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A weighted mean Hausdorff type operator and its summability matrix domain

Merve İlkhan Kara[1] and Hadi Roopaei[2*]

*Correspondence:
h.roopaei@gmail.com
[2]Department of Mathematics and Statistics, University of Victoria, Victoria, Canada
Full list of author information is available at the end of the article

Abstract

In this research, we firstly introduce a criterion for factorizing an operator based on the gamma operator, and as a result we present two factorizations for the Cesàro and Hilbert operators. As another point of view, we study some properties of the matrix domain associated with the gamma matrix of order n and compute the duals of this space. Moreover, we compute the norm of well-known operators into, from, and on the gamma sequence space.

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Keywords: Gamma matrix; Cesàro matrix; Hilbert matrix; Hausdorff matrix; Sequence space

1 Introduction

By ω we denote the set of all real-valued sequences, and a sequence space is a subspace of ω. The following sets are some examples for the sequence spaces.

$$\ell_p = \left\{ x = (x_k) \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty \right\} \quad (1 \leq p < \infty),$$

$$c = \left\{ x = (x_k) \in \omega : \lim_{k\to\infty} x_k \text{ exists} \right\},$$

$$\ell_\infty = \left\{ x = (x_k) \in \omega : \sup_k |x_k| < \infty \right\},$$

$$cs = \left\{ x = (x_k) \in \omega : \left(\sum_{k=0}^{n} x_k \right) \in c \right\},$$

$$bs = \left\{ x = (x_k) \in \omega : \left(\sum_{k=0}^{n} x_k \right) \in \ell_\infty \right\}.$$

In this paper, the supremum is taken over all $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. Also, we use the notion $\mathbb{N} = \{1, 2, 3, \dots\}$.

For two sequence spaces X, Y and an infinite matrix $A = (a_{j,k})$, we define a matrix transformation from X into Y as $Ax = ((Ax)_j) = (\sum_{k=0}^{\infty} a_{j,k}x_k)$ provided that the series is convergent for each $j \in \mathbb{N}_0$. The class of all infinite matrices from X into Y is denoted by (X, Y).

The matrix domain of an infinite matrix A in a sequence space X is defined as

$$X_A = \{x \in \omega : Ax \in X\}$$

which is also a sequence space.

In the literature, there are many papers related to new sequence spaces constructed by means of the matrix domain of a special triangle. In order to give full knowledge on the domains of triangular matrices in classical sequence spaces and related topics, the articles [1–11] and the monographs [12] and [13] are recommended.

Let X and Y be subsets of ω. The set $M(X,Y) = \{a \in \omega : ax \in Y \text{ for all } x \in X\}$ is called the multiplier space of X and Y. In the special cases $Y = \ell_1$, $Y = cs$, and $Y = bs$, the multiplier spaces $X^\alpha = M(X,\ell_1)$, $X^\beta = M(X,cs)$, and $X^\gamma = M(X,bs)$ are called the α-, β-, and γ- *duals* of the space X, respectively, that is,

$$X^\alpha = \left\{a = (a_k) \in \omega : \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for all } x = (x_k) \in X\right\},$$

$$X^\beta = \left\{a = (a_k) \in \omega : \left(\sum_{k=1}^{n} a_k x_k\right) \in c \text{ for all } x = (x_k) \in X\right\},$$

$$X^\gamma = \left\{a = (a_k) \in \omega : \left(\sum_{k=1}^{n} a_k x_k\right) \in \ell_\infty \text{ for all } x = (x_k) \in X\right\}.$$

Hausdorff matrices. Consider the Hausdorff matrix $H^\mu = (h_{j,k})_{j,k=0}^{\infty}$, with entries of the form

$$h_{j,k} = \begin{cases} \binom{j}{k} \int_0^1 \theta^k (1-\theta)^{j-k}\, d\mu(\theta), & j \geq k, \\ 0, & j < k, \end{cases}$$

where μ is a probability measure on $[0,1]$. The Hausdorff matrix contains the famous classes of matrices. For positive integer n, these classes are as follows:

- The choice $d\mu(\theta) = n(1-\theta)^{n-1}\, d\theta$ gives the Cesàro matrix of order n,
- The choice $d\mu(\theta) = n\theta^{n-1}\, d\theta$ gives the gamma matrix of order n,
- The choice $d\mu(\theta) = \frac{|\log\theta|^{n-1}}{\Gamma(n)}\, d\theta$ gives the Hölder matrix of order n,
- The choice $d\mu(\theta) =$ *point evaluation at* $\theta = n$ gives the Euler matrix of order n.

Hardy's formula ([14], Theorem 216) states that the Hausdorff matrix is a bounded operator on ℓ_p if and only if $\int_0^1 \theta^{\frac{-1}{p}}\, d\mu(\theta) < \infty$ and

$$\|H^\mu\|_{\ell_p \to \ell_p} = \int_0^1 \theta^{\frac{-1}{p}}\, d\mu(\theta). \tag{1.1}$$

Cesàro matrix of order n. The measure $d\mu(\theta) = n(1-\theta)^{n-1}\, d\theta$ gives the Cesàro matrix $\mathcal{C}^n = (c^n_{j,k})$ of order n, which is defined by

$$\mathcal{C}^n_{j,k} = \begin{cases} \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}}, & 0 \leq k \leq j, \\ 0, & \text{otherwise}. \end{cases}$$

Hence, according to Hardy's formula, $\mathcal{C}^n$ has the ℓ_p-norm

$$\left\|\mathcal{C}^n\right\|_{\ell_p\to\ell_p}=\frac{\Gamma(n+1)\Gamma(1/p^*)}{\Gamma(n+1/p^*)},\tag{1.2}$$

where p^* is the conjugate of p i.e. $\frac{1}{p}+\frac{1}{p^*}=1$.

Note that $\mathcal{C}^0=\mathcal{I}$, where $\mathcal{I}$ is the identity matrix, and

$$\mathcal{C}^1=\mathcal{C}=\begin{pmatrix}1 & 0 & 0 & \cdots\\ 1/2 & 1/2 & 0 & \cdots\\ 1/3 & 1/3 & 1/3 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}$$

is the classical Cesàro matrix which has the ℓ_p-norm $\|\mathcal{C}\|_{\ell_p\to\ell_p}=\frac{p}{p-1}$.

Gamma matrix of order n. By letting $d\mu(\theta)=n\theta^{n-1}\,d\theta$ in the definition of the Hausdorff matrix, the gamma matrix $\Gamma^n=(\gamma^n_{j,k})$ of order n is given by

$$\gamma^n_{j,k}=\begin{cases}\frac{\binom{n+k-1}{k}}{\binom{n+j}{j}}, & 0\le k\le j,\\ 0, & \text{otherwise},\end{cases}$$

which according to relation (1.1) has the ℓ_p-norm

$$\left\|\Gamma^n\right\|_{\ell_p\to\ell_p}=\frac{np}{np-1}.\tag{1.3}$$

While Γ^1 is the classical Cesàro matrix $\mathcal{C}$, other values of n give birth to different matrices. For example,

$$\mathcal{C}^2=\begin{pmatrix}1 & 0 & 0 & \cdots\\ 2/3 & 1/3 & 0 & \cdots\\ 3/6 & 2/6 & 1/6 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix},\qquad \Gamma^2=\begin{pmatrix}1 & 0 & 0 & \cdots\\ 1/3 & 2/3 & 0 & \cdots\\ 1/6 & 2/6 & 3/6 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{pmatrix}.\tag{1.4}$$

In this study, we firstly introduce a criterion for factorizing an operator based on the gamma operator. Secondly we investigate the matrix domain of the gamma matrix of order n in the spaces ℓ_p $(1\le p<\infty)$ and ℓ_∞ and compute the duals of the resulting spaces.

We shall deal with the spaces ℓ_p of absolutely p-summable real sequences with $1<p<\infty$ endowed with the norm

$$\|x\|_{\ell_p}=\left(\sum_{k=0}^{\infty}|x_k|^p\right)^{\frac{1}{p}}<\infty.$$

Motivation. Let us recall the definition of Nörlund and weighted mean matrices. Suppose that $a=(a_j)_{j=0}^{\infty}$ is a nonnegative sequence with $a_0>0$ and $A_j=a_0+a_1+\cdots+a_j$. The Nörlund and weighted mean matrices $N_a = (a_{j,k})$ and $M_a = (a_{j,k})$ are lower triangular matrices which are defined by

$$a_{j,k} = \begin{cases} \frac{a_{j-k}}{A_j}, & 0 \leq k \leq j, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad a_{j,k} = \begin{cases} \frac{a_k}{A_j}, & 0 \leq k \leq j, \\ 0, & \text{otherwise,} \end{cases}$$

respectively. The sequence $(a_j)_{j=0}^{\infty}$ is called the "symbol" of Nörlund and weighted mean matrices.

It must be mentioned that the Cesàro and gamma matrices of order n are the Nörlund and weighted mean matrices with symbol $a_j^n = \binom{n+j-1}{j}$. For example, for $n = 1$ and $n = 2$, the sequences $a_j^1 = 1$ and $a_j^2 = 1 + j$ are the symbols of the Cesàro and gamma matrices of order 1 (the well-known Cesàro matrix) and the Cesàro and gamma matrices of order 2, with the entries presented in (1.4). As we mentioned in the introduction, both these matrices are also in the classes of Hausdorff matrices with different choosing of their probability measures.

From both aspects of being a Hausdorff operator or a weighted mean matrix, the gamma operator has not been seen as it deserves. Many mathematicians have done and still publish numerous articles about the Cesàro matrix, Cesàro matrix domain, and Cesàro function spaces [15–19], while the importance of the gamma operator or its associated matrix domain has been ignored under the shadow of its rival Cesàro matrix. In this present study, the authors try to reveal some brilliant characteristics of this operator as an independent matrix.

2 Factorization based on gamma operator

In this section, we introduce the necessary and sufficient conditions for factorizing an operator based on the gamma matrix, then as the result we obtain two factorizations for the Cesàro and Hilbert matrices.

Theorem 2.1 *Let $T = (t_{j,k})_{j,k=0}^{\infty}$ be a matrix and $\Gamma^n = (\gamma_{j,k}^n)$ be the gamma matrix of order n. There exists a factor $\mathcal{S}^n = (s_{j,k}^n)_{j,k=0}^{\infty}$ such that $T = \mathcal{S}^n\Gamma^n$ if and only if*

(i) $\frac{t_{j,k}}{\binom{n+k-1}{k}} \to 0$ *as* $k \to \infty$ $\forall j \geq 0$,

(ii) $s_{j,k}^n = \binom{n+k}{k}\{\frac{t_{j,k}}{\binom{n+k-1}{k}} - \frac{t_{j,k+1}}{\binom{n+k}{k+1}}\}$ $\forall j, k \geq 0$.

Proof Let $j, k \geq 0$. Since $T = \mathcal{S}^n\Gamma^n$, we have that

$$t_{j,k} = \sum_{i=0}^{\infty} s_{j,i}^n \gamma_{i,k}^n = \sum_{i=k}^{\infty} s_{j,i}^n \frac{\binom{n+k-1}{k}}{\binom{n+i}{i}},$$

hence

$$\frac{t_{j,k}}{\binom{n+k-1}{k}} = \sum_{i=k}^{\infty} s_{j,i}^n \frac{1}{\binom{n+i}{i}} \to 0$$

as $k \to \infty$, which gives (i). Also $\{\frac{t_{j,k}}{\binom{n+k-1}{k}} - \frac{t_{j,k+1}}{\binom{n+k}{k+1}}\} = s_{j,k}^n \frac{1}{\binom{n+k}{k}}$, which gives (ii).

Now, let $j,k \geq 0$, $N \geq k$. Then we have that

$$\begin{aligned}
\sum_{i=0}^{N} s_{j,i}^{n}\gamma_{i,k}^{n} &= \sum_{i=k}^{N} s_{j,i}^{n}\frac{\binom{n+k-1}{k}}{\binom{n+i}{i}} \\
&= \binom{n+k-1}{k}\sum_{i=k}^{N}\left\{\frac{t_{j,i}}{\binom{n+i-1}{i}} - \frac{t_{j,i+1}}{\binom{n+i}{i+1}}\right\} \quad \text{(by (ii))} \\
&= \binom{n+k-1}{k}\left\{\frac{t_{j,k}}{\binom{n+k-1}{k}} - \frac{t_{j,N+1}}{\binom{n+N}{N+1}}\right\} \\
&\to \binom{n+k-1}{k}\frac{t_{j,k}}{\binom{n+k-1}{k}} = t_{j,k} \quad (\text{as } N\to\infty \text{ by (i)}).
\end{aligned}$$

This implies that $\mathcal{S}^n\Gamma^n$ exists and $T = \mathcal{S}^n\Gamma^n$. □

Recall the definition of the Hilbert matrix $\mathcal{H} = (h_{j,k}) = (\frac{1}{j+k+1})$ of nonnegative integers j,k. The Hilbert operator has the matrix representation

$$\mathcal{H} = \begin{pmatrix} 1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which is a bounded operator on ℓ_p and $\|\mathcal{H}\|_{\ell_p} = \pi\csc(\pi/p)$, ([20], Theorem 323). Bennett in [21] showed that the Hilbert matrix admits a factorization of the form $\mathcal{H} = \mathcal{B}\mathcal{C}$, where $\mathcal{C}$ is the Cesàro matrix and $\mathcal{B} = (b_{j,k})$ is defined by

$$b_{j,k} = \frac{k+1}{(j+k+1)(j+k+2)} \quad (j,k = 0,1,\ldots). \tag{2.1}$$

The matrix $\mathcal{B}$ is a bounded operator on ℓ_p and $\|\mathcal{B}\|_{\ell_p} = \frac{\pi}{p^*}\csc(\pi/p)$.

For obtaining our first result, we need the Hellinger–Toeplitz theorem.

Theorem 2.2 ([15], Proposition 7.2) *Suppose that $1 < p, q < \infty$. A matrix A maps ℓ_p into ℓ_q if and only if the transposed matrix A^t maps ℓ_{q^*} into ℓ_{p^*}. We then have $\|A\|_{\ell_p\to\ell_q} = \|A^t\|_{\ell_{q^*}\to\ell_{p^*}}$.*

Corollary 2.3 *The Hilbert matrix has a factorization of the form $\mathcal{H} = \mathcal{S}^n\Gamma^n$, where $\mathcal{S}^n = (s_{j,k}^n)$ with the entries*

$$s_{j,k}^{n} = \frac{(1-1/n)(j+1)+(k+1)}{(j+k+1)(j+k+2)} \quad (j,k = 0,1,\ldots)$$

is a bounded operator on ℓ_p and $\|\mathcal{S}^n\|_{\ell_p\to\ell_p} = \pi(1-\frac{1}{np})\csc(\pi/p)$.

In particular, for $n = 1$, $\mathcal{H} = \mathcal{B}\mathcal{C}$, where $\mathcal{C}$ is the Cesàro matrix and $\mathcal{B}$ is the matrix defined by relation (2.1).

Proof Let $\mathcal{H} = (h_{j,k})$ be the Hilbert matrix. Since

$$\frac{h_{j,k}}{\binom{n+k-1}{k}} = \frac{1}{(j+k+1)\binom{n+k-1}{k}} \to 0 \quad \text{as } k \to \infty,$$

therefore $\mathcal{H}$ has a factorization $\mathcal{H} = \mathcal{S}^n\Gamma^n$, where the factor $\mathcal{S}^n = (s^n_{j,k})$ in Theorem 2.1 is

$$\begin{aligned} s^n_{j,k} &= \binom{n+k}{k}\left\{\frac{1}{(j+k+1)\binom{n+k-1}{k}} - \frac{1}{(j+k+2)\binom{n+k}{k+1}}\right\} \\ &= \frac{(1-1/n)(j+1)+(k+1)}{(j+k+1)(j+k+2)} \quad (j,k = 0,1,\ldots). \end{aligned}$$

Now, our factorization and relation (1.3) result in

$$\|\mathcal{S}^n\|_{\ell_p\to\ell_p} \geq \pi\left(1-\frac{1}{np}\right)\csc(\pi/p).$$

For proving the other side of the above inequality, by using the definition of the matrix $\mathcal{B}$, as in relation (2.1), we can rewrite

$$\mathcal{S}^n = \left(1-\frac{1}{n}\right)\mathcal{B}^t + \mathcal{B}.$$

Hence, by applying the Hellinger–Toeplitz theorem, we have

$$\|\mathcal{S}^n\|_{\ell_p\to\ell_p} \leq \left(1-\frac{1}{n}\right)\|\mathcal{B}^t\|_{\ell_p\to\ell_p} + \|\mathcal{B}\|_{\ell_p\to\ell_p} = \pi\left(1-\frac{1}{np}\right)\csc(\pi/p),$$

which completes the proof. In a special case $n=1$, $\mathcal{S}^1 = \mathcal{B}$, $\|\mathcal{S}^1\|_{\ell_p\to\ell_p} = \|\mathcal{B}\|_{\ell_p\to\ell_p} = \frac{\pi}{p^*}\csc(\pi/p)$, $\Gamma^1 = \mathcal{C}$ and the Hilbert matrix has the factorization $\mathcal{H} = \mathcal{B}\mathcal{C}$. □

Corollary 2.4 *The Cesàro matrix of order n has a factorization of the form*

$$\mathcal{C}^n = \mathcal{C}^{n-1}\Gamma^n = \Gamma^n\mathcal{C}^{n-1}, \tag{2.2}$$

where $\mathcal{C}^{n-1}$ is the Cesàro matrix of order $n-1$.

Proof In this case, for $t^n_{j,k} = c^n_{j,k} = \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}}$, we have

$$\frac{c^n_{j,k}}{\binom{n+k-1}{k}} = \frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}\binom{n+k-1}{k}} \to 0 \quad \text{as } k\to\infty.$$

Hence $\mathcal{C}^n$ has the factorization of the form $\mathcal{C}^n = \mathcal{S}^n\Gamma^n$. Now, the factor $\mathcal{S}^n = (\mathcal{S}^n_{j,k})$ in Theorem 2.1 is

$$\begin{aligned} s^n_{j,k} &= \binom{n+k}{k}\left\{\frac{\binom{n+j-k-1}{j-k}}{\binom{n+j}{j}\binom{n+k-1}{k}} - \frac{\binom{n+j-k-2}{j-k-1}}{\binom{n+j}{j}\binom{n+k}{k+1}}\right\} \\ &= \frac{\binom{n+j-k-2}{j-k}}{\binom{n+j-1}{j}} = \mathcal{C}^{n-1}_{j,k}. \end{aligned}$$

Hence we have proved $\mathcal{C}^n = \mathcal{C}^{n-1}\Gamma^n$. But, since every two Hausdorff matrices commute ([14], Theorem 197), we have the desired result. □

Roopaei in [22, 23] introduced more factorizations for the Hilbert and Cesàro operators based on gamma matrices of order n.

3 Matrix domain of gamma matrix in ℓ_p

The sequence space associated with Γ^n is the set $\{x = (x_k) \in \omega : \Gamma^n x \in \ell_p\}$. That is,

$$G^n(p) = \left\{ x = (x_k) \in \omega : \sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right|^p < \infty \right\},$$

which is called the gamma space of order n. In a special case $n = 1$, we show the gamma sequence space $G^1(p)$ by the notation $\mathrm{ces}(p)$.

Also, we define the following sequence space $G^n(\infty)$:

$$G^n(\infty) = \left\{ x = (x_k) \in \omega : \sup_j \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right| < \infty \right\}.$$

In the study, by $y = (y_j)$, we mean the Γ^n-transform of a sequence $x = (x_j)$, that is,

$$y_j = \left(\Gamma^n x\right)_j = \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \tag{3.1}$$

for all $j \in \mathbb{N}_0$.

The gamma matrix of order n, Γ^n, is invertible and its inverse $\Gamma^{-n} = (\gamma_{j,k}^{-n})$ is defined by

$$\gamma_{j,k}^{-n} = \begin{cases} 1 + \frac{j}{n}, & k = j, \\ -\frac{j}{n}, & k = j-1, \\ 0, & \text{otherwise}. \end{cases}$$

Theorem 3.1 *The spaces $G^n(p)$ and $G^n(\infty)$ are Banach spaces with the norms*

$$\|x\|_{G^n(p)} = \left(\sum_{j=0}^{\infty} \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right|^p \right)^{\frac{1}{p}}$$

and

$$\|x\|_{G^n(\infty)} = \sup_j \left| \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \right|,$$

respectively.

Proof We omit the proof which is a routine verification. □

Remark 3.2 If $n = 1$, the Cesàro sequence spaces $\mathrm{ces}(p)$ and $\mathrm{ces}(\infty)$ defined in [24] are obtained. $\mathrm{ces}(p)$ and $\mathrm{ces}(\infty)$ are Banach spaces of nonabsolute type with the norm

$$\|x\|_{\mathrm{ces}(p)} = \left(\sum_{j=0}^{\infty}\left|\frac{1}{j+1}\sum_{k=0}^{j}x_k\right|^p\right)^{1/p} \quad (1 \le p < \infty)$$

and

$$\|x\|_{\mathrm{ces}(\infty)} = \sup_j\left|\frac{1}{j+1}\sum_{k=0}^{j}x_k\right|,$$

respectively.

Theorem 3.3 *The spaces $G^n(p)$ and $G^n(\infty)$ are linearly isomorphic to ℓ_p and ℓ_∞, respectively.*

Proof We only prove the first one and the other one can be proved in a similar way. Since Γ^n is invertible, the map $x \mapsto \Gamma^n x$ defines a bijection between $G^n(p)$ and ℓ_p. Also, since $\|x\|_{G^n(p)} = \|\Gamma^n x\|_{\ell_p}$ holds, the defined map preserves the norm, which completes the proof. □

Remark 3.4 If $n = 1$, the Cesàro sequence spaces $\mathrm{ces}(p)$ and $\mathrm{ces}(\infty)$ are linearly isomorphic to ℓ_p and ℓ_∞.

Remark 3.5 The space Γ_2^n is an inner product space with the inner product defined as $\langle x, \tilde{x}\rangle_{\Gamma_2^n} = \langle \Gamma^n x, \Gamma^n \tilde{x}\rangle_2$, where $\langle \cdot, \cdot\rangle_2$ is the inner product on ℓ_2.

Theorem 3.6 *The space $G^n(p)$ is not an inner product space for $p \neq 2$. Then the space $G^n(p)$ is not a Hilbert space for $p \neq 2$.*

Proof Consider the sequences $x = (1, 1, -\frac{2}{n}, 0, 0, \ldots)$ and $\tilde{x} = (1, -1 - \frac{2}{n}, \frac{2}{n}, 0, 0, \ldots)$. We observe that $\Gamma^n x = (1, 1, 0, \ldots, 0, \ldots) \in \ell_p$ and $\Gamma^n \tilde{x} = (1, -1, 0, \ldots, 0, \ldots) \in \ell_p$. Hence, we obtain that

$$\|x + \tilde{x}\|_{G^n(p)}^2 + \|x - \tilde{x}\|_{G^n(p)}^2 = 8 \neq 4.2^{2/p} = 2\left[\|x\|_{G^n(p)}^2 + \|\tilde{x}\|_{G^n(p)}^2\right]$$

for $p \neq 2$. We conclude that the space $G^n(p)$ is not an inner product space except for $p = 2$, and so it is not a Hilbert space since the norm does not satisfy the parallelogram equality. □

Remark 3.7 If $n = 1$, the Cesàro sequence space $\mathrm{ces}(p)$ is not a Hilbert space for $p \neq 2$.

Theorem 3.8 *The inclusion $G^n(p) \subset G^n(q)$ strictly holds, where $1 \le p < q < \infty$.*

Proof Given any $x \in G^n(p)$, we have $\Gamma^n x \in \ell_p$. Since the inclusion $\ell_p \subset \ell_q$ holds for $1 \le p < q < \infty$, we have $\Gamma^n x \in \ell_q$. This implies that $x \in G^n(q)$. Hence, we conclude that the inclusion $G^n(p) \subset G^n(q)$ holds.

Also, since the inclusion $\ell_p \subset \ell_q$ is strict, we can choose $y = (y_j) \in \ell_q \backslash \ell_p$. If we define a sequence $x = (x_j)$ as

$$x_j = \left(1 + \frac{j}{n}\right) y_j - \frac{k}{n} y_{j-1} \quad (j \in \mathbb{N}_0),$$

then we have

$$\left(\Gamma^n x\right)_j = y_j$$

for every $j \in \mathbb{N}_0$. This means $\Gamma^n x = y$, and so $\Gamma^n x \in \ell_q \backslash \ell_p$. Since we have $x \in G^n(q) \backslash G^n(p)$, we conclude that the inclusion $G^n(p) \subset G^n(q)$ strictly holds. □

Remark 3.9 If $n = 1$, then the inclusion $\operatorname{ces}(p) \subset \operatorname{ces}(q)$ is strict, where $1 \leq p < q < \infty$.

Theorem 3.10 *The inclusion* $G^n(p) \subset G^n(\infty)$ *strictly holds, where* $1 \leq p < \infty$.

Proof Choose any $x \in G^n(p)$. Then we have $\Gamma^n x \in \ell_p$. Since the inclusion $\ell_p \subset \ell_\infty$ holds for $1 \leq p < \infty$, we have $\Gamma^n x \in \ell_\infty$. This implies that $x \in G^n(\infty)$. Hence, we conclude that the inclusion $G^n(p) \subset G^n(\infty)$ holds. Consider the sequence $x = (x_j)$ with

$$x_j = (-1)^j \left(\frac{2j}{n} + 1\right) \quad (j \in \mathbb{N}_0).$$

It follows that

$$\begin{aligned}
\left(\Gamma^n x\right)_j &= \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} x_k \\
&= \frac{1}{\binom{n+j}{j}} \sum_{k=0}^{j} \binom{n+k-1}{k} (-1)^k \left(\frac{2k}{n} + 1\right) \\
&= (-1)^j.
\end{aligned}$$

Since $\Gamma^n x = ((-1)^j) \in \ell_\infty \backslash \ell_p$ holds, we have $x \in G^n(\infty) \backslash G^n(p)$. This means that the inclusion $G^n(p) \subset G^n(\infty)$ is strict. □

Remark 3.11 If $n = 1$, then we obtain that the inclusion $\operatorname{ces}(p) \subset \operatorname{ces}(\infty)$ is strict, where $1 \leq p < \infty$.

Theorem 3.12 *The space* $G^n(p)$ $(1 \leq p < \infty)$ *has a basis* (c^k) *defined as*

$$c_j^k = \begin{cases} 1 + \frac{k}{n}, & j = k, \\ -\frac{k+1}{n}, & j = k+1, \\ 0, & \text{otherwise} \end{cases} \quad (j, k \in \mathbb{N}_0). \tag{3.2}$$

Further, every $x \in G^n(p)$ *is written in the form*

$$x = \sum_k \left(\Gamma^n x\right)_k c^k \tag{3.3}$$

uniquely.

Proof From (3.2), we have that $\Gamma^n(c^k) = e^k \in \ell_p$, and so $c^k \in G^n(p)$ for each $k \in \mathbb{N}$. Now, let $x \in G^n(p)$. For every $m \in \mathbb{N}_0$, write

$$x^m = \sum_{k=0}^{m} (\Gamma^n x)_k c^k.$$

Then we have that

$$\Gamma^n(x^m) = \sum_{k=0}^{m} (\Gamma^n x)_k \Gamma^n(c^k) = \sum_{k=0}^{m} (\Gamma^n x)_k e^k,$$

and it follows that

$$(\Gamma^n(x - x^m))_j = \begin{cases} 0, & 0 \leq j \leq m, \\ (\Gamma^n x)_j, & j > m. \end{cases}$$

Given any $\varepsilon > 0$, there exists $M \in \mathbb{N}_0$ such that

$$\sum_{j=M+1}^{\infty} |(\Gamma^n x)_j|^p \leq \left(\frac{\varepsilon}{2}\right)^p.$$

We have that

$$\|x - x^m\|_{G^n(p)} = \left(\sum_{j=m+1}^{\infty} |(\Gamma^n x)_j|^p\right)^{1/p} \leq \left(\sum_{j=M+1}^{\infty} |(\Gamma^n x)_j|^p\right)^{1/p} \leq \frac{\varepsilon}{2} < \epsilon$$

for every $m \geq M$, which implies that $\lim_{m\to\infty} \|x - x^m\|_{G^n(p)} = 0$. This implies that $x = \sum_k (\Gamma^n x)_k c^k$.

Finally, we show that the representation (3.3) of $x \in G^n(p)$ is unique. On the contrary, we suppose that $x = \sum_k (\Lambda x)_k c^k$. By using the continuity of the mapping $L : G^n(p) \to \ell_p$ defined in the proof of Theorem 3.3, we have

$$(\Gamma^n x)_j = \sum_k (\Lambda x)_k (\Gamma^n c^k)_j = \sum_k (\Lambda x)_k \epsilon_{j,k} = (\Lambda x)_j \quad (j \in \mathbb{N}_0),$$

where

$$\epsilon_{j,k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Hence, the representation (3.3) of $x \in G^n(p)$ is unique, which completes the proof. □

The following lemma is essential to determine the dual spaces. Throughout the paper, $\mathcal{N}$ is the collection of all finite subsets of $\mathbb{N}$. Firstly, we list some conditions:

$$\sup_k \sum_{j=0}^{\infty} |a_{j,k}| < \infty, \tag{3.4}$$

$$\sum_{k=0}^{\infty}\left(\sum_{j=0}^{\infty}|a_{j,k}|\right)^{p*}<\infty, \tag{3.5}$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}|a_{j,k}|<\infty, \tag{3.6}$$

$$\lim_{j\to\infty} a_{j,k} \text{ exists for each } k\in\mathbb{N}, \tag{3.7}$$

$$\sup_{j,k}|a_{j,k}|<\infty, \tag{3.8}$$

$$\sup_{j}\sum_{k=0}^{\infty}|a_{j,k}|^{p*}<\infty, \tag{3.9}$$

$$\lim_{j\to\infty}\sum_{k=0}^{\infty}|a_{j,k}|=\sum_{k=0}^{\infty}\left|\lim_{j\to\infty}a_{j,k}\right|, \tag{3.10}$$

$$\sup_{j}\sum_{k=0}^{\infty}|a_{j,k}|<\infty. \tag{3.11}$$

Lemma 3.13 ([25]) *The necessary and sufficient conditions for $A=(a_{j,k})\in(X,Y)$ with $X\in\{\ell_1,\ell_p,\ell_\infty\}$ $(1<p<\infty)$ and $Y\in\{\ell_1,c,\ell_\infty\}$ are given in Table 1.*

1. *(3.4) holds.*
2. *(3.5) holds.*
3. *(3.6) holds.*
4. *(3.7) and (3.8) hold.*
5. *(3.7) and (3.9) hold.*
6. *(3.7) and (3.10) hold.*
7. *(3.8) holds.*
8. *(3.9) holds.*
9. *(3.11) holds.*

Theorem 3.14 *The α-dual, β-dual, and γ-dual of the space $G^n(1)$ are as follows:*

$$\begin{aligned}
\left(G^n(1)\right)^{\alpha} &= \left\{a=(a_k)\in\omega:\sup_k\left\{\left(1+\frac{k}{n}\right)|a_k|+\frac{k+1}{n}|a_{k+1}|\right\}<\infty\right\},\\
\left(G^n(1)\right)^{\beta} &= \left\{a=(a_k)\in\omega:\lim_{j\to\infty}\left[\left(1+\frac{k}{n}\right)a_k-\frac{k+1}{n}a_{k+1}\right]\ \textit{exists for each } k\in\mathbb{N}_0\right\}\\
&\quad\cap\left\{a=(a_k)\in\omega:\sup_k\left\{\left(1+\frac{k}{n}\right)a_k-\frac{k+1}{n}a_{k+1},\left(1+\frac{k}{n}\right)a_k\right\}<\infty\right\},\\
\left(G^n(1)\right)^{\gamma} &= \left\{a=(a_k)\in\omega:\sup_k\left\{\left(1+\frac{k}{n}\right)a_k-\frac{k+1}{n}a_{k+1},\left(1+\frac{k}{n}\right)a_k\right\}<\infty\right\}.
\end{aligned}$$

Table 1 The characterization of the class (X,Y)

From\To	ℓ_1	c	ℓ_∞
ℓ_1	**1.**	**4.**	**7.**
ℓ_p	**2.**	**5.**	**8.**
ℓ_∞	**3.**	**6.**	**9.**

Proof Firstly, we compute the α-dual. Given any $a = (a_k) \in \omega$, we define a matrix $B = (b_{j,k})$ as

$$b_{j,k} = \begin{cases} (1 + \frac{j}{n})a_j, & k = j, \\ -\frac{j}{n}a_j, & k = j - 1, \\ 0, & \text{otherwise.} \end{cases}$$

If we choose any $x = (x_j) \in G^n(1)$, we obtain that

$$a_j x_j = \left(1 + \frac{j}{n}\right)a_j y_j - \frac{j}{n}a_j y_{j-1} = (By)_j$$

for all $j \in \mathbb{N}$. Hence, it follows that $ax \in \ell_1$ for $x \in G^n(1)$ if and only if $By \in \ell_1$ for $y \in \ell_1$. Thus, we have $a \in (G^n(1))^\alpha$ if and only if $B \in (\ell_1, \ell_1)$. By using Lemma 3.13, we conclude that

$$\sup_k \sum_{j=0}^{\infty} |b_{j,k}| = \sup_k \left\{ \left(1 + \frac{k}{n}\right)|a_k| + \frac{k+1}{n}|a_{k+1}| \right\} < \infty$$

holds.

Now, we compute the β-dual. $a = (a_k) \in (G^n(1))^\beta$ if and only if the series $\sum_{k=0}^{\infty} a_k x_k$ is convergent for all $x = (x_k) \in G^n(1)$. Then, from the equality

$$\begin{aligned} \sum_{k=0}^{j} a_k x_k &= \sum_{k=0}^{j} a_k \left[\left(1 + \frac{k}{n}\right) y_k - \frac{k}{n} y_{k-1} \right] \\ &= \left[\sum_{k=0}^{j-1} \left(\left(1 + \frac{k}{n}\right) a_k - \frac{k+1}{n} a_{k+1} \right) y_k \right] + \left(1 + \frac{j}{n}\right) a_j y_j \\ &= \sum_{k=0}^{j} c_{jk} y_k, \end{aligned} \tag{3.12}$$

we have $a = (a_k) \in (G^n(1))^\beta$ if and only if the matrix $C = (c_{j,k})$ is in the class (ℓ_1, c), where

$$c_{j,k} = \begin{cases} (1 + \frac{k}{n})a_k - \frac{k+1}{n}a_{k+1}, & 0 \le k \le j - 1, \\ (1 + \frac{k}{n})a_k, & k = j, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, we conclude from Lemma 3.13 that

$$\lim_{j\to\infty} c_{j,k} = \lim_{j\to\infty} \left[\left(1 + \frac{k}{n}\right) a_k - \frac{k+1}{n} a_{k+1} \right]$$

exists and

$$\sup_{j,k} |c_{j,k}| = \sup_k \left| \left(1 + \frac{k}{n}\right) a_k - \frac{k+1}{n} a_{k+1}, \left(1 + \frac{k}{n}\right) a_k \right| < \infty$$

holds.

Finally, the γ-dual can be computed by utilizing the same technique with the β-dual. From equality (3.12), we deduce that $a=(a_k)\in(G^n(1))^\gamma$ if and only if the matrix $C=(c_{j,k})$ is in the class (ℓ_1,ℓ_∞). Hence, the proof follows from Lemma 3.13. □

Theorem 3.15 *The α-dual, β-dual, and γ-dual of the space $G^n(p)$ for $1<p<\infty$ are as follows*:

$$\bigl(G^n(p)\bigr)^\alpha=\left\{a=(a_k)\in\omega:\sum_{k=0}^{\infty}\left(\left(1+\frac{k}{n}\right)|a_k|+\frac{k+1}{n}|a_{k+1}|\right)^{p*}<\infty\right\},$$

$$\begin{aligned}&\bigl(G^n(p)\bigr)^\beta\\&\quad=\left\{a=(a_k)\in\omega:\lim_{j\to\infty}\left[\left(1+\frac{k}{n}\right)a_k-\frac{k+1}{n}a_{k+1}\right]\text{ exists for each }k\in\mathbb{N}_0\right\}\\&\qquad\cap\left\{a=(a_k)\in\omega:\sup_j\left\{\sum_{k=0}^{j-1}\left|\left(1+\frac{k}{n}\right)a_k-\frac{k+1}{n}a_{k+1}\right|^{p*}+\left|\left(1+\frac{j}{n}\right)a_j\right|^{p*}\right\}<\infty\right\},\\&\bigl(G^n(p)\bigr)^\gamma\\&\quad=\left\{a=(a_k)\in\omega:\sup_j\left\{\sum_{k=0}^{j-1}\left|\left(1+\frac{k}{n}\right)a_k-\frac{k+1}{n}a_{k+1}\right|^{p*}+\left|\left(1+\frac{j}{n}\right)a_j\right|^{p*}\right\}<\infty\right\}.\end{aligned}$$

Proof Consider the matrices $B=(b_{j,k})$ and $C=(c_{j,k})$ defined in the proof of Theorem 3.14. Let $1<p<\infty$. The α-, β-, and γ-dual of the space $G^n(p)$ can be computed from the facts that $B\in(\ell_p,\ell_1)$, $C\in(\ell_p,c)$, and $C\in(\ell_p,\ell_\infty)$, respectively. □

Theorem 3.16 *The α-dual, β-dual, and γ-dual of the space $G^n(\infty)$ are as follows*:

$$\begin{aligned}\bigl(G^n(\infty)\bigr)^\alpha&=\left\{a=(a_k)\in\omega:\sum_{k=0}^{\infty}\left(\left(1+\frac{k}{n}\right)|a_k|+\frac{k+1}{n}|a_{k+1}|\right)<\infty\right\},\\\bigl(G^n(\infty)\bigr)^\beta&=\left\{a=(a_k)\in\omega:\lim_{j\to\infty}\left[\left(1+\frac{k}{n}\right)a_k-\frac{k+1}{n}a_{k+1}\right]\text{ exists for each }k\in\mathbb{N}_0\right\}\\&\qquad\cap\left\{a=(a_k)\in\omega:\lim_{j\to\infty}\left\{\sum_{k=0}^{j-1}\left|\left(1+\frac{k}{n}\right)a_k-\frac{k+1}{n}a_{k+1}\right|+\left|\left(1+\frac{j}{n}\right)a_j\right|\right\}\right.\\&\qquad\left.=\sum_{k=0}^{\infty}\left|\left(1+\frac{k}{n}\right)a_k-\frac{k+1}{n}a_{k+1}\right|\right\},\\\bigl(G^n(\infty)\bigr)^\gamma&=\left\{a=(a_k)\in\omega:\sup_j\left\{\sum_{k=0}^{j-1}\left|\left(1+\frac{k}{n}\right)a_k-\frac{k+1}{n}a_{k+1}\right|+\left|\left(1+\frac{j}{n}\right)a_j\right|\right\}<\infty\right\}.\end{aligned}$$

Proof Consider the matrices $B=(b_{j,k})$ and $C=(c_{j,k})$ defined in the proof of Theorem 3.14. The α-, β-, and γ-dual of the space $G^n(\infty)$ can be computed from the facts that $B\in(\ell_\infty,\ell_1)$, $C\in(\ell_\infty,c)$, and $C\in(\ell_\infty,\ell_\infty)$, respectively. □

4 Norm of operators on a gamma sequence space

In this section, we investigate the norm of well-known operators from/into the gamma matrix domain, and we firstly start with the Hausdorff operators. For computing the norm of these operators on the gamma sequence space, we use the commutativity and the norm separating property of the Hausdorff matrices, which states the following.

Theorem 4.1 ([26], Theorem 9) *Let $p \geq 1$ and H^μ, H^ω, and H^ν be Hausdorff matrices such that $H^\mu = H^\omega H^\nu$. Then H^μ is bounded on ℓ_p if and only if both H^ω and H^ν are bounded on ℓ_p. Moreover, we have*

$$\|H^\mu\|_{\ell_p\to\ell_p} = \|H^\omega\|_{\ell_p\to\ell_p}\|H^\nu\|_{\ell_p\to\ell_p}.$$

The problem of finding the norm of operators on sequence spaces has been studied before in some articles [27–31].

Theorem 4.2 *Suppose that the Hausdorff matrix H^μ has a factorization of the form $H^\mu = H^\omega\Gamma^n$, then*

(a) *H^μ is a bounded operator from ℓ_p into $G^n(p)$ and*

$$\|H^\mu\|_{\ell_p\to G^n(p)} = \frac{np}{np-1}\|H^\mu\|_{\ell_p\to\ell_p}.$$

(b) *H^μ is a bounded operator from $G^n(p)$ into ℓ_p and*

$$\|H^\mu\|_{G^n(p)\to\ell_p} = \left(1-\frac{1}{np}\right)\int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta).$$

(c) *H^μ is a bounded operator on $G^n(p)$ and*

$$\|H^\mu\|_{G^n(p)\to G^n(p)} = \|H^\mu\|_{\ell_p\to\ell_p} = \int_0^1 \theta^{\frac{-1}{p}}\,d\mu(\theta).$$

Proof (a) Applying Theorem 4.1 and relation (1.1) results in

$$\begin{aligned}\|H^\mu\|_{\ell_p\to G^n(p)} &= \sup_{x\in\ell_p}\frac{\|H^\mu x\|_{G^n(p)}}{\|x\|_{\ell_p}} = \sup_{x\in\ell_p}\frac{\|\Gamma^n H^\mu x\|_{\ell_p}}{\|x\|_{\ell_p}} \\ &= \|\Gamma^n\|_{\ell_p\to\ell_p}\|H^\mu\|_{\ell_p\to\ell_p} = \frac{np}{np-1}\|H^\mu\|_{\ell_p\to\ell_p},\end{aligned}$$

which provides the proof.

(b) Let H^μ have a factorization of the form $H^\mu = H^\omega\Gamma^n$. Since $G^n(p)$ and ℓ_p are isomorphic spaces, applying Theorem 4.1 and relations (1.1) and (1.3) results in

$$\begin{aligned}\|H^\mu\|_{G^n(p)\to\ell_p} &= \sup_{x\in G^n(p)}\frac{\|H^\mu x\|_{\ell_p}}{\|x\|_{G^n(p)}} = \sup_{x\in\ell_p}\frac{\|H^\omega\Gamma^n x\|_{\ell_p}}{\|\Gamma^n x\|_{\ell_p}} \\ &= \|H^\omega\|_{\ell_p\to\ell_p} = \|H^\mu\|_{\ell_p\to\ell_p}/\|\Gamma^n\|_{\ell_p\to\ell_p} \\ &= \frac{np-1}{np}\|H^\mu\|_{\ell_p\to\ell_p},\end{aligned}$$

which provides the proof.

(c) Regarding the commutative property of Hausdorff matrices, the proof is similar. □

Theorem 4.3 *For the well-known Hilbert and Cesàro operators we have*:

(a) *The Hilbert operator $\mathcal{H}$ is a bounded operator from $G^n(p)$ into ℓ_p and*

$$\|\mathcal{H}\|_{G^n(p)\to\ell_p} = \pi\left(1 - \frac{1}{np}\right)\csc(\pi/p).$$

In particular, the Hilbert operator $\mathcal{H}$ is a bounded operator from $\mathrm{ces}(p)$ *into ℓ_p and* $\|\mathcal{H}\|_{\mathrm{ces}(p)\to\ell_p} = \frac{\pi}{p^*}\csc(\pi/p)$.

(b) *The Cesàro operator of order n, $\mathcal{C}^n$, is a bounded operator from $G^n(p)$ into ℓ_p and*

$$\left\|\mathcal{C}^n\right\|_{G^n(p)\to\ell_p} = \frac{\Gamma(n)\Gamma(1/p^*)}{\Gamma(n-1/p)}.$$

In particular, the Cesàro operator is a bounded operator from $\mathrm{ces}(p)$ *into ℓ_p and* $\|\mathcal{C}\|_{\mathrm{ces}(p)\to\ell_p} = 1$.

Proof (a) The Hilbert operator has a factorization of the from $\mathcal{H} = \mathcal{S}^n\Gamma^n$, which was introduced in Corollary 2.3. Now,

$$\begin{aligned}\|\mathcal{H}\|_{G^n(p)\to\ell_p} &= \sup_{x\in G^n(p)}\frac{\|\mathcal{H}x\|_{\ell_p}}{\|x\|_{G^n(p)}} = \sup_{x\in\ell_p}\frac{\|\mathcal{S}^n\Gamma^n x\|_{\ell_p}}{\|\Gamma^n x\|_{\ell_p}}\\ &= \left\|\mathcal{S}^n\right\|_{\ell_p\to\ell_p} = \pi\left(1 - \frac{1}{np}\right)\csc(\pi/p),\end{aligned}$$

which provides the proof.

(b) Applying Corollary 2.4, the proof is similar to the previous part. □

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Author details
[1]Department of Mathematics, Düzce University, Düzce, Turkey. [2]Department of Mathematics and Statistics, University of Victoria, Victoria, Canada.

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