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A new iterative approximation scheme for Reich–Suzuki-type nonexpansive operators with an application

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Abstract

In this article, we propose a faster iterative scheme, called the AH iterative scheme, for approximating fixed points of contractive-like mappings and Reich–Suzuki-type nonexpansive mappings. We show that the AH iterative scheme converges faster than a number of existing iterative schemes for contractive-like mappings. The w^2-stability result of the new iterative scheme is established and a supportive example is provided to illustrate the notion of w^2-stability. Then, we prove weak and several strong convergence results of our new iterative scheme for fixed points of Reich–Suzuki-type nonexpansive mappings. Further, we carry out a numerical experiment to illustrate the efficiency of our new iterative scheme. As an application, we use our main result to prove the existence of a solution of a mixed-type nonlinear integral equation. An illustrative example to validate the result in our application is also given. Our results extend and generalize several related results in the existing literature.

MSC: 47H05; 47H09; 39B82

Keywords: Banach space; w^2-stability; Contractive-like mapping; Iterative scheme; Nonlinear integral equation

1 Introduction

Throughout the paper, let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{R}$ a set of all real numbers, $\mathcal{V}$ a nonempty subset of a Banach space $\mathcal{M}$. A mapping $\mathcal{U}:\mathcal{V}\to\mathcal{V}$ is called a contraction if there exists $\zeta\in[0,1)$ such that $\|\mathcal{U}p-\mathcal{U}q\|\leq\zeta\|p-q\|$, for all $p,q\in\mathcal{V}$. If $\zeta=1$, then $\mathcal{U}$ is called a nonexpansive mapping. A point $p^\star\in\mathcal{V}$ is said to be a fixed point of $\mathcal{U}$ if it satisfies $\mathcal{U}p^\star=p^\star$. We denote by $\Im(\mathcal{U})$ the set of all fixed points of $\mathcal{U}$.

The major ideas of fixed-point theory can be divided into two categories. One is to find the necessary and sufficient conditions under which an operator admits fixed points. The other is to determine such fixed points by using some schematic algorithms. Formally, the first category is usually referred to as the existence part and the second one is known as the computation or approximation part. Another important concept of fixed-point theory that is less well known is the study of the behaviors of fixed points such as stability, data dependency, etc.

For some decades now, the fixed-point theory has been revealed as a very powerful and useful tool in the study of nonlinear phenomenon. In particular, fixed-point techniques have been applied in diverse areas of biology, chemistry, economics, engineering, game theory, physics, etc., (see [4, 5, 13, 23–26, 30, 31] and the references therein).

In [6], Berinde provided the class of weak contractions that properly includes the class of Zamfirescu operators [47]. This class of mappings is also known by many authors as almost contraction mappings.

Definition 1.1 A mapping $\mathcal{U}:\mathcal{V}\to\mathcal{V}$ is called a weak contraction if there exist $\zeta\in(0,1)$ and $L\geq 0$ such that

$$\|\mathcal{U}p-\mathcal{U}q\|\leq\zeta\|p-q\|+L\|p-\mathcal{U}p\|,\quad\text{for all } p,q\in\mathcal{V}. \tag{1.1}$$

In [17], Imoru and Olantiwo gave a definition of a class of mapping considered to be a generalization of the classes of mappings studied by Berinde [6], Osilike and Udomene [28] and some other existing classes of contraction mappings as follows.

Definition 1.2 ([17]) A mapping $\mathcal{U}:\mathcal{V}\to\mathcal{V}$ is called contractive-like if there exists a constant $\zeta\in[0,1)$ and a strictly increasing continuous function $\psi:[0,\infty)\to[0,\infty)$ with $\psi(0)=0$ such that

$$\|\mathcal{U}p-\mathcal{U}q\|\leq\zeta\|p-q\|+\psi\big(\|p-\mathcal{U}p\|\big), \tag{1.2}$$

for all $p,q\in\mathcal{V}$.

Remark 1.3 If $\psi(p)=Lp$, then (1.2) reduces to (1.1).

In recent years, many extensions and generalizations of nonexpansive mappings have been studied by several authors due to their importance in terms of applications.

In 2008, Suzuki [35] introduced an interesting generalization of nonexpansive mappings and obtained some existence and convergence results. Such mappings are commonly known as mappings satisfying condition (C).

Definition 1.4 A mapping $\mathcal{U}:\mathcal{V}\to\mathcal{V}$ is said to satisfy condition (C) if

$$\frac{1}{2}\|p-\mathcal{U}p\|\leq\|p-q\|\quad\text{implies}\quad\|\mathcal{U}p-\mathcal{U}q\|\leq\|p-q\|,\quad\text{for all } p,q\in\mathcal{V}. \tag{1.3}$$

In 2019, Pant and Pandey [29] considered the class of Reich–Suzuki-type nonexpansive mappings as follows.

Definition 1.5 A mapping $\mathcal{U}:\mathcal{V}\to\mathcal{V}$ is said to be Reich–Suzuki-type nonexpansive if there exists a real number $\varpi\in[0,1)$ such that for each $p,q\in\mathcal{V}$,

$$\begin{aligned}&\frac{1}{2}\|p-\mathcal{U}p\|\leq\|p-q\|\quad\text{implies}\\&\|\mathcal{U}p-\mathcal{U}q\|\leq\varpi\|p-\mathcal{U}p\|+\varpi\|q-\mathcal{U}q\|+(1-2\varpi)\|p-q\|.\end{aligned} \tag{1.4}$$

Clearly, every mapping satisfying condition (C) is a Reich–Suzuki-type nonexpansive mapping with $\varpi = 0$. However, the converse is not true, as shown in [29].

On the other hand, many iterative schemes have been constructed to approximate the fixed points of different classes of mappings. Some of these iterative schemes are: Mann [21], Ishikawa [19], Noor [22], Agarwal et al. [2], Abbas and Nazir [1], CR [11], Normal-S [32], Picard-S [13], Thakur et al. [37], and M [41] iterative schemes.

Very recently, Ahmad et al. [3] introduced a novel iterative scheme known as the JK iterative scheme as follows:

$$\begin{cases} m_1 \in \mathcal{V}, \\ \vartheta_\nu = (1-r_\nu)m_\nu + r_\nu \mathcal{U} m_\nu, \\ \xi_\nu = \mathcal{U}\vartheta_\nu, \\ m_{\nu+1} = \mathcal{U}((1-k_\nu)\mathcal{U}\vartheta_\nu + k_\nu \mathcal{U}\xi_\nu), \end{cases} \quad \nu \in \mathbb{N}, \tag{1.5}$$

where $\{r_\nu\}$ and $\{k_\nu\}$ are sequences in $(0,1)$. The authors established some weak and strong convergence results for mappings satisfying condition (C). They further showed numerically that the JK iterative scheme converges faster than the S [2] and Thakur [37] iterative schemes.

To see recent results concerning iterative schemes and the existence theory of fixed points one can refer to, for example, [10, 36, 43–46].

Motivated and inspired by the research in this direction, we propose a new four-step iterative scheme called the AH iterative scheme, to approximate the fixed points of contractive-like mappings and Reich–Suzuki-type nonexpansive mappings as follows:

$$\begin{cases} p_1 \in \mathcal{V}, \\ z_\nu = (1-r_\nu)p_\nu + r_\nu \mathcal{U} p_\nu, \\ w_\nu = \mathcal{U}(\mathcal{U}z_\nu), \\ q_\nu = \mathcal{U}(\mathcal{U}w_\nu), \\ p_{\nu+1} = (1-k_\nu)q_\nu + k_\nu \mathcal{U} q_\nu, \end{cases} \quad \nu \in \mathbb{N}, \tag{1.6}$$

where $\{r_\nu\}$ and $\{k_\nu\}$ are sequences in $(0,1)$.

The purpose of this article is to prove that the AH iterative scheme (1.6) converges faster than the JK iterative scheme (1.5) for contractive-like mappings. Numerically, we further show that the AH iterative scheme (1.6) converges faster than a number of existing iterative schemes. Also, we prove that our proposed iterative scheme defined by (1.6) is w^2-stable and the stability result is supported with an example. Again, we establish weak and strong convergence results of the AH iterative scheme (1.6) for Reich–Suzuki-type nonexpansive mappings. Further, we use a new example of Reich–Suzuki-type nonexpansive mappings to show that the AH iterative scheme (1.6) outperforms some existing prominent iterative schemes. Finally, we use our main results to establish the existence of the solution of a nonlinear integral equation in Banach spaces.

2 Preliminaries

Let $\mathcal{M}^*$ be the dual of a Banach space $\mathcal{M}$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between $\mathcal{M}$ and $\mathcal{M}^*$. Then, the multivalued mapping $J : \mathcal{M} \to 2^{\mathcal{M}^*}$ is the normalized duality mapping defined for each $p \in \mathcal{M}$ by

$$J(p) = \{q \in \mathcal{M}^* : \langle p, q\rangle = \|p\|^2 = \|q\|^2\}. \tag{2.1}$$

Let $D = \{p \in \mathcal{M} : \|p\| = 1\}$. Then, a Banach space $\mathcal{M}$ is said to be smooth if the limit

$$\lim_{c\to 0} \frac{\|p + cq\| - \|p\|}{c} \tag{2.2}$$

exists for each $p, q \in D$. In this case, the norm of $\mathcal{M}$ is called Gâteaux differentiable. It is well known that J is single valued if $\mathcal{M}$ is smooth [12]. Suppose for each $p \in D$, the limit of (2.2) exists and is attained uniformly for $q \in D$, the norm of $\mathcal{M}$ is called Fréchet differentiable in this case. It is also well known that

$$\langle q, J(p)\rangle + \frac{1}{2}\|p\|^2 \leq \frac{1}{2}\|p + q\|^2 \leq \langle q, J(p)\rangle + \frac{1}{2}\|p\|^2 + b(q)$$

for all $p, q \in \mathcal{M}$, where J is the Fréchet derivative of the functional $\frac{1}{2}\|\cdot\|^2$ at $p \in \mathcal{M}$ and b is an increasing function defined on $[0,\infty)$ such that $\lim_{\nu\downarrow 0} \frac{b(\nu)}{\nu} = 0$.

A Banach space $\mathcal{M}$ is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for $p, q \in \mathcal{M}$ satisfying $\|p\| \leq 1$, $\|q\| \leq 1$ and $\|p - q\| > \epsilon$, we have $\|\frac{p+q}{2}\| < 1 - \delta$.

A Banach space $\mathcal{M}$ is said to satisfy Opial's condition if for any sequence $\{p_\nu\}$ in $\mathcal{M}$ that converges weakly to $p \in \mathcal{M}$ implies

$$\limsup_{\nu\to\infty} \|p_\nu - p\| < \limsup_{\nu\to\infty} \|p_\nu - q\|, \quad \forall q \in \mathcal{M} \text{ with } q \neq p.$$

Let $\mathcal{V}$ be a nonempty closed convex subset of a Banach space $\mathcal{M}$, and $\{p_\nu\}$ is a bounded sequence in $\mathcal{M}$. For $p \in \mathcal{M}$, we put

$$r(p, \{p_\nu\}) = \limsup_{\nu\to\infty} \|p_\nu - p\|.$$

The asymptotic radius of $\{p_\nu\}$ relative to $\mathcal{V}$ is defined by

$$r(\mathcal{V}, \{p_\nu\}) = \inf\{r(p, \{p_\nu\}) : p \in \mathcal{V}\}.$$

The asymptotic center of $\{p_\nu\}$ relative to $\mathcal{V}$ is given as:

$$A(\mathcal{V}, \{p_\nu\}) = \{p \in \mathcal{V} : r(p, \{p_\nu\}) = r(\mathcal{V}, \{p_\nu\})\}.$$

In a uniformly convex Banach space, it is well known that $A(\mathcal{V}, \{p_\nu\})$ consists of exactly one point.

Let $\mathcal{V}$ be a nonempty closed convex subset of a Banach space $\mathcal{M}$. A mapping $\mathcal{U} : \mathcal{V} \to \mathcal{V}$ is said to be demiclosed with respect to $p \in \mathcal{M}$, if for each sequence $\{p_\nu\}$ that is weakly convergent to $p \in \mathcal{V}$ and $\{\mathcal{U}p_\nu\}$ converges strongly to q implies that $\mathcal{U}p = q$.

Definition 2.1 ([7]) Let $\{\delta_\nu\}$ and $\{\gamma_\nu\}$ be two sequences of real numbers that converge to δ and γ, respectively, and assume that there exists

$$\ell = \lim_{\nu\to\infty} \frac{\|\delta_\nu - \delta\|}{\|\gamma_\nu - \gamma\|}.$$

Then,

(Θ_1) if $\ell = 0$, we say that $\{\delta_\nu\}$ converges to δ faster than $\{\gamma_\nu\}$ does to γ.

(Θ_2) If $0 < \ell < \infty$, we say that $\{\delta_\nu\}$ and $\{\gamma_\nu\}$ have the same rate of convergence.

Definition 2.2 ([7]) Let $\{\eta_\nu\}$ and $\{\phi_\nu\}$ be two fixed-point iteration processes that converge to the same point $p^\star$, the error estimates

$$\begin{aligned} &\|\eta_\nu - p^\star\| \le \delta_\nu, \quad \nu \in \mathbb{N}, \\ &\|\phi_\nu - p^\star\| \le \gamma_\nu, \quad \nu \in \mathbb{N} \end{aligned}$$

are available, where $\{\delta_\nu\}$ and $\{\gamma_\nu\}$ are two sequences of positive numbers converging to zero. Then, we say that $\{\eta_\nu\}$ converges faster to $p^\star$ than $\{\phi_\nu\}$ does if $\{\delta_\nu\}$ converges faster than $\{\gamma_\nu\}$.

Definition 2.3 A sequence $\{p_\nu\}$ in $\mathcal{V}$ is said to be an approximate fixed-point sequence (a.f.p.s. for short) for a mapping $\mathcal{U}: \mathcal{V} \to \mathcal{V}$ if

$$\lim_{\nu\to\infty} \|\mathcal{U}p_\nu - p_\nu\| = 0. \tag{2.3}$$

Definition 2.4 ([34]) A mapping $\mathcal{U}: \mathcal{V} \to \mathcal{V}$ is said to be a satisfied condition (I) if a nondecreasing function $f : [0,\infty) \to [0,\infty)$ exists with $f(0) = 0$ and for all $s > 0$, then $f(s) > 0$ such that $\|p - \mathcal{U}p\| \ge f(d(p, \Im(\mathcal{U})))$ for all $p \in \mathcal{V}$, where $d(p, \Im(\mathcal{U})) = \inf_{p^\star\in\Im(\mathcal{U})} \|p - p^\star\|$.

Lemma 2.5 ([42]) *Let $\{\theta_\nu\}$ and $\{\lambda_\nu\}$ be nonnegative real sequences satisfying the following inequalities*:

$$\theta_{\nu+1} \le (1 - \sigma_\nu)\theta_\nu + \lambda_\nu,$$

where $\sigma_\nu \in (0,1)$ for all $\nu \in \mathbb{N}$, $\sum_{\nu=0}^{\infty} \sigma_\nu = \infty$ and $\lim_{\nu\to\infty} \frac{\lambda_\nu}{\sigma_\nu} = 0$, then $\lim_{\nu\to\infty} \theta_\nu = 0$.

Lemma 2.6 ([33]) *Suppose $\mathcal{M}$ is a uniformly convex Banach space and $\{\iota_\nu\}$ is any sequence satisfying $0 < p \le \iota_\nu \le q < 1$ for all $\nu \ge 1$. Suppose $\{p_\nu\}$ and $\{q_\nu\}$ are any sequences of $\mathcal{M}$ such that*

$$\begin{aligned} &\limsup_{\nu\to\infty} \|p_\nu\| \le x, \\ &\limsup_{\nu\to\infty} \|q_\nu\| \le x \quad \text{and} \\ &\limsup_{\nu\to\infty} \|\iota_\nu p_\nu + (1 - \iota_\nu)q_\nu\| = x \end{aligned}$$

hold for some $x \ge 0$. Then, $\lim_{\nu\to\infty} \|p_\nu - q_\nu\| = 0$.

Lemma 2.7 ([39]) *Let $\mathcal{U}: \mathcal{V} \to \mathcal{V}$ be a mapping. If $\mathcal{U}$ is a Reich–Suzuki-type nonexpansive mapping with $\Im(\mathcal{U}) \ne \emptyset$, then the following hold*:

(i) *If $\mathcal{U}$ is a Reich–Suzuki-type nonexpansive mapping, then for every choice of $p \in \mathcal{V}$ and $p^\star \in \Im(\mathcal{U})$, it follows that $\|\mathcal{U}p - \mathcal{U}p^\star\| \le \|p - p^\star\|$.*

(ii) *If $\mathcal{U}$ satisfies condition (C), then $\mathcal{U}$ is a Reich–Suzuki-type nonexpansive mapping.*

Lemma 2.8 ([40]) *Let $\mathcal{U}: \mathcal{V} \to \mathcal{V}$ be a mapping. If $\mathcal{U}$ is a Reich–Suzuki-type nonexpansive mapping, then for all $p, q \in \mathcal{V}$, the following inequality holds*:

$$\|p - \mathcal{U}q\| \leq \left(\frac{3+\varpi}{1-\varpi}\right)\|p - \mathcal{U}p\| + \|p - q\|. \tag{2.4}$$

We now offer a numerical example that satisfies the inequality of the above lemma but does not satisfy the condition (C).

Example 2.9 Let $(\mathbb{R}, \|\cdot\|)$ be a Banach space with the usual norm and $\mathcal{V} = [-1,1]$. Define $\mathcal{U}: \mathcal{V} \to \mathcal{V}$ by

$$\mathcal{U}p = \begin{cases} \frac{-p}{3}, & \text{if } p \in [-1,0), \\ -p, & \text{if } p \in [0,1]\backslash\{\frac{1}{3}\}, \\ 0, & \text{if } p \in \{\frac{1}{3}\}. \end{cases}$$

(1) The mapping $\mathcal{U}$ does not satisfy the condition (C) and hence is not a nonexpansive mapping. If we take $p = \frac{1}{3}$ and $q = 1$, then

$$\frac{1}{2}|p - \mathcal{U}p| = \frac{1}{2}\left|\frac{1}{3} - \mathcal{U}\left(\frac{1}{3}\right)\right| = \frac{1}{6} \leq \frac{2}{3} = |p - q|.$$

On the other hand,

$$|\mathcal{U}p - \mathcal{U}q| = 1 > \frac{2}{3} = |p - q|.$$

(2) Now, we show that $\mathcal{U}$ satisfies condition (2.4). For this, the following conditions are considered:

Case I: If $p, q \in [-1,0)$, we have

$$\begin{aligned} |p - \mathcal{U}q| &\leq |p - \mathcal{U}p| + |\mathcal{U}p - \mathcal{U}q| \\ &= |p - \mathcal{U}p| + \frac{1}{3}|p - q|. \end{aligned}$$

Case II: If $p, q \in [0,1]\backslash\{\frac{1}{3}\}$, then we obtain

$$\begin{aligned} |p - \mathcal{U}q| &\leq |p - \mathcal{U}p| + |\mathcal{U}p - \mathcal{U}q| \\ &\leq |p - \mathcal{U}p| + |p - q|. \end{aligned}$$

Case III: If $p \in [-1,0)$ and $q \in [0,1]\backslash\{\frac{1}{3}\}$, we obtain

$$\begin{aligned} |p - \mathcal{U}q| &= |p + q| \leq |p| + |q| \\ &\leq \frac{4}{3}|p| + |p - q| \quad (\text{as } p < 0 \text{ and } q \geq 0) \\ &= \left|p - \left(-\frac{p}{3}\right)\right| + |p - q| \\ &= |p - \mathcal{U}q| + |p - q|. \end{aligned}$$

Case IV: If $p \in [-1,0)$ and $q = \frac{1}{3}$, we have

$$\begin{aligned}|p - \mathcal{U}q| = |p| &\leq \frac{4}{3}|p| + \left|p - \frac{1}{3}\right| \\ &= |p - \mathcal{U}p| + |p - q|.\end{aligned}$$

Case V: If $p \in [0,1]\backslash\{\frac{1}{3}\}$ and $q = \frac{1}{3}$, we obtain

$$\begin{aligned}|p - \mathcal{U}q| = |p| &\leq 2|p| + \left|p - \frac{1}{3}\right| \\ &= |p - \mathcal{U}q| + \left|p - \frac{1}{3}\right|.\end{aligned}$$

Hence, $\mathcal{U}$ satisfies the condition (2.4) with $\frac{3+\varpi}{1-\varpi} \geq 1$.

3 Rate of convergence

In this section, we show both analytically and numerically that the AH iterative scheme (1.6) converges faster than the JK iterative scheme (1.5) for contractive-like mappings.

Theorem 3.1 *Let $\mathcal{U}$ be a mapping satisfying (1.2) defined on a nonempty closed convex subset $\mathcal{V}$ of a Banach space $\mathcal{M}$. Then, the sequence $\{p_v\}$ generated by the AH iterative scheme (1.6) converges strongly to a unique fixed point of $\mathcal{U}$.*

Proof Using (1.2) and (1.6), we have

$$\begin{aligned}\|z_v - p^\star\| &= \|(1-r_v)p_v + r_v\mathcal{U}p_v - \mathcal{U}p^\star\| \\ &\leq (1-r_v)\|p_v - p^\star\| + r_v\|\mathcal{U}p_v - \mathcal{U}p^\star\| \\ &\leq (1-r_v)\|p_v - p^\star\| + r_v\big[\zeta\|p_v - p^\star\| + L\|p^\star - \mathcal{U}p^\star\|\big] \\ &= \big(1 - r_v(1-\zeta)\big)\|p_v - p^\star\|. \end{aligned} \tag{3.1}$$

Using (1.6) and (3.1), we have

$$\begin{aligned}\|w_v - p^\star\| &= \|\mathcal{U}(\mathcal{U}z_v) - p^\star\| \\ &\leq \zeta\|\mathcal{U}z_v - p^\star\| \\ &\leq \zeta^2\|z_v - p^\star\| \\ &\leq \zeta^2\big(1 - r_v(1-\zeta)\big)\|p_v - p^\star\|. \end{aligned} \tag{3.2}$$

Now, from (1.6) and (3.2), we obtain

$$\begin{aligned}\|q_v - p^\star\| &= \|\mathcal{U}(\mathcal{U}w_v) - p^\star\| \\ &\leq \zeta\|\mathcal{U}w_v - p^\star\| \\ &\leq \zeta^2\|w_v - p^\star\| \\ &\leq \zeta^4\big(1 - r_v(1-\zeta)\big)\|p_v - p^\star\|. \end{aligned} \tag{3.3}$$

Finally, using (1.6) and (3.3), we obtain

$$\begin{aligned}\|p_{\nu+1}-p^\star\| &= \|(1-k_\nu)q_\nu + k_\nu\mathcal{U}q_\nu - \mathcal{U}p^\star\| \\ &\leq (1-k_\nu)\|q_\nu - p^\star\| + k_\nu\|\mathcal{U}q_\nu - \mathcal{U}p^\star\| \\ &\leq \bigl(1-k_\nu(1-\zeta)\bigr)\|q_\nu - p^\star\| \\ &\leq \zeta^4\bigl(1-r_\nu(1-\zeta)\bigr)\bigl(1-k_\nu(1-\zeta)\bigr)\|p_\nu - p^\star\|. \end{aligned} \tag{3.4}$$

Since $0<\zeta<1$ and $r_\nu, k_\nu \in (0,1)$, it implies that $(1-r_\nu(1-\zeta))<1$ and $(1-k_\nu(1-\zeta))<1$. It follows that $(1-k_\nu(1-\zeta))(1-k_\nu(1-\zeta))<1$. □

Thus, from (3.4), we have

$$\|p_{\nu+1}-p^\star\| \leq \zeta^4\|p_\nu - p^\star\|. \tag{3.5}$$

Inductively, we obtain:

$$\|p_{\nu+1}-p^\star\| \leq \zeta^{4(\nu+1)}\|p_0 - p^\star\|. \tag{3.6}$$

Since $0<\zeta<1$, $p_\nu \to p^\star$ as $\nu\to\infty$. This completes the proof.

Theorem 3.2 *Let $\mathcal{U}$ be a mapping satisfying (1.2) defined on a nonempty closed convex subset $\mathcal{V}$ of a Banach space $\mathcal{M}$. If $\{p_\nu\}$ is a sequence generated by the AH iterative scheme (1.6), then $\{p_\nu\}$ converges faster than $\{m_\nu\}$ generated by the JK iterative scheme (1.5).*

Proof Recalling (3.6) in Theorem 3.1, we have

$$\|p_{\nu+1}-p^\star\| \leq \zeta^{4(\nu+1)}\|p_0 - p^\star\|, \quad \nu\in\mathbb{N}.$$

Also, from (1.5), we obtain

$$\begin{aligned}\|\vartheta_\nu-p^\star\| &= \|(1-r_\nu)m_\nu + r_\nu\mathcal{U}m_\nu - \mathcal{U}p^\star\| \\ &\leq (1-r_\nu)\|m_\nu - p^\star\| + r_\nu\|\mathcal{U}m_\nu - \mathcal{U}p^\star\| \\ &= \bigl(1-r_\nu(1-\zeta)\bigr)\|m_\nu - p^\star\|. \end{aligned} \tag{3.7}$$

Using (1.5) and (3.7), we have

$$\begin{aligned}\|\xi_\nu-p^\star\| &= \|\mathcal{U}\vartheta_\nu - p^\star\| \\ &\leq \zeta\|\vartheta_\nu - p^\star\| \\ &\leq \zeta\bigl(1-r_\nu(1-\zeta)\bigr)\|m_\nu - p^\star\|. \end{aligned} \tag{3.8}$$

Again, from (1.5) and (3.8), we obtain

$$\|m_{\nu+1}-p^\star\| = \bigl\|\mathcal{U}\bigl((1-k_\nu)\mathcal{U}\vartheta_\nu + k_\nu\mathcal{U}\xi_\nu\bigr) - \mathcal{U}p^\star\bigr\|$$

$$\begin{aligned}
&\leq \zeta\big((1-k_\nu)\big\|\mathcal{U}\vartheta_\nu - \mathcal{U}p^\star\big\| + k_\nu\big\|\mathcal{U}\xi_\nu - \mathcal{U}p^\star\big\|\big)\\
&\leq \zeta^2\big((1-k_\nu)\big\|\vartheta_\nu - p^\star\big\| + k_\nu\big\|\xi_\nu - p^\star\big\|\big)\\
&\leq \zeta^2\big((1-k_\nu)\big(1-r_\nu(1-\zeta)\big)\big\|m_\nu - p^\star\big\|\\
&\quad + k_\nu\zeta\big(1-r_\nu(1-\zeta)\big)\big\|m_\nu - p^\star\big\|\big)\\
&= \zeta^2\big(\big(1-r_\nu(1-\zeta)\big)\big(1-k_\nu(1-\zeta)\big)\big)\big\|m_\nu - p^\star\big\|\\
&\leq \zeta^2\big\|m_\nu - p^\star\big\|.
\end{aligned} \tag{3.9}$$

Inductively, we obtain:

$$\big\|m_{\nu+1} - p^\star\big\| \leq \zeta^{2(\nu+1)}\big\|m_0 - p^\star\big\|. \tag{3.10}$$

Let $\Lambda_\nu = \zeta^{4(\nu+1)}\|p_0 - p^\star\|$ and $\beta_\nu = \zeta^{2(\nu+1)}\|m_0 - p^\star\|$, then we have that

$$\frac{\Lambda_\nu}{\beta_\nu} = \frac{\zeta^{4(\nu+1)}\|p_0 - p^\star\|}{\zeta^{2(\nu+1)}\|m_0 - p^\star\|} = \zeta^{2(\nu+1)}\frac{\|p_0 - p^\star\|}{\|m_0 - p^\star\|} \to 0 \quad \text{as } \nu \to \infty. \tag{3.11}$$

Hence, the sequence $\{p_\nu\}$ converges faster to $p^\star$ than $\{m_\nu\}$. □

Now, we give a nontrivial example to compare the rate of convergence of the AH iterative scheme (1.6) with some leading iterative schemes in the literature.

Example 3.3 Let $\mathcal{M} = \mathbb{R}^3$ and $\mathcal{V} = \{p = (p_1,p_2,p_3) : (p_1,p_2,p_3) \in [0,8] \times [0,8] \times [0,8]\}$ be a subset of $\mathcal{M}$ with norm $\|p\| = \|(p_1,p_2,p_3)\| = |p_1| + |p_2| + |p_3|$. Let $\mathcal{U} : \mathcal{V} \to \mathcal{V}$ be defined by

$$\mathcal{U}(p_1,p_2,p_3) = \begin{cases} (\frac{p_1}{4}, \frac{p_2}{4}, \frac{p_3}{4}), & \text{if } (p_1,p_2,p_3) \in [0,4) \times [0,4) \times [0,4),\\ (\frac{p_1}{8}, \frac{p_2}{8}, \frac{p_3}{8}), & \text{if } (p_1,p_2,p_3) \in [4,8] \times [4,8] \times [4,8].\end{cases}$$

Clearly, the only fixed point of $\mathcal{U}$ is $(0,0,0)$. We will now show that $\mathcal{U}$ is a contractive-like mapping. To see this, we define a function $\psi : [0,\infty) \to [0,\infty)$ by $\psi(p) = \frac{p}{6}$. Clearly, ψ is a strictly increasing continuous function with $\psi(0) = 0$. We show that

$$\|\mathcal{U}p - \mathcal{U}p\| = \zeta\|p - q\| + \psi\big(\|p - \mathcal{U}p\|\big), \tag{3.12}$$

for all $p,q \in \mathcal{V}$ and $\zeta \in (0,1]$. It will be useful to note the following. If $p = (p_1,p_2,p_3) \in [0,4) \times [0,4) \times [0,4)$, then

$$\|p - \mathcal{U}p\| = \left\|(p_1,p_2,p_3) - \left(\frac{p_1}{4}, \frac{p_2}{4}, \frac{p_3}{4}\right)\right\| = \left\|\left(\frac{3p_1}{4}, \frac{3p_2}{4}, \frac{3p_3}{4}\right)\right\|$$

and

$$\begin{aligned}
\psi\big(\|p - \mathcal{U}p\|\big) &= \psi\left(\left\|\left(\frac{3p_1}{4}, \frac{3p_2}{4}, \frac{3p_3}{4}\right)\right\|\right)\\
&= \left\|\left(\frac{p_1}{8}, \frac{p_2}{8}, \frac{p_3}{8}\right)\right\| = \left|\frac{p_1}{8}\right| + \left|\frac{p_2}{8}\right| + \left|\frac{p_3}{8}\right|.
\end{aligned} \tag{3.13}$$

Similarly, if $p = (p_1, p_2, p_3) \in [4,8] \times [4,8] \times [4,8]$, we have

$$\|p - \mathcal{U}p\| = \left\|(p_1, p_2, p_3) - \left(\frac{p_1}{8}, \frac{p_2}{8}, \frac{p_3}{8}\right)\right\| = \left\|\left(\frac{7p_1}{8}, \frac{7p_2}{8}, \frac{7p_3}{8}\right)\right\|$$

and

$$\begin{aligned} \psi\left(\|p - \mathcal{U}p\|\right) &= \psi\left(\left\|\left(\frac{7p_1}{8}, \frac{7p_2}{8}, \frac{7p_3}{8}\right)\right\|\right) \\ &= \left\|\left(\frac{7p_1}{48}, \frac{7p_2}{48}, \frac{7p_3}{48}\right)\right\| = \left|\frac{7p_1}{48}\right| + \left|\frac{7p_2}{48}\right| + \left|\frac{7p_3}{48}\right|. \end{aligned} \tag{3.14}$$

Next, we consider the following cases:

Case 1: If $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in [0,4) \times [0,4) \times [0,4)$, then using (3.13), we have

$$\begin{aligned} \|\mathcal{U}p - \mathcal{U}q\| &= \left\|\left(\frac{p_1}{4}, \frac{p_2}{4}, \frac{p_3}{4}\right) - \left(\frac{q_1}{4}, \frac{q_2}{4}, \frac{q_3}{4}\right)\right\| \\ &= \left|\frac{p_1}{4} - \frac{q_1}{4}\right| + \left|\frac{p_2}{4} - \frac{q_2}{4}\right| + \left|\frac{p_3}{4} - \frac{q_3}{4}\right| \\ &= \frac{1}{4}|p_1 - q_1| + \frac{1}{4}|p_2 - q_2| + \frac{1}{4}|p_3 - q_3| \\ &= \frac{1}{4}\left\|(p_1, p_2, p_3) - (q_1, q_2, q_3)\right\| \\ &\leq \frac{1}{4}\|p - q\| + \left|\frac{p_1}{8}\right| + \left|\frac{p_2}{8}\right| + \left|\frac{p_3}{8}\right| \\ &= \frac{1}{4}\|p - q\| + \psi\left(\|p - \mathcal{U}p\|\right). \end{aligned}$$

Case 2: If $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in [4,8] \times [4,8] \times [4,8]$, then using (3.14), we obtain

$$\begin{aligned} \|\mathcal{U}p - \mathcal{U}q\| &= \left\|\left(\frac{p_1}{8}, \frac{p_2}{8}, \frac{p_3}{8}\right) - \left(\frac{q_1}{8}, \frac{q_2}{8}, \frac{q_3}{8}\right)\right\| \\ &= \left|\frac{p_1}{8} - \frac{q_1}{8}\right| + \left|\frac{p_2}{8} - \frac{q_2}{8}\right| + \left|\frac{p_3}{8} - \frac{q_3}{8}\right| \\ &= \frac{1}{8}|p_1 - q_1| + \frac{1}{8}|p_2 - q_2| + \frac{1}{8}|p_3 - q_3| \\ &= \frac{1}{8}\left\|(p_1, p_2, p_3) - (q_1, q_2, q_3)\right\| \\ &\leq \frac{1}{4}\|p - q\| + \left|\frac{7p_1}{48}\right| + \left|\frac{7p_2}{48}\right| + \left|\frac{7p_3}{48}\right| \\ &= \frac{1}{4}\|p - q\| + \psi\left(\|p - \mathcal{U}p\|\right). \end{aligned}$$

Case 3: If $p = (p_1, p_2, p_3) \in [0,4) \times [0,4) \times [0,4)$ and $q = (q_1, q_2, q_3) \in [4,8] \times [4,8] \times [4,8]$, then using (3.13), we have

$$\|\mathcal{U}p - \mathcal{U}q\| = \left\|\left(\frac{p_1}{4}, \frac{p_2}{4}, \frac{p_3}{4}\right) - \left(\frac{p_1}{8}, \frac{p_2}{8}, \frac{p_3}{8}\right)\right\|$$

$$
\begin{aligned}
&= \left\| \left(\frac{p_1}{4} - \frac{q_1}{8}\right), \left(\frac{p_2}{4} - \frac{q_2}{8}\right), \left(\frac{p_3}{4} - \frac{q_3}{8}\right) \right\| \\
&= \left\| \left(\frac{p_1}{8} + \frac{p_1}{8} - \frac{q_1}{8}\right), \left(\frac{p_2}{8} + \frac{p_2}{8} - \frac{q_2}{8}\right), \left(\frac{p_3}{8} + \frac{p_3}{8} - \frac{q_3}{8}\right) \right\| \\
&= \left|\frac{p_1}{8} + \frac{p_1}{8} - \frac{q_1}{8}\right| + \left|\frac{p_2}{8} + \frac{p_2}{8} - \frac{q_2}{8}\right| + \left|\frac{p_3}{8} + \frac{p_3}{8} - \frac{q_3}{8}\right| \\
&\leq \left|\frac{p_1}{8}\right| + \left|\frac{p_2}{8}\right| + \left|\frac{p_3}{8}\right| + \left|\frac{p_1}{8} - \frac{q_1}{8}\right| + \left|\frac{p_2}{8} - \frac{q_2}{8}\right| + \left|\frac{p_3}{8} - \frac{q_3}{8}\right| \\
&= \frac{1}{8}\big(|p_1 - q_1| + |p_2 - q_2| + |p_3 - q_3|\big) + \psi\big(\|p - \mathcal{U}p\|\big) \\
&\leq \frac{1}{4}\big\|(p_1, p_2, p_3) - (q_1, q_2, q_3)\big\| + \psi\big(\|p - \mathcal{U}p\|\big) \\
&= \frac{1}{4}\|p - q\| + \psi\big(\|p - \mathcal{U}p\|\big).
\end{aligned}
$$

Case 4: If $p = (p_1, p_2, p_3) \in [4,8] \times [4,8] \times [4,8]$ and $q = (q_1, q_2, q_3) \in [0,4) \times [0,4) \times [0,4)$, then using (3.13), we obtain

$$
\begin{aligned}
\|\mathcal{U}p - \mathcal{U}q\| &= \left\| \left(\frac{p_1}{8}, \frac{p_2}{8}, \frac{p_3}{8}\right) - \left(\frac{p_1}{4}, \frac{p_2}{4}, \frac{p_3}{4}\right) \right\| \\
&= \left\| \left(\frac{p_1}{8} - \frac{q_1}{4}\right), \left(\frac{p_2}{8} - \frac{q_2}{4}\right), \left(\frac{p_3}{8} - \frac{q_3}{4}\right) \right\| \\
&= \left\| \left(\frac{p_1}{4} - \frac{p_1}{8} - \frac{q_1}{4}\right), \left(\frac{p_2}{4} - \frac{p_2}{8} - \frac{q_2}{4}\right), \left(\frac{p_3}{4} - \frac{p_3}{8} - \frac{q_3}{4}\right) \right\| \\
&= \left|\frac{p_1}{4} - \frac{p_1}{8} - \frac{q_1}{4}\right| + \left|\frac{p_2}{4} - \frac{p_2}{8} - \frac{q_2}{4}\right| + \left|\frac{p_3}{4} - \frac{p_3}{8} - \frac{q_3}{4}\right| \\
&\leq \left|\frac{p_1}{8}\right| + \left|\frac{p_2}{8}\right| + \left|\frac{p_3}{8}\right| + \left|\frac{p_1}{4} - \frac{q_1}{4}\right| + \left|\frac{p_2}{4} - \frac{q_2}{4}\right| + \left|\frac{p_3}{4} - \frac{q_3}{4}\right| \\
&= \frac{1}{4}\big(|p_1 - q_1| + |p_2 - q_2| + |p_3 - q_3|\big) + \psi\big(\|p - \mathcal{U}p\|\big) \\
&= \frac{1}{4}\big\|(p_1, p_2, p_3) - (q_1, q_2, q_3)\big\| + \psi\big(\|p - \mathcal{U}p\|\big) \\
&= \frac{1}{4}\|p - q\| + \psi\big(\|p - \mathcal{U}p\|\big).
\end{aligned}
$$

Hence, (5.1) is fulfilled with $\zeta = \frac{1}{4}$. Thus, $\mathcal{U}$ is a contractive-like mapping.

Using MATLAB R2015a, we obtain Tables 1–3 and Fig. 1. From Tables 1–3, we can easily see that all the iterative schemes with control parameters $r_\nu = 0.8$, $k_\nu = 0.6$, $l_\nu = 0.5$, $\nu \in \mathbb{N}$ and starting point $(2, 2.5, 3)$ converge to $p^\star = (0,0,0)$. Obviously, our iterative scheme (1.6) requires the least number of iterations as compared to other iterative schemes. Also, from the graphical point of view in Fig. 1, it is evident that the AH iterative scheme (1.6) converges faster than other iterative schemes.

4 Stability results

The concept of stability of a fixed-point iteration process was rigorously studied by Harder in her Ph.D thesis that was published in 1987.

Table 1 Convergence behavior of various iterative schemes

Step	Noor	S
1	(2.000000, 2.500000, 3.000000)	(2.000000, 2.500000, 3.000000)
2	(0.597500, 0.746875, 0.896250)	(0.320000, 0.400000, 0.480000)
3	(0.178503, 0.223129, 0.267755)	(0.051200, 0.064000, 0.076800)
4	(0.053328, 0.066660, 0.079992)	(0.008192, 0.010240, 0.012288)
5	(0.015932, 0.019915, 0.023898)	(0.001311, 0.001638, 0.001966)
6	(0.004760, 0.005949, 0.007139)	(0.000210, 0.000262, 0.000315)
7	(0.001422, 0.001777, 0.002133)	(0.000034, 0.000042, 0.000050)
8	(0.000425, 0.000531, 0.000637)	(0.000005, 0.000007, 0.000008)
9	(0.000127, 0.000159, 0.000190)	(0.000001, 0.000001, 0.000001)
10	(0.000038, 0.000047, 0.000057)	(0.000000, 0.000000, 0.000000)
11	(0.000011, 0.000014, 0.000017)	(0.000000, 0.000000, 0.000000)
12	(0.000003, 0.000004, 0.000005)	(0.000000, 0.000000, 0.000000)
13	(0.000001, 0.000001, 0.000002)	(0.000000, 0.000000, 0.000000)
14	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.000000)

Table 2 Convergence behavior of various iterative schemes

Step	Abbas	Thakur
1	(2.000000, 2.500000, 3.000000)	(2.000000, 2.500000, 3.000000)
2	(0.269375, 0.336719, 0.404062)	(0.080000, 0.100000, 0.120000)
3	(0.036281, 0.045352, 0.054422)	(0.003200, 0.004000, 0.004800)
4	(0.004887, 0.006108, 0.007330)	(0.000128, 0.000160, 0.000192)
5	(0.000658, 0.000823, 0.000987)	(0.000005, 0.000006, 0.000008)
6	(0.000089, 0.000111, 0.000133)	(0.000000, 0.000000, 0.000000)
7	(0.000012, 0.000015, 0.000018)	(0.000000, 0.000000, 0.000000)
8	(0.000002, 0.000002, 0.000002)	(0.000000, 0.000000, 0.000000)
9	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.000000)
10	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.000000)
11	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.000000)
12	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.000000)
13	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.000000)
14	(0.000000, 0.000000, 0.000000)	(0.000000, 0.000000, 0.000000)

Table 3 Convergence behavior of various iterative schemes

Step	JK	AH
1	(2.000000, 2.500000, 3.000000)	(2.000000 , 2.500000, 3.000000)
2	(0.027500, 0.034375, 0.041250)	(0.001719 , 0.002148, 0.002578)
3	(0.000378, 0.000473, 0.000567)	(0.000000 , 0.000000, 0.000000)
4	(0.000005, 0.000006, 0.000008)	(0.000000 , 0.000000, 0.000000)
5	(0.000000, 0.000000, 0.000000)	(0.000000 , 0.000000, 0.000000)
6	(0.000000, 0.000000, 0.000000)	(0.000000 , 0.000000, 0.000000)
7	(0.000000, 0.000000, 0.000000)	(0.000000 , 0.000000, 0.000000)
8	(0.000000, 0.000000, 0.000000)	(0.000000 , 0.000000, 0.000000)
9	(0.000000, 0.000000, 0.000000)	(0.000000 , 0.000000, 0.000000)
10	(0.000000, 0.000000, 0.000000)	(0.000000 , 0.000000, 0.000000)
11	(0.000000, 0.000000, 0.000000)	(0.000000 , 0.000000, 0.000000)
12	(0.000000, 0.000000, 0.000000)	(0.000000 , 0.000000, 0.000000)
13	(0.000000, 0.000000, 0.000000)	(0.000000 , 0.000000, 0.000000)
14	(0.000000, 0.000000, 0.000000)	(0.000000 , 0.000000, 0.000000)

Definition 4.1 ([14, 15]) Let $\mathcal{U} : \mathcal{V} \to \mathcal{V}$ be a mapping. Define a fixed-point iteration method by $p_{\nu+1} = f(\mathcal{U}, p_\nu)$ such that $\{p_\nu\}$ converges to a fixed point $p^\star$ of $\mathcal{U}$. Let $\{h_\nu\}$ be an arbitrary sequence in $\mathcal{M}$. Define

$$\epsilon_\nu = \left\| h_\nu - f(\mathcal{U}, h_\nu) \right\|, \quad \forall \nu \in \mathbb{N}. \tag{4.1}$$

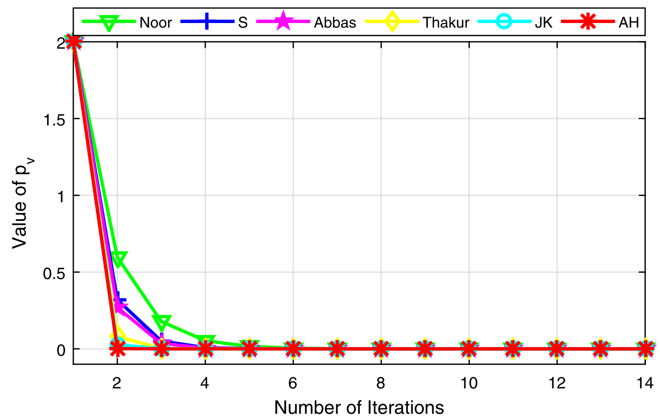

Figure 1 Graph corresponding to Tables 1–3

A fixed-point iterative method is said to be $\mathcal{U}$-stable if the following condition is fulfilled:

$$\lim_{\nu\to\infty} \epsilon_\nu = 0 \quad \text{if and only if} \quad \lim_{\nu\to\infty} h_\nu = p^\star. \tag{4.2}$$

The notion of stability in Definition 4.1 has recently been studied by several authors for different classes of contraction mappings (see [16, 18, 27] and the references in them).

In [8], Berinde observed that the concept of stability in Definition 4.1 is not precise because of the sequence $\{h_\nu\}$ that is arbitrarily taken. To overcome this limitation, Berinde [8] observed that it would be more natural if $\{h_\nu\}$ were an approximate sequence of $\{p_\nu\}$. Therefore, any iteration process that is stable will also be weakly stable but the converse is generally not true.

Definition 4.2 ([8]) Let $\{p_\nu\} \subset \mathcal{V}$ be a given sequence. Then, a sequence $\{h_\nu\} \subset \mathcal{V}$ is an approximate sequence of $\{p_\nu\}$ if, for any $c \in \mathbb{N}$, there exists $\eta = \eta(c)$ such that

$$\|p_\nu - h_\nu\| \leq \eta, \quad \forall \nu \geq c.$$

Definition 4.3 ([8]) Let $\mathcal{U} : \mathcal{V} \to \mathcal{V}$ be a mapping and $\{p_\nu\}$ be an iterative procedure defined for $p_1 \in \mathcal{V}$ and

$$p_{\nu+1} = f(\mathcal{U}, p_\nu), \quad \nu \geq 0. \tag{4.3}$$

Let $\{p_\nu\}$ converge to a fixed point $p^\star$ of $\mathcal{U}$. Suppose for any approximate sequence $\{h_\nu\} \subset \mathcal{V}$ of $\{p_\nu\}$

$$\lim_{\nu\to\infty} \epsilon_\nu = \lim_{\nu\to\infty} \left\| h_{\nu+1} - f(\mathcal{U}, h_\nu) \right\| = 0 \quad \Rightarrow \quad \lim_{\nu\to\infty} h_\nu = p^\star,$$

then we shall say that (4.3) is weakly $\mathcal{U}$-stable or weakly stable with respect to $\mathcal{U}$.

In 2010, Timis [38] studied a new concept of weak stability that is obtained from Definition 4.3 by replacing the approximate sequence with the notion of the equivalent sequence that is more general.

Definition 4.4 ([9]) Let $\{p_\nu\}$ and $\{h_\nu\}$ be two sequences. We say that these sequences are equivalent if

$$\lim_{\nu\to\infty} \|p_\nu - h_\nu\| = 0.$$

Definition 4.5 ([38]) Let $\mathcal{U} : \mathcal{V} \to \mathcal{V}$ be a mapping. Let $\{p_\nu\}$ be an iterative procedure defined for $p_1 \in \mathcal{V}$ and

$$p_{\nu+1} = f(\mathcal{U}, p_\nu), \quad \nu \geq 0. \tag{4.4}$$

Let $\{p_\nu\}$ converge to a fixed point $p^\star$ of $\mathcal{U}$. Suppose for any equivalent sequence $\{h_\nu\} \subset \mathcal{V}$ of $\{p_\nu\}$

$$\lim_{\nu\to\infty} \epsilon_\nu = \lim_{\nu\to\infty} \big\| h_{\nu+1} - f(\mathcal{U}, h_\nu)\big\| = 0 \quad \Rightarrow \quad \lim_{\nu\to\infty} h_\nu = p^\star,$$

then we shall say that (4.4) is weakly w^2-stable with respect to $\mathcal{U}$.

It is shown in [38] with an example that any equivalent sequence is an approximative sequence but the reverse is not true.

In this section, we prove that the AH iterative scheme (1.6) is w^2-stable with respect to $\mathcal{U}$ for contractive-like mappings.

Theorem 4.6 *Suppose all the conditions in Theorem 3.1 hold. Then, the AH iterative scheme (1.6) is w^2-stable with respect to $\mathcal{U}$.*

Proof Let $\{h_\nu\} \subset \mathcal{V}$ be an equivalent sequence of $\{p_\nu\}$. Put $\epsilon_\nu = \|h_{\nu+1} - (1-k_\nu)a_\nu - k_\nu \mathcal{U}a_\nu\|$, where $a_\nu = \mathcal{U}(\mathcal{U}b_\nu)$, $b_\nu = \mathcal{U}(\mathcal{U}c_\nu)$, $c_\nu = (1-r_\nu)h_\nu + r_\nu \mathcal{U}h_\nu$. Suppose that $\lim_{\nu\to\infty} \epsilon_\nu = 0$. Then, applying the triangular inequality, (1.2) and (1.6) give

$$\begin{aligned}
\big\|h_{\nu+1} - p^\star\big\| &\leq \|h_{\nu+1} - p_{\nu+1}\| + \big\|p_{\nu+1} - p^\star\big\| \\
&\leq \big\|h_{\nu+1} - (1-k_\nu)a_\nu - k_\nu \mathcal{U}a_\nu\big\| \\
&\quad + \big\|(1-k_\nu)a_\nu + k_\nu \mathcal{U}a_\nu - p_{\nu+1}\big\| + \big\|p_{\nu+1} - p^\star\big\| \\
&= \epsilon_\nu + \big\|(1-k_\nu)a_\nu - k_\nu \mathcal{U}a_\nu - (1-k_\nu)q_\nu - k_\nu \mathcal{U}q_\nu\big\| \\
&\quad + \big\|p_{\nu+1} - p^\star\big\| \\
&\leq \epsilon_\nu + (1-k_\nu)\|q_\nu - a_\nu\| + k_\nu\|\mathcal{U}q_\nu - \mathcal{U}a_\nu\| + \big\|p_{\nu+1} - p^\star\big\| \\
&\leq \epsilon_\nu + (1-k_\nu)\|q_\nu - a_\nu\| + k_\nu\big[\zeta\|q_\nu - a_\nu\| + \psi(\|q_\nu - \mathcal{U}q_\nu)\big] \\
&\quad + \big\|p_{\nu+1} - p^\star\big\| \\
&\leq \epsilon_\nu + \big(1 - (1-\zeta)k_\nu\big)\|q_\nu - a_\nu\| \\
&\quad + k_\nu\psi\big(\big\|q_\nu - p^\star\big\| + \big\|\mathcal{U}p^\star - \mathcal{U}q_\nu\big\|\big) + \big\|p_{\nu+1} - p^\star\big\| \\
&\leq \epsilon_\nu + \big(1 - (1-\zeta)k_\nu\big)\|q_\nu - a_\nu\| \\
&\quad + k_\nu\psi\big((1+\zeta)\big\|q_\nu - p^\star\big\|\big) + \big\|p_{\nu+1} - p^\star\big\|.
\end{aligned} \tag{4.5}$$

Since $(1-(1-\zeta)k_\nu) < 1$ for all $\nu \in \mathbb{N}$, then from (4.5), we obtain

$$\left\|h_{\nu+1}-p^\star\right\| \le \epsilon_\nu + \|q_\nu - a_\nu\| + k_\nu\psi\left((1+\zeta)\left\|q_\nu - p^\star\right\|\right) + \left\|p_{\nu+1}-p^\star\right\|. \tag{4.6}$$

Also,

$$\begin{aligned}
\|q_\nu - a_\nu\| &= \left\|\mathcal{U}(\mathcal{U}w_\nu) - \mathcal{U}(\mathcal{U}b_\nu)\right\| \\
&\le \zeta\|\mathcal{U}w_\nu - \mathcal{U}b_\nu\| + \psi\left(\left\|\mathcal{U}w_\nu - \mathcal{U}(\mathcal{U}w_\nu)\right\|\right) \\
&\le \zeta\left(\zeta\|w_\nu - b_\nu\| + \psi\left(\|w_\nu - \mathcal{U}w_\nu\|\right)\right) \\
&\quad + \psi\left(\zeta\left\|w_\nu - p^\star\right\| + \zeta\left\|\mathcal{U}w_\nu - p^\star\right\|\right) \\
&\le \zeta^2\|w_\nu - b_\nu\| + \zeta\psi\left((1+\zeta)\left\|w_\nu - p^\star\right\|\right) \\
&\quad + \psi\left(\zeta\left\|w_\nu - p^\star\right\| + \zeta^2\left\|w_\nu - p^\star\right\|\right) \\
&= \zeta^2\|w_\nu - b_\nu\| + \zeta\psi\left((1+\zeta)\left\|w_\nu - p^\star\right\|\right) \\
&\quad + \psi\left(\zeta(1+\zeta)\left\|w_\nu - p^\star\right\|\right).
\end{aligned} \tag{4.7}$$

Similarly,

$$\|w_\nu - b_\nu\| \le \zeta^2\|z_\nu - c_\nu\| + \zeta\psi\left((1+\zeta)\left\|z_\nu - p^\star\right\|\right) + \psi\left(\zeta(1+\zeta)\left\|z_\nu - p^\star\right\|\right). \tag{4.8}$$

Finally, since $(1-(1-\zeta)r_\nu) < 1$ for all $\nu \in \mathbb{N}$, we have

$$\begin{aligned}
\|z_\nu - c_\nu\| &\le (1-r_\nu)\|p_\nu - h_\nu\| + r_\nu\|\mathcal{U}p_\nu - \mathcal{U}h_\nu\| \\
&\le (1-r_\nu)\|p_\nu - h_\nu\| + r_\nu\zeta\|p_\nu - h_\nu\| + r_\nu\psi\left(\|p_\nu - \mathcal{U}p_\nu\|\right) \\
&\le \left(1-(1-\zeta)r_\nu\right)\|p_\nu - h_\nu\| + r_\nu(1+\zeta)\left\|p_\nu - p^\star\right\| \\
&\le \|p_\nu - h_\nu\| + r_\nu\psi\left((1+\zeta)\left\|p_\nu - p^\star\right\|\right).
\end{aligned} \tag{4.9}$$

Using (4.6), (4.7), (4.8) and (4.9), we obtain

$$\begin{aligned}
\left\|h_{\nu+1}-p^\star\right\| &\le \epsilon_\nu + \zeta^4\|p_\nu - h_\nu\| + \zeta^4 r_\nu\psi\left((1+\zeta)\left\|p_\nu - p^\star\right\|\right) \\
&\quad + \zeta^2\psi\left((1+\zeta)\left\|z_\nu - p^\star\right\|\right) + \zeta^2\psi\left(\zeta(1+\zeta)\left\|w_\nu - p^\star\right\|\right) \\
&\quad + \psi\left((1+\zeta)\|w_\nu - p^\star)\right) + \psi\left(\zeta(1+\zeta)\left\|w_\nu - p^\star\right\|\right) \\
&\quad + k_\nu\psi\left((1+\zeta)\left\|q_\nu - p^\star\right\|\right) + \left\|p_{\nu+1} - p^\star\right\|.
\end{aligned} \tag{4.10}$$

We established in Theorem 3.1 that $\lim_{\nu\to\infty}\|p_\nu - p^\star\| = 0$ and since ψ is a strictly increasing continuous function with $\psi(0) = 0$, consequently $\lim_{\nu\to\infty}\|p_{\nu+1} - p^\star\| = 0$. Following the equivalence of $\{p_\nu\}$ and $\{h_\nu\}$, we have that $\lim_{\nu\to\infty}\|p_\nu - h_\nu\| = 0$. Since $\lim_{\nu\to\infty}\epsilon_\nu = 0$, then taking the limits of both sides of (4.10) yields $\lim_{\nu\to\infty}\|h_\nu - p^\star\| = 0$. Hence, the AH iterative scheme (1.6) is w^2-stable with respect to $\mathcal{U}$. □

In order to support the analytical proof of Theorem 4.6, we provide the following illustrative example.

Example 4.7 Let $(\mathbb{R}, \|\cdot\|)$ be a Banach space with the usual norm and $\mathcal{V} = [0,1]$. Let $\mathcal{U} : \mathcal{V} \to \mathcal{V}$ be a mapping defined by $\mathcal{U}p = \frac{p}{6}$. Then clearly, zero is the only fixed point of $\mathcal{U}$ and $\mathcal{U}$ satisfies (1.1) with $\zeta = \frac{1}{6}$.

Next, we show that the sequence $\{p_\nu\}$ generated by the iterative scheme (1.6) converges to $p^\star = 0 \in \Im(\mathcal{U})$. For this, let $r_\nu = k_\nu = \frac{1}{\nu+3}$ and $p_1 \in [0,1]$, then from (1.6), we obtain

$$\begin{aligned}
z_\nu &= \left(1 - \frac{1}{\nu+3} + \frac{1}{6(\nu+3)}\right)p_\nu = \left(1 - \frac{5}{6(\nu+3)}\right)p_\nu, \\
w_\nu &= \frac{1}{36}\left(1 - \frac{5}{6(\nu+3)}\right)p_\nu, \\
q_\nu &= \frac{1}{1296}\left(1 - \frac{5}{6(\nu+3)}\right)p_\nu, \\
p_{\nu+1} &= \frac{1}{1296}\left(1 - \frac{10}{6(\nu+3)} + \frac{25}{36(\nu+3)^2}\right)p_\nu \\
&= \left[1 - \left(\frac{1295}{1296} + \frac{10}{6^5(\nu+3)} + \frac{25}{6^7(\nu+3)^2}\right)\right]p_\nu.
\end{aligned}$$

Set $y_\nu = \frac{1295}{1296} + \frac{10}{6^5(\nu+3)} + \frac{25}{6^7(\nu+3)^2}$. Note that $y_\nu \in (0,1)$ for all $\nu \in \mathbb{N}$ and $\sum_{\nu=0}^{\infty} y_\nu = \infty$. Thus, by Lemma 2.5, we obtain $\lim_{\nu\to\infty} p_\nu = 0$.

It is not difficult to see that $\lim_{\nu\to\infty} \|p_\nu\| = \|\lim_{\nu\to\infty} p_\nu\| = 0$. Then, if we take $h_\nu = \frac{1}{\nu+4}$ for all $\nu \in \mathbb{N}$, we obtain

$$0 \le \lim_{\nu\to\infty} \|p_\nu - h_\nu\| \le \lim_{\nu\to\infty} \|p_\nu\| + \lim_{\nu\to\infty} \|h_\nu\| = 0,$$

which shows that $\lim_{\nu\to\infty} \|p_\nu - h_\nu\| = 0$. If follows that $\{p_\nu\}$ and $\{h_\nu\}$ are equivalent sequences.

Suppose that ϵ_ν is the sequence associated with the iterative sequence $\{p_\nu\}$, then we obtain

$$\begin{aligned}
\epsilon_\nu &= \left| h_{\nu+1} - \left(\frac{1}{1296} - \frac{10}{6^5(\nu+3)} + \frac{25}{6^7(\nu+3)^2}\right)h_\nu \right| \\
&= \left| \frac{1}{\nu+5} - \frac{1}{1296(\nu+4)} + \frac{10}{6^5(\nu+3)(\nu+4)} - \frac{25}{6^7(\nu+3)^2(\nu+4)} \right|.
\end{aligned}$$

Clearly, $\lim_{\nu\to\infty} \epsilon_\nu = 0$. Therefore, the sequence $\{p_\nu\}$ generated by the AH iterative scheme (1.6) is w^2-stable with respect $\mathcal{U}$.

5 Convergence results

In this section, we prove weak and strong convergence theorems of the AH iterative scheme (1.6) for Reich–Suzuki-type nonexpansive mappings.

Lemma 5.1 *Let $\mathcal{U}$ be a self Reich–Suzuki-type nonexpansive mapping defined on a nonempty closed convex subset $\mathcal{V}$ of a Banach space $\mathcal{M}$ with $\Im(\mathcal{U}) \neq \emptyset$. Let $\{p_\nu\}$ be the sequence generated by the AH iterative scheme (1.6), then $\lim_{\nu\to\infty} \|p_\nu - p^\star\|$ exists for each $p^\star \in \Im(\mathcal{U})$.*

Proof Let $p^\star \in \Im(\mathcal{U})$. By Lemma 2.7, we obtain

$$\begin{aligned}
\|z_\nu - p^\star\| &= \|(1-r_\nu)p_\nu + r_\nu \mathcal{U}p_\nu - \mathcal{U}p^\star\| \\
&\le (1-r_\nu)\|p_\nu - p^\star\| + r_\nu\|\mathcal{U}p_\nu - \mathcal{U}p^\star\| \\
&\le (1-r_\nu)\|p_\nu - p^\star\| + r_\nu\|p_\nu - p^\star\| \\
&= \|p_\nu - p^\star\|, \qquad (5.1)
\end{aligned}$$

$$\begin{aligned}
\|w_\nu - p^\star\| &= \|\mathcal{U}(\mathcal{U}z_\nu) - p^\star\| \\
&\le \|\mathcal{U}z_\nu - p^\star\| \\
&\le \|z_\nu - p^\star\| \\
&\le \|p_\nu - p^\star\|, \qquad (5.2)
\end{aligned}$$

$$\begin{aligned}
\|q_\nu - p^\star\| &= \|\mathcal{U}(\mathcal{U}w_\nu) - p^\star\| \\
&\le \|\mathcal{U}w_\nu - p^\star\| \\
&\le \|w_\nu - p^\star\| \\
&\le \|p_\nu - p^\star\|, \qquad (5.3)
\end{aligned}$$

$$\begin{aligned}
\|p_{\nu+1} - p^\star\| &= \|(1-k_\nu)q_\nu + k_\nu \mathcal{U}q_\nu - \mathcal{U}p^\star\| \\
&\le (1-k_\nu)\|q_\nu - p^\star\| + k_\nu\|\mathcal{U}q_\nu - \mathcal{U}p^\star\| \\
&\le \|q_\nu - p^\star\| \\
&\le \|p_\nu - p^\star\|. \qquad (5.4)
\end{aligned}$$

Thus, $\{\|p_\nu - p^\star\|\}$ is a bounded and decreasing sequence of reals and hence $\lim_{\nu\to\infty}\|p_\nu - p^\star\|$ exists. □

Lemma 5.2 *Let $\mathcal{U}$ be a self Reich–Suzuki-type nonexpansive mapping defined on a nonempty closed convex subset $\mathcal{V}$ of a uniformly convex Banach space $\mathcal{M}$. Let $\{p_\nu\}$ be the iterative sequence defined by the AH iterative scheme (1.6). Then, $\Im(\mathcal{U}) \neq \emptyset$ if and only if $\{p_\nu\}$ is bounded and* $\lim_{\nu\to\infty}\|\mathcal{U}p_\nu - p_\nu\| = 0$.

Proof Suppose that $\Im(\mathcal{U}) \neq \emptyset$ and $p^\star \in \Im(\mathcal{U})$. Then, by Lemma 5.1, $\lim_{\nu\to\infty}\|p_\nu - p^\star\|$ exists and $\{p_\nu\}$ is bounded. Now, we set

$$\lim_{\nu\to\infty}\|p_\nu - p^\star\| = d. \qquad (5.5)$$

From (5.1), (5.2) and (5.5), we have

$$\limsup_{\nu\to\infty}\|z_\nu - p^\star\| \le d, \qquad (5.6)$$

$$\limsup_{\nu\to\infty}\|w_\nu - p^\star\| \le d. \qquad (5.7)$$

Recalling Lemma 2.7, we have

$$\limsup_{\nu\to\infty}\|\mathcal{U}p_\nu - p^\star\| \le \limsup_{\nu\to\infty}\|p_\nu - p^\star\| = d. \qquad (5.8)$$

Also, from (1.6) and Lemma 5.1, we obtain

$$\begin{aligned}
\|p_{\nu+1}-p^\star\| &= \|(1-k_\nu)q_\nu + k_\nu \mathcal{U}q_\nu - \mathcal{U}p^\star\| \\
&\le (1-k_\nu)\|q_\nu - p^\star\| + k_\nu\|\mathcal{U}q_\nu - \mathcal{U}p^\star\| \\
&\le (1-k_\nu)\|p_\nu - p^\star\| + k_\nu\|q_\nu - p^\star\| \\
&\le (1-k_\nu)\|p_\nu - p^\star\| + k_\nu\|\mathcal{U}(\mathcal{U}w_\nu) - p^\star\| \\
&\le (1-k_\nu)\|p_\nu - p^\star\| + k_\nu\|\mathcal{U}w_\nu - p^\star\| \\
&= (1-k_\nu)\|p_\nu - p^\star\| + k_\nu\|w_\nu - p^\star\| \\
&\le (1-k_\nu)\|p_\nu - p^\star\| + k_\nu\|\mathcal{U}(\mathcal{U}z_\nu) - p^\star\| \\
&\le (1-k_\nu)\|p_\nu - p^\star\| + k_\nu\|\mathcal{U}z_\nu - p^\star\| \\
&\le (1-k_\nu)\|p_\nu - p^\star\| + k_\nu\|z_\nu - p^\star\| \\
&= \|p_\nu - p^\star\| - k_\nu\|p_\nu - p^\star\| + k_\nu\|z_\nu - p^\star\|.
\end{aligned}$$

This implies that

$$\|p_{\nu+1}-p^\star\| - \|p_\nu - p^\star\| \le \frac{\|p_{\nu+1}-p^\star\| - \|p_\nu - p^\star\|}{k_\nu} \le \|z_\nu - p^\star\| - \|p_\nu - p^\star\|.$$

Therefore,

$$d \le \liminf_{\nu\to\infty}\|z_\nu - p^\star\|. \tag{5.9}$$

From (5.6) and (5.9), we obtain

$$d = \lim_{\nu\to\infty}\|z_\nu - p^\star\|. \tag{5.10}$$

Using (1.6), we have

$$d = \lim_{\nu\to\infty}\|z_\nu - p^\star\| = \lim_{\nu\to\infty}\|(1-r_\nu)(p_\nu - p^\star) + r_\nu(\mathcal{U}p_\nu - p^\star)\|.$$

Since $0 < r_\nu < 1$ for all $\nu \in \mathbb{N}$, then from Lemma 2.6, we have

$$\lim_{\nu\to\infty}\|\mathcal{U}p_\nu - p_\nu\| = 0.$$

Conversely, suppose that $\{p_\nu\}$ is bounded and $\lim_{\nu\to\infty}\|\mathcal{U}p_\nu - p_\nu\| = 0$. Let $p^\star \in A(\mathcal{V}, \{p_\nu\})$. By Lemma 2.8, we have

$$\begin{aligned}
r(\mathcal{U}p^\star, \{p_\nu\}) &= \limsup_{\nu\to\infty}\|p_\nu - \mathcal{U}p^\star\| \\
&\le \left(\frac{3+\varpi}{1-\varpi}\right)\limsup_{\nu\to\infty}\|\mathcal{U}p_\nu - p_\nu\| + \limsup_{\nu\to\infty}\|p_\nu - p^\star\| \\
&= \limsup_{\nu\to\infty}\|p_\nu - p^\star\| \\
&= r(p^\star, \{p_\nu\}).
\end{aligned}$$

This implies that $\mathcal{U}p^\star \in A(\mathcal{V},\{p_\nu\})$. Since $\mathcal{M}$ is uniformly convex, then $A(\mathcal{V},\{p_\nu\})$ has only one element, therefore we obtain $\mathcal{U}p^\star = p^\star$. □

Now, we establish the weak convergence result. For this, the following Lemma will be useful:

Lemma 5.3 *If all the assumptions in Theorem 5.4 are fulfilled, then* $\lim_{\nu\to\infty}\langle p_\nu, J(p_1^\star - p_2^\star)\rangle$ *exists for any* $p_1^\star, p_2^\star \in \Im(\mathcal{U})$; *in particular* $\lim_{\nu\to\infty}\langle p-q, J(p_1^\star - p_2^\star)\rangle = 0$ *for all* $p,q \in \omega_w(p_\nu)$, *where* $\omega_w(p_\nu)$ *denotes the set of all weak limit points of* $\{p_\nu\}$.

Proof The conclusion follows from Lemma 2.3 in [20]. □

Theorem 5.4 *Let* $\mathcal{U}$, $\mathcal{V}$ *and* p_ν *be as in Lemma 5.2. Let* $\mathcal{M}$ *be a uniformly convex Banach space. Suppose that either of the following assumptions holds*:

(a) $\mathcal{M}$ *satisfies Opial's condition and* $I - \mathcal{U}$ *is demiclosed with respect to zero*;

(b) $\mathcal{M}$ *has a Fréchet differential norm.*

If $\Im(\mathcal{U}) \neq \emptyset$, *then the sequence* $\{p_\nu\}$ *converges weakly to a point of* $\mathcal{U}$.

Proof By Lemma 5.1, we have that $\lim_{\nu\to\infty}\|p_\nu - p^\star\|$ exists. Now, it is sufficient to prove that $\{p_\nu\}$ have a unique weak subsequential limit in $\Im(\mathcal{U})$. Suppose that $\{p_{\nu_i}\}$ and $\{p_{\nu_k}\}$ are two subsequences of $\{p_\nu\}$, which converge weakly to g and y, respectively. Now, suppose that (a) is true. Then, from Lemma 5.2, $\lim_{\nu\to\infty}\|\mathcal{U}p_\nu - p_\nu\| = 0$ and by the demiclosedness of $I - \mathcal{U}$ with respect to zero, we have that $(1-\mathcal{U})g = 0$. That is, $g = \mathcal{U}g$; similarly $y = \mathcal{U}y$.

Next, we prove uniqueness. Since $g, y \in \Im(\mathcal{U})$, then $\lim_{\nu\to\infty}\|p_\nu - g\|$ and $\lim_{\nu\to\infty}\|p_\nu - y\|$ exists. If $g \neq y$, then from Opial's condition, we have

$$\begin{aligned}\lim_{\nu\to\infty}\|p_\nu - g\| &= \lim_{\nu_i\to\infty}\|p_{\nu_i} - g\| < \lim_{\nu_i\to\infty}\|p_{\nu_i} - y\| = \lim_{\nu\to\infty}\|p_\nu - y\| \\ &= \lim_{\nu_k\to\infty}\|p_{\nu_k} - y\| < \lim_{\nu_k\to\infty}\|p_{\nu_k} - g\| = \lim_{\nu\to\infty}\|p_\nu - g\|,\end{aligned}$$

which is a contradiction, so $g = y$. Again, assume that (b) holds. Recalling Lemma 5.3, we have $\langle p_\nu, J(p_1^\star - p_2^\star)\rangle = 0$ for all $p,q \in \omega_w(p_\nu)$. Thus, $\|g-y\|^2 = \langle g-y, J(g-y)\rangle$ implies $g = y$. □

We now establish the following strong convergence results:

Theorem 5.5 *Let* $\mathcal{U}$, $\mathcal{V}$ *and* $\mathcal{M}$ *be as in Lemma 5.2. The sequence* $\{p_\nu\}$ *generated by the AH iterative scheme (1.6) converges to an element of* $\Im(\mathcal{U})$ *if and only if* $\liminf_{\nu\to\infty} d(p_\nu, \Im(\mathcal{U})) = 0$, *where* $d(p_\nu, \Im(\mathcal{U})) = \inf\{\|p_\nu - p^\star\| : p^\star \in \Im(\mathcal{U})\}$.

Proof The necessity is obvious.

Conversely, suppose $\liminf_{\nu\to\infty} d(p_\nu, \Im(\mathcal{U})) = 0$ and $p^\star \in \Im(\mathcal{U})$. By Lemma 5.1, $\lim_{\nu\to\infty}\|p_\nu - p^\star\|$ exists, for any $p^\star \in \Im(\mathcal{U})$. It is sufficient to prove that the sequence $\{p_\nu\}$ is Cauchy in $\mathcal{V}$. Since $\lim_{\nu\to\infty} d(p_\nu, \Im(\mathcal{U})) = 0$, then given $\varepsilon > 0$, there exists $\rho_0 \in \mathbb{N}$ such that for all $\nu \geq \rho_0$

$$\begin{aligned}&d(p_\nu, \Im(\mathcal{U})) < \frac{\varepsilon}{2}, \\ &\inf\{\|p_\nu - p^\star\| : p^\star \in \Im(\mathcal{U})\} < \frac{\varepsilon}{2}.\end{aligned}$$

In particular, $\inf\{\|p_{\rho_0} - p^\star\| : p^\star \in \Im(\mathcal{U})\} < \frac{\varepsilon}{2}$. Therefore, there exists $p^\star \in \Im(\mathcal{U})$ such that

$$\|p_{\rho_0} - p^\star\| < \frac{\varepsilon}{2}.$$

Now for $\rho, \nu \geq \rho_0$, we have

$$\begin{aligned}
\|p_{\nu+\rho} - p_\nu\| &\leq \|p_{\nu+\rho} - p^\star\| + \|p_\nu - p^\star\| \\
&\leq \|p_{\rho_0} - p^\star\| + \|p_{\rho_0} - p^\star\| \\
&= 2\|p_{\rho_0} - p^\star\| < \varepsilon.
\end{aligned}$$

This implies that the sequence $\{p_\nu\}$ is Cauchy in $\mathcal{V}$. Since $\mathcal{V}$ is closed, there must be an element $q \in \mathcal{V}$ such that $\lim_{\nu\to\infty} p_\nu = q$. Now, $\lim_{\nu\to\infty} d(p_\nu, \Im(\mathcal{U})) = 0$ gives that $d(q, \Im(\mathcal{U}) = 0$, that is $q \in \Im(\mathcal{U})$. □

A strong convergence on a compact domain is established in the following way:

Theorem 5.6 *Let $\mathcal{U}$ and $\mathcal{M}$ be as in Lemma 5.2 and $\mathcal{V}$ be a nonempty compact convex subset of $\mathcal{M}$. Then, the sequence $\{p_\nu\}$ generated by the iterative scheme (1.6) converges strongly to a fixed point of $\mathcal{U}$.*

Proof According to Lemma 5.2, $\lim_{\nu\to\infty} \|\mathcal{U}p_\nu - p_\nu\| = 0$. Since $\mathcal{V}$ is convex and compact, the iterative sequence $\{p_\nu\}$ contained in the set $\mathcal{V}$ has a convergent subsequence, namely, $\{p_{\nu_i}\}$ endowed with a strong limit, namely, $q \in \mathcal{V}$. Putting $p = p_{\nu_i}$ and $q = q$, we apply Lemma 2.8, to obtain

$$\|p_{\nu_i} - \mathcal{U}q\| \leq \left(\frac{3+\varpi}{1-\varpi}\right)\|p_{\nu_i} - \mathcal{U}p_{\nu_i}\| + \|p_{\nu_i} - q\|.$$

As $i \to \infty$, one can see that $p_{\nu_i} \to \mathcal{U}q$. It follows that $\mathcal{U}q = q$, i.e., $q \in \Im(\mathcal{U})$. According to Lemma 5.1, $\lim_{\nu\to\infty} \|p_\nu - q\|$ exists, that is, q forms a strong limit for $\{p_\nu\}$. □

A strong convergence theorem using a condition (I) of the operators is the following:

Theorem 5.7 *Let $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{M}$ be as in Lemma 5.2. If $\mathcal{U}$ satisfies condition (I), then the sequence $\{p_\nu\}$ generated by the AH iterative scheme (1.6) converges strongly to a fixed point of $\mathcal{U}$.*

Proof In Lemma 5.2, we have shown that

$$\lim_{\nu\to\infty} \|\mathcal{U}p_\nu - p_\nu\| = 0. \tag{5.11}$$

By Definition 2.4 and (5.11), we have

$$0 \leq \lim_{\nu\to\infty} f\bigl(d\bigl(p_\nu, \Im(\mathcal{U})\bigr)\bigr) \leq \lim_{\nu\to\infty} \|p_\nu - \mathcal{U}p_\nu\| = 0 \quad \Rightarrow \quad f\bigl(d\bigl(p_\nu, \Im(\mathcal{U})\bigr)\bigr) = 0.$$

Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function that satisfies the condition $f(0) = 0$ and $f(s) > 0$, for all $s > 0$, we obtain

$$\lim_{\nu\to\infty} d\bigl(p_\nu, \Im(\mathcal{U})\bigr) = 0.$$

Since all the requirements of Theorem 5.5 are shown, one concludes that the sequence $\{p_v\}$ is strongly convergent in the fixed-point set of $\mathcal{U}$. □

6 Numerical result

In this section, we give an example of a Reich–Suzuki-type nonexpansive mapping that does not satisfy the condition (C). Further, we compare the convergence of the AH iterative scheme with some leading iterative schemes in the literature.

Example 6.1 Let $(\mathbb{R}, \|\cdot\|)$ be a Banach space with the usual norm and $\mathcal{V} = [5,7]$. Let $\mathcal{U} : \mathcal{V} \to \mathcal{V}$ be a mapping defined by

$$\mathcal{U}p = \begin{cases} \frac{p+20}{5}, & \text{if } p < 7, \\ 4, & \text{if } p = 7. \end{cases} \tag{6.1}$$

(i) The mapping $\mathcal{U}$ does not satisfy the condition (C). For this, let $p = 6$ and $q = 7$, we have

$$\frac{1}{2}|p - \mathcal{U}p| = \frac{1}{2}\big|6 - \mathcal{U}(6)\big| = \frac{2}{5} < 1 = |p - q|.$$

On the other hand,

$$|\mathcal{U}p - \mathcal{U}q| = \big|\mathcal{U}(6) - \mathcal{U}(7)\big| = \frac{6}{5} > 1 = |p - q|.$$

(ii) Now, to demonstrate that $\mathcal{U}$ is a Reich–Suzuki-type nonexpasive mapping, the following cases are considered.

Case 1: If $p, q < 7$, then

$$\begin{aligned}
&\varpi|p - \mathcal{U}p| + \varpi|q - \mathcal{U}q| + (1 - 2\varpi)|p - q| \\
&\quad = \frac{1}{2}\left|p - \left(\frac{p+20}{5}\right)\right| + \frac{1}{2}\left|q - \left(\frac{p+20}{5}\right)\right| \\
&\quad = \frac{1}{2}\left|\frac{4p-20}{6}\right| + \frac{1}{2}\left|\frac{4q-20}{5}\right| \\
&\quad \geq \frac{1}{2}\left|\left(\frac{4p-20}{5}\right) - \left(\frac{4q-20}{5}\right)\right| \\
&\quad = \frac{1}{2}\left|\frac{4p}{5} - \frac{4p}{5}\right| = \frac{2}{5}|p - q| \\
&\quad \geq \frac{1}{5}|p - q| = |\mathcal{U}p - \mathcal{U}q|.
\end{aligned}$$

Case 2: If $p < 7$ and $q = 7$, then we have

$$\begin{aligned}
\varpi|p - \mathcal{U}p| + \varpi|q - \mathcal{U}q| + (1 - 2\varpi)|p - q| &= \frac{1}{2}\left|\frac{4p-20}{5}\right| + \frac{1}{2}|7 - 4| \\
&= \frac{1}{2}\left|\frac{4p-20}{5}\right| + \frac{3}{2} \\
&\geq \left|\frac{p}{5}\right| = |\mathcal{U}p - \mathcal{U}q|.
\end{aligned}$$

Case 3: If $q < 7$ and $p = 7$, then we obtain

$$\begin{aligned}\varpi|p-\mathcal{U}p|+\varpi|q-\mathcal{U}q|+(1-2\varpi)|p-q| &= \frac{1}{2}|7-4|+\frac{1}{2}\left|\frac{4q-20}{5}\right| \\ &= \frac{3}{2}+\frac{1}{2}\left|\frac{4q-20}{5}\right| \\ &\geq \left|\frac{q}{5}\right| = |\mathcal{U}p-\mathcal{U}q|.\end{aligned}$$

Case 4: If $p = q = 7$, then we obtain

$$\varpi|p-\mathcal{U}p|+\varpi|q-\mathcal{U}q|+(1-2\varpi)|p-q| \geq 0 = |4-4| = |\mathcal{U}p-\mathcal{U}q|.$$

Hence, $\mathcal{U}$ is a Reich–Suzuki-type nonexpansive mapping and has fixed point 5.

From Table 4 and Fig. 2, it can be clearly seen that the AH iterative converges faster to the fixed point of $\mathcal{U}$ than other iterative schemes.

Table 4 Comparison of convergence behavior of AH (1.6) with S, Abbas, Thakur and JK iterative schemes

Step	S	Abbas	Thakur	JK	AH
1	5.5000000	5.5000000	5.5000000	5.5000000	5.5000000
2	5.0993388	5.0602104	5.0198678	5.0171967	5.0006879
3	5.0197364	5.0072506	5.0007895	5.0005915	5.0000009
4	5.0039212	5.0008731	5.0000314	5.0000203	5.0000000
5	5.0007791	5.0001051	5.0000012	5.0000007	5.0000000
6	5.0001548	5.0000127	5.0000000	5.0000000	5.0000000
7	5.0000308	5.0000015	5.0000000	5.0000000	5.0000000
8	5.0000061	5.0000002	5.0000000	5.0000000	5.0000000
9	5.0000012	5.0000000	5.0000000	5.0000000	5.0000000
10	5.0000002	5.0000000	5.0000000	5.0000000	5.0000000
11	5.0000000	5.0000000	5.0000000	5.0000000	5.0000000

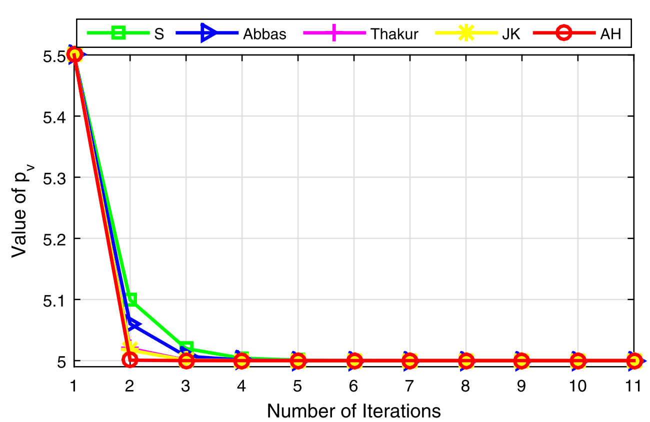

Figure 2 Graph corresponding to Table 4

7 Application

In this section, we consider the application of our main results to the following nonlinear mixed Volterra–Fredholm-type integral equation:

$$p(t) = \vartheta(t) + \lambda \int_a^t \int_a^b \varphi(r,s)\Upsilon\bigl(s,p(s)\bigr)\,ds\,dr, \quad t \in [a,b]. \tag{7.1}$$

Let $C([a,b])$ denote the space of all real-valued continuous functions on $I = [a,b]$. It is well known that $C([a,b])$ is a Banach space with the maximum norm,

$$\|p-q\|_\infty = \max_{t\in[a,b]} \bigl|p(t)-q(t)\bigr|, \quad \forall p,q \in C\bigl([a,b]\bigr).$$

Theorem 7.1 *Let $\mathcal{H}$ be a nonempty closed convex subset of $\mathcal{G} = C([a,b])$ and $\mathcal{U} : \mathcal{H} \to \mathcal{H}$ be defined by*

$$\mathcal{U}p(t) = \vartheta(t) + \lambda \int_a^t \int_a^b \varphi(r,s)\Upsilon\bigl(s,p(s)\bigr)\,ds\,dr, \quad t \in [a,b], \lambda \geq 0.$$

Assume that the following conditions hold:

(Z_1) $\vartheta : I \to \mathbb{R}$ *is continuous*;

(Z_2) $\varphi : I \times I \to \mathbb{R}$ *is continuous for all* $(r,s) \in I \times I$ *such that* $|\varphi(s,r)| \leq M$;

(Z_3) $\Upsilon : I \times \mathcal{H} \to \mathcal{H}$ *is continuous and there exists a constant* $L_\Upsilon > 0$ *such that*

$$\bigl|\Upsilon(s,p_1) - \Upsilon(s,p_2)\bigr| \leq L_\Upsilon |p_1 - q_2|,$$

for all $s \in I$ *and* $p_1, q_2 \in \mathbb{R}$;

(Z_4) $\lambda L_\Upsilon M(b-a)^2 < 1$.

Then, the mixed Volterra–Fredholm integral equation (7.1) *has a unique solution in* $C([a,b])$ *if and only if* $\mathcal{U}$ *admits an a.f.p.s.*

Proof Let $p,q \in C([a,b])$, then

$$\begin{aligned}
\|p-\mathcal{U}q\|_\infty &= \max_{t\in[a,b]} \bigl|p(t)-\mathcal{U}q(t)\bigr| \\
&= \max_{t\in[a,b]} \biggl|p(t) - \vartheta(t) - \lambda \int_a^t \int_a^b \varphi(r,s)\Upsilon\bigl(s,p(s)\bigr)\,ds\,dr + \vartheta(t) \\
&\quad + \lambda \int_a^t \int_a^b \varphi(r,s)\Upsilon\bigl(s,p(s)\bigr)\,ds\,dr - \vartheta(t)\lambda \int_a^t \int_a^b \varphi(r,s)\Upsilon\bigl(s,q(s)\bigr)\,ds\,dr\biggr| \\
&\leq \max_{t\in[a,b]} \bigl|p(t)-\mathcal{U}p(t)\bigr| \\
&\quad + \lambda \max_{t\in[a,b]} \biggl|\int_a^t \int_a^b \varphi(r,s)\Upsilon\bigl(s,p(s)\bigr)\,ds\,dr - \int_a^t \int_a^b \varphi(r,s)\Upsilon\bigl(s,q(s)\bigr)\,ds\,dr\biggr| \\
&\leq \max_{t\in[a,b]} \bigl|p(t)-\mathcal{U}p(t)\bigr| + \lambda L_\Upsilon \max_{t\in[a,b]} \int_a^t \int_a^b \bigl|\varphi(r,s)\bigr|\bigl|p(s)-q(s)\bigr|\,ds\,dr \\
&\leq \|p-\mathcal{U}p\|_\infty + \lambda L_\Upsilon \|p-q\|_\infty \max_{t\in[a,b]} \int_a^t \int_a^b \bigl|\varphi(r,s)\bigr|\,ds\,dr \\
&\leq \|p-\mathcal{U}p\|_\infty + \lambda L_\Upsilon M(b-a)^2 \|p-q\|_\infty.
\end{aligned} \tag{7.2}$$

By assumption (Z_3), we have $\lambda L_\Upsilon M(b-a)^2 < 1$, then (7.2) yields

$$\|p - \mathcal{U}q\|_\infty \le \|p - \mathcal{U}p\|_\infty + \|p - q\|_\infty.$$

Therefore, by Lemma 2.8, $\mathcal{U}$ is a Reich–Suzuki-type nonexpansive mapping since it satisfies the condition (2.4) on $\mathcal{H}$ with $\frac{3+\varpi}{1-\varpi} = 1$.

Take $\mathcal{H} = \mathcal{V}$ and $\mathcal{G} = \mathcal{M}$. Then, all the conditions of Lemma 5.2 are fulfilled, thus, (7.1) has a solution in $\mathcal{H} \subseteq C([a,b])$. □

Example 7.2 Consider the following mixed-type nonlinear integral equation:

$$p(t) = \frac{\pi}{2}t - \frac{t^2}{7\pi} + \frac{2}{7}\int_0^t \int_0^1 r\frac{\sin(p(s))}{2}\,dt\,dr, \quad t \in [0,1]. \tag{7.3}$$

Clearly, the above integral equation is a special case of (7.1) with

$$\vartheta(t) = \frac{\pi}{2}t - \frac{t^2}{7\pi}, \qquad \varphi(r,s) = r \quad \text{and} \quad \Upsilon\bigl(t,p(s)\bigr) = \frac{\sin(p(s))}{2}.$$

Then, for any $t \in [0,1]$ and $p_1, q_1 \in \mathbb{R}$, we obtain

$$\bigl|\Upsilon(t,p_1) - \Upsilon(t,q_1)\bigr| = \frac{1}{2}|\sin p_1 - \sin q_1|. \tag{7.4}$$

Next, for any $p, q \in \mathbb{R}$ with $p < q$, then by the mean-value theorem, there exists f, $p < f < q$ such that $\frac{\cos p - \cos q}{p-q} = -\sin f \Rightarrow |\cos p - \cos q| \le |p-q|$. Therefore, (7.4) yields

$$\bigl|\Upsilon(t,p_1) - \Upsilon(t,q_1)\bigr| \le \frac{1}{2}|p_1 - q_1|.$$

It is not difficult to see that $\vartheta(t)$ is continuous on $[0,1]$. Also, $\Upsilon(t,p(s))$ is continuous and $(r,s) \in [0,1] \times [0,1]$, thus

$$M = \max_{(r,s)\in[0,1]\times[0,1]} |r| = 1.$$

Consequently, all the assumptions in Theorem 7.1 are performed with $\lambda L_\Upsilon M(b-a)^2 = \frac{2}{7} \cdot \frac{1}{2} \cdot 1 \cdot (1-0)^2 = \frac{2}{14} < 1$.

Hence, there exists a solution of the mixed-type nonlinear integral equation (7.3). Further, one can easily verify that the function $p(t) = \frac{\pi}{2}t$ is the exact solution of (7.3).

8 Conclusion

In this work, we have introduced a four-step iterative scheme, called the AH iterative scheme (1.6) for approximating the fixed points of contractive-like mappings and Reich–Suzuki-type nonexpansive mappings. The new iterative scheme has been shown to converge faster than the JK iterative scheme (1.5) analytically for contractive-like mappings. Furthermore, we have illustrated numerically that our new iterative scheme converges faster than many prominent iterative schemes in the literature for contractive-like mappings. The w^2-stability result of the AH iterative scheme has also been established for contractive-like mappings. We have provided an example to illustrate the notion of w^2-stability of the AH iterative scheme with respect to $\mathcal{U}$. Also, we have proved weak and several strong convergence theorems for Reich–Suzuki-type nonexpansive mappings in uniformly convex Banach spaces. A new example of Reich–Suzuki-type nonexpansive mappings has been provided to compare the convergence behavior of the AH iterative method (1.6) with some well-known iterative schemes. As an application, we used our main results to establish the existence of solution of a mixed-type nonlinear integral equation. Finally, we illustrated the result in our application with an interesting example.

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Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AEO analyzed the results in the literature and made the original draft preparation. HI carried out the formal analysis, writing review and editing, and project administration. FA performed the validation and formal analysis. JA conducted the formal analysis, writing review and editing. All the authors read and approved the final manuscript.

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