Li *Journal of Inequalities and Applications* (2022) 2022:20
https://doi.org/10.1186/s13660-022-02756-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Stabilization of stochastic regime-switching Poisson jump equations by delay feedback control

Guangjie Li[1*]

*Correspondence:
scutliguangjie@163.com
[1]School of Mathematics and Statistics, Guangdong University of Foreign Studies, Guangzhou, China

Abstract

This paper is concerned with the stabilization of stochastic regime-switching Poisson jump equations (also known as stochastic differential equations with Markovian switching and Poisson jumps, abbreviated as SDEwMJs). The aim of this paper is to design a feedback controller with delay δ ($\delta > 0$) to make an unstable SDEwMJ become stable. It is proved that the delay δ is bounded by a constant $\bar{\delta}$. Moreover, an implicit lower bound for $\bar{\delta}$, which can be computed numerically, is provided. As a product, the almost sure exponential stability of the controlled SDEwMJ is obtained. Besides, an example is given to demonstrate the theoretical results.

Keywords: Almost sure exponential stability; Markov's chain; Poisson jumps; Delay feedback control

1 Introduction

Stochastic hybrid systems have been widely used to model such systems where they may experience abrupt changes in their structure and parameters (see [1]). One of the important types of stochastic hybrid systems is the SDEs with Markovian switching (SDEwMSs). In the study of the SDEwMSs, stability analysis has received a great deal of attention (see [2–7] and the references therein).

As we know, a Brownian motion is a continuous stochastic process. However, in real life, some systems may suffer from jump-type stochastic abrupt perturbations, such as financial crisis, earthquakes, and hurricanes. In these cases, employing the Brown motions to depict these systems cannot meet the needs of reality. At the same time, it is found that Poisson jumps can describe these jump-type phenomena. It is, therefore, reasonable to use Poisson jump processes to cope with such jump-type discontinuous systems (see, e.g., [8–13]). Results on the stability of stochastic systems with the Markovian switching and Poisson jumps have been obtained, see, e.g., [14, 15].

Given an unstable SDEwMJ

$$dx(t) = f\big(x(t), t, r(t)\big)\,dt + g\big(x(t), t, r(t)\big)\,d\omega(t) + h\big(x(t), t, r(t)\big)\,dN(t), \tag{1}$$

where $x(t) \in R^n$ is the state. The details of (1) can be found in (4) below. As we know, it is classical to design a feedback control $u(x(t), t, r(t))$ (based on the current state $x(t)$) to make the controlled system

$$\begin{aligned} dx(t) = {} & \bigl[f\bigl(x(t), t, r(t)\bigr) + u\bigl(x(t), t, r(t)\bigr)\bigr]\,dt + g\bigl(x(t), t, r(t)\bigr)\,d\omega(t) \\ & + h\bigl(x(t), t, r(t)\bigr)\,dN(t) \end{aligned} \tag{2}$$

stable. When $h \equiv 0$, (2) becomes an SDEwMS. Many results on the stabilization problem of SDEwMSs by state feedback controls without delay have been well studied (see [1, 16, 17] and the references therein). However, due to the fact that there exists a time lag δ between the time when the observation of the state is made and the time when the feedback control reaches the system; therefore, the feedback control should be designed depending on a past state $x(t-\delta)$ (see, e.g., [18–20]). Then the controlled system can be described by

$$\begin{aligned} dx(t) = {} & \bigl[f\bigl(x(t), t, r(t)\bigr) + u\bigl(x(t-\delta), t, r(t)\bigr)\bigr]\,dt + g\bigl(x(t), t, r(t)\bigr)\,d\omega(t) \\ & + h\bigl(x(t), t, r(t)\bigr)\,dN(t). \end{aligned} \tag{3}$$

Paper by Mao et al. [18] was the first one to design a delay feedback control in the drift part for an SDEwMS to make the unstable system become stable by using the linear matrix inequalities and Lyapunov functionals. Based on such a feedback control, this paper is concerned with the exponential mean-square stabilization of SDEwMSs. Fei et al. [21] focused on the stabilization problem for a class Markovian jump linear systems with time delay by the linear matrix inequalities. Along this line, Chen et al. [19, 20, 22] investigated the neutral stochastic delay differential equations with Markovian switching by the Lyapunov functions technique and the linear matrix inequalities. Hu et al. [23] investigated the stabilization of SDEwMSs by a new theorem. Authors in [24–26] successfully used the method of the Lyapunov functionals to study the stabilization by delay feedback controls of highly nonlinear neutral stochastic delay differential equations with the Markovian switching. Subsequently, Mei et al. [27] studied the exponential stabilization by delay feedback controls for highly nonlinear hybrid stochastic functional differential equations with infinite delay, which generalizes the results of [24–26].

It is a pity that the above-mentioned work did not discuss the stabilization by delay feedback controls for SDEwMJs. In reality, because of the movement of a Markov chain, a stochastic system including several regimes may switch from one to the others. Moreover, the increment of a Brownian motion is a zero mean, but for a Poisson jump, its increment is a nonzero mean. Therefore, due to the difficulty stemming from the presence of the Markovian switching and Poisson jump, more techniques are needed to study the stability of SDEwMJs. The aim of this paper is to investigate the exponential stabilization of SDEwMJs by designing a feedback control in the drift term with the delay δ. Moreover, it is not only illustrated that the δ is bounded by $\bar{\delta}$ but also presented an implicit lower bound for $\bar{\delta}$.

The remainder of this paper is arranged as follows: Sect. 2 introduces some preliminaries. Section 3 is devoted to presenting the main results. In Sect. 4, an illustrative example is provided to show the effectiveness of the obtained theory.

2 Preliminaries

Throughout this paper, unless otherwise specified, the following notations are used. Let $R = (-\infty, +\infty)$, $R^+ = [0, +\infty)$. R^n represents the n-dimensional Euclidean space, and $|x|$ denotes its norm of a vector $x \in R^n$. $(\Omega, \mathcal{F}, P)$ indicating a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. For $\delta > 0$, $C([-\delta, 0]; R^n)$ represents the family of all continuous R^n-valued functions φ defined on $[-\delta, 0]$. Denote by $C^b_{\mathcal{F}_0}(\Omega; R^n)$ the family of all $\mathcal{F}_0$-measurable bounded $C([-\delta, 0]; R^n)$-valued random variables equipped with the norm $\|\varphi\| = \sup_{-\delta\leq\theta\leq 0} |\varphi(\theta)|$. For $\forall x, y \in R^n$, $\langle x, y\rangle$ or $x^T y$ represents the inner product. Let $\{r(t)\}_{t\geq 0}$ be a right-continuous Markov chain on the complete probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$P\bigl\{r(t+\Delta) = j | r(t) = i\bigr\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases}$$

where $\Delta > 0$, $\lim_{\Delta\to 0} o(\Delta)/\Delta = 0$, $\gamma_{ij} \geq 0$ is the transition rate from i to j for $i \neq j$, and $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$ (see [1]).

Consider the following n-dimensional unstable stochastic regime-switching jump equation, that is, the n-dimensional SDEwMJ

$$dx(t) = f\bigl(x(t), t, r(t)\bigr)\, dt + g\bigl(x(t), t, r(t)\bigr)\, d\omega(t) + h\bigl(x(t), t, r(t)\bigr)\, dN(t) \tag{4}$$

on $t \geq t_0 \geq 0$, with the initial data

$$x(t_0) = x_0 \in C^b_{\mathcal{F}_0}\bigl(\Omega; R^n\bigr) \quad \text{and} \quad r(t_0) = i_0 \in S, \tag{5}$$

where $f, g, h : R^n \times R^+ \times S \to R^n$, $\omega(t)$ is a scalar Brownian motion, and $N(t)$ is a scalar Poisson process with intensity $\lambda > 0$. $\tilde{N}(t) = N(t) - \lambda t$ is a compensated Poisson process. Moreover, $B(t)$, $N(t)$, and $r(t)$ are assumed to be mutually independent of each other. To achieve the goal of the stability, it is assumed that $f(0, t, i) = g(0, t, i) = h(0, t, i) = 0$ for all $t \geq 0$ and $i \in S$.

The aim of this paper is to design a control function $u : R^n \times R^+ \times S \to R^n$ making the unstable SDEwMJ (4) become stable. The form of the corresponding controlled SDEwMJ is described by

$$\begin{aligned} dx(t) = {} & \bigl[f\bigl(x(t), t, r(t)\bigr) + u\bigl(x(t-\delta), t, r(t)\bigr)\bigr]\, dt + g\bigl(x(t), t, r(t)\bigr)\, d\omega(t) \\ & + h\bigl(x(t), t, r(t)\bigr)\, dN(t) \end{aligned} \tag{6}$$

on $t \geq t_0$, with the initial value

$$x_{t_0} = \varphi = \bigl\{x(t_0 + \theta) : -\delta \leq \theta \leq 0\bigr\} \in C^b_{\mathcal{F}_0}\bigl(\Omega; R^n\bigr) \quad \text{and} \quad r(t_0) = i_0 \in S. \tag{7}$$

The coefficients of SDEwMJ (4) are assumed to satisfy the following global Lipschitz condition.

(A1) There are four positive constants L_i $(i = 1, 2, 3)$ and K such that

$$\begin{aligned}
&\bigl|f(x_1,t,i)-f(x_2,t,i)\bigr| \le L_1|x_1-x_2|,\\
&\bigl|g(x_1,t,i)-g(x_2,t,i)\bigr| \le L_2|x_1-x_2|,\\
&\bigl|h(x_1,t,i)-h(x_2,t,i)\bigr| \le L_3|x_1-x_2|,\\
&\bigl|u(x_1,t,i)-u(x_2,t,i)\bigr| \le K|x_1-x_2|,
\end{aligned} \tag{8}$$

for all $x_1, x_2 \in R^n$, $t \in R^+$ and $i \in S$. Moreover, for all $(t, i) \in R^+ \times S$, $f(0, t, i) = g(0, t, i) = h(0, t, i) = 0$, one can thus obtain the following linear growth condition

$$\begin{aligned}
&\bigl|f(x,t,i)\bigr| \le L_1|x|, \qquad \bigl|g(x,t,i)\bigr| \le L_2|x|, \qquad \bigl|h(x,t,i)\bigr| \le L_3|x|,\\
&\bigl|u(x,t,i)\bigr| \le K|x|.
\end{aligned} \tag{9}$$

Remark 1 Under Assumption (A1), the controlled SDEwMJ (6) admits a unique solution $x(t)$ satisfying $E|x(t)|^2 < \infty$ for all $t \ge 0$ corresponding to the initial value (7). The proof is standard, and one thus omits it.

The auxiliary controlled SDEwMJ is presented as follows:

$$\begin{aligned}
dy(t) = {}&\bigl[f\bigl(y(t),t,r(t)\bigr) + u\bigl(y(t),t,r(t)\bigr)\bigr]\,dt + g\bigl(y(t),t,r(t)\bigr)\,d\omega(t)\\
&+ h\bigl(y(t),t,r(t)\bigr)\,dN(t).
\end{aligned} \tag{10}$$

Similar to the proof of Theorem 3.8 in [1], one can obtain that under (A1), the auxiliary controlled SDEwMJ (10) with the initial data $y(t_0) \in C^b_{\mathcal{F}_{t_0}}(R^n)$ and $r(t_0) \in S$ has a unique solution $y(t)$ $(t \ge t_0 \ge 0)$, which satisfies $E|y(t)|^2 < \infty$ for all $t \ge t_0 \ge 0$.

Let $C^{2,1}(R^n \times R^+ \times S; R^+)$ be the family of non-negative functions $V(x, t, i)$ defined on $(x, t, i) \in R^n \times R^+ \times S$, which are continuously twice differentiable in x and once in t. For $V \in C^{2,1}(R^n \times R^+ \times S; R^+)$, define $LV : R^n \times R^+ \times S \to R$ by

$$\begin{aligned}
LV(x,t,i) = {}& V_t(x,t,i) + V_x(x,t,i)\bigl[f(x,t,i) + u(x,t,i)\bigr]\\
&+ \frac{1}{2}\operatorname{trace}\bigl[g^T(x,t,i)V_{xx}(x,t,i)g(x,t,i)\bigr]\\
&+ \lambda\bigl[V\bigl(x + h(x,t,i),t,i\bigr) - V(x,t,i)\bigr]\\
&+ \sum_{j=1}^{N} \gamma_{ij} V(x,t,j),
\end{aligned} \tag{11}$$

where

$$V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t}, \qquad V_x(x,t,i) = \left(\frac{\partial V(x,t,i)}{\partial x_1}, \frac{\partial V(x,t,i)}{\partial x_2}, \dots, \frac{\partial V(x,t,i)}{\partial x_n}\right)$$

and

$$V_{xx}(x,t,i) = \left(\frac{\partial^2 V(x,t,i)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Next, according to the Theorem 5.8 in Mao and Yuan [1], one can acquire the following lemma in parallel, which will be used in the sequel.

Lemma 2.1 *Let* (A1) *hold. Let* p, c_1, c_2, *and* λ_1 *be positive constants. For all* $(y,t,i) \in R^n \times R^+ \times S$, *assume that there is a function* $V \in C^{2,1}(R^n \times R^+ \times S; R^+)$ *satisfying*

$$c_1|y|^p \le V(y,t,i) \le c_2|y|^p \tag{12}$$

and

$$LV(y,t,i) \le -\lambda_1|y|^p. \tag{13}$$

Then

$$E\big|y(t)\big|^p \le \frac{c_2}{c_1}e^{-\frac{\lambda_1}{c_2}(t-t_0)}E\big|y(t_0)\big|^p, \tag{14}$$

for all $y(t_0) \in C^b_{\mathcal{F}_{t_0}}(R^n)$, $(t_0, r(t_0)) \in R^+ \times S$. *In other words, the trivial solution of the auxiliary controlled SDEwMJ* (10) *is the pth moment exponentially stable.*

From Lemma 2.1, one can obtain the auxiliary controlled SDEwMJ (10) is the pth moment exponentially stable. While the pth moment exponential stability of SDEwMJ (10) was well studied (see, e.g., [1, 17]). Therefore, one can assume that (10) is the pth moment exponentially stable in this paper, which is given as follows.

(A2) Let $p > 0$. Assume that there is a pair of positive constants M and γ such that the solution of the auxiliary controlled SDEwMJ (10) satisfies

$$E\big|y\big(t;t_0,r(t_0)\big)\big|^p \le Me^{-\gamma(t-t_0)}E\big|y(t_0)\big|^p, \forall t \ge t_0 \ge 0 \tag{15}$$

for all $y(t_0) \in C^b_{\mathcal{F}_{t_0}}(R^n)$, $(t_0, r(t_0)) \in R^+ \times S$.

3 Main results

The aim of this paper is to show the controlled SDEwMJ (6) is almost surely exponentially stable. Before proving this result, a number of lemmas should be proved.

Lemma 3.1 *Let* (A1) *hold, and* $p \in (0,1)$. *Then, for any* $t_0 \ge 0$ *and* $T \ge 0$,

$$\sup_{t_0\le t\le t_0+T+\delta} E\big|x(t)\big|^p \le M_1^{\frac{p}{2}}E\|\varphi\|^p, \tag{16}$$

$$\sup_{t_0\le t\le t_0+T} E\Big(\sup_{0\le v\le\delta}\big|x(t+v)-x(t)\big|^p\Big) \le M_2^{\frac{p}{2}}E\|\varphi\|^p, \tag{17}$$

and

$$E\Big(\sup_{t_0\le t\le t_0+T+\delta}\big|x(t)\big|^p\Big) \le M_3^{\frac{p}{2}}E\|\varphi\|^p, \tag{18}$$

where

$$M_1 \triangleq: M_1(p,\delta,T) = (1+k\delta)e^{[2(L_1+\lambda L_3)+L_2^2+\lambda L_3^2+2K](T+\delta)}, \tag{19}$$

$$M_2 \triangleq: M_2(p,\delta,T) = 4\big(L_1^2\delta + 4L_2^2 + 2\lambda^2 L_3^2\delta + 8\lambda L_3^2 + K^2\delta\big)M_1\delta, \tag{20}$$

$$\begin{aligned} M_3 \triangleq: M_3(p,\delta,T) = 5\big[&1 + L_1^2(T+\delta) + K^2(T+\delta) + 4L_2^2 + 2\lambda^2 L_3^2(T+\delta) \\ &+ 8\lambda L_3^2\big]\big)M_1(T+\delta). \end{aligned} \tag{21}$$

Proof (1) Applying the Itô's formula to $|x(t)|^2$ and taking the expectation, one obtains

$$\begin{aligned} E\big|x(t)\big|^2 = {}& E\big|x(t_0)\big|^2 + 2E\int_{t_0}^t x^T(s)f\big(x(s),s,r(s)\big)\,ds \\ &+ 2E\int_{t_0}^t x^T(s)u\big(x(s-\delta),s,r(s)\big)\,ds \\ &+ E\int_{t_0}^t \big|g\big(x(s),s,r(s)\big)\big|^2\,ds \\ &+ \lambda E\int_{t_0}^t \big[h^2\big(x(s),s,r(s)\big) + 2x^T(s)h\big(x(s),s,r(s)\big)\big]\,ds. \end{aligned} \tag{22}$$

According to (9), one gains

$$\begin{aligned} E\big|x(t)\big|^2 \le {}& E\big|x(t_0)\big|^2 + 2L_1E\int_{t_0}^t \big|x(s)\big|^2\,ds + 2KE\int_{t_0}^t \big|x(s)\big|\big|x(s-\delta)\big|\,ds \\ &+ L_2^2E\int_{t_0}^t \big|x(s)\big|^2\,ds + \lambda L_3^2E\int_{t_0}^t \big|x(s)\big|^2\,ds \\ &+ 2\lambda L_3E\int_{t_0}^t \big|x(s)\big|^2\,ds. \end{aligned} \tag{23}$$

By basic inequality, one can get

$$\begin{aligned} &2KE\int_{t_0}^t \big|x(s)\big|\big|x(s-\delta)\big|\,ds \\ &\quad\le KE\int_{t_0}^t \big|x(s)\big|^2\,ds + KE\int_{t_0}^t \big|x(s-\delta)\big|^2\,ds \\ &\quad= KE\int_{t_0}^t \big|x(s)\big|^2\,ds + KE\int_{t_0-\delta}^{t-\delta} \big|x(\nu)\big|^2\,d\nu \\ &\quad= KE\int_{t_0}^t \big|x(s)\big|^2\,ds + KE\int_{t_0-\delta}^{t_0} \big|x(\nu)\big|^2\,d\nu + KE\int_{t_0}^{t-\delta} \big|x(\nu)\big|^2\,d\nu \\ &\quad\le 2KE\int_{t_0}^t \big|x(s)\big|^2\,ds + K\delta E\|\varphi\|^2. \end{aligned} \tag{24}$$

According to (6) and substituting (24) into (23), one can gain

$$\begin{aligned} &E\big|x(t)\big|^2 \\ &\quad\le E\big|x(t_0)\big|^2 + \big[2(L_1+\lambda L_3) + L_2^2 + \lambda L_3^2 + 2K\big]\int_{t_0}^t E\big|x(s)\big|^2\,ds + K\delta E\|\varphi\|^2 \\ &\quad\le (1+K\delta)E\|\varphi\|^2 + \big[2(L_1+\lambda L_3) + L_2^2 + \lambda L_3^2 + 2K\big]\int_{t_0}^t \Big(\sup_{t_0\le\nu\le s} E\big|x(\nu)\big|^2\Big)\,ds. \end{aligned} \tag{25}$$

Therefore,

$$\begin{aligned}
&\sup_{t_0\le\nu\le t} E\left|x(\nu)\right|^2 \\
&\quad\le (1+K\delta)E\|\varphi\|^2 + \left[2(L_1+\lambda L_3)+L_2^2+\lambda L_3^2+2K\right] \\
&\qquad\times \int_{t_0}^{t}\Big(\sup_{t_0\le\nu\le s} E\left|x(\nu)\right|^2\Big)\,ds.
\end{aligned} \tag{26}$$

By the Gronwall inequality, one obtains

$$\begin{aligned}
\sup_{t_0\le\nu\le t_0+T+\delta} E\left|x(\nu)\right|^2 &\le (1+K\delta)e^{[2(L_1+\lambda L_3)+L_2^2+\lambda L_3^2+2K](T+\delta)}E\|\varphi\|^2 \\
&\triangleq: M_1 E\|\varphi\|^2.
\end{aligned} \tag{27}$$

Applying the Hölder inequality, one has

$$\sup_{t_0\le t\le t_0+T+\delta} E\left|x(t)\right|^p \le M_1^{\frac{p}{2}} E\|\varphi\|^p. \tag{28}$$

(2) For $0\le\nu\le\delta$, it follows from (6) that one can get

$$\begin{aligned}
x(t+\nu)-x(t) &= \int_t^{t+\nu}\left[f\big(x(s),s,r(s)\big)+u\big(x(s-\delta),s,r(s)\big)\right]ds \\
&\quad+\int_t^{t+\nu} g\big(x(s),s,r(s)\big)\,d\omega(s) \\
&\quad+\int_t^{t+\nu} h\big(x(s),s,r(s)\big)\,dN(s).
\end{aligned} \tag{29}$$

Then

$$\begin{aligned}
\left|x(t+\nu)-x(t)\right|^2 &\le 4\left|\int_t^{t+\nu} f\big(x(s),s,r(s)\big)\,ds\right|^2 \\
&\quad+4\left|\int_t^{t+\nu} u\big(x(s-\delta),s,r(s)\big)\,ds\right|^2 \\
&\quad+4\left|\int_t^{t+\nu} g\big(x(s),s,r(s)\big)\,d\omega(s)\right|^2 \\
&\quad+4\left|\int_t^{t+\nu} h\big(x(s),s,r(s)\big)\,dN(s)\right|^2 \\
&\le 4\nu\int_t^{t+\nu} f^2\big(x(s),s,r(s)\big)\,ds \\
&\quad+4\nu\int_t^{t+\nu} u^2\big(x(s-\delta),s,r(s)\big)\,ds \\
&\quad+4\left|\int_t^{t+\nu} g\big(x(s),s,r(s)\big)\,d\omega(s)\right|^2 \\
&\quad+8\lambda^2\nu\int_t^{t+\nu} h^2\big(x(s),s,r(s)\big)\,ds
\end{aligned}$$

$$+ 8\left|\int_t^{t+\nu} h\bigl(x(s),s,r(s)\bigr)\,d\tilde{N}(s)\right|^2. \tag{30}$$

For $\int_t^{t+\nu} g(x(s),s,r(s))\,d\omega(s)$ and $\int_t^{t+\nu} h(x(s),s,r(s))\,d\tilde{N}(s)$ are martingales, thus by the Doob martingale inequality, one gains

$$\begin{aligned}
E\Bigl(\sup_{0\le\nu\le\delta}\bigl|x(t+\nu)-x(t)\bigr|^2\Bigr) \le{}& 4\delta E\int_t^{t+\delta} f^2\bigl(x(s),s,r(s)\bigr)\,ds \\
&+ 4\delta E\int_t^{t+\delta} u^2\bigl(x(s-\delta),s,r(s)\bigr)\,ds \\
&+ 16E\int_t^{t+\delta} g^2\bigl(x(s),s,r(s)\bigr)\,ds \\
&+ 8\lambda^2\delta E\int_t^{t+\delta} h^2\bigl(x(s),s,r(s)\bigr)\,ds \\
&+ 32\lambda E\int_t^{t+\delta} h^2\bigl(x(s),s,r(s)\bigr)\,ds.
\end{aligned} \tag{31}$$

Applying (A1), one can further get

$$\begin{aligned}
&E\Bigl(\sup_{0\le\nu\le\delta}\bigl|x(t+\nu)-x(t)\bigr|^2\Bigr) \\
&\quad\le 4\delta L_1^2 E\int_t^{t+\delta}\bigl|x(s)\bigr|^2\,ds + 4\delta K^2 E\int_t^{t+\delta}\bigl|x(s-\delta)\bigr|^2\,ds \\
&\qquad+ 16L_2^2 E\int_t^{t+\delta}\bigl|x(s)\bigr|^2\,ds + 8\lambda^2 L_3^2\delta E\int_t^{t+\delta}\bigl|x(s)\bigr|^2\,ds \\
&\qquad+ 32\lambda L_3^2 E\int_t^{t+\delta}\bigl|x(s)\bigr|^2\,ds \\
&\quad\triangleq: \bigl(4L_1^2\delta + 16L_2^2 + 8\lambda^2 L_3^2\delta + 32\lambda L_3^2\bigr)E\int_t^{t+\delta}\bigl|x(s)\bigr|^2\,ds \\
&\qquad+ 4K^2\delta E\int_t^{t+\delta}\bigl|x(s-\delta)\bigr|^2\,ds.
\end{aligned} \tag{32}$$

In fact, for $t\in[t_0-\delta,t_0]$, $E|x(t)|^2 \le E\|\varphi\|^2 \le M_1 E\|\varphi\|^2$. Together with (27), one can get $E|x(t)|^2 \le M_1 E\|\varphi\|^2$ for $t\in[t_0-\delta,t_0+T+\delta]$. Then

$$\begin{aligned}
&\sup_{t_0\le t\le t_0+T}\Bigl(E\Bigl(\sup_{0\le\nu\le\delta}\bigl|x(t+\nu)-x(t)\bigr|^2\Bigr)\Bigr) \\
&\quad\le \bigl(4L_1^2\delta + 16L_2^2 + 8\lambda^2 L_3^2\delta + 32\lambda L_3^2\bigr)M_1\delta E\|\varphi\|^2 + 4\delta K^2 M_1\delta E\|\varphi\|^2 \\
&\quad= 4\bigl(L_1^2\delta + 4L_2^2 + 2\lambda^2 L_3^2\delta + 8\lambda L_3^2 + K^2\delta\bigr)M_1\delta E\|\varphi\|^2 \triangleq: M_2 E\|\varphi\|^2.
\end{aligned} \tag{33}$$

By the Hölder inequality, one has

$$\sup_{t_0\le t\le t_0+T}\Bigl(E\Bigl(\sup_{0\le\nu\le\delta}\bigl|x(t+\nu)-x(t)\bigr|^p\Bigr)\Bigr) \le M_2^{\frac{p}{2}} E\|\varphi\|^p. \tag{34}$$

(3) It follows from (6) that

$$x(t) = x(t_0) + \int_{t_0}^{t} \bigl[f\bigl(x(s),s,r(s)\bigr) + u\bigl(x(s-\delta),s,r(s)\bigr)\bigr]\,ds + \int_{t_0}^{t} g\bigl(x(s),s,r(s)\bigr)\,d\omega(s) + \int_{t_0}^{t} h\bigl(x(s),s,r(s)\bigr)\,dN(s). \tag{35}$$

Using the Hölder inequality and (A1), one gains

$$\begin{aligned} \bigl|x(t)\bigr|^2 &\le 5\bigl|x(t_0)\bigr|^2 + 5\biggl|\int_{t_0}^{t} f\bigl(x(s),s,r(s)\bigr)\,ds\biggr|^2 \\ &\quad + 5\biggl|\int_{t_0}^{t} u\bigl(x(s-\delta),s,r(s)\bigr)\,ds\biggr|^2 \\ &\quad + 5\biggl|\int_{t_0}^{t} g\bigl(x(s),s,r(s)\bigr)\,d\omega(s)\biggr|^2 \\ &\quad + 5\biggl|\int_{t_0}^{t} h\bigl(x(s),s,r(s)\bigr)\,dN(s)\biggr|^2 \\ &\le 5\bigl|x(t_0)\bigr|^2 + 5(t-t_0)\int_{t_0}^{t} f^2\bigl(x(s),s,r(s)\bigr)\,ds \\ &\quad + 5(t-t_0)\int_{t_0}^{t} u^2\bigl(x(s-\delta),s,r(s)\bigr)\,ds \\ &\quad + 5\biggl|\int_{t_0}^{t} g\bigl(x(s),s,r(s)\bigr)\,d\omega(s)\biggr|^2 \\ &\quad + 10\lambda^2(t-t_0)\int_{t_0}^{t} h^2\bigl(x(s),s,r(s)\bigr)\,ds \\ &\quad + 10\biggl|\int_{t_0}^{t} h\bigl(x(s),s,r(s)\bigr)\,d\tilde{N}(s)\biggr|^2. \end{aligned} \tag{36}$$

By the Doob martingale inequality, one can gain

$$\begin{aligned} &E\Bigl(\sup_{t_0\le t\le t_0+T+\delta}\bigl|x(t)\bigr|^2\Bigr) \\ &\quad\le 5E\bigl|x(t_0)\bigr|^2 + 5(T+\delta)E\int_{t_0}^{t_0+T+\delta} f^2\bigl(x(s),s,r(s)\bigr)\,ds \\ &\quad\quad + 5(T+\delta)E\int_{t_0}^{t_0+T+\delta} u^2\bigl(x(s-\delta),s,r(s)\bigr)\,ds \\ &\quad\quad + 20E\int_{t_0}^{t_0+T+\delta} g^2\bigl(x(s),s,r(s)\bigr)\,ds \\ &\quad\quad + 10\lambda^2(T+\delta)E\int_{t_0}^{t_0+T+\delta} h^2\bigl(x(s),s,r(s)\bigr)\,ds \\ &\quad\quad + 40\lambda E\int_{t_0}^{t_0+T+\delta} h^2\bigl(x(s),s,r(s)\bigr)\,ds \\ &\quad\le 5E\|\varphi\|^2 + 5L_1^2(T+\delta)\int_{t_0}^{t_0+T+\delta} E\bigl|x(s)\bigr|^2\,ds \end{aligned}$$

$$
\begin{aligned}
&+5K^2(T+\delta)\int_{t_0}^{t_0+T+\delta} E\big|x(s-\delta)\big|^2\,ds \\
&+20L_2^2\int_{t_0}^{t_0+T+\delta} E\big|x(s)\big|^2\,ds \\
&+10\lambda^2L_3^2(T+\delta)E\int_{t_0}^{t_0+T+\delta} E\big|x(s)\big|^2\,ds \\
&+40\lambda L_3^2E\int_{t_0}^{t_0+T+\delta} E\big|x(s)\big|^2\,ds. \qquad (37)
\end{aligned}
$$

By (27), one obtains

$$
\begin{aligned}
&E\Big(\sup_{t_0\le t\le t_0+T+\delta}\big|x(t)\big|^2\Big) \\
&\quad\le 5\big[1+L_1^2(T+\delta)+K^2(T+\delta)+4L_2^2+2\lambda^2L_3^2(T+\delta)+8\lambda L_3^2\big] \\
&\qquad\times(T+\delta)M_1E\|\varphi\|^2 \\
&\quad\triangleq: M_3E\|\varphi\|^2. \qquad (38)
\end{aligned}
$$

Applying the Hölder inequality, one can compute

$$
E\Big(\sup_{t_0\le t\le t_0+T+\delta}\big|x(t)\big|^2\Big)\le M_3^{\frac{p}{2}}E\|\varphi\|^p. \qquad (39)
$$

The proof is complete. □

Lemma 3.2 *Let* (A1) *hold and* $p\in(0,1)$. *Given* $t_0\ge\delta$ *and* $T\ge 0$ *arbitrarily. Denote* $y(t;x(t_0),r(t_0),t_0)=y(t)$ *for all* $t\ge t_0$. *Then*

$$
E\big|x(t)-y(t)\big|^p\le M_4^{\frac{p}{2}}E\|\varphi|^p,\quad \textit{for } \forall t\in[t_0,t_0+T+\delta], \qquad (40)
$$

where

$$
M_4\triangleq: M_4(p,\delta,T)=M_2K(T+\delta)e^{[2L_1+L_2^2+\lambda L_3^2+2\lambda L_3+3K](T+\delta)}. \qquad (41)
$$

Proof It follows from (6) and (10) that one gains

$$
\begin{aligned}
d\big[x(t)-y(t)\big]&=\big[f\big(x(t),t,r(t)\big)-f\big(y(t),t,r(t)\big)\big]\,dt \\
&\quad+\big[u\big(x(t-\delta),t,r(t)\big)-u\big(y(t),t,r(t)\big)\big]\,dt \\
&\quad+\big[g\big(x(t),t,r(t)\big)-g\big(y(t),t,r(t)\big)\big]\,d\omega(t) \\
&\quad+\big[h\big(x(t),t,r(t)\big)-h\big(y(t),t,r(t)\big)\big]\,dN(t). \qquad (42)
\end{aligned}
$$

Applying the Itô's formula to $|x(t)-y(t)|^2$, one gets

$$
\begin{aligned}
&E\big|x(t)-y(t)\big|^2 \\
&\quad=E\big|x(t_0)-y(t_0)\big|^2+2E\int_{t_0}^{t}\big(x(s)-y(s)\big)^T\big(f\big(x(s),s,r(s)\big)-f\big(y(s),s,r(s)\big)\big)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+ 2E\int_{t_0}^{t}\bigl(x(s)-y(s)\bigr)^T\bigl(u\bigl(x(s-\delta),s,r(s)\bigr)-u\bigl(y(s),s,r(s)\bigr)\bigr)\,ds\\
&+ E\int_{t_0}^{t}\bigl|g\bigl(x(s),s,r(s)\bigr)-g\bigl(y(s),s,r(s)\bigr)\bigr|^2\,ds\\
&+ \lambda E\int_{t_0}^{t}\bigl|h\bigl(x(s),s,r(s)\bigr)-h\bigl(y(s),s,r(s)\bigr)\bigr|^2\,ds\\
&+ 2\lambda E\int_{t_0}^{t}\bigl(x(s)-y(s)\bigr)^T\bigl(h\bigl(x(s),s,r(s)\bigr)-h\bigl(y(s),s,r(s)\bigr)\bigr)\,ds.
\end{aligned}
\tag{43}
$$

Using (A1), one can further obtain

$$
\begin{aligned}
&E\bigl|x(t)-y(t)\bigr|^2\\
&\quad\le 2L_1\int_{t_0}^{t}E\bigl|x(s)-y(s)\bigr|^2\,ds+2KE\int_{t_0}^{t}\bigl|x(s)-y(s)\bigr|\bigl|x(s-\delta)-y(s)\bigr|\,ds\\
&\qquad + L_2^2\int_{t_0}^{t}E\bigl|x(s)-y(s)\bigr|^2\,ds+\lambda L_3^2\int_{t_0}^{t}E\bigl|x(s)-y(s)\bigr|^2\,ds\\
&\qquad + 2\lambda L_3\int_{t_0}^{t}E\bigl|x(s)-y(s)\bigr|^2\,ds\\
&\quad\triangleq: \bigl(2L_1+L_2^2+\lambda L_3^2+2\lambda L_3\bigr)\int_{t_0}^{t}E\bigl|x(s)-y(s)\bigr|^2\,ds\\
&\qquad + 2KE\int_{t_0}^{t}\bigl|x(s)-y(s)\bigr|\bigl|x(s-\delta)-y(s)\bigr|\,ds.
\end{aligned}
\tag{44}
$$

Noting that

$$
\begin{aligned}
&E\int_{t_0}^{t}\bigl|x(s)-y(s)\bigr|\bigl|x(s-\delta)-y(s)\bigr|\,ds\\
&\quad= E\int_{t_0}^{t}\bigl|x(s)-y(s)\bigr|\bigl|x(s-\delta)-x(s)+x(s)-y(s)\bigr|\,ds\\
&\quad\le E\int_{t_0}^{t}\bigl|x(s)-y(s)\bigr|\bigl|x(s-\delta)-x(s)\bigr|\,ds+E\int_{t_0}^{t}\bigl|x(s)-y(s)\bigr|^2\,ds\\
&\quad\le \frac{3}{2}\int_{t_0}^{t}E\bigl|x(s)-y(s)\bigr|^2\,ds+\frac{1}{2}\int_{t_0}^{t}E\bigl|x(s-\delta)-x(s)\bigr|^2\,ds.
\end{aligned}
\tag{45}
$$

Substituting (45) into (44), one has

$$
\begin{aligned}
E\bigl|x(t)-y(t)\bigr|^2 \le{}& \bigl(2L_1+L_2^2+\lambda L_3^2+2\lambda L_3+3K\bigr)\int_{t_0}^{t}E\bigl|x(s)-y(s)\bigr|^2\,ds\\
&+K\int_{t_0}^{t}E\bigl|x(s-\delta)-x(s)\bigr|^2\,ds.
\end{aligned}
\tag{46}
$$

By (33), one can get for $t\in[t_0,t_0+T+\delta]$,

$$
\begin{aligned}
E\bigl|x(t)-y(t)\bigr|^2 \le{}& \bigl(2L_1+L_2^2+\lambda L_3^2+2\lambda L_3+3K\bigr)\int_{t_0}^{t}E\bigl|x(s)-y(s)\bigr|^2\,ds\\
&+M_2K(T+\delta)E\|\varphi\|^2.
\end{aligned}
\tag{47}
$$

According to the Gronwall inequality, one gets

$$E\left|x(t)-y(t)\right|^2 \le M_4 E\|\varphi\|^2. \tag{48}$$

Further, using the Hölder inequality, it follows that

$$E\left|x(t)-y(t)\right|^p \le M_4^{\frac{p}{2}} E\|\varphi\|^p. \tag{49}$$

The proof is complete. □

Theorem 3.3 *Let* (A1) *and* (A2) *hold. There is a positive number* $\bar{\delta}$ *such that the solution of the controlled SDEwMJ* (6) *is almost surely exponentially stable provided* $\delta < \bar{\delta}$, *that is*,

$$\limsup_{t\to\infty} \frac{\log(|x(t;x_{t_0},r(t_0),t_0)|)}{t} < 0 \quad a.s. \tag{50}$$

In practice, one can choose a constant $\varepsilon \in (0,1)$ *and set* $T = \frac{1}{\gamma}\log(\frac{4^p M}{\varepsilon})$. *Let* $\bar{\delta}$ *is the unique root to the following equation*

$$2^p\left(M_2^{\frac{p}{2}} + 2^p M_4^{\frac{p}{2}}\right) = 1-\varepsilon, \tag{51}$$

where M_2 *and* M_4 *have been defined in* (20) *and* (41), *respectively.*

Remark 2 Note that the left side of (51) is an increasing continuous function of δ. When $\delta = 0$, one can obtain $M_2 = M_4 = 0$. Therefore, the left side of (51) is 0 when $\delta = 0$. Moreover, the left side of (51) tends to $+\infty$ when $\delta \to +\infty$. Then, one can assert that the equation (51) admits a unique positive root $\bar{\delta}$. The root $\bar{\delta}$ can be obtained numerically, but its explicit form can not be expressed.

Proof Step 1: Denote $x(t;x_{t_0},r(t_0),t_0) = x(t)$ and $r(t;r(t_0),t_0) = r(t)$. Fix $\delta \in (0,\bar{\delta})$. Let us consider $x(t)$ on $[\delta, 2\delta+T]$, which can be regarded as the solution of (6) with the initial value x_δ and $r(\delta)$ at $t=\delta$. Also consider the solution $y(t;x(\delta),r(\delta),\delta)$ of (10) on $t \in [\delta,\delta+T]$ with the initial value $x(\delta)$ and $r(\delta)$ at $t=\delta$. Denote $y(\delta+T;x(\delta),r(\delta),\delta) = y(\delta+T)$. By (A2), one can get

$$E\left|y(\delta+T)\right|^p \le Me^{-\gamma T}E\left|x(\delta)\right|^p. \tag{52}$$

For

$$E\left|x(\delta+T)\right|^p \le 2^p E\left|x(\delta+T)-y(\delta+T)\right|^p + 2^p E\left|y(\delta+T)\right|^p, \tag{53}$$

then it follows from Lemma 3.2 and (52) that one gets

$$\begin{aligned} E\left|x(\delta+T)\right|^p &\le 2^p M_4^{\frac{p}{2}} E\|x_\delta\|^p + 2^p Me^{-\gamma T}E\left|x(\delta)\right|^p \\ &\le 2^p\left(Me^{-\gamma T} + M_4^{\frac{p}{2}}\right)E\|x_\delta\|^p. \end{aligned} \tag{54}$$

Applying (17) and (54), one can further get

$$\begin{aligned} E\|x_{2\delta+T}\|^p &= E\Big(\sup_{-\delta\le\theta\le 0}\big|x(2\delta+T+\theta)\big|^p\Big) = E\Big(\sup_{0\le\nu\le\delta}\big|x(\delta+T+\nu)\big|^p\Big) \\ &= E\Big(\sup_{0\le\nu\le\delta}\big|x(\delta+T+\nu)-x(\delta+T)+x(\delta+T)\big|^p\Big) \\ &\le 2^pE\Big(\sup_{0\le\nu\le\delta}\big|x(\delta+T+\nu)-x(\delta+T)\big|^p\Big)+2^pE\big|x(\delta+T)\big|^p \\ &\le 2^pM_2^{\frac{p}{2}}E\|x_\delta\|^p+4^p\big(Me^{-\gamma T}+M_4^{\frac{p}{2}}\big)E\|x_\delta\|^p. \end{aligned} \tag{55}$$

Choose $T=\frac{1}{\gamma}\log(\frac{4^pM}{\varepsilon})$. Therefore,

$$E\|x_{2\delta+T}\|^p\le\big[\varepsilon+2^p\big(M_2^{\frac{p}{2}}+2^pM_4^{\frac{p}{2}}\big)\big]E\|x_\delta\|^p. \tag{56}$$

For $\delta<\bar{\delta}$, then it follows from (51) that $\varepsilon+2^p(M_2^{\frac{p}{2}}+2^pM_4^{\frac{p}{2}})<1$, one can assert that there exists a $\tilde{\lambda}>0$ such that $\varepsilon+2^p(M_2^{\frac{p}{2}}+2^pM_4^{\frac{p}{2}})=e^{-\tilde{\lambda}(\delta+T)}$. Then

$$E\|x_{2\delta+T}\|^p\le e^{-\tilde{\lambda}(\delta+T)}E\|x_\delta\|^p. \tag{57}$$

Step 2: Consider the solution $x(t)$ on $t\in[2\delta+T,\delta+2(\delta+T)]$, which can be regarded as the solution of (6) with the initial data $x_{2\delta+T}$ and $r(2\delta+T)$ at $t=2\delta+T$. Similar to proof of (57), one can prove

$$E\|x_{\delta+2(\delta+T)}\|^p\le e^{-\tilde{\lambda}(\delta+T)}E\|x_{2\delta+T}\|^p\le e^{-2\tilde{\lambda}(\delta+T)}E\|x_\delta\|^p. \tag{58}$$

By induction, one gains

$$E\|x_{\delta+k(\delta+T)}\|^p\le e^{-\tilde{\lambda}(\delta+T)}E\|x_{\delta+(k-1)(\delta+T)}\|^p\le\cdots\le e^{-k\tilde{\lambda}(\delta+T)}E\|x_\delta\|^p, \tag{59}$$

for all $k=1,2,\ldots$. Using (18) and (59), one gets

$$\begin{aligned} E\Big(\sup_{\delta+k(\delta+T)\le t\le\delta+(k+1)(T+\delta)}\big|x(t)\big|^p\Big) &\le M_3^{\frac{p}{2}}E\|x_{\delta+k(\delta+T)}\|^p \\ &\le M_3^{\frac{p}{2}}e^{-k\tilde{\lambda}(\delta+T)}E\|x_\delta\|^p, \end{aligned} \tag{60}$$

for all $k=0,1,2,\ldots$. Hence, for $t\in[\delta+k(\delta+T),\delta+(k+1)(\delta+T)]$, applying the Chebyshev inequality, one gets

$$\begin{aligned} &E\Big(\sup_{\delta+k(\delta+T)\le t\le\delta+(k+1)(T+\delta)}\big|x(t)\big|^p\ge e^{-\frac{1}{2}k\tilde{\lambda}(\delta+T)}\Big) \\ &\quad\le M_3^{\frac{p}{2}}e^{-\frac{1}{2}k\tilde{\lambda}(\delta+T)}E\|x_\delta\|^p. \end{aligned} \tag{61}$$

By the Borel–Cantelli lemma, one obtains for almost all $\omega\in\Omega$ that there exists a positive integer k_0 satisfying

$$\sup_{\delta+k(\delta+T)\le t\le\delta+(k+1)(T+\delta)}\big|x(t)\big|^p<e^{-\frac{1}{2}k\tilde{\lambda}(\delta+T)},\forall k\ge k_0,$$

which implies

$$\limsup_{t\to\infty}\frac{1}{t}\log\bigl(|x(t)|\bigr)\le-\frac{\tilde{\lambda}}{2p}a.s.$$

The proof is complete. □

Remark 3 It follows from Theorem 3.3 that an unstable SDEwMJ can be stabilized by a delay feedback control. It is well known that SDEs, SDEwMSs, and SDEs with Poisson jumps are the special cases of SDEwMJs. Though [1, 16, 17] well studied the stabilization problem of SDEwMSs by state feedback controls, authors in these papers neither designed the feedback controls with delay nor took Poisson jumps into account. Authors in [18–20] designed the feedback controls depending on a past state, but they did not consider the Poisson jumps. Therefore, the results in this paper cover part of the results in [1, 16–20].

4 An example

In this section, an example is provided to illustrate the obtained results. Before giving an example, a lemma is presented as follows.

Lemma 4.1 ([28]) *The Markov chain $r(t)$ is irreducible, that is, $\pi\Gamma = 0$, and $\sum_{i\in S}\pi_i = 1$ has a unique stationary distribution $\pi = (\pi_1,\pi_2,\dots,\pi_N)\in R^{1\times N}$ satisfying $\pi_i > 0$ for each $i = 1,2,\dots,N$.*

Example Let $B(t)$ be a scalar Brown motion and $N(t)$ be a Poisson process with intensity $\lambda = 1$. Let $r(t)$ be a right-continuous Markov chain taking values $S = \{1,2\}$ with generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}.$$

It follows from Lemma 4.1 that this Markov chain $r(t)$ is irreducible and $\pi = (1/2, 1/2)$. Consider the following 1-dimensional Black-Scholes model with the Markovian switching and Poisson jumps

$$dx(t) = a\bigl(r(t)\bigr)x(t)\,dt + b\bigl(r(t)\bigr)x(t)\,d\omega(t) + c\bigl(r(t)\bigr)x(t)\,dN(t),\quad t\ge 0, \tag{62}$$

with the initial value $y(0) = 1$ and $r(0) = 1$, and

$$\begin{aligned}&\text{if } i = 1,\quad a(1) = 0.1,\qquad b(1) = 0.2,\qquad c(1) = 0.02,\\ &\text{if } i = 2,\quad a(2) = 0.12,\qquad b(2) = 0.1,\qquad c(2) = 0.01.\end{aligned}$$

It is easy to check that $L_1 = 0.12$, $L_2 = 0.2$, $L_3 = 0.02$. It follows from Theorem 3.1 in [28] that one can get

$$\lim_{t\to\infty}\frac{1}{t}\log\bigl(|x(t)|\bigr) = \Sigma_{i=1}^{2}\pi_i\left(a(i)-\frac{1}{2}b^2(i)+\lambda\log\bigl|1+c(i)\bigr|\right) = 0.1124 > 0\quad\text{a.s.}$$

That is, the system (62) is not almost surely exponentially stable.

Let us now design a delay feedback control to stabilize the system (62). Consider a linear delay feedback controller of the form $u(x,t,i) = d(i)x$ $(i = 1,2)$. Therefore, the controlled system is given by

$$dx(t) = \left[a\bigl(r(t)\bigr)x(t) + d\bigl(r(t)\bigr)x(t-\delta)\right]dt + b\bigl(r(t)\bigr)x(t)\,dB(t) + c\bigl(r(t)\bigr)x(t)\,dN(t), \tag{63}$$

where $d(1) = -1.3$ and $d(2) = -1.2$. So, the condition (A1) is satisfied with $L_1 = 0.12$, $L_2 = 0.2$, $L_3 = 0.02$ and $K = 1.3$.

The auxiliary controlled system is of the form

$$dy(t) = \left[a\bigl(r(t)\bigr)y(t) + d\bigl(r(t)\bigr)y(t)\right]dt + b\bigl(r(t)\bigr)y(t)\,dB(t) + c\bigl(r(t)\bigr)y(t)\,dN(t). \tag{64}$$

Set $V(y,i,t) = |y|^p$ $(i = 1,2)$ and $p = 0.999$. One can compute for $\forall i = 1,2$,

$$LV(y,i,t) \le -1.0689|y|^p. \tag{65}$$

Therefore, $c_1 = c_2 = 1$, $\lambda_1 = 1.0689$. It follows from Lemma 2.1 that the auxiliary controlled system (64) is the pth moment exponentially stable with $M = c_2/c_1 = 1$ and $\gamma = \lambda_1/c_2 = 1.0689$, that is, the condition (A2) is true. Choose $\varepsilon = 0.95$, thus $T = \frac{1}{\gamma}\log(\frac{4^pM}{\varepsilon}) = 1.3436$. By computing, one can choose $\delta < \bar{\delta} = 4.5262 \times 10^{-7}$, where $\bar{\delta}$ is the unique positive root of (51). Then all the conditions of Theorem 3.3 are satisfied. Therefore, the controlled system (63) is almost surely exponentially stable.

5 Conclusion

So far, there are few results on the stabilization analysis of SDEwMJs. This paper discussed that an unstable SDEwMJ could be stabilized by a delayed feedback control $u(x(t-\delta),t,r(t))$ designed in the drift term, and the delay δ is bounded by a constant $\bar{\delta}$. Moreover, the implicit lower bound for $\bar{\delta}$ can be computed numerically. As a product, the almost sure exponential stability of the controlled SDEwMJ is obtained. An example is also given to verify the obtained results.

Acknowledgements
The author would like to thank anonymous referees and editors for their helpful comments and suggestions, which greatly improved the quality of this paper.

Funding
This research was supported by the National Natural Science Foundation of China (No. 11901398).

Availability of data and materials
Data sharing not applicable to this article as no dataries were generated or analyzed during the current study.

Declarations

Competing interests
The author declares that there is no competing interests.

Authors' contributions
The author read and approved the final manuscript.

Publisher's Note

Received: 12 August 2021 Accepted: 23 January 2022 Published online: 28 January 2022

References

1. Mao, X., Yuan, C.: Stochastic Differential Equations with Markovian Switching. Imperial College Press, London (2006)
2. Yuan, C., Lygeros, J.: Stabilization of a class of stochastic differential equations with Markovian switching. Syst. Control Lett. **54**(9), 819–833 (2005)
3. Pang, S., Deng, F., Mao, X.: Almost sure and moment exponential stability of Euler–Maruyama discretizations for hybrid stochastic differential equations. J. Comput. Appl. Math. **213**, 127–141 (2008)
4. Liu, L., Shen, Y.: The asymptotic stability and exponential stability of nonlinear stochastic differential systems with Markovian switching and with polynomial growth. J. Math. Anal. Appl. **391**(1), 323–334 (2012)
5. Teel, A.R., Subbaraman, A., Sferlazza, A.: Stability analysis for stochastic hybrid systems: a survey. Automatica **50**(10), 2435–2456 (2014)
6. Nguyen, D.H., Yin, G.: Stability of stochastic functional differential equations with regime-switching: analysis using Dupire's functional Itô formula. Potential Anal. **53**, 247–265 (2020)
7. Sheng, G., Xiao, R., Yin, X., Zhang, J.: Stabilization for hybrid stochastic systems by aperiodically intermittent control. Nonlinear Anal. Hybrid Syst. **39**, 100990 (2021)
8. Rong, S.: Theory of Stochastic Differential Equations with Jumps and Applications: Mathematical and Analytical Techniques with Applications to Engineering. Springer, Berlin (2006)
9. Mo, H., Deng, F., Zhang, C.: Exponential stability of the split-step θ-method for neutral stochastic delay differential equations with jumps. Appl. Math. Comput. **315**, 85–95 (2017)
10. Wei, Y., Yang, Q.: Dynamics of the stochastic low concentration trimolecular oscillatory chemical system with jumps. Nonlinear Sci. Numer. Simul. **59**, 396–408 (2018)
11. Zhang, X., Wang, K.: Stochastic SEIR model with jumps. Appl. Math. Comput. **239**, 133–143 (2014)
12. Li, G., Yang, Q.: Stability analysis of the split-step theta method for nonlinear regime-switching jump systems. J. Comput. Math. **39**(2), 192–206 (2021)
13. Ahmadian, D., Farkhondehrouz, O.: Exponential mean-square stability of numerical solutions for stochastic delay integro-differential equations with Poisson jump. J. Inequal. Appl. **2020**, 186 (2020)
14. Luo, J.: Comparison principle and stability of Ito stochastic differential delay equations with Poisson jump and Markovian switching. Nonlinear Anal. **64**(2), 253–262 (2006)
15. Wang, M., Meng, Q., Shen, Y.: H_2/H_∞ control for stochastic jump-diffusion systems with Markovian switching. Syst. Sci. Complex. **34**(3), 924–954 (2021)
16. Yuan, C., Mao, X.: Robust stability and controllability of stochastic differential delay equations with Markovian switching. Automatica **40**(3), 343–354 (2004)
17. Mao, X., Yin, G.G., Yuan, C.: Stabilization and destabilization of hybrid systems of stochastic differential equations. Automatica **43**(2), 264–273 (2007)
18. Mao, X., Lam, J., Huang, L.: Stabilisation of hybrid stochastic differential equations by delay feedback control. Syst. Control Lett. **57**(11), 927–935 (2008)
19. Chen, W., Xu, S., Zhang, B., Zhang, Q.: Stability and stabilisation of neutral stochastic delay Markovian jump systems. IET Control Theory Appl. **10**(15), 1798–1807 (2016)
20. Chen, W., Xu, S., Yun, Z.: Stabilization of hybrid neutral stochastic differential delay equations by delay feedback control. Syst. Control Lett. **88**(1), 1–13 (2016)
21. Fei, Z., Gao, H., Shi, P.: New results on stabilization of Markovian jump systems with time delay. Automatica **45**(10), 2300–2306 (2009)
22. Chen, W., Ma, Q., Wang, L., Xu, H.: Stabilisation and H_∞ control of neutral stochastic delay Markovian jump systems. Int. J. Syst. Sci. **49**(1–4), 58–67 (2018)
23. Hu, J., Liu, W., Deng, F., Mao, X.: Advances in stabilisation of hybrid stochastic differential equations by delay feedback control. SIAM J. Control Optim. **58**(2), 735–754 (2020)
24. Lu, Z., Hu, J., Mao, X.: Stabilisation by delay feedback control for highly nonlinear hybrid stochastic differential equation. Discrete Contin. Dyn. Syst., Ser. B **24**(8), 4099–4116 (2019)
25. Li, X., Mao, X.: Stabilisation of highly nonlinear hybrid stochastic differential delay equations by delay feedback control. Automatica **112**, 108657 (2020)
26. Shen, M., Fei, C., Fei, W., Mao, X.: Stabilisation by delay feedback control for highly nonlinear neutral stochastic differential equations. Syst. Control Lett. **137**, 104645 (2020)
27. Mei, C., Fei, C., Fei, W., Mao, X.: Exponential stabilization by delay feedback control for highly nonlinear hybrid stochastic functional differential equations with infinite delay. Nonlinear Anal. Hybrid Syst. **40**(9), 101026 (2021)
28. Zong, X., Wu, F., Yin, G., Jin, Z.: Almost sure and pth-moment stability and stabilization of regime-switching jump diffusion systems. SIAM J. Control Optim. **52**(4), 2595–2622 (2014)