On existence of global classical solutions to the 3D compressible MHD equations with vacuum

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Abstract

In this paper, the existence of global classical solutions is justified for the three-dimensional compressible magnetohydrodynamic (MHD) equations with vacuum. The main goal of this paper is to obtain a unique global classical solution on $\mathbb{R}^3 \times [0, T]$ with any $T \in (0, \infty)$, provided that the initial magnetic field in the L^3-norm and the initial density are suitably small. Note that the first result is obtained under the condition of $\rho_0 \in L^\gamma \cap W^{2,q}$ with $q \in (3, 6)$ and $\gamma \in (1, 6)$. It should be noted that the initial total energy can be arbitrarily large, the initial density allowed to vanish, and the system does not satisfy the conservation law of mass (i.e., $\rho_0 \notin L^1$). Thus, the results obtained particularly extend the one due to Li–Xu–Zhang (Li et al. in SIAM J. Math. Anal. 45:1356–1387, 2013), where the global well-posedness of classical solutions with small energy was proved.

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1 Introduction

One of the important problems in the theory of magnetohydrodynamics (MHD) is that of existence of global solutions to the equations of motion for a viscous compressible fluid. In this paper, we consider the MHD system of equations for a compressible isentropic MHD flows which in the case of 3D motion has the form (cf. [1, 7]):

$$\begin{cases} \rho_t + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\rho) = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u} + (\nabla \times \mathbf{B}) \times \mathbf{B}, \\ \mathbf{B}_t + \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \cdot \nabla \mathbf{u} + \mathbf{B} \operatorname{div} \mathbf{u} = \nu \Delta \mathbf{B}, \\ \operatorname{div} \mathbf{B} = 0, \end{cases} \tag{1.1}$$

where $t \geq 0$ is the time, $x \in \mathbb{R}^3$ is the spatial coordinate, and $\rho \geq 0$, $\mathbf{u} = (u^1, u^2, u^3)$, $\mathbf{B} = (B^1, B^2, B^3)$ are the fluid density, velocity, and magnetic field, respectively. The pressure

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$P = P(\rho)$ satisfies the condition

$$P(\rho) = A\rho^{\gamma} \quad \text{with } A > 0, \gamma > 1, \tag{1.2}$$

where $\gamma > 1$ is the adiabatic exponent, and $A > 0$ is a physical constant. The viscosity coefficients μ and λ satisfy

$$\mu > 0, \qquad \lambda + \frac{2}{3}\mu \geq 0. \tag{1.3}$$

Positive constant ν is the resistivity coefficient acting as a magnetic diffusivity of magnetic field. In the second equation in (1.1), the circled times $\otimes$ means matrix multiplication, namely if $\mathbf{a} = (a^1, a^2, a^3)$, $\mathbf{b} = (b^1, b^2, b^3)$, then

$$\mathbf{a} \otimes \mathbf{b} = \begin{pmatrix} a^1 b^1 & a^1 b^2 & a^1 b^3 \\ a^2 b^1 & a^2 b^2 & a^2 b^3 \\ a^3 b^1 & a^3 b^2 & a^3 b^3 \end{pmatrix}.$$

Now, we consider the Cauchy problems of (1.1)–(1.3) with $(\rho, \mathbf{u}, \mathbf{B})(x, t)$ vanishing at infinity:

$$(\rho, \mathbf{u}, \mathbf{B})(x, t) \to 0 \quad \text{as } |x| \to \infty, \tag{1.4}$$

and the initial conditions:

$$(\rho, \mathbf{u}, \mathbf{B})(x, 0) = (\rho_0, \mathbf{u}_0, \mathbf{B}_0)(x) \quad \text{with } x \in \mathbb{R}^3. \tag{1.5}$$

A great number of works have been devoted to the well-posedness theory of the multidimensional compressible MHD equations. The system of equations (1.1) describes the interaction between fluid flow and magnetic field. If we ignore the magnetic effects in (1.1) (i.e., $\mathbf{B} = 0$), then the MHD system reduces to the Navier–Stokes system, which has been discussed by many mathematicians (see, for example, [11, 17, 18]). In [5], Huang et al. (2012) established the global existence and uniqueness of classical solutions to the Cauchy problem for the compressible Navier–Stokes equations in 3D with smooth initial data that are of small energy. Then, in [6], Huang et al. (2014) considered the two-dimensional density-dependent Navier–Stokes equations over bounded domains, and they derived a new blow-up criterion for strong solutions with vacuum. In [10], Li et al. (2019) were concerned with the global well-posedness and large time asymptotic behavior of strong solutions to the Cauchy problems of the Navier–Stokes equations for viscous compressible barotropic flows in 2D and 3D. However, if we consider the influence of magnetic field, the physical phenomena and mathematical structure of these equations make it more complex than the Navier–Stokes system, which makes more and more researchers begin to study the equations with magnetic field (see [2, 14, 16] and the references therein). Moreover, in [3], Fan and Li obtained the global strong solutions to the 3D compressible nonisentropic MHD equations with zero resistivity, and the results do not need the positivity of initial density, thus, it may vanish in an open subset of the domain. Hu and Wang in [4] considered the equations of the three-dimensional viscous, compressible, and

heat-conducting magnetohydrodynamic flows in a bounded domain, and they obtained a solution to the initial-boundary value problem through an approximation scheme and a weak convergence method, and then, the existence of a global variational weak solution to the three-dimensional full magnetohydrodynamic equations with large data was established. Later, in [4], they got the global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows. In [15], Zhang et al. (2009) studied the initial boundary value problems of MHD equations in plasma physics, and obtained the global existence of weak solutions with cylindrical symmetry. Recently, for the Cauchy problem, Li et al. (2013) in [9] considered the three-dimensional isentropic compressible magnetohydrodynamic equations, and they proved the global well-posedness of a classical solution with small energy but possibly large oscillations, where the flow density was allowed to contain vacuum states. Later, Si et al. (2018) in [13] improved the result of [17], and excluded the unsatisfactory restriction on the adiabatic exponent (i.e., $\gamma \in (1, 3/2)$), and they obtained the global classical solutions of compressible isentropic Navier–Stokes equations with small density and the adiabatic exponent $\gamma \in (1, 6)$ and $\gamma \in (1, \infty)$, respectively.

The main purpose of this paper is to obtain the global existence and uniqueness of classical solution of the problem (1.1)–(1.5). Before stating the main results, we explain the notation and conventions used throughout this paper. We denote

$$\int f(x)\,dx = \int_{\mathbb{R}^3} f(x)\,dx.$$

For $1 < r < \infty$ and $k \in \mathbb{Z}$, we denote the standard homogeneous and inhomogeneous Sobolev spaces:

$$\begin{cases} L^r = L^r(\mathbb{R}^3), \qquad D^{k,r} = \{\mathbf{u} \in L^1_{\mathrm{loc}} | \|\nabla^k \mathbf{u}\|_{L^r} < \infty\}, \qquad \|\mathbf{u}\|_{D^{k,r}} = \|\nabla^k \mathbf{u}\|_{L^r}, \\ W^{k,r} = L^r \cap D^{k,r}, \qquad H^k = W^{k,2}, \qquad D^k = D^{k,2}, \qquad D^1 = \{\mathbf{u} \in L^6 | \|\nabla \mathbf{u}\|_{L^2} < \infty\}. \end{cases}$$

The total energy is defined as follows:

$$E(t) = \int \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{A}{\gamma - 1}\rho^\gamma + \frac{1}{2}|\mathbf{B}|^2 \right)(x, t)\,dx, \tag{1.6}$$

and the initial energy is denoted by E_0, i.e.,

$$E_0 \triangleq E(0) = \int \left(\frac{1}{2}\rho_0|\mathbf{u}_0|^2 + \frac{A}{\gamma - 1}\rho_0^\gamma + \frac{1}{2}|\mathbf{B}_0|^2 \right)(x)\,dx. \tag{1.7}$$

The first result of this paper is formulated in the following theorem.

Theorem 1.1 *For any given numbers $M_0, M_1 > 0$ and $q \in (3, 6)$, suppose that*

$$\begin{cases} \rho_0|\mathbf{u}_0|^2 + \rho_0^\gamma + |\mathbf{B}_0|^2 \in L^1, \qquad (\mathbf{u}_0, \mathbf{B}_0) \in D^1 \cap D^2, \\ 0 \le \inf \rho_0 \le \rho_0(x) \le \sup \rho_0 \le M_0, \qquad (\rho_0, P(\rho_0)) \in H^2 \cap W^{2,q}, \\ \|\nabla \mathbf{u}_0\|_{L^2}^2 + \|\nabla \mathbf{B}_0\|_{H^1}^2 \le M_1, \end{cases} \tag{1.8}$$

and the compatibility condition holds

$$-\mu \Delta \mathbf{u}_0 - (\lambda + \mu)\nabla \operatorname{div} \mathbf{u}_0 + \nabla P(\rho_0) - (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 = \rho_0 g \quad with$$

$$\|\sqrt{\rho_0}g\|_{L^2}^2 \le M_2, \tag{1.9}$$

for some $g \in D^1$. *Then, there exists a positive constant* ε *depending only on* μ, ν, λ, A, γ, M_0, M_1, *and* M_2 *such that if*

$$\|\mathbf{B}_0\|_{L^3}^{\frac{1}{2\gamma}} \le M_0 \le \varepsilon \quad and \quad 1 < \gamma < 6, \tag{1.10}$$

then for any $0 < T < \infty$, *there exists a unique global classical solution* $(\rho, \mathbf{u}, \mathbf{B})$ *of the problem* (1.1)–(1.5) *on* $\mathbb{R}^3 \times [0, T]$, *satisfying*

$$0 \le \rho \le 2M_0 \quad for\ all\ x \in \mathbb{R}^3, t \ge 0, \tag{1.11}$$

and

$$\begin{cases} (\rho, P(\rho)) \in C([0, T]; H^2 \cap W^{2,q}), \quad \sqrt{\rho}\mathbf{u}_t \in L^\infty(0, T; L^2), \\ \mathbf{u} \in C([0, T]; D^1 \cap D^2) \cap L^\infty(\tau, T; D^3 \cap W^{3,q}), \\ \mathbf{u}_t \in L^\infty(\tau, T; D^1 \cap D^2) \cap H^1(\tau, T; D^1), \\ \mathbf{B} \in C([0, T]; H^2) \cap L^\infty(\tau, T; H^3), \\ \mathbf{B}_t \in C([0, T]; L^2) \cap H^1(\tau, T; H^1) \end{cases} \tag{1.12}$$

for any $0 < \tau < T < \infty$.

Remark 1.1 In Theorem 1.1, the classical solution of (1.1)–(1.5) is justified under the condition that the initial density and the L^3-norm of the initial magnetic field are sufficiently small, and this solution is far away from the initial time. It is worth noting that the total initial energy E_0 can be arbitrarily large and the vacuum states are allowed.

Remark 1.2 The proof of Theorem 1.1 is based on a new t-weighted estimate of $\|(\nabla \dot{\mathbf{u}}, \nabla \mathbf{B}_t)\|_{L^2}$ (see (3.56)), and the $L^1(0, T; L^\infty)$-estimate of the effective viscous flux F will be achieved by making full use of (3.56). It is worth pointing out that the effective viscous flux F plays an important role in applying the Zlotnik's inequality (see Lemma 2.3) to finish the proof of the (a priori) upper bound of the density.

Remark 1.3 Indeed, if, in addition, the conservation law of the total mass holds (i.e., $\|\rho(t)\|_{L^1} = \|\rho_0\|_{L^1}$ for all $t > 0$), then Theorem 1.1 is similar to the results of [12] for all $\gamma > 1$.

The rest of the paper is organized as follows. In Sect. 2, we recall some known facts and elementary inequalities which will be frequently used later. Section 3 is devoted to the global a priori estimates, which are necessary for the proof of Theorem 1.1.

2 Preliminaries

In this section, we will recall some known facts and elementary inequalities which will be used frequently later. We start with the well-known Gagliardo–Nirenberg inequality [8].

Lemma 2.1 *For $p \in [2,6]$, $q \in (1,\infty)$, and $r \in (3,\infty)$, assume that $f \in H^1(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3) \cap D^{1,r}(\mathbb{R}^3)$. Then there exists a generic constant $C > 0$, depending only on q and r, such that*

$$\|f\|_{L^p} \leq C\|f\|_{L^2}^{\frac{6-p}{2p}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2p}}, \tag{2.1}$$

$$\|g\|_{L^\infty} \leq C\|g\|_{L^q}^{\frac{q(r-3)}{3r+q(r-3)}} \|\nabla g\|_{L^r}^{\frac{3r}{3r+q(r-3)}}. \tag{2.2}$$

As in [9], we introduce the effective viscous flux F, the vorticity ω, and the material derivative "$\cdot$", which are defined as follows:

$$F \triangleq (2\mu + \lambda)\operatorname{div}\mathbf{u} - P(\rho) - \frac{1}{2}|\mathbf{B}|^2, \qquad \omega \triangleq \nabla \times \mathbf{u}, \qquad \dot{\mathbf{u}} \triangleq \mathbf{u}_t + \mathbf{u} \cdot \nabla\dot{\mathbf{u}}. \tag{2.3}$$

Then it is easily derived from (1.1) that

$$\Delta F = \operatorname{div}(\rho\dot{\mathbf{u}}) - \operatorname{div}\operatorname{div}(\mathbf{B} \otimes \mathbf{B}) \quad \text{and} \quad \mu\Delta\omega = \nabla \times \left(\rho\dot{\mathbf{u}} - \operatorname{div}(\mathbf{B} \otimes \mathbf{B})\right). \tag{2.4}$$

Thus, it follows from Lemma 2.1 and the standard L^p-estimates of elliptic equations that we have the following lemma.

Lemma 2.2 *Let $(\rho, \mathbf{u}, \mathbf{B})$ be a smooth solution of (1.1)–(1.4). Then there exists a generic constant $C > 0$ such that for any $p \in [2,6]$,*

$$\|\nabla F\|_{L^p} + \|\nabla\omega\|_{L^p} \leq C\left(\|\rho\dot{\mathbf{u}}\|_{L^p} + \|\nabla\mathbf{B} \cdot \mathbf{B}\|_{L^p}\right), \tag{2.5}$$

$$\|F\|_{L^p} \leq C\left(\|\nabla\mathbf{u}\|_{L^2} + \left\|P(\rho)\right\|_{L^2} + \left\||\mathbf{B}|^2\right\|_{L^2}\right)^{(6-p)/2p}$$
$$\times \left(\|\rho\dot{\mathbf{u}}\|_{L^2} + \|\nabla\mathbf{B} \cdot \mathbf{B}\|_{L^2}\right)^{(3p-6)/2p}, \tag{2.6}$$

$$\|\omega\|_{L^p} \leq C\|\nabla\mathbf{u}\|_{L^2}^{(6-p)/2p}\left(\|\rho\dot{\mathbf{u}}\|_{L^2} + \|\nabla\mathbf{B} \cdot \mathbf{B}\|_{L^2}\right)^{(3p-6)/2p}, \tag{2.7}$$

$$\|\nabla\mathbf{u}\|_{L^p} \leq C\left(\|F\|_{L^p} + \left\|P(\rho)\right\|_{L^p} + \left\||\mathbf{B}|^2\right\|_{L^p} + \|\omega\|_{L^p}\right), \tag{2.8}$$

$$\|\nabla\mathbf{u}\|_{L^p} \leq C\|\nabla\mathbf{u}\|_{L^2}^{(6-p)/2p}\left(\|\rho\dot{\mathbf{u}}\|_{L^2} + \left\|P(\rho)\right\|_{L^6} + \|\nabla\mathbf{B} \cdot \mathbf{B}\|_{L^2}\right)^{(3p-6)/2p}. \tag{2.9}$$

The proof of Lemma 2.2 can be found in [9, 13], hence, we skip it for simplicity.

Lemma 2.3 *Let $y \in W^{1,1}(0,T)$ satisfy the ODE system*:

$$y' = g(y) + b'(t) \quad on\ [0,T], \qquad y(0) = y_0,$$

where $b \in W^{1,1}(0,T)$, $g \in C(\mathbb{R})$, and $g(+\infty) = -\infty$. Assume that there are two constants $N_0 \geq 0$ and $N_1 \geq 0$ such that for all $0 \leq t_1 < t_2 \leq T$,

$$b(t_2) - b(t_1) \leq N_0 + N_1(t_2 - t_1). \tag{2.10}$$

Then

$$y(t) \leq \max\{y_0, \xi^*\} + N_0 < +\infty \quad on\ [0, T],$$

where $\xi^* \in \mathbb{R}$ is a constant such that

$$g(\xi) \leq -N_1 \quad for\ \xi \geq \xi^*. \tag{2.11}$$

We can use the above Zlotnik's inequality (see [19]) to prove the t-independent upper bound of the density.

Finally, we state the local existence result of classical solutions to the problem (1.1)–(1.5) with large initial data which may contain vacuum states (see [9]).

Lemma 2.4 *Assume that the initial data $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$ satisfy the conditions (1.8) and (1.9) of Theorem 1.1. Then there exists a positive time $T_0 > 0$ and a unique classical solution $(\rho, \mathbf{u}, \mathbf{B})$ of (1.1)–(1.5) on $\mathbb{R}^3 \times (0, T_0]$, satisfying $\rho \geq 0$, and for any $\tau \in (0, T_0)$,*

$$\begin{cases} (\rho, P(\rho)) \in C([0, T_0]; H^1 \cap W^{1,q}) \cap L^\infty(0, T_0; H^2 \cap W^{2,q}), \\ \mathbf{u} \in C([0, T_0]; D^1 \cap D^2) \cap L^\infty(\tau, T_0; D^3 \cap W^{3,q}), \\ \mathbf{u}_t \in L^\infty(\tau, T_0; D^1 \cap D^2) \cap H^1(\tau, T_0; D^1), \\ \mathbf{B} \in C([0, T_0]; H^2) \cap L^\infty(\tau, T_0; H^3), \\ \mathbf{B}_t \in C([0, T_0]; L^2) \cap H^1(\tau, T_0; H^1). \end{cases} \tag{2.12}$$

3 Proof of Theorem 1.1

In the section, we will establish the uniform a priori bounds of local solutions $(\rho, \mathbf{u}, \mathbf{B})$ to the Cauchy problems (1.1)–(1.5) whose existence is guaranteed by Lemma 2.4. Thus, let $T > 0$ be a fixed time and $(\rho, \mathbf{u}, \mathbf{B})$ be the smooth solution of (1.1)–(1.5) on $\mathbb{R}^3 \times [0, T]$ with smooth initial data $(\rho_0, \mathbf{u}_0, \mathbf{B}_0)$ satisfying (1.8). To estimate this solution, we define

$$A_1(T) \triangleq \sup_{t \in [0,T]} \left\| (\nabla \mathbf{u}, \nabla \mathbf{B}) \right\|_{L^2}^2 + \int_0^T \left\| \left(\sqrt{\rho} \dot{\mathbf{u}}, \nabla^2 \mathbf{B}, \mathbf{B}_t \right) \right\|_{L^2}^2 dt,$$

$$A_2(T) \triangleq \sup_{t \in [0,T]} \left\| \left(\sqrt{\rho} \dot{\mathbf{u}}, \nabla^2 \mathbf{B}, \mathbf{B}_t \right) \right\|_{L^2}^2 + \int_0^T \left\| (\nabla \dot{\mathbf{u}}, \nabla \mathbf{B}_t) \right\|_{L^2}^2 dt.$$

Here, $\|(f, g)\|_{L^p} \triangleq \|f\|_{L^p} + \|g\|_{L^p}$.

The proof of Theorem 1.1 is based on the following key a priori estimates of $(\rho, \mathbf{u}, \mathbf{B})$.

Proposition 3.1 *Let the conditions (1.8) and (1.9) be in force. Assume that $(\rho, \mathbf{u}, \mathbf{B})$ is a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3 \times [0, T]$ with $T > 0$. Then there exist positive constants K and ε, depending only on μ, λ, ν, γ, A, E_0, M_1, and M_2, such that if*

$$\begin{cases} 0 \leq \rho(x, t) \leq 2M_0, \quad \forall (x, t) \in \mathbb{R}^3 \times [0, T], \\ A_1(t) + A_2(t) \leq 2K, \end{cases} \tag{3.1}$$

then one has

$$
\begin{cases}
0 \le \rho(x,t) \le \frac{7}{4}M_0, \quad \forall (x,t) \in \mathbb{R}^3 \times [0,T], \\
A_1(t) + A_2(t) \le K,
\end{cases}
\tag{3.2}
$$

provided

$$
\|\mathbf{B}_0\|_{L^3}^{\frac{1}{2\gamma}} \le M_0 \le \varepsilon \quad and \quad \varepsilon \in (1,6).
\tag{3.3}
$$

The proof of Proposition 3.1 will be presented by a series of lemmas below. For simplicity, we will use the conventions that C and C_i $(i = 1,2,\dots)$ denote various positive constants, which may depend on μ, λ, ν, γ, A, E_0, M_1, and M_2, but are independent of T and M_0. Sometimes we also write $C(\alpha)$ to emphasize the dependence on α.

We first begin with the following standard energy estimates, which can be easily deduced from (1.1)–(1.5).

Lemma 3.1 *Let $(\rho, \mathbf{u}, \mathbf{B})$ be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3 \times [0,T]$. Then*

$$
E(t) + \int_0^T \left(\mu \|\nabla \mathbf{u}\|_{L^2}^2 + (\mu + \lambda)\|\operatorname{div}\mathbf{u}\|_{L^2}^2 + \nu \|\nabla \mathbf{B}\|_{L^2}^2 \right) dt \le E_0,
\tag{3.4}
$$

where $E(t) \ge 0$ and E_0 are as given in (1.6) and (1.7).

Proof Multiplying $(1.1)_1$, $(1.1)_2$, and $(1.1)_3$ by $\frac{\gamma}{\gamma-1}A\rho^{\gamma-1}$, $\mathbf{u}$, and $\mathbf{B}$, respectively, and integrating the resulting equations by parts over $\mathbb{R}^3$, we obtain after adding them together that

$$
\frac{d}{dt}E(t) + \mu \|\nabla \mathbf{u}\|_{L^2}^2 + (\mu + \lambda)\|\operatorname{div}\mathbf{u}\|_{L^2}^2 + \nu \|\nabla \mathbf{B}\|_{L^2}^2 = 0,
$$

which, integrated over $(0,t)$, $\forall t \in [0,T]$, immediately leads to (3.4). $\qquad\square$

By virtue of (3.1) and (3.4), we infer from Lemma 2.1 ($p = 6$ in (2.1)) that

$$
\begin{aligned}
&\int_0^T \left(\|\mathbf{u}\|_{L^6}^4 + \|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla \mathbf{B}\|_{L^2}^4 \right) dt \\
&\le C \int_0^T \left(\|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla \mathbf{B}\|_{L^2}^4 \right) dt \\
&\le C \sup_{t \in [0,T]} \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 \right) \int_0^T \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla \mathbf{B}\|_{L^2}^2 \right) dt \le C(E_0)K.
\end{aligned}
\tag{3.5}
$$

Here the constant $C > 0$ comes from the Gagliardo–Nirenberg–Sobolev inequality in (2.1), and we use the constant $C(E_0) > 0$ to emphasize the dependence on E_0. Thus, combining (3.4) and (3.5) yields the following lemma:

Lemma 3.2 *Let $(\rho, \mathbf{u}, \mathbf{B})$ be a smooth solution of (1.1)–(1.5) on $\mathbb{R}^3 \times [0,T]$ satisfying (3.1). Then*

$$
\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3}^3 + \int_0^T \|\mathbf{B}\|_{L^9}^3 \, dt \le e^{CK} \|\mathbf{B}_0\|_{L^3}^3,
\tag{3.6}
$$

where the constant $C > 0$ depends on ν, E_0, and the coefficients of the Gagliardo–Nirenberg–Sobolev inequality in Lemma 2.1, *but is independent of* M_0.

Proof Multiplying the third equation of (1.1) by $3|\mathbf{B}|\mathbf{B}$ and integrating by parts over $\mathbb{R}^3$, we have

$$
\frac{d}{dt}\|\mathbf{B}\|_{L^3}^3 + 3\nu\int\left(|\mathbf{B}||\nabla\mathbf{B}|^2 + |\mathbf{B}|\left|\nabla\left(|\mathbf{B}|\right)\right|^2\right)dx
$$
$$
\leq \nu\int|\mathbf{B}||\nabla\mathbf{B}|^2\,dx + C\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{B}\|_{L^{9/2}}^3, \tag{3.7}
$$

where the last term on the right-hand in (3.7) comes from the following inequality:

$$
\int|\nabla\mathbf{u}||\mathbf{B}|^3\,dx \leq C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{B}\|_{L^{9/2}}^{3/2}\|\mathbf{B}\|_{L^9}^{3/2}
$$
$$
\leq C\|\nabla\mathbf{u}\|_{L^2}\|\mathbf{B}\|_{L^{9/2}}^{3/2}\left\||\mathbf{B}|^{3/2}\right\|_{L^6}
$$
$$
\leq C\left(\|\nabla\mathbf{u}\|_{L^2}^2\|\mathbf{B}\|_{L^{9/2}}^3\right)^{1/2}\left\||\nabla\mathbf{B}||\mathbf{B}|^{1/2}\right\|_{L^2}.
$$

To deal with the right-hand side of (3.7), we notice that

$$
\|\mathbf{B}\|_{L^9}^3 \leq C\left\||\mathbf{B}|^{3/2}\right\|_{L^6}^2 \leq C\left\||\nabla\mathbf{B}||\mathbf{B}|^{1/2}\right\|_{L^2}^2,
$$

then

$$
\|\mathbf{B}\|_{L^{9/2}} \leq C\|\mathbf{B}\|_{L^3}^{1/2}\|\mathbf{B}\|_{L^9}^{1/2} \leq C\|\mathbf{B}\|_{L^3}^{1/2}\left\|\nabla|\mathbf{B}|^{3/2}\right\|_{L^2}^{1/3},
$$

which, together with (3.7), yields

$$
\frac{d}{dt}\|\mathbf{B}\|_{L^3}^3 + \|\mathbf{B}\|_{L^9}^3 \leq C\|\nabla\mathbf{u}\|_{L^2}^4\|\mathbf{B}\|_{L^3}^3.
$$

As a result, we deduce from (3.5) and the Gronwall's inequality that (3.6) holds.   □

Now, to estimate $A_1(T)$ and $A_2(T)$, we first prove the following lemma.

Lemma 3.3 *Let* $(\rho, \mathbf{u}, \mathbf{B})$ *be a smooth solution of* (1.1)–(1.5) *on* $\mathbb{R}^3 \times [0,T]$ *satisfying* (3.1). *Then*

$$
A_1(T) \leq C\left(\sup_{t\in[0,T]}\|\mathbf{B}\|_{L^3}^2\right)A_1(T) + C\left(1 + M_0^\gamma\right) + C\int_0^T\|\nabla\mathbf{u}\|_{L^4}^4\,dt, \tag{3.8}
$$

$$
A_2(T) \leq C + \frac{1}{4}A_1(T) + C\left[\left(M_0^{\frac{1}{2}}K + M_0^{\frac{5\gamma}{6}}K^{1/2} + \left(\sup_{t\in[0,T]}\|\mathbf{B}\|_{L^3}\right)K\right]A_1(T)
$$
$$
+ C\left(\sup_{t\in[0,T]}\|\mathbf{B}\|_{L^3}^2 + K\sup_{t\in[0,T]}\|\mathbf{B}\|_{L^3}^4\right)A_2(T) \tag{3.9}
$$
$$
+ C\int_0^T\left(\|\nabla\mathbf{u}\|_{L^4}^4 + \left\|P(\rho)\right\|_{L^4}^4\right)dt,
$$

where the constant $C > 0$ depends on μ, λ, ν, γ, A, E_0, M_1, M_2, and the coefficients of the Gagliardo–Nirenberg–Sobolev inequality in Lemma 2.1, *but is independent of* M_0.

Proof In order to prove (3.8), using Lemma 2.1 and equation $(1.1)_3$, we have

$$
\nu\big(\|\nabla\mathbf{B}\|_{L^2}^2\big)_t + \nu^2\big\|\nabla^2\mathbf{B}\big\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2
$$

$$
= \int (\mathbf{B}_t - \nu\Delta\mathbf{B})^2\,dx
$$

$$
= \int |\mathbf{B}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{B} - \mathbf{B}\operatorname{div}\mathbf{u}|^2\,dx
$$

$$
\le C\big(\|\mathbf{B}\|_{L^6}^2\|\nabla\mathbf{u}\|_{L^3}^2 + \|\mathbf{u}\|_{L^\infty}^2\|\nabla\mathbf{B}\|_{L^2}^2\big)
$$

$$
\le C\|\mathbf{B}\|_{L^2}\big\|\nabla^2\mathbf{B}\big\|_{L^2}\big(\|\nabla\mathbf{u}\|_{L^3}^2 + \|\mathbf{u}\|_{L^\infty}^2\big)
$$

$$
\le \frac{\nu^2}{2}\big\|\nabla^2\mathbf{B}\big\|_{L^2}^2 + C(E_0)\big(\|\nabla\mathbf{u}\|_{L^3}^4 + \|\mathbf{u}\|_{L^\infty}^4\big),
$$

which, together with the Cauchy–Schwarz inequality, one obtains

$$
\big(\nu\|\nabla\mathbf{B}\|_{L^2}^2\big)_t + \nu^2\big\|\nabla^2\mathbf{B}\big\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 \le C\big(\|\nabla\mathbf{u}\|_{L^3}^4 + \|\mathbf{u}\|_{L^\infty}^4\big). \tag{3.10}
$$

Next, multiplying $(1.1)_2$ by $\dot{\mathbf{u}}$ and integrating by parts, we deduce

$$
\int \rho|\dot{\mathbf{u}}|^2\,dx = \int\bigg(-\nabla P\cdot\dot{\mathbf{u}} + \mu\Delta\mathbf{u}\cdot\dot{\mathbf{u}} + (\lambda+\mu)\nabla\operatorname{div}\mathbf{u}\cdot\dot{\mathbf{u}}
$$
$$
+ \mathbf{B}\cdot\nabla\mathbf{B}\cdot\dot{\mathbf{u}} - \frac{1}{2}\dot{\mathbf{u}}\cdot\nabla\big(|\mathbf{B}|^2\big)\bigg)\,dx \triangleq \sum_{i=1}^{5} I_i. \tag{3.11}
$$

The right-hand side of (3.11) can be estimated as follows. It holds by $(1.1)_1$ that

$$
P(\rho)_t + \mathbf{u}\cdot\nabla P(\rho) + \gamma P(\rho)\operatorname{div}\mathbf{u} = 0, \tag{3.12}
$$

which, together with (3.1), yields

$$
I_1 = -\int \nabla P\cdot(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u})\,dx
$$

$$
= \int\big(P(\rho)\operatorname{div}\mathbf{u}_t - (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\nabla P(\rho)\big)\,dx
$$

$$
= \bigg(\int P(\rho)\operatorname{div}\mathbf{u}\,dx\bigg)_t - \int\big(P(\rho)_t\operatorname{div}\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u}\cdot\nabla P(\rho)\big)\,dx \tag{3.13}
$$

$$
= \bigg(\int P(\rho)\operatorname{div}\mathbf{u}\,dx\bigg)_t + \int\big((\gamma-1)P(\rho)(\operatorname{div}\mathbf{u})^2 + P(\rho)|\nabla\mathbf{u}|^2\big)\,dx
$$

$$
\le C\bigg(\int P(\rho)\operatorname{div}\mathbf{u}\,dx\bigg)_t + C(A,\gamma)M_0^\gamma\|\nabla\mathbf{u}\|_{L^2}^2.
$$

Thanks to (3.4), we find (keep in mind that $0\le\rho\le 2M_0$)

$$
\int P(\rho)\operatorname{div}\mathbf{u}\,dx \le \frac{\mu}{8}\|\nabla\mathbf{u}\|_{L^2}^2 + C\big\|P(\rho)\big\|_{L^\infty}\big\|P(\rho)\big\|_{L^1}
$$
$$
\le \frac{\mu}{8}\|\nabla\mathbf{u}\|_{L^2}^2 + C(E_0)M_0^\gamma. \tag{3.14}
$$

Integrating by parts, one has

$$
\begin{aligned}
I_2 = \mu \int \Delta \mathbf{u} \cdot (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u})\, dx &= -\frac{\mu}{2}\big(\|\nabla \mathbf{u}\|_{L^2}^2\big)_t + \mu \int \partial_i u^j \partial_i\big(u^k \partial_k u^j\big)\, dx \\
&\le -\frac{\mu}{2}\big(\|\nabla \mathbf{u}\|_{L^2}^2\big)_t + \mu \int \partial_i u^j \partial_i u^k \partial_k u^j\, dx - \frac{1}{2}\int |\nabla \mathbf{u}|^2 \operatorname{div} \mathbf{u}\, dx \\
&\le -\frac{\mu}{2}\big(\|\nabla \mathbf{u}\|_{L^2}^2\big)_t + C\|\nabla \mathbf{u}\|_{L^3}^3.
\end{aligned}
\tag{3.15}
$$

Similarly,

$$
I_3 = (\lambda + \mu) \int \dot{\mathbf{u}} \cdot \nabla \operatorname{div} \mathbf{u}\, dx \le -\frac{\lambda + \mu}{2}\big(\|\operatorname{div} \mathbf{u}\|_{L^2}^2\big)_t + C\|\nabla \mathbf{u}\|_{L^3}^3.
\tag{3.16}
$$

Integration by parts also gives

$$
\begin{aligned}
I_4 &= \int \mathbf{B} \cdot \nabla \mathbf{B} \cdot (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u})\, dx \\
&= -\int \mathbf{B} \cdot \nabla \mathbf{u}_t \cdot \mathbf{B}\, dx + \int B^i \partial_i B^j u^k \partial_k u^j\, dx \\
&= -\left(\int \mathbf{B} \cdot \nabla \mathbf{u} \cdot \mathbf{B}\, dx\right)_t + \int \big(\mathbf{B}_t \cdot \nabla \mathbf{u} \cdot \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{u} \cdot \mathbf{B}_t + B^i \partial_i B^j u^k \partial_k u^j\big)\, dx \\
&\le -\left(\int \mathbf{B} \cdot \nabla \mathbf{u} \cdot \mathbf{B}\, dx\right)_t \\
&\quad + C\|\mathbf{B}_t\|_{L^2}\|\nabla \mathbf{u}\|_{L^3}\|\mathbf{B}\|_{L^6} + C\|\mathbf{B}\|_{L^2}\|\nabla \mathbf{B}\|_{L^6}\|\mathbf{u}\|_{L^\infty}\|\nabla \mathbf{u}\|_{L^3} \\
&\le -\left(\int \mathbf{B} \cdot \nabla \mathbf{u} \cdot \mathbf{B}\, dx\right)_t \\
&\quad + \frac{1}{4}\big(\|\mathbf{B}_t\|_{L^2}^2 + \nu^2\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\big) + C(E_0)\big(\|\nabla \mathbf{u}\|_{L^3}^4 + \|\mathbf{u}\|_{L^\infty}^4\big),
\end{aligned}
\tag{3.17}
$$

where we also used (2.1), (3.4), and the simple fact

$$
\|\mathbf{B}\|_{L^6} \le C\|\mathbf{B}\|_{L^2}^{1/2}\|\nabla \mathbf{B}\|_{L^6}^{1/2} \le C\|\mathbf{B}\|_{L^2}^{1/2}\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^{1/2},
$$

which, together with Cauchy–Schwarz inequality, gives

$$
\int \mathbf{B} \cdot \nabla \mathbf{u} \cdot \mathbf{B}\, dx \le \frac{\mu}{8}\|\nabla \mathbf{u}\|_{L^2}^2 + C\|\mathbf{B}\|_{L^3}^2\|\nabla \mathbf{B}\|_{L^2}^2.
\tag{3.18}
$$

Similarly,

$$
\begin{aligned}
I_5 &= -\frac{1}{2}\int \nabla\big(|\mathbf{B}|^2\big) \cdot (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u})\, dx \\
&\le \frac{1}{2}\left(\int |\mathbf{B}|^2 \operatorname{div} \mathbf{u}\, dx\right)_t \\
&\quad + \frac{1}{4}\big(\|\mathbf{B}_t\|_{L^2}^2 + \nu^2\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\big) + C(E_0)\big(\|\nabla \mathbf{u}\|_{L^3}^4 + \|\mathbf{u}\|_{L^\infty}^4\big)
\end{aligned}
\tag{3.19}
$$

and

$$\frac{1}{2}\int |\mathbf{B}|^2 \operatorname{div} \mathbf{u}\,dx \le \frac{\mu}{8}\|\nabla \mathbf{u}\|_{L^2}^2 + C\|\mathbf{B}\|_{L^3}^2\|\nabla \mathbf{B}\|_{L^2}^2. \tag{3.20}$$

Thus, substituting (3.13), (3.15)–(3.17), and (3.19) into (3.11), using (3.10), we obtain

$$\begin{aligned}
&\left(\frac{\mu}{2}\|\nabla \mathbf{u}\|_{L^2}^2 + \frac{\mu+\lambda}{2}\|\operatorname{div}\mathbf{u}\|_{L^2}^2 + \nu\|\nabla \mathbf{B}\|_{L^2}^2\right)_t \\
&\quad + \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \nu^2\left\|\nabla^2 \mathbf{B}\right\|_{L^2}^2 \\
&\le \left(\int\left[\left(P(\rho)\operatorname{div}\mathbf{u} - \mathbf{B}\cdot\nabla\mathbf{u}\cdot\mathbf{B} + \frac{1}{2}|\mathbf{B}|^2\operatorname{div}\mathbf{u}\right]dx\right)_t + CM_0^\gamma\|\nabla\mathbf{u}\|_{L^2}^2 \\
&\quad + C\left(\|\nabla\mathbf{u}\|_{L^3}^3 + \|\nabla\mathbf{u}\|_{L^3}^4 + \|\mathbf{u}\|_{L^\infty}^4\right).
\end{aligned} \tag{3.21}$$

Thanks to (2.1) and (2.2), we find from (3.21) that

$$\begin{aligned}
&\int_0^T \left(\|\nabla\mathbf{u}\|_{L^3}^3 + \|\nabla\mathbf{u}\|_{L^3}^4 + \|\mathbf{u}\|_{L^\infty}^4\right)dt \\
&\le C\int_0^T \left(\|\nabla\mathbf{u}\|_{L^2}^2 + \|\nabla\mathbf{u}\|_{L^4}^4 + \|\nabla\mathbf{u}\|_{L^2}^{4/3}\|\nabla\mathbf{u}\|_{L^4}^{8/3}\right)dt \\
&\le C + \delta\int_0^T \|\nabla\mathbf{u}\|_{L^2}^4\,dt + C(\delta)\int_0^T \|\nabla\mathbf{u}\|_{L^4}^4\,dt \\
&\le C + \delta E_0/\mu \sup_{t\in[0,T]}\|\nabla\mathbf{u}\|_{L^2}^2 + C(\delta)\int_0^T \|\nabla\mathbf{u}\|_{L^4}^4\,dt,
\end{aligned} \tag{3.22}$$

where $\delta > 0$ is an undetermined small constant and the constant $C(\delta) > 0$ depends on δ and the coefficients in (2.1). Moreover, the derivation of inequality (3.22) also uses the following inequalities:

$$\|\nabla\mathbf{u}\|_{L^3} \le C\|\nabla\mathbf{u}\|_{L^2}^{1/3}\|\nabla\mathbf{u}\|_{L^4}^{2/3} \quad\text{and}\quad \|\mathbf{u}\|_{L^\infty} \le C\|\mathbf{u}\|_{L^6}^{1/3}\|\nabla\mathbf{u}\|_{L^4}^{2/3}.$$

Combining (3.14), (3.18), and (3.20), then integrating (3.21) over $(0, T)$, and choosing $\delta > 0$ suitably small in (3.22), by virtue of the Gronwall's inequality, we obtain (3.8).

To prove (3.9), applying $\dot{u}^j[\partial_t + \operatorname{div}(\mathbf{u}\cdot)]$ to $(1.1)_2^j$, summing with respect to j, and integrating the resulting equation over $\mathbb{R}^3$, we obtain after integration by parts

$$\begin{aligned}
\left(\frac{1}{2}\int \rho|\dot{\mathbf{u}}|^2\,dx\right)_t &= -\int \dot{u}^j\left[\partial_j P_t + \operatorname{div}(\mathbf{u}\partial_j P)\right]dx \\
&\quad + \mu\int \dot{u}^j\left[\Delta u_t^j + \operatorname{div}\left(\mathbf{u}\Delta u^j\right)\right]dx \\
&\quad + (\lambda+\mu)\int \dot{u}^j\left[\partial_j\partial_t(\operatorname{div}\mathbf{u}) + \operatorname{div}\left(\mathbf{u}\partial_j(\operatorname{div}\mathbf{u})\right)\right]dx \\
&\quad + \int \dot{u}^j\left[\partial_t\left(\mathbf{B}\cdot\nabla B^j\right) + \operatorname{div}\left(\mathbf{u}\mathbf{B}\cdot\nabla B^j\right)\right]dx \\
&\quad - \frac{1}{2}\int \dot{u}^j\left[\partial_t\partial_j\left(|\mathbf{B}|^2\right) + \operatorname{div}\left(\mathbf{u}\partial_j\left(|\mathbf{B}|^2\right)\right)\right]dx \triangleq \sum_{i=1}^5 J_i,
\end{aligned} \tag{3.23}$$

where the first term on the right-hand side can be estimated as follows, based on integration by parts and (3.12):

$$
\begin{aligned}
J_1 &= \int \left(\partial_j \dot{u}^j P(\rho)_t + \partial_k \dot{u}^j u^k \partial_j P(\rho) \right) dx \\
&= \int \left(-\gamma P(\rho) \operatorname{div} \mathbf{u} \partial_j \dot{u}^j - \partial_j \dot{u}^j u^k \partial_k P(\rho) - \partial_j \left(\partial_k \dot{u}^j u^k \right) P(\rho) \right) dx \\
&= \int \left(-\gamma P(\rho) \operatorname{div} \mathbf{u} \partial_j \dot{u}^j + \partial_j \dot{u}^j \partial_k u^k P(\rho) - \partial_k \dot{u}^j \partial_j u^k P(\rho) \right) dx \\
&\leq \frac{\mu}{8} \| \nabla \dot{\mathbf{u}} \|_{L^2}^2 + C \left(\| \nabla \mathbf{u} \|_{L^4}^4 + \| P(\rho) \|_{L^4}^4 \right).
\end{aligned}
\tag{3.24}
$$

Similarly,

$$
J_2 = \mu \int \dot{u}^j \left[\Delta u_t^j + \operatorname{div} \left(\mathbf{u} \Delta u^j \right) \right] dx \leq -\frac{3\mu}{4} \| \nabla \dot{\mathbf{u}} \|_{L^2}^2 + C \| \nabla \mathbf{u} \|_{L^4}^4
\tag{3.25}
$$

and

$$
J_3 \leq -\frac{\lambda + \mu}{2} \| \operatorname{div} \dot{\mathbf{u}} \|_{L^2}^2 + \frac{\mu}{4} \| \nabla \dot{\mathbf{u}} \|_{L^2}^2 + C \| \nabla \mathbf{u} \|_{L^4}^4.
\tag{3.26}
$$

Next, integrating by parts, one has (keeping in mind that div $\mathbf{B} = 0$)

$$
\begin{aligned}
J_4 &= \int \dot{u}^j \left[\partial_t \left(\mathbf{B} \cdot \nabla B^j \right) + \operatorname{div} \left(\mathbf{u} \mathbf{B} \cdot \nabla B^j \right) \right] dx \\
&= \int \left(\dot{u}^j \left(B_t^i \partial_i B^j + B^i \partial_i B_t^j \right) - \partial_k \dot{u}^j u^k B^i \partial_i B^j \right) dx \\
&\leq -\int \left(B^j \partial_i \dot{u}^j B_t^i + B_t^j \partial_i \dot{u}^j B^i \, dx + \partial_k \dot{u}^j u^k B^i \partial_i B^j \right) dx \\
&\leq \frac{\mu}{8} \| \nabla \dot{\mathbf{u}} \|_{L^2}^2 + C \| \mathbf{B} \|_{L^3}^2 \| \nabla \mathbf{B}_t \|_{L^2}^2 + C \| \mathbf{B} \|_{L^3}^4 \| \nabla^2 \mathbf{B} \|_{L^2}^4 + \| \mathbf{u} \|_{L^\infty}^4.
\end{aligned}
\tag{3.27}
$$

Similarly,

$$
J_5 \leq \frac{\mu}{8} \| \nabla \dot{\mathbf{u}} \|_{L^2}^2 + C \| \mathbf{B} \|_{L^3}^2 \| \nabla \mathbf{B}_t \|_{L^2}^2 + C \| \mathbf{B} \|_{L^3}^4 \| \nabla^2 \mathbf{B} \|_{L^2}^4 + \| \mathbf{u} \|_{L^\infty}^4.
\tag{3.28}
$$

Substituting (3.24)–(3.28) into (3.23), we obtain

$$
\begin{aligned}
\left(\| \sqrt{\rho} \dot{\mathbf{u}} \|_{L^2}^2 \right)_t &+ \| \nabla \dot{\mathbf{u}} \|_{L^2}^2 \\
&\leq C \left(\| \nabla \mathbf{u} \|_{L^4}^4 + \| P(\rho) \|_{L^4}^4 \right) + C \| \mathbf{B} \|_{L^3}^2 \| \nabla \mathbf{B}_t \|_{L^2}^2 + C \| \mathbf{B} \|_{L^3}^4 \| \nabla^2 \mathbf{B} \|_{L^2}^4 + \| \mathbf{u} \|_{L^\infty}^4.
\end{aligned}
\tag{3.29}
$$

On the other hand, it follows from $(1.1)_3$ that

$$
\mathbf{B}_{tt} - \nu \Delta \mathbf{B}_t = (\mathbf{B} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{B} - \mathbf{B} \operatorname{div} \mathbf{u})_t.
\tag{3.30}
$$

Multiplying (3.30) by $\mathbf{B}_t$ and integrating over $\mathbb{R}^3$ yields

$$
\begin{aligned}
\left(\frac{1}{2}\|\mathbf{B}_t\|_{L^2}^2\right)_t &+ \nu\|\nabla\mathbf{B}_t\|_{L^2}^2 \\
&= \int (\mathbf{B}_t \cdot \nabla\mathbf{u} - \mathbf{u} \cdot \nabla\mathbf{B}_t - \mathbf{B}_t \cdot \operatorname{div}\mathbf{u}) \cdot \mathbf{B}_t \, dx \\
&\quad + \int \left(-\mathbf{B} \cdot \nabla(\mathbf{u} \cdot \nabla\mathbf{u}) + (\mathbf{u} \cdot \nabla\mathbf{u}) \cdot \nabla\mathbf{B} + \mathbf{B}\operatorname{div}(\mathbf{u} \cdot \nabla\mathbf{u})\right) \cdot \mathbf{B}_t \, dx \\
&\quad + \int (\mathbf{B} \cdot \nabla\dot{\mathbf{u}} - \dot{\mathbf{u}} \cdot \nabla\mathbf{B} - \mathbf{B} \cdot \operatorname{div}\dot{\mathbf{u}}) \cdot \mathbf{B}_t \, dx \triangleq \sum_{i=1}^{3} N_i.
\end{aligned}
\tag{3.31}
$$

Now, we estimate N_i as follows. By using (2.1), (2.2), and integrating by parts, we obtain

$$
\begin{aligned}
N_1 &= \int (\mathbf{B}_t \cdot \nabla\mathbf{u} - \mathbf{u} \cdot \nabla\mathbf{B}_t - \mathbf{B}_t \cdot \operatorname{div}\mathbf{u}) \cdot \mathbf{B}_t \, dx \\
&\leq C\|\nabla\mathbf{B}_t\|_{L^2}\|\mathbf{B}_t\|_{L^2}\left(\|\nabla\mathbf{u}\|_{L^3} + \|\mathbf{u}\|_{L^\infty}\right) \\
&\leq \frac{\nu}{8}\|\nabla\mathbf{B}_t\|_{L^2}^2 + C\|\mathbf{B}_t\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^6},
\end{aligned}
\tag{3.32}
$$

and

$$
N_2 = \int (\mathbf{B} \cdot \nabla\dot{\mathbf{u}} - \dot{\mathbf{u}} \cdot \nabla\mathbf{B} - \mathbf{B} \cdot \operatorname{div}\dot{\mathbf{u}}) \cdot \mathbf{B}_t \, dx \leq \frac{1}{4}\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\mathbf{B}\|_{L^3}^2\|\nabla\mathbf{B}_t\|_{L^2}^2,
\tag{3.33}
$$

and

$$
\begin{aligned}
N_3 &= \int \left(-\mathbf{B} \cdot \nabla(\mathbf{u} \cdot \nabla\mathbf{u}) + (\mathbf{u} \cdot \nabla\mathbf{u}) \cdot \nabla\mathbf{B} + \mathbf{B}\operatorname{div}(\mathbf{u} \cdot \nabla\mathbf{u})\right) \cdot \mathbf{B}_t \, dx \\
&= \int \left(u^k\partial_k u^j B^i\partial_i B_t^j + u^k\partial_k u^i\partial_i B^j B_t^j - u^k\partial_k u^i\partial_i B^j B_t^j - u^k\partial_k u^i B^j\partial_i B_t^j\right) dx \\
&= \int \left(u^k\partial_k u^j B^i\partial_i B_t^j - u^k\partial_k u^i B^j\partial_i B_t^j\right) dx \\
&\leq \frac{\nu}{8}\|\nabla\mathbf{B}_t\|_{L^2}^2 + C\|\nabla\mathbf{B}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^6}^2.
\end{aligned}
\tag{3.34}
$$

Thus, substituting (3.32)–(3.34) into (3.31), we infer that

$$
\begin{aligned}
\left(\|\mathbf{B}_t\|_{L^2}^2\right)_t &+ \|\nabla\mathbf{B}_t\|_{L^2}^2 \leq \frac{1}{4}\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + C\|\mathbf{B}\|_{L^3}^2\|\nabla\mathbf{B}_t\|_{L^2}^2 \\
&+ C\left(\|\mathbf{B}_t\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^6} + \|\nabla\mathbf{B}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^6}^2\right).
\end{aligned}
\tag{3.35}
$$

The combination of (3.29) and (3.35) gives

$$
\begin{aligned}
\left(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2\right)_t &+ \|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla\mathbf{B}_t\|_{L^2}^2 \\
&\leq C\left(\|\nabla\mathbf{u}\|_{L^4}^4 + \|P(\rho)\|_{L^4}^4\right) + C\|\mathbf{B}\|_{L^3}^2\|\nabla\mathbf{B}_t\|_{L^2}^2 + C\|\mathbf{B}\|_{L^3}^4\|\nabla^2\mathbf{B}\|_{L^2}^4 + \|\mathbf{u}\|_{L^\infty}^4 \\
&\quad + C\left(\|\mathbf{B}_t\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}\|\nabla\mathbf{u}\|_{L^6} + \|\nabla\mathbf{B}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^2}^2\|\nabla\mathbf{u}\|_{L^6}^2\right).
\end{aligned}
\tag{3.36}
$$

The right-hand side of (3.36) on $[0, T]$ can be estimated as follows. It holds by (3.1) that

$$
\begin{aligned}
\int_0^T \|\mathbf{B}\|_{L^3}^4 \|\nabla^2 \mathbf{B}\|_{L^2}^4 \, dt &\leq C \sup_{t \in [0,T]} \left(\|\mathbf{B}\|_{L^3}^4 \|\nabla^2 \mathbf{B}\|_{L^2}^2 \right) \int_0^T \|\nabla^2 \mathbf{B}\|_{L^2}^2 \, dt \\
&\leq C K A_2(T) \Big(\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3}^4 \Big).
\end{aligned}
\tag{3.37}
$$

An application of the L^p-theory for the elliptic equation $(1.1)_2$ leads to

$$
\begin{aligned}
\|\nabla \mathbf{u}\|_{L^6} &\leq C \left(\|\rho \dot{\mathbf{u}}\|_{L^2} + \|P(\rho)\|_{L^6} + \|\mathbf{B}\|_{L^\infty} \|\nabla \mathbf{B}\|_{L^2} \right) \\
&\leq C \left(M_0^{1/2} \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2} + M_0^{5\gamma/6} \|P(\rho)\|_{L^1}^{1/6} + \|\mathbf{B}\|_{L^3} \|\nabla^2 \mathbf{B}\|_{L^2} \right) \\
&\leq C \left(M_0^{1/2} \|\sqrt{\rho} \dot{\mathbf{u}}\|_{L^2} + M_0^{5\gamma/6} E_0^{1/6} + \|\mathbf{B}\|_{L^3} \|\nabla^2 \mathbf{B}\|_{L^2} \right) \\
&\leq C \left(M_0^{1/2} K^{1/2} + M_0^{5\gamma/6} + \|\mathbf{B}\|_{L^3} K^{1/2} \right),
\end{aligned}
\tag{3.38}
$$

where we have used (3.1). Thus

$$
\begin{aligned}
\int_0^T &\left(\|\mathbf{B}_t\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^6} + \|\nabla \mathbf{B}\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{u}\|_{L^6}^2 \right) dt \\
&\leq C K^{1/2} \left(M_0^{1/2} K^{1/2} + M_0^{5\gamma/6} + \Big(\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3} \Big) K^{1/2} \right) \int_0^T \|\mathbf{B}_t\|_{L^2}^2 \, dt \\
&\quad + C \Big(\sup_{t \in [0,T]} \|\nabla \mathbf{u}\|_{L^2}^2 \Big) \Big(M_0 K + M_0^{5\gamma/3} + \Big(\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3}^2 \Big) K \Big) \int_0^T \|\nabla \mathbf{B}\|_{L^2}^2 \, dt \\
&\leq C \Big(M_0^{1/2} K + M_0^{5\gamma/6} K^{1/2} + \Big(\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3} \Big) K \Big) A_1(T).
\end{aligned}
\tag{3.39}
$$

In addition, it holds by $(1.1)_3$, (2.1), (2.2), and (3.1) that

$$
\|\nabla^2 \mathbf{B}\|_{L^2}^2 \leq C \left(\|\mathbf{B}_t\|_{L^2}^2 + K \|\mathbf{B}\|_{L^3}^2 \|\nabla \mathbf{u}\|_{L^2}^2 \right),
$$

which, together with (3.36)–(3.39), (3.22), and Gronwall's inequality, gives rise to (3.9). □

We are now in a position of providing the concluding estimates of $A_1(T)$ and $A_2(T)$.

Lemma 3.4 *Let $(\rho, \mathbf{u}, \mathbf{B})$ be a smooth solution of* (1.1)–(1.5) *on* $\mathbb{R}^3 \times [0, T]$, *satisfying* $0 \leq \rho \leq 2M_0$. *Then there exist positive numbers* ε_1 *and* K, *depending only on* $\mu, \lambda, \nu, \gamma, A, E_0,$ $M_1,$ *and* M_2, *such that*

$$
A_1(T) + A_2(T) \leq K,
\tag{3.40}
$$

provided

$$
A_1(T) + A_2(T) \leq 2K \quad \text{and} \quad \|\mathbf{B}_0\|_{L^3}^{\frac{1}{2\gamma}} \leq M_0 \leq \varepsilon_1.
\tag{3.41}
$$

Proof Indeed, it follows from (2.8) that

$$\int_0^T \|\nabla \mathbf{u}\|_{L^4}^4 \, dt \le C \int_0^T \big(\|F\|_{L^4}^4 + \big\|P(\rho)\big\|_{L^4}^4 + \big\||\mathbf{B}|^2\big\|_{L^4}^4 + \|\omega\|_{L^4}^4 \big) \, dt. \tag{3.42}$$

To estimate $\|P(\rho)\|_{L^4}$, we multiply (3.12) by $3(P(\rho))^2$, integrate it over $\mathbb{R}^3$ and utilize (2.3) to get that

$$\begin{aligned}
&\left(\int (P(\rho))^3 \, dx \right)_t + \frac{3\gamma - 1}{2\mu + \lambda} \big\|P(\rho)\big\|_{L^4}^4 + \frac{3\gamma - 1}{2(2\mu + \lambda)} \int |\mathbf{B}|^2 P^3 \, dx \\
&= -\frac{3\gamma - 1}{2\mu + \lambda} \int F \cdot P^3 \, dx \le \frac{3\gamma - 1}{2(2\mu + \lambda)} \big\|P(\rho)\big\|_{L^4}^4 + C\|F\|_{L^4}^4,
\end{aligned} \tag{3.43}$$

thus

$$\int_0^T \big\|P(\rho)\big\|_{L^4}^4 \, dt \le C \int_0^T \|F\|_{L^4}^4 \, dt + C M_0^{2\gamma}. \tag{3.44}$$

Using (2.1), (2.2), (2.5)–(2.7), and the fact that $0 \le \rho \le 2M_0$, we deduce

$$\begin{aligned}
&\int_0^T \big(\|F\|_{L^4}^4 + \big\||\mathbf{B}|^2\big\|_{L^4}^4 + \|\omega\|_{L^4}^4 \big) \, dt \\
&\le \int_0^T \big(\|F\|_{L^2}^2 + \|\omega\|_{L^2}^2 \big) \big(\|F\|_{L^\infty}^2 + \|\omega\|_{L^\infty}^2 \big) \, dt + \int_0^T \|\mathbf{B}\|_{L^8}^8 \, dt \\
&\le \int_0^T \big(\|\nabla \mathbf{u}\|_{L^2}^2 + \big\|P(\rho)\big\|_{L^2}^2 + \big\||\mathbf{B}|^2\big\|_{L^2}^2 \big) \big(\|\rho\dot{\mathbf{u}}\|_{L^2} + \|\nabla\mathbf{B} \cdot \mathbf{B}\|_{L^2} \big) \\
&\quad \times \big(\|\rho\dot{\mathbf{u}}\|_{L^6} + \|\nabla\mathbf{B} \cdot \mathbf{B}\|_{L^6} \big) \, dt + \int_0^T \|\mathbf{B}\|_{L^8}^8 \, dt
\end{aligned} \tag{3.45}$$

Now, we estimate the right-hand side of (3.45) as follows:

$$\begin{aligned}
&\int_0^T \big(\|\nabla \mathbf{u}\|_{L^2}^2 + \big\|P(\rho)\big\|_{L^2}^2 \big) \|\rho\dot{\mathbf{u}}\|_{L^2} \|\rho\dot{\mathbf{u}}\|_{L^6} \, dt \\
&\le C M_0^{3/2} \big(A_1(T) + M_0^\gamma \big) \left(\int_0^T \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 \, dt \right)^{1/2} \left(\int_0^T \|\nabla\dot{\mathbf{u}}\|_{L^2}^2 \, dt \right)^{1/2} \\
&\le C M_0^{3/2} \big(A_1(T) + M_0^\gamma \big) A_1^{1/2}(T) A_2^{1/2}(T).
\end{aligned} \tag{3.46}$$

It follows from (2.1) and (2.2) that

$$\|\nabla\mathbf{B} \cdot \mathbf{B}\|_{L^6} \le C \big\|\nabla(\nabla\mathbf{B} \cdot \mathbf{B})\big\|_{L^2} \le C \|\nabla\mathbf{B}\|_{L^2}^{1/2} \big\|\nabla^2\mathbf{B}\big\|_{L^2}^{3/2}. \tag{3.47}$$

Hence

$$\begin{aligned}
&\int_0^T \big(\|\nabla \mathbf{u}\|_{L^2}^2 + \big\|P(\rho)\big\|_{L^2}^2 \big) \|\rho\dot{\mathbf{u}}\|_{L^2} \|\nabla\mathbf{B} \cdot \mathbf{B}\|_{L^6} \, dt \\
&\le C M_0^{1/2} \big(A_1(T) + M_0^\gamma \big) A_2^{1/2}(T) \left(\int_0^T \|\nabla\mathbf{B}\|_{L^2}^2 \, dt \right)^{1/4} \left(\int_0^T \big\|\nabla^2\mathbf{B}\big\|_{L^2}^2 \, dt \right)^{3/4} \\
&\le C M_0^{1/2} \big(A_1(T) + M_0^\gamma \big) A_1^{3/4}(T) A_2^{1/2}(T),
\end{aligned} \tag{3.48}$$

and

$$\int_0^T \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|P(\rho)\|_{L^2}^2 \right) \|\rho \dot{\mathbf{u}}\|_{L^6} \|\nabla \mathbf{B} \cdot \mathbf{B}\|_{L^2} \, dt$$

$$\leq C M_0 \left(A_1(T) + M_0^\gamma \right) \int_0^T \left\| \nabla^2 \mathbf{B} \right\|_{L^2} \|\mathbf{B}\|_{L^3} \|\nabla \dot{\mathbf{u}}\|_{L^2}$$

$$\leq C M_0 \left(\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3} \right) \left(A_1(T) + M_0^\gamma \right) \left(\int_0^T \left\| \nabla^2 \mathbf{B} \right\|_{L^2}^2 \, dt \right)^{1/2} \qquad (3.49)$$

$$\times \left(\int_0^T \|\nabla \dot{\mathbf{u}}\|_{L^2}^2 \, dt \right)^{1/2}$$

$$\leq C M_0 \left(\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3} \right) \left(A_1(T) + M_0^\gamma \right) A_1^{1/2}(T) A_2^{1/2}(T).$$

It follows from (3.47) that

$$\int_0^T \left(\|\nabla \mathbf{u}\|_{L^2}^2 + \|P(\rho)\|_{L^2}^2 \right) \|\nabla \mathbf{B} \cdot \mathbf{B}\|_{L^6} \|\nabla \mathbf{B} \cdot \mathbf{B}\|_{L^2} \, dt$$

$$\leq C \left(A_1(T) + M_0^\gamma \right) \int_0^T \left\| \nabla^2 \mathbf{B} \right\|_{L^2}^{5/2} \|\mathbf{B}\|_{L^3} \|\nabla \mathbf{B}\|_{L^2}^{1/2}$$

$$\leq C \left(\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3} \right) \left(A_1(T) + M_0^\gamma \right) A_2^{1/2} \left(\int_0^T \left\| \nabla^2 \mathbf{B} \right\|_{L^2}^2 \, dt \right)^{3/4} \left(\int_0^T \|\nabla \mathbf{B}\|_{L^2}^2 \, dt \right)^{1/4}$$

$$\leq C \left(\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3} \right) \left(A_1(T) + M_0^\gamma \right) A_1^{3/4}(T) A_2^{1/2}(T).$$

$$(3.50)$$

Similarly,

$$\int_0^T \left\| |\mathbf{B}|^2 \right\|_{L^2}^2 \|\rho \dot{\mathbf{u}}\|_{L^2} \left(\|\rho \dot{\mathbf{u}}\|_{L^6} + \|\nabla \mathbf{B} \cdot \mathbf{B}\|_{L^6} \right) dt$$

$$\leq C \left(\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3}^2 \right) \left(M_0^{3/2} A_1(T) + M_0^{1/2} A_1^{5/4}(T) \right) A_1^{1/2}(T) A_2^{1/2}(T), \qquad (3.51)$$

and

$$\int_0^T \left[\left\| |\mathbf{B}|^2 \right\|_{L^2}^2 \|\nabla \mathbf{B} \cdot \mathbf{B}\|_{L^2} \left(\|\rho \dot{\mathbf{u}}\|_{L^6} + \|\nabla \mathbf{B} \cdot \mathbf{B}\|_{L^6} \right) + \|\mathbf{B}\|_{L^8}^8 \right] dt$$

$$\leq C \left(\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3}^3 \right) \left(M_0 A_1(T) + A_1^{3/4}(T) A_2^{1/4}(T) \right) A_1^{1/2}(T) A_2^{1/2}(T). \qquad (3.52)$$

Substituting (3.46)–(3.52) into (3.45), by virtue of (3.42) and (3.44), one obtains

$$\int_0^T \left(\|\nabla \mathbf{u}\|_{L^4}^4 + \|P(\rho)\|_{L^4}^4 \right) dt$$

$$\leq C M_0^{2\gamma} + C \left(A_1(T) + M_0^\gamma \right) \left(M_0^{3/2} + M_0^{1/2} A_1^{1/4} \right) A_1^{1/2}(T) A_2^{1/2}(T)$$

$$+ C \left(A_1(T) + M_0^\gamma \right) \left(\sup_{t \in [0,T]} \|\mathbf{B}\|_{L^3} \right) \left(M_0 + A_1^{1/4}(T) \right) A_1^{1/2}(T) A_2^{1/2}(T) \qquad (3.53)$$

$$+ C\Big(\sup_{t\in[0,T]} \|\mathbf{B}\|_{L^3}^2\Big)\big(M_0^{3/2} A_1(T) + M_0^{1/2} A_1^{5/4}(T)\big) A_1^{1/2}(T) A_2^{1/2}(T)$$

$$+ C\Big(\sup_{t\in[0,T]} \|\mathbf{B}\|_{L^3}^3\Big)\big(M_0 A_1(T) + A_1^{3/4}(T) A_2^{1/4}(T)\big) A_1^{1/2}(T) A_2^{1/2}(T).$$

Collecting (3.6), (3.8), (3.9), and (3.53) together, and choosing $K \geq 1$, $M_0 \leq 1/2$, we immediately obtain

$$
\begin{aligned}
A_1(T) &+ A_2(T) \\
&\leq C\big(M_0^{1/2} K + M_0^{5\gamma/6} K^{1/2} + e^{CK} M_0^{2\gamma+1} + e^{CK} M_0^{2\gamma}\big) A_1(T) \\
&\quad + C\big(e^{2CK} M_0^{4\gamma} + e^{4CK} M_0^{8\gamma}\big) A_2(T) + C\big(1 + M_0^{\gamma} + M_0^{2\gamma}\big) \\
&\quad + C\big(M_0^{1/2} + e^{CK} M_0^{2\gamma}\big)\big(A_1(T) + M_0^{\gamma}\big)\big(M_0 + A_1^{1/4}(T)\big) A_1^{1/2}(T) A_2^{1/2}(T) \\
&\quad + C e^{2CK} M_0^{4\gamma}\big(M_0^{3/2} A_1(T) + M_0^{1/2} A_1^{5/4}(T)\big) A_1^{1/2}(T) A_2^{1/2}(T) \\
&\quad + C e^{3CK} M_0^{6\gamma}\big(M_0 A_1(T) + A_1^{3/4}(T) A_2^{1/4}(T)\big) A_1^{1/2}(T) A_2^{1/2}(T) \\
&\leq C_1 e^{4C_0 K} M_0^{1/2}\big[A_1(T) + A_2(T)\big] + C_2\big(1 + M_0^{\gamma} + M_0^{2\gamma}\big) \\
&\quad + C_3 e^{3C_0 K} M_0^{1/2}\big(A_1^{5/4}(T) + A_2^{5/8}(T)\big) A_1^{1/2}(T) A_2^{1/2}(T),
\end{aligned}
\tag{3.54}
$$

where C_i ($i = 0, 1, 2, 3$) > 0 depend on μ, λ, ν, A, E_0, M_1, and M_2, but not on M_0 and T.

To continue, set

$$K \triangleq \max\{1, 8C_2, M_1 + M_2\}.$$

Thus, if it holds that

$$M_0 \leq \varepsilon_1 \triangleq \min\left\{\frac{1}{2}, \left(\frac{1}{2C_1 e^{4C_0 K}}\right)^2, \left(\frac{1}{16C_3 e^{3C_0 K^{5/4}}}\right)^2\right\},$$

then one infers from (3.54) that

$$
\begin{aligned}
A_1(T) + A_2(T) &\leq 4C_2 + 16C_3 e^{3C_0 K} M_0^{1/2} K^{5/4} A_1^{1/2}(T) A_2^{1/2}(T) \\
&\leq \frac{K}{2} + \frac{1}{2}\big[A_1(T) + A_2(T)\big],
\end{aligned}
\tag{3.55}
$$

provided (3.41) holds. Thus, the desired estimate of (3.40) immediately follows from (3.55) by choosing K and ε_1 as above. □

In order to derive a uniform upper bound of the density, we still need the following t-weighted estimate.

Lemma 3.5 *Let the conditions of Lemma* 3.4 *be in force. Then there exists a positive constant* ε_2, *depending only on* μ, λ, ν, γ, A, E_0, M_1, *and* M_2, *such that if* $\|\mathbf{B}_0\|_{L^3}^{1/2\gamma} \leq M_0 \leq \varepsilon_2$, *then for any* $0 \leq t_1 < t_2 \leq T$,

$$
\begin{aligned}
\sup_{t_1 \leq t \leq t_2} &\big[(t - t_1)\big(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \|\nabla^2 \mathbf{B}\|_{L^2}^2\big)\big] \\
&+ \int_{t_1}^{t_2} (t - t_1)\big(\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla\mathbf{B}_t\|_{L^2}^2\big)\,dt \leq C\big[1 + M_0^{\frac{5\gamma}{2}}(t_2 - t_1)\big],
\end{aligned}
\tag{3.56}
$$

where the constant $C > 0$ depends on $\mu, \lambda, \nu, \gamma, A, E_0, M_1, M_2, K$, and the coefficients of the Gagliardo–Nirenberg–Sobolev inequality in Lemma 2.1, *but is independent of M_0.*

Proof Indeed, it follows from (3.6), (3.38), and (3.40) that

$$\|\nabla \mathbf{u}\|_{L^6} \le C\big(M_0^{1/2}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2} + M_0^{5\gamma/6} + M_0^{2\gamma}\big\|\nabla^2 \mathbf{B}\big\|_{L^2}\big), \tag{3.57}$$

thus

$$\begin{aligned}
&\|\nabla \mathbf{u}\|_{L^4}^4 + \big\|P(\rho)\big\|_{L^4}^4 \\
&\quad \le C\|\nabla \mathbf{u}\|_{L^2}\|\nabla \mathbf{u}\|_{L^6}^3 + CM_0^{3\gamma} \\
&\quad \le C\big(M_0^{5\gamma/2} + M_0^{3/2}\|\nabla \mathbf{u}\|_{L^2}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^3 + M_0^{6\gamma}\|\nabla \mathbf{u}\|_{L^2}\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^3\big).
\end{aligned} \tag{3.58}$$

Multiplying (3.36) by $(t - t_1)$, integrating it over (t_1, t) with $t_1 \le t \le t_2$, using (3.6) and (3.40), we get

$$\begin{aligned}
&\sup_{t_1 \le t \le t_2}\Big[(t - t_1)\big(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2\big)\Big] + \int_{t_1}^{t_2}(t - t_1)\big(\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2\big)\,dt \\
&\quad \le C + \int_{t_1}^{t_2}(t - t_1)\big(\|\nabla \mathbf{u}\|_{L^4}^4 + \big\|P(\rho)\big\|_{L^4}^4 + M_0^{8\gamma}\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^4\big)\,dt \\
&\qquad + C\int_{t_1}^{t_2}(t - t_1)\big(\|\mathbf{B}_t\|_{L^2}^2\|\nabla \mathbf{u}\|_{L^2}\|\nabla \mathbf{u}\|_{L^6}\big)\,dt \\
&\qquad + C\int_{t_1}^{t_2}(t - t_1)\big(\|\nabla \mathbf{B}\|_{L^2}^2\|\nabla \mathbf{u}\|_{L^2}^2\|\nabla \mathbf{u}\|_{L^6}^2 + \|\mathbf{u}\|_{L^\infty}^4\big)\,dt \\
&\quad \triangleq C + J_1 + J_2 + J_3,
\end{aligned} \tag{3.59}$$

where the second term on the right-hand side can be estimated as follows based on (3.58):

$$\begin{aligned}
J_1 &\le C\int_{t_1}^{t_2}(t - t_1)\big(M_0^{5\gamma/2} + M_0^{3/2}\|\nabla \mathbf{u}\|_{L^2}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^3 + M_0^{6\gamma}\|\nabla \mathbf{u}\|_{L^2}\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^3 \\
&\qquad + M_0^{8\gamma}\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^4\big)\,dt \\
&\le CM_0^{3/2}\sup_{t_1 \le t \le t_2}\Big[(t - t_1)\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2\Big]\bigg(\int_{t_1}^{t_2}\|\nabla \mathbf{u}\|_{L^2}^2\,dt\bigg)^{1/2}\bigg(\int_{t_1}^{t_2}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2\,dt\bigg)^{1/2} \\
&\quad + CM_0^{6\gamma}\sup_{t_1 \le t \le t_2}\Big[(t - t_1)\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\Big]\bigg(\int_{t_1}^{t_2}\|\nabla \mathbf{u}\|_{L^2}^2\,dt\bigg)^{1/2}\bigg(\int_{t_1}^{t_2}\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\,dt\bigg)^{1/2} \\
&\quad + CM_0^{8\gamma}\sup_{t_1 \le t \le t_2}\Big[(t - t_1)\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\Big]\int_{t_1}^{t_2}\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\,dt + CM_0^{5\gamma/2}(t_1 - t_2)^2 \\
&\le CM_0^{3/2}\sup_{t_1 \le t \le t_2}\Big[(t - t_1)\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2\Big] + CM_0^{6\gamma}\sup_{t_1 \le t \le t_2}\Big[(t - t_1)\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\Big] \\
&\quad + CM_0^{5\gamma/2}(t_1 - t_2)^2.
\end{aligned}$$

By virtue of (3.57) and Cauchy–Schwarz inequality, we also have

$$J_2 \le C\int_{t_1}^{t_2}(t - t_1)\|\mathbf{B}_t\|_{L^2}^2\|\nabla \mathbf{u}\|_{L^2}\big(M_0^{5\gamma/6} + M_0^{1/2}\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2} + M_0^{2\gamma}\big\|\nabla^2 \mathbf{B}\big\|_{L^2}\big)\,dt$$

$$\leq CM_0^{1/2} \sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\|\mathbf{B}_t\|_{L^2}^2\big] \bigg(\int_{t_1}^{t_2} \|\nabla \mathbf{u}\|_{L^2}^2 \, dt\bigg)^{1/2} \bigg(\int_{t_1}^{t_2} \|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 \, dt\bigg)^{1/2}$$

$$+ CM_0^{5\gamma/6} \sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\|\mathbf{B}_t\|_{L^2}\big] \bigg(\int_{t_1}^{t_2} \|\nabla \mathbf{u}\|_{L^2}^2 \, dt\bigg)^{1/2} \bigg(\int_{t_1}^{t_2} \|\mathbf{B}_t\|_{L^2}^2 \, dt\bigg)^{1/2}$$

$$+ CM_0^{2\gamma} \sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\|\mathbf{B}_t\|_{L^2}^2\big] \bigg(\int_{t_1}^{t_2} \big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2 \, dt\bigg)^{1/2} \bigg(\int_{t_1}^{t_2} \|\nabla \mathbf{u}\|_{L^2}^2 \, dt\bigg)^{1/2}$$

$$\leq C\big(M_0^{1/2} + M_0^{5\gamma/12}\big) \sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\|\mathbf{B}_t\|_{L^2}^2\big] + CM_0^{5\gamma/2}(t_1 - t_2)^2 + C.$$

Similarly, we infer from (3.40) that

$$J_3 \leq C \int_{t_1}^{t_2} (t - t_1)\|\nabla \mathbf{u}\|_{L^2}^2 \big(M_0^{5\gamma/3} + M_0\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + M_0^{4\gamma} \big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\big) \, dt$$

$$\leq CM_0 \sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2\big] + CM_0^{4\gamma} \sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\big]$$

$$+ CM_0^{10\gamma/3}(t_1 - t_2)^2 + C.$$

In addition, it holds by $(1.1)_3$ and (3.40) that

$$\sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\big] \leq C \sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\big(\|\mathbf{B}_t\|_{L^2}^2 + \|\mathbf{B}\|_{L^3}^3 \|\nabla \mathbf{u}\|_{L^2}^2\big)\big]$$

$$\leq C \sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\|\mathbf{B}_t\|_{L^2}^2\big] + CM_0^{8\gamma}(t_1 - t_2)^2 + C. \tag{3.60}$$

Substituting J_1, J_2, and J_3 into (3.59) and using (3.60), we obtain

$$\sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\big(\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 + \big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\big)\big]$$

$$+ \int_{t_1}^{t_2} (t - t_1)\big(\|\nabla \dot{\mathbf{u}}\|_{L^2}^2 + \|\nabla \mathbf{B}_t\|_{L^2}^2\big) \, dt$$

$$\leq C_4 M_0 \sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\|\sqrt{\rho}\dot{\mathbf{u}}\|_{L^2}^2\big] \tag{3.61}$$

$$+ C_5\big(M_0^{1/2} + M_0^{5\gamma/12}\big) \sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\|\mathbf{B}_t\|_{L^2}^2\big]$$

$$+ C_6 M_0^{4\gamma} \sup_{t_1 \leq t \leq t_2} \big[(t - t_1)\big\|\nabla^2 \mathbf{B}\big\|_{L^2}^2\big] + CM_0^{5\gamma/2}(t_1 - t_2)^2 + C.$$

Thus, if M_0 is chosen to be suitably small such that

$$M_0 \leq \varepsilon_2 \triangleq \min\bigg\{\frac{1}{2C_4}, \bigg(\frac{1}{2C_5}\bigg)^{\frac{12}{5\gamma+6}}, \bigg(\frac{1}{2C_6}\bigg)^{\frac{1}{4\gamma}}\bigg\},$$

then we immediately obtain (3.56) from (3.61).　　　　　　　　　　　　　　　□

We are now in a position of estimating an upper bound of the density.

Lemma 3.6 *Let the conditions of Lemma* 3.4 *and* 3.5 *be in force. Then there exists a positive constant ε, depending only on μ, λ, ν, γ, A, E_0, M_1, and M_2, such that if $\|\mathbf{B}_0\|_{L^3}^{1/2\gamma} \leq M_0 \leq \varepsilon$,*

then

$$\rho(x,t) \le \frac{7M_0}{4} \quad \text{for all } (x,t) \in \mathbb{R}^3 \times [0,T].$$ (3.62)

Proof In view of $(2.3)_1$, we can rewrite $(1.1)_1$ in the form:

$$D_t\rho = g(\rho) + b'(t),$$ (3.63)

where

$$g(\rho) \triangleq -\frac{A\rho}{2\mu+\lambda}\rho^\gamma, \qquad b(t) \triangleq -\frac{1}{2\mu+\lambda}\int_0^t \left(\rho F + \frac{1}{2}\rho|\mathbf{B}|^2\right)ds.$$ (3.64)

Obviously, it holds that $g(\infty) = -\infty$. So, to apply Lemma 2.3, we still need to deal with $b(t)$. To do this, we first utilize (2.2), (2.5), (2.6), and the fact that $0 \le \rho \le 2M_0$ to deduce that for any $0 \le t_1 < t_2 \le T$,

$$
\begin{aligned}
\left|b(t_2) - b(t_1)\right| &\le CM_0 \int_{t_1}^{t_2} \left(\|F\|_{L^\infty} + \|\mathbf{B}\|_{L^\infty}^2\right)dt \\
&\le CM_0 \int_{t_1}^{t_2} \left(\|F\|_{L^2}^{1/4}\|\nabla F\|_{L^6}^{3/4} + \|\mathbf{B}\|_{L^3}^{2/3}\|\nabla \mathbf{B}\|_{L^6}^{4/3}\right)dt \\
&\le CM_0 \int_{t_1}^{t_2} \left(\|\nabla\mathbf{u}\|_{L^2} + M_0^{\gamma/2} + \|\mathbf{B}\|_{L^4}^2\right)^{1/4} \\
&\quad \times \left(M_0\|\nabla\dot{\mathbf{u}}\|_{L^2} + \|\mathbf{B}\|_{L^3}^{1/3}\|\nabla\mathbf{B}\|_{L^6}^{5/3}\right)^{3/4}dt \\
&\quad + CM_0 \int_{t_1}^{t_2} \|\mathbf{B}\|_{L^3}^{2/3}\|\nabla\mathbf{B}\|_{L^6}^{4/3}dt \triangleq \sum_{i=1}^7 N_i.
\end{aligned}
$$ (3.65)

The right-hand side of (3.65) can be estimated as follows. We deduce from (3.4), (3.56), and Hölder inequality that

$$
\begin{aligned}
N_1 &= CM_0^{7/4} \int_{t_1}^{t_2} (t-t_1)^{-3/8}\left(\|\nabla\mathbf{u}\|_{L^2}^2\right)^{1/8}\left[(t-t_1)\|\nabla\dot{\mathbf{u}}\|_{L^2}^2\right]^{3/8}dt \\
&\le CM_0^{7/4}\left(\int_{t_1}^{t_2}(t-t_1)^{-3/4}dt\right)^{1/2}\left(\int_{t_1}^{t_2}\|\nabla\mathbf{u}\|_{L^2}^2 dt\right)^{1/8}\left(\int_{t_1}^{t_2}(t-t_1)\|\nabla\dot{\mathbf{u}}\|_{L^2}^2 dt\right)^{3/8} \\
&\le CM_0^{7/4}(t_2-t_1)^{1/8}\left[1 + M_0^{15\gamma/16}(t_2-t_1)^{3/4}\right] \\
&\le \frac{AM_0^{\gamma+1}}{8(2\mu+\lambda)}(t_2-t_1) + C\left(M_0^{\frac{13-\gamma}{7}} + M_0^{\frac{14+\gamma}{2}}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
N_2 &= CM_0 \int_{t_1}^{t_2} \|\nabla\mathbf{u}\|_{L^2}^{1/4}\|\mathbf{B}\|_{L^3}^{1/4}\|\nabla^2\mathbf{B}\|_{L^2}^{3/4}\left[(t-t_1)\|\nabla^2\mathbf{B}\|_{L^2}^2\right]^{1/4}(t-t_1)^{-1/4}dt \\
&\le CM_0^{(2+\gamma)/2}\left[\sup_{t_1\le t\le t_2}\left((t-t_1)\|\nabla^2\mathbf{B}\|_{L^2}^2\right)\right]^{1/4}\left(\int_{t_1}^{t_2}\|\nabla\mathbf{u}\|_{L^2}^2 dt\right)^{1/8} \\
&\quad \times \left(\int_{t_1}^{t_2}\|\nabla\mathbf{B}\|_{L^2}^2 dt\right)^{3/8}\left(\int_{t_1}^{t_2}(t-t_1)^{-1/2}dt\right)^{1/2}
\end{aligned}
$$

$$\leq CM_0^{(2+\gamma)/2}(t_2 - t_1)^{1/4}\big[1 + M_0^{5\gamma/8}(t_2 - t_1)^{1/2}\big]$$

$$\leq \frac{AM_0^{\gamma+1}}{8(2\mu + \lambda)}(t_2 - t_1) + C\big(M_0^{\frac{3+\gamma}{3}} + M_0^{\frac{2+3\gamma}{2}}\big).$$

Similarly, using (3.56) and Hölder inequality, we have

$$N_3 = CM_0^{(14+\gamma)/8}\int_{t_1}^{t_2}(t - t_1)^{-3/8}\big[(t - t_1)\|\nabla\dot{\mathbf{u}}\|_{L^2}^2\big]^{3/8}\,dt$$

$$\leq CM_0^{(14+\gamma)/8}\bigg(\int_{t_1}^{t_2}(t - t_1)^{-3/5}\,dt\bigg)^{5/8}\bigg(\int_{t_1}^{t_2}(t - t_1)\|\nabla\dot{\mathbf{u}}\|_{L^2}^2\,dt\bigg)^{3/8}$$

$$\leq CM_0^{(14+\gamma)/8}(t_2 - t_1)^{1/4}\big[1 + M_0^{15\gamma/16}(t_2 - t_1)^{3/4}\big]$$

$$\leq \frac{AM_0^{\gamma+1}}{2\mu + \lambda}\bigg(\frac{1}{8} + CM_0^{\frac{12+\gamma}{16}}\bigg)(t_2 - t_1) + CM_0^{\frac{12-\gamma}{6}},$$

and

$$N_4 = CM_0^{(8+\gamma)/8}\int_{t_1}^{t_2}\|\mathbf{B}\|_{L^3}^{1/4}\|\nabla^2\mathbf{B}\|_{L^2}^{3/4}\,dt$$

$$\leq CM_0^{(8+5\gamma)/8}\Big[\sup_{t_1\leq t\leq t_2}\big((t - t_1)\|\nabla^2\mathbf{B}\|_{L^2}^2\big)\Big]^{1/4}\bigg(\int_{t_1}^{t_2}\|\nabla^2\mathbf{B}\|_{L^2}^2\,dt\bigg)^{3/8}$$

$$\times\bigg(\int_{t_1}^{t_2}(t - t_1)^{-2/5}\,dt\bigg)^{5/8}$$

$$\leq CM_0^{(8+5\gamma)/8}(t_2 - t_1)^{3/8}\big[1 + M_0^{5\gamma/8}(t_2 - t_1)^{1/2}\big]$$

$$\leq \frac{AM_0^{\gamma+1}}{8(2\mu + \lambda)}(t_2 - t_1) + C\big(M_0^{\frac{5+2\gamma}{5}} + M_0^{1+3\gamma}\big).$$

By virtue of (3.6) and (3.40), we infer from Cauchy–Schwarz inequality that

$$N_5 = CM_0^{7/4}\int_{t_1}^{t_2}\|\mathbf{B}\|_{L^3}^{1/4}\|\nabla\mathbf{B}\|_{L^2}^{1/4}\|\nabla\dot{\mathbf{u}}\|_{L^2}^{3/4}\,dt$$

$$\leq CM_0^{(7+2\gamma)/4}\bigg(\int_{t_1}^{t_2}(t - t_1)\|\nabla\dot{\mathbf{u}}\|_{L^2}^2\,dt\bigg)^{3/8}\bigg(\int_{t_1}^{t_2}(t - t_1)^{-3/5}\,dt\bigg)^{5/8}$$

$$\leq CM_0^{(7+2\gamma)/4}(t_2 - t_1)^{1/4}\big[1 + M_0^{15\gamma/16}(t_2 - t_1)^{3/4}\big]$$

$$\leq \frac{AM_0^{\gamma+1}}{2\mu + \lambda}\bigg(\frac{1}{8} + CM_0^{\frac{12+7\gamma}{16}}\bigg)(t_2 - t_1) + CM_0^{\frac{6+\gamma}{3}},$$

and

$$N_6 = CM_0\int_{t_1}^{t_2}\|\mathbf{B}\|_{L^3}^{1/2}\|\nabla\mathbf{B}\|_{L^2}^{1/4}\|\nabla^2\mathbf{B}\|_{L^2}^{5/4}\,dt$$

$$\leq CM_0^{1+\gamma}\Big[\sup_{t_1\leq t\leq t_2}\big((t - t_1)\|\nabla^2\mathbf{B}\|_{L^2}^2\big)\Big]^{1/4}\big(\int_{t_1}^{t_2}\big(\|\nabla^2\mathbf{B}\|_{L^2}^2\,dt\big)^{3/8}$$

$$\times\bigg(\int_{t_1}^{t_2}(t - t_1)^{-2/5}\,dt\bigg)^{5/8}$$

$$\leq CM_0^{(1+\gamma)}(t_2 - t_1)^{3/8}\big[1 + M_0^{5\gamma/8}(t_2 - t_1)^{1/2}\big]$$

$$\leq \frac{AM_0^{\gamma+1}}{8(2\mu + \lambda)}(t_2 - t_1) + C\big(M_0^{1+\gamma} + M_0^{1+2\gamma}\big),$$

and

$$N_7 = CM_0 \int_{t_1}^{t_2} \|\mathbf{B}\|_{L^3}^{2/3}\|\nabla^2\mathbf{B}\|_{L^2}^{4/3} \, dt$$

$$\leq CM_0^{\frac{3+4\gamma}{3}}\left(\int_{t_1}^{t_2}\big(\|\nabla^2\mathbf{B}\|_{L^2}^2 \, dt\big)^{2/3}\left(\int_{t_1}^{t_2} dt\right)^{1/3}\right.$$

$$\leq CM_0^{\frac{3+4\gamma}{3}}(t_2 - t_1)^{1/3}$$

$$\leq \frac{AM_0^{\gamma+1}}{8(2\mu + \lambda)}(t_2 - t_1) + CM_0^{\frac{2+3\gamma}{2}}.$$

Due to $1 < \gamma < 6$, it is obvious that

$$\frac{13 - \gamma}{7} < \frac{12 - \gamma}{6}$$

and hence, putting N_i ($i = 1, 2, \ldots, 7$) into (3.65), we obtain

$$|b(t_2) - b(t_1)| \leq C_7\big(M_0^{\frac{13-\gamma}{7}} + M_0^{\frac{3+\gamma}{3}}\big) + \frac{AM_0^{\gamma+1}}{2\mu + \lambda}\bigg(\frac{7}{8} + C_8 M_0^{\frac{12+\gamma}{16}}\bigg)(t_2 - t_1). \tag{3.66}$$

Now, in view of (2.10) and (2.11), we set

$$N_0 \triangleq C_7\big(M_0^{\frac{13-\gamma}{7}} + M_0^{\frac{3+\gamma}{3}}\big), \qquad N_1 \triangleq \frac{AM_0^{\gamma+1}}{2\mu + \lambda}\bigg(\frac{7}{8} + C_8 M_0^{\frac{12+\gamma}{16}}\bigg), \quad \text{and} \quad \xi^* \triangleq M_0.$$

Thus, if M_0 is chosen to be small enough such that (noting that $1 < \gamma < 6$)

$$M_0 \triangleq \left\{\varepsilon_2, \left(\frac{1}{4C_7}\right)^{\frac{7}{6-\gamma}}, \left(\frac{1}{2C_7}\right)^{\frac{3}{\gamma}}, \left(\frac{1}{8C_8}\right)^{\frac{16}{12+\gamma}}\right\},$$

then it is easy to check that

$$g(\xi) = -\frac{A\xi^{\gamma+1}}{2\mu + \lambda} \leq -\frac{AM_0^{\gamma+1}}{2\mu + \lambda} \leq -N_1, \quad \forall \xi \geq \xi^* = M_0.$$

As a result, we conclude from (3.66) and Lemma 2.3 that

$$\sup_{0 \leq t \leq T}\|\rho(t)\|_{L^\infty} \leq M_0 + N_0 \leq M_0 + \frac{3M_0}{4} = \frac{7M_0}{4}.$$

The proof of Lemma 3.6 is therefore complete. ☐

Proof of Proposition 3.1 In view of (3.40) and (3.62), we obtain the desired estimate of (3.2) with K and ε being the same as in Lemmas 3.4 and 3.6. ☐

With the help of Proposition 3.1, the higher-order estimates of the solution $(\rho, \mathbf{u}, \mathbf{B})$ can be shown in a manner similar to that in [9] (see Lemmas 4.1–4.6). For completeness and for convenience, we collect these estimates in the following lemma without proofs.

Proposition 3.2 *Let the conditions of Proposition* 3.1 *be in force. Then for any* $T > 0$ *and* $0 < \tau < T$,

$$\sup_{t \in [0,T]} \big(\big\|(\nabla \rho, \nabla P)\big\|_{H^1 \cap W^{1,q}} + \big\|(\rho_t, P_t)\big\|_{H^1} \big) + \int_0^T \big\|(\rho_{tt}, P_{tt})\big\|_{L^2}^2 \, dt \le C(T), \quad q \in (3,6),$$

$$\sup_{t \in [0,T]} \big(\big\|(\sqrt{\rho}\,\mathbf{u}_t, \mathbf{B}_t)\big\|_{L^2}^2 + \big\|(\nabla \mathbf{u}, \nabla \mathbf{B})\big\|_{H^1}^2 \big) + \int_0^T \big(\big\|(\nabla \mathbf{u}_t, \nabla \mathbf{B}_t)\big\|_{L^2}^2 + \big\|\nabla^2 \mathbf{u}\big\|_{H^1}^2 \big) \, dt$$

$$\le C(T),$$

$$\sup_{t \in [\tau,T]} \big(\big\|\nabla^2 \mathbf{u}\big\|_{H^1 \cap W^{1,q}}^2 + \big\|\nabla^2 \mathbf{B}\big\|_{H^2}^2 + \big\|(\nabla \mathbf{u}_t, \nabla \mathbf{B}_t)\big\|_{H^1}^2 \big) + \int_\tau^T \big\|(\nabla \mathbf{u}_{tt}, \nabla \mathbf{B}_{tt})\big\|_{L^2}^2 \, dt$$

$$\le C(\tau, T),$$

where $C(\tau, T)$ *denotes a positive constant depending on* τ *and* T*, in addition to* $\mu, \lambda, \nu, \gamma,$ *$A, E_0, M_1, M_2, K,$ and the coefficients of the Gagliardo–Nirenberg–Sobolev inequality in Lemma* 2.1.

With Propositions 3.1–3.2 at hand, we can extend the local solutions obtained in Lemma 2.4 to be a global one in a similar manner as that in [9].

Proof of Theorem 1.1 By Lemma 2.4, there exists a $T_* > 0$ such that the problem (1.1)–(1.5) has a classical solution $(\rho, \mathbf{u}, \mathbf{B})$ on $(0, T_*]$. Noting that

$$A_1(0) + A_2(0) = M_1 + M_2 \le K, \quad 0 \le \rho_0 \le M_0,$$

and using the continuity arguments, one infers that there exists a $T_1 \in (0, T_*]$ such that (3.1) holds for $T = T_1$. Next, let

$$T^* \triangleq \sup\{T \mid (3.1) \text{ holds}\}. \tag{3.67}$$

Then $T^* \ge T_1 > 0$. We claim that

$$T^* = \infty. \tag{3.68}$$

Indeed, if we had $T^* < \infty$, then it would follow from Proposition 3.1 that (3.2) holds for $T = T^*$, provided $\|\mathbf{B}_0\|_{L^3}^{1/2\gamma} \le M_0 \le \varepsilon$ and $1 < \gamma < 6$. It would also follow from Proposition 3.2 that for any $0 < \tau < T \le T^*$,

$$\begin{cases} (\nabla \mathbf{u}_t, \nabla \mathbf{B}_t, \nabla^2 \mathbf{u}, \nabla^3 \mathbf{B}) \in C([\tau, T]; L^2 \cap L^4), \\ \nabla^3 \mathbf{u} \in C([\tau, T]; L^p) \quad \text{with } p \in [2, q), q \in (3, 6), \\ (\nabla \mathbf{u}, \nabla \mathbf{B}, \nabla^2 \mathbf{u}, \nabla^2 \mathbf{B}) \in C([\tau, T]; L^2 \cap C(\mathbb{R}^3)), \end{cases} \tag{3.69}$$

where we have also used the following embedding:

$$L^\infty\big(\tau, T; H^1\big) \cap H^1\big(\tau, T; H^{-1}\big) \hookrightarrow C\big(\tau, T; L^\alpha\big), \quad \forall \alpha \in [2, 6).$$

Moreover, for any $0 < \tau < T \leq T^*$, it then follows from $(1.1)_1$, Propositions 3.1 and 3.2 that

$$\int_\tau^T \int \big|\partial_t\big(\rho |\mathbf{u}_t|^2\big)\big| \, dx \, dt + \int_\tau^T \int \big|\partial_t\big(\rho |\mathbf{u} \nabla \mathbf{u}|^2\big)\big| \, dx \, dt \leq C(\tau, T),$$

which, together Proposition 3.2, yields

$$\sqrt{\rho}\mathbf{u}_t, \sqrt{\rho}\mathbf{u} \cdot \nabla \mathbf{u} \in C\big(\tau, T; L^2\big),$$

and, consequently,

$$\sqrt{\rho}\dot{\mathbf{u}} \in C\big(\tau, T; L^2\big).$$

In a similar manner, we can also show using (3.69) that

$$\nabla \dot{\mathbf{u}} \in C\big(\tau, T; L^2\big).$$

This particularly implies that $(\rho, \mathbf{u}, \mathbf{B})(x, T^*)$ satisfies the compatibility condition (1.9) with $g(x) \triangleq \dot{\mathbf{u}}(x, T^*)$ at $t = T^*$. Thus, using Lemma 2.4, (3.2), and the continuity arguments, we know that there exists a $T^{**} > T^*$ such that (3.1) holds for $T = T^{**}$, which contradicts (3.67). Hence, (3.68) holds. This, together with Proposition 3.2 again, shows that the solution $(\rho, \mathbf{u}, \mathbf{B})$ is in fact the unique classical solution on $\mathbb{R}^3 \times (0, T]$ for any $0 < T < T^* = \infty$. The proof of Theorem 1.1 is therefore complete. □

Availability of data and materials
Not applicable.

Declarations

Competing interests
The author declares that she has no competing interests.

Authors' contributions
This entire work has been completed by the author, Dr. MZ. The author read and approved the finial manuscript.

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The author of this article is Dr. Zhang. She comes from the School of Mathematics and Information Science, Weifang University.

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