(\xi, \wp_n(\xi)) - \mathfrak{w}(\xi, \wp(\xi))) d_q \xi \right|.
 \end{aligned}$$

Hence,

$$\|\mathfrak{H}\wp_n - \mathfrak{H}\wp\| = \sup_{t \in J} \left| \frac{\mathfrak{H}\wp_n(t) - \mathfrak{H}\wp(t)}{1+t^{\zeta-1}} \right| \rightarrow 0,$$

as  $n \rightarrow \infty$ . So,  $\mathfrak{H}$  is continuous.

**STEP B:** We show that  $\mathfrak{H}$  is uniformly bounded. For this case, let  $\mathbb{Q}$  be any bounded subset of  $\mathbb{P}$ , i.e.,  $\exists \mu > 0$ ,  $\|\wp\| \leq \mu$  for any  $\wp \in \mathbb{Q}$ . It is enough to prove that  $\mathfrak{H}(\wp)$  is bounded in  $\mathbb{P}$ . For  $\wp \in \mathbb{Q}$ , we obtain:

$$\begin{aligned}
 \|\mathfrak{H}\wp\| &= \sup_{t \in J} \left( \int_0^\infty \frac{{}_1\mathbb{G}_q(t, \xi)}{1+t^{\zeta-1}} \mathfrak{w}(t, \wp(\xi)) d_q \xi \right. \\
 &\quad + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_1\eta_i, \xi)}{1+t^{\zeta-1}} \mathfrak{w}(t, \wp(\xi)) d_q \xi \\
 &\quad \left. + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_2\eta_i, \xi)}{1+t^{\zeta-1}} \mathfrak{w}(t, \wp(\xi)) d_q \xi \right) \\
 &\leq \frac{1}{\Gamma_q(\zeta)} \int_0^\infty |\mathfrak{w}(t, \wp(\xi))| d_q \xi
 \end{aligned}$$

$$\begin{aligned}
& + \frac{\sum_{i=1}^m {}_1a_{i1}\eta_i^{\zeta-1}}{\Lambda\Gamma_q(\zeta)} \int_0^\infty |\mathfrak{w}(\mathfrak{t}, \wp(\xi))| \, d_q\xi \\
& + \frac{\sum_{i=1}^m {}_2a_{i2}\eta_i^{\zeta-\nu-1}}{\Lambda\Gamma_q(\zeta-\nu)} \int_0^\infty \mathfrak{w}(\mathfrak{t}, \wp(\xi)) \, d_q\xi \\
& = \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_{i1}\eta_i^{\zeta-1}}{\Lambda\Gamma_q(\zeta)} \right. \\
& \quad \left. + \frac{\sum_{i=1}^m {}_2a_{i2}\eta_i^{\zeta-\nu-1}}{\Lambda\Gamma_q(\zeta-\nu)} \right) \int_0^\infty \left| \mathfrak{w}\left(\xi, \frac{\wp(\xi)(1+\xi^{\zeta-1})}{1+\xi^{\zeta-1}}\right) \right| \, d_q\xi \\
& = \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_{i1}\eta_i^{\zeta-1}}{\Lambda\Gamma_q(\zeta)} \right. \\
& \quad \left. + \frac{\sum_{i=1}^m {}_2a_{i2}\eta_i^{\zeta-\nu-1}}{\Lambda\Gamma_q(\zeta-\nu)} \right) \int_0^\infty \left| \mathfrak{w}\left(\xi, \frac{\wp(\xi)}{1+\xi^{\zeta-1}}\right) \right| \, d_q\xi \\
& \leq \mathfrak{p}(\mu) \nabla \int_0^\infty \psi(\xi) \, d_q\xi < \infty.
\end{aligned}$$

Hence,  $\mathfrak{H}(\mathbb{Q})$  is uniformly bounded. Now, we show that  $\mathfrak{H}(\mathbb{Q})$  is equicontinuous on any compact interval. First, for each given  $\bar{s} > 0$ ,  $\mathfrak{t}_1, \mathfrak{t}_2 \in [0, \bar{s}]$ , and  $\wp \in \mathbb{P}$ , and  $\mathfrak{t}_2 > \mathfrak{t}_1$ , we deduce:

$$\begin{aligned}
\left| \frac{\mathfrak{H}\wp(\mathfrak{t}_2)}{1+\mathfrak{t}_2^{\zeta-1}} - \frac{\mathfrak{H}\wp(\mathfrak{t}_1)}{1+\mathfrak{t}_1^{\zeta-1}} \right| & \leq \left| \int_0^\infty \frac{{}_1\mathbb{G}_q(\mathfrak{t}_2, \xi)}{1+\mathfrak{t}_2^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \right. \\
& \quad \left. - \int_0^\infty \frac{{}_1\mathbb{G}_q(\mathfrak{t}_1, \xi)}{1+\mathfrak{t}_1^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \right| \\
& \quad + \left| \left( \frac{\mathfrak{t}_2^{\zeta-1}}{1+\mathfrak{t}_2^{\zeta-1}} - \frac{\mathfrak{t}_1^{\zeta-1}}{1+\mathfrak{t}_1^{\zeta-1}} \right) \right. \\
& \quad \times \sum_{i=1}^m {}_1a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_1\eta_i, \xi)}{\Lambda} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \left. \right| \\
& \quad + \left| \left( \frac{\mathfrak{t}_2^{\zeta-1}}{1+\mathfrak{t}_2^{\zeta-1}} - \frac{\mathfrak{t}_1^{\zeta-1}}{1+\mathfrak{t}_1^{\zeta-1}} \right) \right. \\
& \quad \times \sum_{i=1}^m {}_2a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_2\eta_i, \xi)}{\Lambda} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \left. \right| \\
& \leq \left| \int_0^\infty \frac{{}_1\mathbb{G}_q(\mathfrak{t}_2, \xi)}{1+\mathfrak{t}_2^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \right. \\
& \quad \left. - \int_0^\infty \frac{{}_1\mathbb{G}_q(\mathfrak{t}_1, \xi)}{1+\mathfrak{t}_2^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \right| \\
& \quad + \left| \int_0^\infty \frac{{}_1\mathbb{G}_q(\mathfrak{t}_1, \xi)}{1+\mathfrak{t}_2^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \right. \\
& \quad \left. - \int_0^\infty \frac{{}_1\mathbb{G}_q(\mathfrak{t}_1, \xi)}{1+\mathfrak{t}_1^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \right| \\
& \quad + \left| \left( \frac{\mathfrak{t}_2^{\zeta-1}}{1+\mathfrak{t}_2^{\zeta-1}} - \frac{\mathfrak{t}_1^{\zeta-1}}{1+\mathfrak{t}_1^{\zeta-1}} \right) \right|
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{i=1}^m {}_1a_i \int_0^\infty \frac{{}_2\mathbb{G}_q(1\eta_i, \xi)}{\Gamma_q(\varsigma) - \Lambda} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \Big| \\
& + \left| \left( \frac{t_2^{\varsigma-1}}{1+t_2^{\varsigma-1}} - \frac{t_1^{\varsigma-1}}{1+t_1^{\varsigma-1}} \right) \right. \\
& \left. \times \sum_{i=1}^m {}_2a_i \int_0^\infty \frac{{}_2\mathbb{G}_q(2\eta_i, \xi)}{\Lambda} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \right|.
\end{aligned}$$

On the other hand, for all  $\wp \in \mathbb{Q}$ ,  $t_1 \rightarrow t_2$ , we obtain

$$\begin{aligned}
& \int_0^\infty \frac{{}_1\mathbb{G}_q(t_2, \xi) - {}_1\mathbb{G}_q(t_1, \xi)}{1+t_2^{\varsigma-1}} |\mathfrak{w}(\xi, \wp(\xi))| \, d_q\xi \\
& \leq \int_0^{t_1} \left| \frac{(t_2^{\varsigma-1} - t_1^{\varsigma-1}) - ((t_2 - q\xi)^{(\varsigma-1)} - (t_1 - q\xi)^{(\varsigma-1)})}{\Gamma_q(\varsigma)(1+t_2^{\varsigma-1})} \right| \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \\
& \quad + \int_{t_1}^{t_2} \left| \frac{t_2^{\varsigma-1} - t_1^{\varsigma-1} - (t_2 - q\xi)^{(\varsigma-1)}}{\Gamma_q(\varsigma)(1+t_2^{\varsigma-1})} \right| \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \\
& \quad + \int_{t_2}^\infty \left| \frac{t_2^{\varsigma-1} - t_1^{\varsigma-1}}{\Gamma_q(\varsigma)(1+t_2^{\varsigma-1})} \right| \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \\
& \leq \int_0^{t_1} \frac{(t_2^{\varsigma-1} - t_1^{\varsigma-1}) + ((t_2 - q\xi)^{(\varsigma-1)} - (t_1 - q\xi)^{(\varsigma-1)})}{\Gamma_q(\varsigma)(1+t_2^{\varsigma-1})} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \\
& \quad + \frac{3\bar{s}^{\varsigma-1}}{\Gamma_q(\varsigma)(1+t_2^{\varsigma-1})} \int_{t_1}^{t_2} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \\
& \quad + \frac{t_2^{\varsigma-1} - t_1^{\varsigma-1}}{\Gamma_q(\varsigma)(1+t_2^{\varsigma-1})} \int_{t_2}^\infty \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \\
& \leq p(\mu) \int_0^{t_1} \frac{(t_2^{\varsigma-1} - t_1^{\varsigma-1}) + ((t_2 - q\xi)^{(\varsigma-1)} - (t_1 - q\xi)^{(\varsigma-1)})}{\Gamma_q(\varsigma)(1+t_2^{\varsigma-1})} \psi(\xi) \, d_q\xi \\
& \quad + p(\mu) \frac{3\bar{s}^{\varsigma-1}}{\Gamma_q(\varsigma)(1+t_2^{\varsigma-1})} \int_{t_1}^{t_2} \psi(\xi) \, d_q\xi \\
& \quad + p(\mu) \frac{t_2^{\varsigma-1} - t_1^{\varsigma-1}}{\Gamma_q(\varsigma)(1+t_2^{\varsigma-1})} \int_{t_2}^\infty \psi(\xi) \, d_q\xi \rightarrow 0.
\end{aligned} \tag{16}$$

Similar to (16), for all  $\wp \in \mathbb{Q}$ ,  $t_1 \rightarrow t_2$ , we obtain

$$\begin{aligned}
& \left| \int_0^\infty \frac{{}_1\mathbb{G}_q(t_1, \xi)}{1+t_2^{\varsigma-1}} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi - \int_0^\infty \frac{{}_1\mathbb{G}_q(t_1, \xi)}{1+t_1^{\varsigma-1}} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \right| \\
& \leq \int_0^\infty \frac{{}_1\mathbb{G}_q(t, \xi) |(1+t_1^{\varsigma-1}) - (1+t_2^{\varsigma-1})|}{(1+t_1^{\varsigma-1})(1+t_2^{\varsigma-1})} \mathfrak{w}(\xi, \wp(\xi)) \, d_q\xi \\
& \leq \frac{|t_2^{\varsigma-1} - t_1^{\varsigma-1}|}{(1+t_2^{\varsigma-1})} \int_0^\infty \frac{{}_1\mathbb{G}_q(t_1, \xi)}{1+t_1^{\varsigma-1} \mathfrak{w}(\xi, \wp(\xi))} \, d_q\xi \\
& \leq \frac{|t_2^{\varsigma-1} - t_1^{\varsigma-1}|}{(1+t_2^{\varsigma-1})} \int_0^\infty \frac{1}{\Gamma_q(\varsigma)} p(\mu) \psi(\xi) \, d_q\xi
\end{aligned}$$

$$+ p(\mu) \frac{|t_2^{\zeta-1} - t_1^{\zeta-1}|}{\Gamma_q(\zeta)(1 + t_2^{\zeta-1})} \int_0^\infty \psi(\xi) d_q \xi \rightarrow 0,$$

and

$$\begin{aligned} & \left| \left( \frac{t_2^{\zeta-1}}{1 + t_2^{\zeta-1}} - \frac{t_1^{\zeta-1}}{1 + t_1^{\zeta-1}} \right) \left( \sum_{i=1}^m {}_1a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_1\eta_i, \xi)}{\Lambda} \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \right. \right. \\ & \quad \left. \left. + \sum_{i=1}^m {}_2a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_2\eta_i, \xi)}{\Lambda} \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \right) \right| \\ & \leq \frac{|t_2^{\zeta-1} - t_1^{\zeta-1}|}{(1 + t_2^{\zeta-1})(1 + t_1^{\zeta-1})} \int_0^\infty \left( \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} \right. \\ & \quad \left. + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta - \nu)} \right) \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \\ & \leq \frac{|t_2^{\zeta-1} - t_1^{\zeta-1}|}{(1 + t_2^{\zeta-1})(1 + t_1^{\zeta-1})} \frac{p(\mu)}{\Lambda} \left( \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Gamma_q(\zeta)} \right. \\ & \quad \left. + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Gamma_q(\zeta - \nu)} \right) \int_0^\infty \psi(\xi) d_q \xi \rightarrow 0. \end{aligned}$$

Hence,  $\mathfrak{H}(\mathbb{Q})$  is equicontinuous on any compact intervals of  $J$ .

**STEP C:** Let us prove that  $\mathfrak{H}$  is equiconvergent at  $\infty$ . For any  $\wp \in \mathbb{Q}$ , by (A5)

$$\int_0^\infty \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \leq p(\mu) \int_0^\infty \psi(\xi) d_q \xi < \infty, \quad (17)$$

we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \left| \frac{\mathfrak{H}\wp}{1 + t^{\zeta-1}} \right| &= \lim_{t \rightarrow \infty} \left| \int_0^\infty \frac{{}_1\mathbb{G}_q(t, \xi)}{1 + t^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \right. \\ & \quad + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_1\eta_i, \xi)}{1 + t^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \\ & \quad \left. + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_2\eta_i, \xi)}{1 + t^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \right| \\ & \leq \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} \right. \\ & \quad \left. + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta - \nu)} \right) \int_0^\infty \left| \mathfrak{w} \left( \xi, \frac{\wp(\xi)(1 + \xi^{\zeta-1})}{1 + \xi^{\zeta-1}} \right) \right| d_q \xi \\ & \leq \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} \right. \\ & \quad \left. + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta - \nu)} \right) \int_0^\infty \left| \mathfrak{w}_* \left( \xi, \frac{\wp(\xi)}{1 + \xi^{\zeta-1}} \right) \right| d_q \xi \\ & \leq p(\mu) \int_0^\infty \psi(\xi) d_q \xi < \infty. \end{aligned}$$

Thus, we obtain

$$\lim_{t \rightarrow +\infty} \left| \frac{\mathfrak{H}\wp(t)}{1+t^{\zeta-1}} \right| \rightarrow 0.$$

Hence,  $\mathfrak{H} : \mathbb{P} \rightarrow \mathbb{P}$  is equiconvergent at infinity. Thus, the operator  $\mathfrak{H}$  is completely continuous. The proof is completed.  $\square$

In the next subsections, the boundary value problem's (4) existence and uniqueness are shown via the Banach fixed-point theorem, the Schauder fixed-point theorem, and the Leray–Schauder nonlinear alternative theorem.

### 3.1 Existence via the Leray–Schauder nonlinear alternative theorem

**Theorem 3.3** *Suppose that  $\exists \mu > 0$ , and functions  $\mathfrak{p}$  and  $\psi$  satisfy:*

$$\mu > \nabla \mathfrak{p}(\mu) \int_0^\infty \psi(\xi) d_q \xi, \quad (18)$$

*and conditions (A5) and (A6) hold. Then, the boundary value problem (4) has an unbounded solution  $\wp(t)$  such that*

$$0 \leq \frac{\wp(t)}{1+t^{\zeta-1}} \leq \mu \quad (t \in J).$$

*Proof* Let

$$\Upsilon = \{\wp \in \mathbb{P} : \|\wp\| \leq \mu\},$$

we have  $\wp \neq \lambda \mathfrak{H}\wp$  for  $\wp \in \partial \Upsilon$  and  $\lambda \in (0, 1)$ . If  $\exists \wp \in \partial \Upsilon$  with  $\wp = \lambda \mathfrak{H}\wp$ , then for  $\lambda \in (0, 1)$ , we obtain

$$\begin{aligned} \|\wp\| &= \sup_{t \in J} \left| \frac{\lambda \mathfrak{H}\wp(t)}{1+t^{\zeta-1}} \right| \leq \sup_{t \in J} \left| \frac{t\wp(t)}{1+t^{\zeta-1}} \right| \\ &\leq \sup_{t \in J} \left( \int_0^\infty \frac{{}_1\mathbb{G}_q(t, \xi)}{1+t^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \right. \\ &\quad \left. + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_1\eta_i, \xi)}{1+t^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \right. \\ &\quad \left. + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_2\eta_i, \xi)}{1+t^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \right) \\ &\leq \frac{1}{\Gamma_q(\zeta)} \int_0^\infty |\mathfrak{w}(\xi, \wp(\xi))| d_q \xi \\ &\quad + \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} \int_0^\infty |\mathfrak{w}(\xi, \wp(\xi))| d_q \xi \\ &\quad + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta-\nu)} \int_0^\infty \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \\ &= \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} \right) \end{aligned}$$

$$\begin{aligned}
& + \frac{\sum_{i=1}^m {}_2a_i {}_2\eta_i \xi^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta-\nu)} \Bigg) \int_0^\infty \left| \mathfrak{w} \left( \xi, \frac{\wp(\xi)(1+\xi^{\zeta-1})}{1+\xi^{\zeta-1}} \right) \right| d_q \xi \\
& = \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_i {}_1\eta_i \xi^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} \right. \\
& \quad \left. + \frac{\sum_{i=1}^m {}_1\eta_i {}_2\eta_i \xi^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta-\nu)} \right) \int_0^\infty \left| \mathfrak{w}_* \left( \xi, \frac{\wp(\xi)}{1+\xi^{\zeta-1}} \right) \right| d_q \xi \\
& \leq \mathfrak{p}(\mu) \nabla \int_0^\infty \Psi(\xi) d_q \xi < \infty.
\end{aligned}$$

Hence, for  $\wp \in \partial \Upsilon$ , we find

$$\mu \leq \mathfrak{p}(\mu) \nabla \int_0^\infty \Psi(\xi) d_q \xi,$$

which contradicts (18). By the fixed-point theorem of Schauder, the problem (4) has an unbounded solution  $\wp = \wp(t)$ ,

$$0 \leq \wp(t) \leq \mu(1+t^{\zeta-1}), \quad \forall t \in J.$$

The proof is completed.  $\square$

### 3.2 Existence via the Schauder FpThm

**Theorem 3.4** *From (A4), the boundary value problem (4) has at least 1 solution on  $t \in J$ .*

*Proof* Assume that an operator  $\mathfrak{H}$  is defined in (14). By testing to see that all Schauder fixed-point theorem's axioms on  $\mathcal{C}_\infty$  are satisfied.  $\mathfrak{w}$  and  ${}_1\mathbb{G}_q, {}_2\mathbb{G}_q$  are given continuous functions indicating that the operator  $\mathfrak{H}$  is continuous. The remainder of the Theorem 3.4 proof is divided as follows:

*STEP A:* Let

$$\Upsilon = \{ \wp \in \mathcal{C}_\infty : \|\wp\| \leq \mu \},$$

closed ball. Then, we prove that  $\mathfrak{H} : \Upsilon \rightarrow \Upsilon$ . We select

$$\mu \geq \mathfrak{p}(\mu) \nabla \int_0^\infty \Psi(\xi) d_q \xi.$$

For any  $\wp \in \mathcal{C}_\infty$ , we prove that  $\mathfrak{H}\Upsilon \subset \Upsilon$ . Hence, for  $t \in J$ , we obtain:

$$\begin{aligned}
\left| \frac{\mathfrak{H}\wp(t)}{1+t^{\zeta-1}} \right| & \leq \sup_{t \in J} \left( \int_0^\infty \frac{{}_1\mathbb{G}_q(t, \xi)}{1+t^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \right. \\
& \quad + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^\infty \frac{{}_1\mathbb{G}_q({}_1\eta_i, \xi)}{1+t^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \\
& \quad \left. + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^\infty \frac{{}_2\mathbb{G}_q({}_2\eta_i, \xi)}{1+t^{\zeta-1}} \mathfrak{w}(\xi, \wp(\xi)) d_q \xi \right) \\
& \leq \frac{1}{\Gamma_q(\zeta)} \int_0^\infty |\mathfrak{w}(\xi, \wp(\xi))| d_q \xi
\end{aligned}$$

$$\begin{aligned}
& + \frac{\sum_{i=1}^m {}_1a_{i1}\eta_i^{\zeta-1}}{\Lambda\Gamma_q(\zeta)} \int_0^\infty |\mathfrak{w}(\xi, \wp(\xi))| d_q\xi \\
& + \frac{\sum_{i=1}^m {}_2a_{i2}\eta_i^{\zeta-\nu-1}}{\Lambda\Gamma_q(\zeta-\nu)} \int_0^\infty \mathfrak{w}(\xi, \wp(\xi)) d_q\xi \\
& = \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_{i1}\eta_i^{\zeta-1}}{\Lambda\Gamma_q(\zeta)} \right. \\
& \quad \left. + \frac{\sum_{i=1}^m {}_2a_{i2}\eta_i^{\zeta-\nu-1}}{\Lambda\Gamma_q(\zeta-\nu)} \right) \int_0^\infty \left| \mathfrak{w} \left( \xi, \frac{\wp(\xi)(1+\xi^{\zeta-1})}{1+\xi^{\zeta-1}} \right) \right| d_q\xi \\
& = \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_{i1}\eta_i^{\zeta-1}}{\Lambda\Gamma_q(\zeta)} \right. \\
& \quad \left. + \frac{\sum_{i=1}^m {}_2a_{i2}\eta_i^{\zeta-\nu-1}}{\Lambda\Gamma_q(\zeta-\nu)} \right) \int_0^\infty \left| \mathfrak{w}_* \left( \xi, \frac{\wp(\xi)}{1+\xi^{\zeta-1}} \right) \right| d_q\xi \\
& \leq \mathfrak{p}(\mu) \nabla \int_0^\infty \Psi(\xi) d_q\xi < \infty.
\end{aligned}$$

As a result,  $\|\mathfrak{H}\wp\|_{C_\infty} \leq \mu$ , which indicates that  $\mathfrak{H}\Upsilon \subset \Upsilon$ , i.e., the operator  $\mathfrak{H}$  maps  $\Upsilon$  into  $\Upsilon$ .

**STEP B:** From Lemma (3.2), we need to prove that  $\mathfrak{H}$  is continuous and completely continuous on  $C_\infty$ . Thus, by the Schauder fixed-point theorem, the operator  $\mathfrak{H}$  has a fixed point  $\wp$  in  $C_\infty$  which is a solution of problem (4).  $\square$

### 3.3 Existence and uniqueness results via the Banach FpThm

**Theorem 3.5** *Let (A5) and a hypothesis hold (A6),  $\exists$  is a positive function  $\varrho(t)$  with*

$$\varrho^* = \int_0^\infty (1 + \xi^{\zeta-1}) \varrho(\xi) d\xi < \infty$$

and

$$|\mathfrak{w}(t, \wp) - \mathfrak{w}(t, \wp')| \leq \varrho(t) |\wp - \wp'|, \quad \forall t \in J, \wp, \wp' \in \mathbb{R}.$$

Then, the boundary value problem (4) has a unique solution:  $\wp(t)$  in  $C_\infty$ . Moreover,  $\exists$  is a monotone iterative sequence  $\{\wp_n(t)\}$  such that  $\wp_n(t) \rightarrow \wp(t)$  as  $n \rightarrow \infty$ , which is uniform on any unbounded subdomain of  $t \in \wp$ , where

$$\wp_n(t) = \int_0^\infty {}_1\mathbb{G}_q(t, \xi) \mathfrak{w}(\xi, \wp_{n-1}(\xi)) d\xi.$$

Furthermore,  $\exists$  is an error estimate for the following:

$$\|\wp_n - \wp\|_\infty \leq \frac{\theta^n}{1-\theta} \|\wp_1 - \wp_0\|_\infty \quad (n = 1, 2, \dots), \quad (19)$$

and

$$\theta = \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_{i1}\eta_i^{\zeta-1}}{\Lambda\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_2a_{i2}\eta_i^{\zeta-\nu-1}}{\Lambda\Gamma_q(\zeta-\nu)} \right) \varrho^* < 1. \quad (20)$$

*Proof* By considering an operator  $\mathfrak{H}$  defined by the equation (14). Assume that  $\wp_1, \wp_2 \in \mathcal{C}_\infty$ . For  $t \in J$ , we have

$$\begin{aligned}
 & \left| \frac{1}{1+t^{\zeta-1}} (\mathfrak{H}\wp_1(t) - \mathfrak{H}\wp_2(t)) \right| \\
 & \leq \sup_{t \in J} \left( \int_0^\infty \frac{{}_1\mathbb{G}_q(t, \xi)}{1+t^{\zeta-1}} |\mathfrak{w}(\xi, \wp_1(\xi)) - \mathfrak{w}(\xi, \wp_2(\xi))| d_q \xi \right. \\
 & \quad + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_1a_i \int_0^\infty \frac{{}_2\mathbb{G}_q(1\eta_i, \xi)}{1+t^{\zeta-1}} |\mathfrak{w}(\xi, \wp_1(\xi)) - \mathfrak{w}(\xi, \wp_2(\xi))| d_q \xi \\
 & \quad \left. + \frac{t^{\zeta-1}}{\Lambda} \sum_{i=1}^m {}_2a_i \int_0^\infty \frac{{}_2\mathbb{G}_q(2\eta_i, \xi)}{1+t^{\zeta-1}} |\mathfrak{w}(\xi, \wp_1(\xi)) - \mathfrak{w}(\xi, \wp_2(\xi))| d_q \xi \right) \\
 & \leq \frac{1}{\Gamma_q(\zeta)} \int_0^\infty |\mathfrak{w}(\xi, \wp_1(\xi)) - \mathfrak{w}(\xi, \wp_2(\xi))| d_q \xi \\
 & \quad + \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} \int_0^\infty |\mathfrak{w}(\xi, \wp_1(\xi)) - \mathfrak{w}(\xi, \wp_2(\xi))| d_q \xi \\
 & \quad + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta-\nu)} \int_0^\infty |\mathfrak{w}(\xi, \wp_1(\xi)) - \mathfrak{w}(\xi, \wp_2(\xi))| d_q \xi \\
 & \leq \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta-\nu)} \right) \\
 & \quad \times \int_0^\infty \left| \mathfrak{w} \left( \xi, \frac{\wp_1(\xi)(1+\xi^{\zeta-1})}{1+\xi^{\zeta-1}} \right) \right. \\
 & \quad \left. - \mathfrak{w} \left( \xi, \frac{\wp_2(\xi)(1+\xi^{\zeta-1})}{1+\xi^{\zeta-1}} \right) \right| d_q \xi \\
 & \leq \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta-\nu)} \right) \\
 & \quad \times \int_0^\infty \left| \mathfrak{w}_* \left( \xi, \frac{\wp_1(\xi)}{1+\xi^{\zeta-1}} \right) - \mathfrak{w}_* \left( \xi, \frac{\wp_2(\xi)}{1+\xi^{\zeta-1}} \right) \right| d_q \xi \\
 & \leq \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta-\nu)} \right) \\
 & \quad \times \left( \int_0^\infty [(1+\xi^{\zeta-1}) \varrho(\xi)] d\xi \right) \|\wp_1 - \wp_2\|_\infty.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 & |(\mathfrak{H}\wp_1)(t) - (\mathfrak{H}\wp_2)(t)| \\
 & \leq \left( \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_1a_i \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} + \frac{\sum_{i=1}^m {}_2a_i \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta-\nu)} \right) \varrho^* \|\wp - \wp\|_\infty \\
 & = \theta \|\wp_1 - \wp_2\|_\infty.
 \end{aligned}$$

Thus, we collect that

$$\|\mathfrak{H}\wp_1 - \mathfrak{H}\wp_2\|_\infty \leq \theta \|\wp_1 - \wp_2\|_\infty, \quad \forall \wp_1, \wp_2 \in \mathcal{C}_\infty. \quad (21)$$

As  $\theta < 1$ , then the Banach fixed-point theorem ensures that  $\mathfrak{H}$  has a unique fixed point  $\wp$  in  $\mathcal{C}_\infty$ . Hence, the problem (4) has a unique solution  $\wp \in \mathcal{C}_\infty$ . Furthermore, for each  $\wp_0 \in \mathcal{C}_\infty$ ,  $\|\wp_n - \wp\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ , where  $\wp_n = \mathfrak{H}\wp_{n-1}$  ( $n = 1, 2, \dots$ ). From (21), we find

$$\|\wp_n - \wp_{n-1}\|_\infty \leq \theta^{n-1} \|\wp_1 - \wp_0\|_\infty,$$

and

$$\begin{aligned} \|\wp_n - \wp_j\|_\infty &\leq \|\wp_n - \wp_{n-1}\|_\infty + \|\wp_{n-1} - \wp_{n-2}\|_\infty \\ &\quad + \dots + \|\wp_{j+1} - \wp_j\|_\infty \\ &= \frac{\theta^n(1 - \theta^{n-j})}{1 - \theta} \|\wp_1 - \wp_0\|_\infty. \end{aligned} \quad (22)$$

By supposing that  $n \rightarrow \infty$  on both sides of (22), we can estimate

$$\|\wp_n - \wp\|_\infty \leq \frac{\theta^n}{1 - \theta} \|\wp_1 - \wp_0\|_\infty.$$

Hence equation (19) holds, and this theorem's proof is completed.  $\square$

#### 4 Applications via numerical results

*Example 4.1* By taking

$$\varsigma = \frac{7}{2} \in (3, 4], \quad \nu = \frac{9}{5} > 0,$$

and  $q = \{\frac{1}{8}, \frac{1}{2}, \frac{10}{13}\}$ , let us have the following boundary value problem for nonlinear fractional  $q$ -difference equations on an unbounded domain:

$$\begin{cases} \mathbb{D}_q^{\frac{7}{2}} \wp(t) = \frac{\ln(1+t)}{(1+t^2)^2} + \frac{e^{-3t}}{1+t^2} \sin(3t^2 + |\wp(t)|), & t \in J, \\ \wp(0) = \wp'(0) = \wp''(0) = 0, \\ \mathbb{D}_q^{\frac{5}{2}} \wp(\infty) = \sum_{i=1}^m {}_1a_i \wp({}_1\eta_i) + \sum_{i=1}^m {}_2a_i \mathbb{D}_q^2 \wp({}_2\eta_i), \end{cases} \quad (23)$$

where

$$\mathfrak{w}(t, \wp) = \frac{\ln(1+t)}{(1+t^2)^2} + \frac{e^{-3t}}{(1+t^2)} \sin(3t^2 + |\wp(t)|),$$

and  ${}_1a_i, {}_2a_i, {}_1\eta_i, {}_2\eta_i$  satisfy

$$\begin{aligned} 0 &\leq {}_1a_i \eta_i^{\frac{5}{2}} + {}_2a_i {}_2\eta_i^{\frac{1}{2}} \\ &= 1 \times \frac{1}{4} + 1 \times \frac{2}{3} = 0.9166 < \Gamma_q(\varsigma) = \begin{cases} 1.2076, & q = \frac{1}{8}, \\ 2.3814, & q = \frac{1}{2}, \\ 5.5035, & q = \frac{10}{13}, \end{cases} \end{aligned}$$

whenever, we take

$${}_1\eta_i = \frac{1}{4}, \quad {}_2\eta_i = \frac{2}{3}, \quad {}_1a_i = {}_2a_i = 1.$$

**Table 1** Numerical results of  $\Lambda$ ,  $\varrho^*$  and  $\theta$  for  $q = \frac{1}{8}, \frac{1}{2}$  in Example 4.1

$n$	$\Gamma_q(\zeta)$	$\Gamma_q(\nu)$	$\Gamma_q(\zeta - \nu)$	$\Lambda$	$\theta_0$	$\varrho^*$	$\theta$
$q = \frac{1}{8}$							
1	1.2102	1.2112	1.2705	0.4618	2.1656	0.2192	0.4747
2	1.2079	1.2124	1.2726	0.4621	2.1642	0.2192	0.4744
3	1.2077	1.2125	1.2729	0.4621	2.1641	0.2192	0.4744
4	1.2076	1.2125	1.2729	0.4621	2.1640	0.2192	0.4744
5	1.2076	1.2126	1.2729	0.4621	2.1640	0.2192	0.4744
$q = \frac{1}{2}$							
1	2.8267	2.1519	2.2893	1.8658	0.5360	0.2192	0.1175
2	2.5877	2.1986	2.3674	1.7335	0.5769	0.2192	0.1265
3	2.4808	2.2206	2.4045	1.6728	0.5978	0.2192	0.1311
4	2.4301	2.2313	2.4226	1.6436	0.6084	0.2192	0.1334
5	2.4054	2.2366	2.4315	1.6294	0.6137	0.2192	0.1345
6	2.3933	2.2392	2.4359	1.6223	0.6164	0.2192	0.1351
7	2.3872	2.2405	2.4381	1.6188	0.6177	0.2192	0.1354
8	2.3842	2.2412	2.4392	1.6170	0.6184	0.2192	0.1356
9	2.3827	2.2415	2.4398	1.6162	0.6188	0.2192	0.1356
10	2.3819	2.2417	2.4401	1.6157	0.6189	0.2192	0.1357
11	2.3816	2.2417	2.4402	1.6155	0.6190	0.2192	0.1357
12	2.3814	2.2418	2.4403	1.6154	0.6190	0.2192	0.1357

We can see the results of  $\Lambda$  and  $\theta$  in Tables 1 and 2. These results are plotted in Fig. 1. Then (A4) holds. Secondly, we obtain

$$|w(t, \varphi) - w(t, \varphi')| = \frac{e^{-3t} |\varphi - \varphi'|}{5(1 + t^{2.5})}.$$

By taking  $\varrho(t) = \frac{e^{-t}}{5(1+t^2)}$ , we directly obtain:

$$\varrho^* = \int_0^{+\infty} \varrho(\xi)(1 + \xi^{5-1}) d\xi = \frac{1}{5} < +\infty.$$

We shall check that condition (20) is satisfied. By using the MATLAB program Additional file 1, Algorithm 1) according to Eq. (14), we find

$$\theta = \theta_0 \varrho^* \simeq \begin{cases} 0.4744, & q = \frac{1}{8}, \\ 0.1357, & q = \frac{1}{2}, \\ 0.0465, & q = \frac{10}{13}, \end{cases} < 1,$$

where

$$\theta_0 = \frac{1}{\Gamma_q(\zeta)} + \frac{\sum_{i=1}^m a_{i1} \eta_i^{\zeta-1}}{\Lambda \Gamma_q(\zeta)} + \frac{\sum_{i=1}^m a_{i2} \eta_i^{\zeta-\nu-1}}{\Lambda \Gamma_q(\zeta - \nu)}.$$

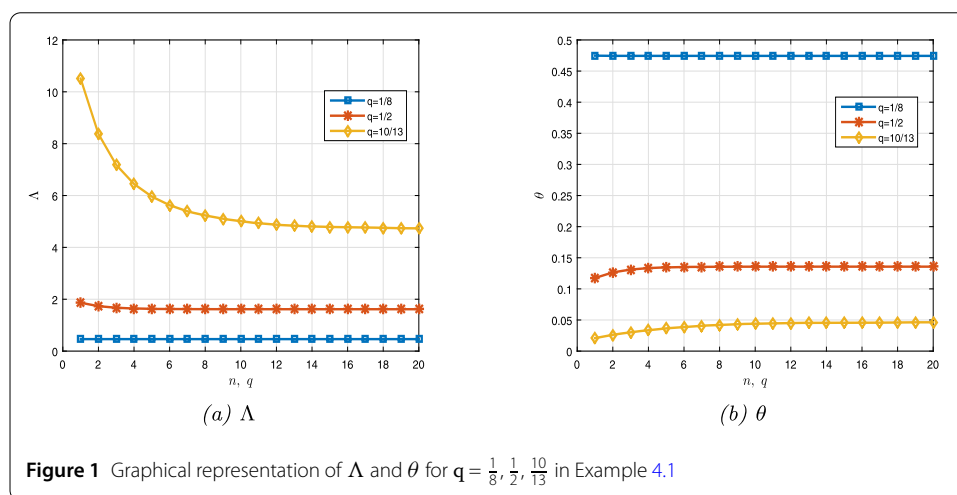
Tables 1 and 2 show the results. Also, We can see a graphical representation of  $\theta$  in Fig. 1. Thus, from Theorem 3.5, the boundary value problem (23) has a unique solution.

**Example 4.2** By taking

$$\zeta = \frac{17}{5} \in (3, 4], \quad \nu = \frac{25}{9} > 0,$$

**Table 2** Numerical results of  $\Lambda$ ,  $\varrho^*$  and  $\theta$  for  $q = \frac{10}{13}$  in Example 4.1

$n$	$\Gamma_q(\zeta)$	$\Gamma_q(\nu)$	$\Gamma_q(\zeta - \nu)$	$\Lambda$	$\theta_0$	$\varrho^*$	$\theta$
	$q = \frac{10}{13}$						
1	12.7438	4.2729	4.3560	10.5099	0.0951	0.2192	0.0209
2	10.0203	4.4743	4.6759	8.3756	0.1194	0.2192	0.0262
3	8.5259	4.6081	4.8917	7.1824	0.1392	0.2192	0.0305
4	7.6129	4.7014	5.0440	6.4453	0.1552	0.2192	0.0340
5	7.0164	4.7686	5.1543	5.9602	0.1678	0.2192	0.0368
6	6.6088	4.8178	5.2355	5.6272	0.1777	0.2192	0.0390
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
18	5.5407	4.9627	5.4771	4.7478	0.2106	0.2192	0.0462
19	5.5312	4.9641	5.4794	4.7399	0.2110	0.2192	0.0463
20	5.5239	4.9652	5.4813	4.7339	0.2112	0.2192	0.0463
21	5.5183	4.9660	5.4827	4.7293	0.2115	0.2192	0.0464
22	5.5140	4.9667	5.4838	4.7257	0.2116	0.2192	0.0464
23	5.5107	4.9672	5.4846	4.7229	0.2117	0.2192	0.0464
24	5.5081	4.9676	5.4852	4.7208	0.2118	0.2192	0.0464
25	5.5062	4.9678	5.4857	4.7192	0.2119	0.2192	0.0465
26	5.5047	4.9681	5.4861	4.7180	0.2120	0.2192	0.0465
27	5.5035	4.9682	5.4864	4.7170	0.2120	0.2192	0.0465

**Figure 1** Graphical representation of  $\Lambda$  and  $\theta$  for  $q = \frac{1}{8}, \frac{1}{2}, \frac{10}{13}$  in Example 4.1

and

$$q = \left\{ \frac{1}{7}, \frac{1}{2}, \frac{8}{9} \right\},$$

let us have the following boundary value problem for nonlinear fractional  $q$ -difference equations on an unbounded domain:

$$\begin{cases} \mathbb{D}_q^{\frac{17}{5}} \wp(t) = \frac{e^{-2t} |\wp(t)|}{7(1 + \sqrt[12]{t^5})}, & t \in J, \\ \wp(0) = \wp'(0) = \wp''(0) = 0, \\ \mathbb{D}_q^{\frac{12}{5}} \wp(\infty) = \sum_{i=1}^m {}_1 a_i \wp({}_1 \eta_i) + \sum_{i=1}^m {}_2 a_i \mathbb{D}_q^{\frac{25}{9}} \wp({}_2 \eta_i), \end{cases} \quad (24)$$

where

$$\mathfrak{w}(t, \wp) = \frac{e^{-2t} |\wp(t)|}{7(1 + \sqrt[12]{t^5})},$$

**Table 3** Numerical results of  $\Lambda$ ,  $\nabla$  and  $\mu$  for  $q = \frac{1}{7}$  in Example 4.2

$n$	$\Gamma_q(\zeta)$	$\Gamma_q(\nu)$	$\Gamma_q(\zeta - \nu)$	$\Lambda$	$\nabla$	$\mu > \nabla p(\mu) \int_0^\infty \psi(\xi) d_q \xi$
$q = \frac{1}{7}$ and calculated $\mu = 0.1$						
1	1.2287	1.1454	-1.0393	2.7995	0.3572	$0.1 > 0.0042$
2	1.2253	1.1428	-1.0823	2.6808	0.3730	$0.1 > 0.0043$
3	1.2249	1.1424	-1.0885	2.6645	0.3753	<u><math>0.1 &gt; 0.0044</math></u>
4	1.2248	1.1423	-1.0893	2.6622	0.3756	$0.1 > 0.0044$
5	1.2248	1.1423	-1.0895	2.6619	0.3757	$0.1 > 0.0044$

**Table 4** Numerical results of  $\Lambda$ ,  $\nabla$  and  $\mu$  for  $q = \frac{1}{2}$  in Example 4.2

$n$	$\Gamma_q(\zeta)$	$\Gamma_q(\nu)$	$\Gamma_q(\zeta - \nu)$	$\Lambda$	$\nabla$	$\mu > \nabla p(\mu) \int_0^\infty \psi(\xi) d_q \xi$
$q = \frac{1}{2}$ and calculated $\mu = 0.1$						
1	2.6975	2.1251	-2.7521	3.8027	0.2630	$0.1 > 0.0047$
2	2.4776	2.0057	-3.5667	2.9254	0.3418	$0.1 > 0.0061$
3	2.3791	1.9515	-3.9922	2.5995	0.3847	$0.1 > 0.0069$
4	2.3324	1.9256	-4.2092	2.4564	0.4071	$0.1 > 0.0073$
5	2.3096	1.9130	-4.3188	2.3891	0.4186	$0.1 > 0.0075$
6	2.2984	1.9067	-4.3739	2.3565	0.4244	$0.1 > 0.0076$
7	2.2928	1.9036	-4.4015	2.3404	0.4273	<u><math>0.1 &gt; 0.0077</math></u>
8	2.2900	1.9020	-4.4153	2.3324	0.4287	$0.1 > 0.0077$
9	2.2886	1.9013	-4.4222	2.3284	0.4295	$0.1 > 0.0077$
10	2.2879	1.9009	-4.4257	2.3264	0.4298	$0.1 > 0.0077$
11	2.2876	1.9007	-4.4274	2.3254	0.4300	$0.1 > 0.0077$
12	2.2874	1.9006	-4.4283	2.3249	0.4301	$0.1 > 0.0077$
13	2.2873	1.9005	-4.4287	2.3247	0.4302	$0.1 > 0.0077$
14	2.2873	1.9005	-4.4289	2.3246	0.4302	$0.1 > 0.0077$

$$\mathfrak{w}_*(t, \wp) = \mathfrak{w}(t, (1 + t^{\zeta-1})\wp) = \frac{e^{-2t} |(1 + t^{\zeta-1})\wp(t)|}{7(1 + \sqrt[12]{t^5})} = \frac{1}{7} |\wp(t)| e^{-2t}.$$

Choose  $p(\wp) = |\wp(t)|$ ,  $\psi(t) = e^{-2t}$ . For example, if we take

$$\begin{aligned} {}_1a_1 &= 1, & {}_1a_2 &= 1.5, & {}_2a_1 &= 1, & {}_2a_2 &= 0.75, \\ {}_1\eta_1 &= \frac{4}{5}, & {}_1\eta_2 &= \frac{2}{3}, & {}_2\eta_1 &= \frac{1}{3}, & {}_2\eta_2 &= \frac{7}{8}, \end{aligned}$$

we have

$$|\mathfrak{w}_*(t, \wp) - \mathfrak{w}_*(t, \wp')| = \frac{1}{7} e^{-2t} |\wp - \wp'| \leq \frac{e^{-2t}}{2} |\wp - \wp'|,$$

and

$$\begin{aligned} 0 &\leq \left\{ \sum_{i=1}^m a_i \xi_i^{\zeta-1} + \sum_{i=1}^m b_i \xi_i^{\zeta-\nu-1} \frac{\Gamma_q(\zeta)}{\Gamma_q(\zeta-\nu)} \right\} \leq \Gamma_q(\zeta) \\ &= \begin{cases} -1.4396, & q = \frac{1}{7}, \\ -0.0476, & q = \frac{1}{2}, \\ -2.4611, & q = \frac{8}{9}. \end{cases} \end{aligned}$$

Tables 3, 4 and 5 show the results. Also, We can see a graphical representation of

$$\nabla p(\mu) \int_0^\infty \psi(\xi) d_q \xi,$$

**Table 5** Numerical results of  $\Lambda$ ,  $\nabla$  and  $\mu$  for  $q = \frac{8}{9}$  in Example 4.2

$n$	$\Gamma_q(\zeta)$	$\Gamma_q(v)$	$\Gamma_q(\zeta - v)$	$\Lambda$	$\nabla$	$\mu > \nabla p(\mu) \int_0^\infty \psi(\xi) d_q \xi$
$q = \frac{8}{9}$ and calculated $\mu = 0.1$						
1	55.9818	21.9886	-3.1623	95.6034	0.0105	0.1 > 0.0000
2	41.2031	18.2245	-5.4122	57.5855	0.0174	0.1 > 0.0000
3	32.8940	15.9102	-7.6526	41.6420	0.0240	0.1 > 0.0000
4	27.6527	14.3446	-9.8069	32.9949	0.0303	0.1 > 0.0000
5	24.0881	13.2182	-11.8371	27.6229	0.0362	0.1 > 0.0000
6	21.5322	12.3725	-13.7253	23.9933	0.0417	0.1 > 0.0001
7	19.6267	11.7174	-15.4655	21.3974	0.0467	0.1 > 0.0001
8	18.1631	11.1978	-17.0587	19.4632	0.0514	0.1 > 0.0001
⋮	⋮	⋮	⋮	⋮	⋮	⋮
79	11.0024	8.3731	-31.1770	10.6630	0.0938	0.1 > 0.0001
80	11.0023	8.3731	-31.1774	10.6629	0.0938	0.1 > 0.0001
81	11.0022	8.3730	-31.1777	10.6627	0.0938	0.1 > 0.0001
82	11.0021	8.3730	-31.1780	10.6626	0.0938	0.1 > 0.0001
83	11.0020	8.3729	-31.1783	10.6625	0.0938	0.1 > 0.0001
84	11.0019	8.3729	-31.1786	10.6624	0.0938	0.1 > 0.0001
85	11.0019	8.3729	-31.1788	10.6624	0.0938	0.1 > 0.0001
86	11.0018	8.3729	-31.1790	10.6623	0.0938	0.1 > 0.0001
87	11.0017	8.3728	-31.1792	10.6622	0.0938	0.1 > 0.0001
88	11.0017	8.3728	-31.1793	10.6622	0.0938	0.1 > 0.0001
89	11.0016	8.3728	-31.1794	10.6621	0.0938	0.1 > 0.0001
90	11.0016	8.3728	-31.1796	10.6621	0.0938	0.1 > 0.0001

in Fig. 2 whenever  $q = \{\frac{1}{7}, \frac{1}{2}, \frac{8}{9}\}$ . Also,  ${}_1G_q(t, s)$  is a continuous function for  $(t, s) \in [1, \infty) \times [1, \infty)$  and

$$|w_*(t, s)| \leq \psi(t)p(|s|),$$

on  $J \times \mathbb{R}$  with  $p \in C(J, \mathbb{R})$  nondecreasing and

$$\int_0^\infty \psi(\xi) d_q \xi < +\infty.$$

Furthermore,

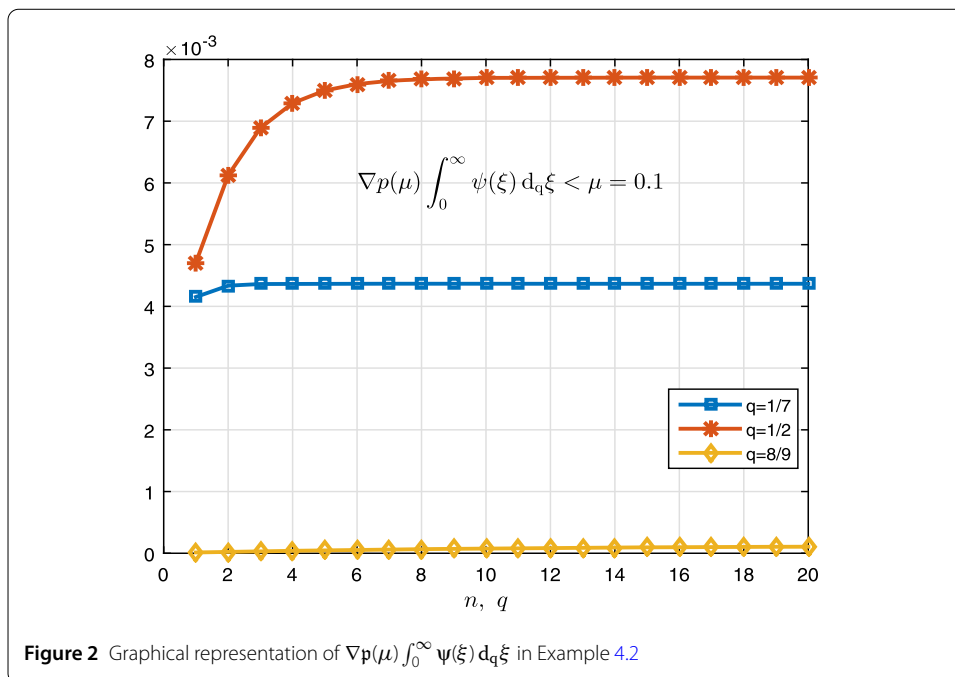
$$p(\mu) \int_0^\infty \psi(\xi) d_q \xi = \begin{cases} 0.0044, & q = \frac{1}{7}, \\ 0.0077, & q = \frac{1}{2}, \\ 0.0001, & q = \frac{8}{9}. \end{cases} < \mu = 0.1.$$

Thus, from Theorem 3.3 we conclude that problem (24) has at least one positive solution  $\varphi(t)$ , for  $t \in J$ ,

$$0 \leq \frac{\varphi(t)}{1 + t^{\frac{12}{5}}} \leq \mu.$$

## 5 Conclusion

A fractional boundary value problem system has been proposed and studied in terms of its unbounded solutions' existence for a class of nonlinear fractional  $q$ -difference equations via the Riemann–Liouville fractional  $q$ -derivative on an infinite interval. The system's solution has been formulated using Green's function. In addition, a compactness criterion



has been obtained in special space. Fixed-point theorems have been utilized to obtain all our results in this work. Illustrative examples have been provided at the end of our study.

### Supplementary information

Supplementary information accompanies this paper at <https://doi.org/10.1186/s13660-022-02764-6>.

**Additional file 1.** Supplementary information (PDF 88 kB)

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#### Ethics approval and consent to participate

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#### Consent for publication

Not applicable.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

AB: Actualization, methodology, formal analysis, validation, investigation, and initial draft. MB: Actualization, validation, methodology, formal analysis, investigation, and initial draft. MKAK: Actualization, methodology, formal analysis, validation, investigation, initial draft, and supervision of the original draft and editing. FM: Actualization, validation, methodology, formal analysis, investigation, and initial draft. MES: Actualization, methodology, formal analysis, validation, investigation, software, simulation, initial draft and was a major contributor in writing the manuscript. MK: Actualization, validation, methodology, formal analysis, investigation, and initial draft. All authors read and approved the final manuscript.

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