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On the improvement of Fickett's theorem on bounded sets

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Abstract

Fickett proved the stability of isometries on bounded subsets of \mathbb{R}^n for $n \geq 2$. Jung then improved Fickett's theorem for $n \geq 3$. In this paper, we improve Fickett's theorem for $n = 2$ and improve Jung's result for $n = 3$, by employing a fundamental analytic method, as it can be used to explain mathematically many practical engineering problems.

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1 Introduction

In 1982 Fickett proved the following stability of isometries:

Fickett's Theorem ([4]) *For a fixed integer $n \geq 2$, let D be a bounded subset of the Euclidean space \mathbb{R}^n and let $\varepsilon > 0$ be given. If a function $f : D \rightarrow \mathbb{R}^n$ satisfies the inequality*

$$|\|f(x) - f(y)\| - \|x - y\|| \leq \varepsilon$$

for all $x, y \in D$, then there exists an isometry $U : D \rightarrow \mathbb{R}^n$ such that

$$\|f(x) - U(x)\| \leq 27\varepsilon^{1/2^n} \quad (1.1)$$

for each $x \in D$.

For any sufficiently small $\varepsilon > 0$, one can see that the upper bound of inequality (1.1) can be very large in comparison to ε . This was the motivation for this paper and we will focus on $n = 2, 3$ as this theorem can be very useful for solving physics problems.

Some mathematicians have tried to improve Fickett's theorem in various ways. Alestalo et al. [1] and Väisälä [13] significantly improved Fickett's result by proving the Hyers–Ulam stability of isometries defined on bounded subsets of \mathbb{R}^n . Very recently, Jung [8] improved the results of the previous two papers [1, 13] significantly in the case of $n \geq 3$. In this paper, we will improve Fickett's results when $n = 2$. Moreover, by improving the results of Jung [8] it will also be improved when $n = 3$. The reason for focusing on this problem in \mathbb{R}^2 and

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\mathbb{R}^3 is that it can be used to explain mathematically many practical engineering problems that occur in 2-dimensional or 3-dimensional Euclidean spaces. For a mathematical tool, we will use a very fundamental analytic method.

In this paper, we use the notation \mathbb{K} to denote either \mathbb{R} or \mathbb{C} . Further, $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ denote the Hilbert spaces over \mathbb{K} . If a function $f: E \rightarrow F$ satisfies the condition

$$\|f(u) - f(v)\| = \|u - v\| \quad (1.2)$$

for all $u, v \in E$, then f is called an *isometry*.

By recalling the definition of Hyers and Ulam [7], for any fixed $\varepsilon \geq 0$, a function $f: E \rightarrow F$ is called an ε -*isometry* if f satisfies the inequality

$$\|\|f(u) - f(v)\| - \|u - v\|\| \leq \varepsilon \quad (1.3)$$

for all $u, v \in E$. If there exists a constant $K > 0$ such that for any $\varepsilon > 0$ and ε -isometry $f: E \rightarrow F$, there is an isometry $U: E \rightarrow F$ that satisfies $\|f(x) - U(x)\| \leq K\varepsilon$ for all $x \in E$, then the functional equation (1.2) is said to have (or satisfy) the *Hyers–Ulam stability*. To understand this topic more broadly, one can refer to [2, 3, 5–7, 9–12].

2 Useful lemmas

An orthogonal matrix \mathbf{Q} is a real square matrix whose columns and rows are orthonormal vectors. In other words, a real square matrix \mathbf{Q} is orthogonal if its transpose is equal to its inverse: $\mathbf{Q}^{\text{tr}} = \mathbf{Q}^{-1}$. As a linear transformation, an orthogonal matrix preserves the inner product of vectors, and therefore acts as an isometry of Euclidean space.

Now, we introduce the familiar theorem called **QR decomposition**. We omit the proof because it can be proved by using the Gram–Schmidt process.

Lemma 2.1 (QR decomposition) *Every real square matrix \mathbf{A} can be decomposed as $\mathbf{A} = \mathbf{QR}$, where \mathbf{Q} is an orthogonal matrix and \mathbf{R} is an upper triangular matrix with non-negative diagonal elements.*

The following lemma will be used to almost halve n^2 unknowns. We will prove it by applying QR decomposition mentioned above. For the proof, one can refer to [8].

Lemma 2.2 *Given an integer $n > 0$, let $\{e_1, e_2, \dots, e_n\}$ be the standard basis for the n -dimensional Euclidean space \mathbb{R}^n , where each e_i is written in a column vector, let D be a subset of \mathbb{R}^n that satisfies $\{e_1, e_2, \dots, e_n\} \subset D$, and let $f: D \rightarrow \mathbb{R}^n$ be a function. Then, there exist an orthogonal matrix \mathbf{Q} and real numbers e'_{ij} for $i, j \in \{1, 2, \dots, n\}$ with $i \geq j$ such that*

$$\mathbf{Q}^{\text{tr}} f(e_i) = (e'_{i1}, e'_{i2}, \dots, e'_{ii}, 0, \dots, 0)^{\text{tr}}$$

for every $i \in \{1, 2, \dots, n\}$. In particular, $e'_{ii} \geq 0$ for all $i \in \{1, 2, \dots, n\}$.

3 Main theorems

From now on, we use the symbol $\|x\|$ to denote the norm of x defined by $\|x\| := \sqrt{x_1^2 + x_2^2}$ or $\|x\| := \sqrt{x_1^2 + x_2^2 + x_3^2}$ for each $x = (x_1, x_2) \in \mathbb{R}^2$ or $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, respectively.

3.1 On bounded subsets of \mathbb{R}^2

To prove the next lemma we will use Lemma 2.2 to almost halve the number of unknowns to consider.

In the following lemma, the parameters c_{ij} are assumed to be positive real numbers, while they were assumed to be positive integers in [8].

Lemma 3.1 *Let $\{e_1, e_2\}$ be the standard basis for the 2-dimensional Euclidean space \mathbb{R}^2 , let D be a subset of \mathbb{R}^2 that includes 0, e_1 , and e_2 , and let $f : D \rightarrow \mathbb{R}^2$ be a function that satisfies $f(0) = 0$ and the inequality (1.3) for all $x, y \in \{0, e_1, e_2\}$ and for some constant ε with $0 < \varepsilon \leq \frac{1}{13}$. In view of Lemma 2.2, it can be assumed that $f(e_1) = (e'_{11}, 0)$ and $f(e_2) = (e'_{21}, e'_{22})$, where $e'_{11} \geq 0$ and $e'_{22} \geq 0$. Then, there exist real numbers $c_{ij} > 0$, $i, j \in \{1, 2\}$ with $j \leq i$, such that*

$$\begin{cases} -c_{ij}\varepsilon \leq e'_{ij} \leq c_{ij}\varepsilon & (\text{for } i > j), \\ 1 - c_{ii}\varepsilon \leq e'_{ii} \leq 1 + \varepsilon & (\text{for } i = j). \end{cases} \quad (3.1)$$

In particular, $c_{11} = 1.0000000$, $c_{21} \approx 3.7403981$, and $c_{22} \approx 1.5978326$ may be selected.

Proof In view of (1.3) and $f(0) = 0$, we have

$$\|f(e_1)\| - 1 \leq \varepsilon, \quad \|f(e_2)\| - 1 \leq \varepsilon, \quad \|f(e_1) - f(e_2)\| - \sqrt{2} \leq \varepsilon$$

for any ε with $0 < \varepsilon \leq \frac{1}{13}$. Hence, it follows from these inequalities that

$$1 - \varepsilon \leq e'_{11} \leq 1 + \varepsilon, \quad (3.2)$$

$$(1 - \varepsilon)^2 \leq e_{21}^2 + e_{22}^2 \leq (1 + \varepsilon)^2, \quad (3.3)$$

$$(\sqrt{2} - \varepsilon)^2 \leq (e'_{11} - e'_{21})^2 + e_{22}^2 \leq (\sqrt{2} + \varepsilon)^2. \quad (3.4)$$

It follows from (3.4) that

$$(\sqrt{2} - \varepsilon)^2 - e_{11}^2 - (e_{21}^2 + e_{22}^2) \leq -2e'_{11}e'_{21} \leq (\sqrt{2} + \varepsilon)^2 - e_{11}^2 - (e_{21}^2 + e_{22}^2).$$

We recall that $0 < \varepsilon \leq \frac{1}{13}$. By (3.2) and (3.3), we obtain

$$-\frac{26 + 13\sqrt{2}}{12}\varepsilon \leq \frac{-(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - \varepsilon)} \leq e'_{21} \leq \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - \varepsilon)} \leq \frac{53 + 26\sqrt{2}}{24}\varepsilon.$$

Now, we select the smallest possible positive real number c_{21} that satisfies

$$-c_{21}\varepsilon \leq -\frac{26 + 13\sqrt{2}}{12}\varepsilon \leq e'_{21} \leq \frac{53 + 26\sqrt{2}}{24}\varepsilon \leq c_{21}\varepsilon.$$

Hence, we may choose

$$c_{21} = \frac{53 + 26\sqrt{2}}{24} \approx 3.7403981 \quad (3.5)$$

as the smallest possible positive number that satisfies the last inequality.

Furthermore, using our assumption that $0 < \varepsilon \leq \frac{1}{13}$, it follows from (3.1) with $i = 2$ and $j = 1$, (3.3), and (3.5) that

$$1 - 2\varepsilon - \frac{3585 + 2756\sqrt{2}}{576}\varepsilon^2 = 1 - 2\varepsilon + (1 - c_{21}^2)\varepsilon^2 \leq e_{22}^{\prime 2} \leq (1 + \varepsilon)^2.$$

By solving the following inequality, where c_{22} is unknown,

$$(1 - c_{22}\varepsilon)^2 \leq 1 - 2\varepsilon - \frac{3585 + 2756\sqrt{2}}{576}\varepsilon^2,$$

we obtain

$$\begin{aligned} \frac{1}{\varepsilon} \left(1 - \sqrt{1 - 2\varepsilon - \frac{3585 + 2756\sqrt{2}}{576}\varepsilon^2} \right) \\ \leq c_{22} \leq \frac{1}{\varepsilon} \left(1 + \sqrt{1 - 2\varepsilon - \frac{3585 + 2756\sqrt{2}}{576}\varepsilon^2} \right). \end{aligned} \quad (3.6)$$

Thus, putting $\varepsilon = \frac{1}{13}$ in the lower bound for c_{22} in (3.6), we may select

$$c_{22} = 13 - \sqrt{\frac{78,783 - 2756\sqrt{2}}{576}} \approx 1.5978326 \quad (3.7)$$

as the smallest possible positive constant that satisfies the inequality (3.6). \square

In the following theorem, we set $e_1 = (1, 0)$ and $e_2 = (0, 1)$ such that $\{e_1, e_2\}$ is the standard basis for the 2-dimensional Euclidean space \mathbb{R}^2 . We denote by $B_d(0)$ the closed ball of radius d and centered at the origin of \mathbb{R}^2 , i.e., $B_d(0) = \{x \in \mathbb{R}^2 : \|x\| \leq d\}$.

Theorem 3.2 *Let D be a subset of the Euclidean space \mathbb{R}^2 such that $\{0, e_1, e_2\} \subset D \subset B_d(0)$ for some $d \geq 1$ and let $f : D \rightarrow \mathbb{R}^2$ be a function that satisfies $f(0) = 0$ and the inequality (1.3) for all $x, y \in D$ and for some constant ε with $0 < \varepsilon \leq \frac{1}{13}$. Then, there exists an isometry $U : D \rightarrow \mathbb{R}^2$ such that*

$$\|f(x) - U(x)\| \leq (8d + 4)\varepsilon \quad (3.8)$$

for all $x \in D$.

Proof Due to Lemma 2.2, we may assume that $f(e_1) = (e'_{11}, 0)$ and $f(e_2) = (e'_{21}, e'_{22})$ with $e'_{11} \geq 0$ and $e'_{22} \geq 0$. For every point $x = (x_1, x_2) \in D$, let $f(x) = x' = (x'_1, x'_2)$. Then, by (1.3), we have

$$|\sqrt{x_1'^2 + x_2'^2} - \sqrt{x_1^2 + x_2^2}| \leq \varepsilon, \quad (3.9)$$

$$|\sqrt{(x'_1 - e'_{11})^2 + x_2'^2} - \sqrt{(x_1 - 1)^2 + x_2^2}| \leq \varepsilon, \quad (3.10)$$

$$|\sqrt{(x'_1 - e'_{21})^2 + (x'_2 - e'_{22})^2} - \sqrt{x_1^2 + (x_2 - 1)^2}| \leq \varepsilon. \quad (3.11)$$

According to (3.9), we obtain

$$\begin{aligned} & |(x_1'^2 + x_2'^2) - (x_1^2 + x_2^2)| \\ &= |\sqrt{x_1'^2 + x_2'^2} - \sqrt{x_1^2 + x_2^2}| |\sqrt{x_1'^2 + x_2'^2} + \sqrt{x_1^2 + x_2^2}| \\ &\leq \varepsilon(d + \varepsilon + d) \leq \left(2d + \frac{1}{13}\right)\varepsilon, \end{aligned} \quad (3.12)$$

since $\sqrt{x_1'^2 + x_2'^2} = \|f(x)\| \leq \|x\| + \varepsilon \leq d + \varepsilon$, $\sqrt{x_1^2 + x_2^2} \leq d$ and $0 < \varepsilon \leq \frac{1}{13}$. Similarly, by (3.10), we obtain

$$|((x_1' - e_{11}')^2 + x_2'^2) - ((x_1 - 1)^2 + x_2^2)| \leq \left(2d + \frac{27}{13}\right)\varepsilon, \quad (3.13)$$

since $\sqrt{(x_1' - e_{11}')^2 + x_2'^2} = \|f(x) - f(e_1)\| \leq \|x - e_1\| + \varepsilon \leq d + 1 + \varepsilon$, $\sqrt{(x_1 - 1)^2 + x_2^2} = \|x - e_1\| \leq d + 1$ and $0 < \varepsilon \leq \frac{1}{13}$. Analogously, in view of (3.11), we have

$$|((x_1' - e_{21}')^2 + (x_2' - e_{22}')^2) - (x_1^2 + (x_2 - 1)^2)| \leq \left(2d + \frac{27}{13}\right)\varepsilon, \quad (3.14)$$

since $\sqrt{(x_1' - e_{21}')^2 + (x_2' - e_{22}')^2} = \|f(x) - f(e_2)\| \leq \|x - e_2\| + \varepsilon \leq d + 1 + \varepsilon$ and $0 < \varepsilon \leq \frac{1}{13}$.

It now follows from (3.13) that

$$|(x_1'^2 + x_2'^2) - (x_1^2 + x_2^2) - 2e_{11}'x_1' + 2x_1 + e_{11}'^2 - 1| \leq \left(2d + \frac{27}{13}\right)\varepsilon.$$

Using (3.12), we obtain

$$-\left(4d + \frac{28}{13}\right)\varepsilon \leq -2e_{11}'x_1' + 2x_1 + e_{11}'^2 - 1 \leq \left(4d + \frac{28}{13}\right)\varepsilon. \quad (3.15)$$

Similarly, using (3.14), we obtain

$$|(x_1'^2 + x_2'^2) - (x_1^2 + x_2^2) - 2e_{21}'x_1' - 2e_{22}'x_2' + 2x_2 + e_{21}'^2 + e_{22}'^2 - 1| \leq \left(2d + \frac{27}{13}\right)\varepsilon.$$

Using (3.12) again, we obtain

$$-\left(4d + \frac{28}{13}\right)\varepsilon \leq -2e_{21}'x_1' - 2e_{22}'x_2' + 2x_2 + e_{21}'^2 + e_{22}'^2 - 1 \leq \left(4d + \frac{28}{13}\right)\varepsilon. \quad (3.16)$$

Moreover, put $x = e_1$ and $y = 0$ in (1.3) and use (3.1) to obtain

$$|e_{11}' - 1| \leq \varepsilon \quad \text{and} \quad |e_{11}'^2 - 1| = |e_{11}' - 1| |e_{11}' + 1| \leq \frac{27}{13}\varepsilon.$$

Hence, it follows from (3.15) that

$$|x_1 - x_1'| \leq \left(3d + \frac{57}{26}\right)\varepsilon, \quad (3.17)$$

since $\max\{|x'_1|, |x'_2|\} \leq \|f(x)\| \leq \|x\| + \varepsilon \leq d + \frac{1}{13}$ and

$$-2e'_{11}x'_1 + 2x_1 + e_{11}^2 - 1 = 2(1 - e'_{11})x'_1 + 2(x_1 - x'_1) + e_{11}^2 - 1.$$

On the other hand, it follows from (3.1) that

$$|2e'_{21}x'_1| \leq 2c_{21}|x'_1|\varepsilon \leq 2c_{21}\left(d + \frac{1}{13}\right)\varepsilon.$$

Due to (3.1), we obtain

$$-\varepsilon \leq 1 - e'_{22} \leq c_{22}\varepsilon.$$

By (3.16) together with (3.3), we obtain

$$-\left((2 + c_{21})d + \frac{55 + 2c_{21}}{26}\right)\varepsilon \leq x_2 - e'_{22}x'_2 \leq \left((2 + c_{21})d + \frac{27 + c_{21}}{13}\right)\varepsilon,$$

or since $x_2 - e'_{22}x'_2 = (x_2 - x'_2) + (1 - e'_{22})x'_2$,

$$\begin{aligned} & -\left((2 + c_{21} + c_{22})d + \frac{55 + 2c_{21} + 2c_{22}}{26}\right)\varepsilon \\ & \leq x_2 - x'_2 \leq \left((2 + c_{21} + c_{22})d + \frac{27 + c_{21} + c_{22}}{13}\right)\varepsilon. \end{aligned}$$

Therefore, we have

$$|x_2 - x'_2| \leq \left((2 + c_{21} + c_{22})d + \frac{55 + 2c_{21} + 2c_{22}}{26}\right)\varepsilon. \quad (3.18)$$

Finally, we define an isometry $U : D \rightarrow \mathbb{R}^2$ by $U(x) = x = (x_1, x_2)$. It then follows from (3.17), (3.18) and Lemma 3.1 that

$$\begin{aligned} \|f(x) - U(x)\| &= \|f(x) - x\| = \|(x'_1 - x_1, x'_2 - x_2)\| \\ &= \sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2} \\ &\leq (8d + 4)\varepsilon, \end{aligned}$$

which completes our proof. \square

By c_{ij} in Lemma 3.1, (3.17) and (3.18), we can express the upper bound of inequality (3.8) more precisely using rational numbers as:

$$\begin{aligned} \|f(x) - U(x)\| &= \sqrt{62.8496299d^2 + 50.2268490d + 11.1869791}\varepsilon \\ &\leq \sqrt{62.8496299d^2 + 51.0138281d + 10.4000000}\varepsilon \\ &\leq (7.9277759d + 3.2249031)\varepsilon. \end{aligned}$$

3.2 On bounded subsets of \mathbb{R}^3

Lemma 3.1 can now be extended without difficulty to the case of the 3-dimensional Euclidean space. It is evident that there exist positive real numbers c_{ij} , $i, j \in \{1, 2, 3\}$ with $j \leq i$ that satisfy the conditions in (3.1) whenever the function f satisfies $f(0) = 0$ and the inequality (1.3) for all $x, y \in \{0, e_1, e_2, e_3\}$.

In the following lemma, the parameters c_{ij} are assumed to be positive real numbers, while they were assumed to be positive integers in [8].

Lemma 3.3 *Let $\{e_1, e_2, e_3\}$ be the standard basis for the 3-dimensional Euclidean space \mathbb{R}^3 , let D be a subset of \mathbb{R}^3 that satisfies $\{0, e_1, e_2, e_3\} \subset D$, and let $f : D \rightarrow \mathbb{R}^3$ be a function that satisfies $f(0) = 0$ and the inequality (1.3) for all $x, y \in \{0, e_1, e_2, e_3\}$ and for some constant ε with $0 < \varepsilon \leq \frac{1}{13}$. By Lemma 2.2, it can be assumed that $f(e_1) = (e'_{11}, 0, 0)$, $f(e_2) = (e'_{21}, e'_{22}, 0)$, and $f(e_3) = (e'_{31}, e'_{32}, e'_{33})$, where $e'_{11} \geq 0$, $e'_{22} \geq 0$, and $e'_{33} \geq 0$. Then, there exist positive real numbers c_{ij} , $i, j \in \{1, 2, 3\}$ with $j \leq i$ that satisfy the inequalities in (3.1). In particular, $c_{11} = 1.0000000$, $c_{21} \approx 3.7403981$, $c_{22} \approx 1.5978326$, $c_{31} \approx 3.7403981$, $c_{32} \approx 5.1635231$, and $c_{33} \approx 2.8340052$ may be selected.*

Proof In view of (1.3) and $f(0) = 0$, we have

$$\begin{aligned} \|f(e_1)\| - 1 &\leq \varepsilon, & \|f(e_2)\| - 1 &\leq \varepsilon, & \|f(e_3)\| - 1 &\leq \varepsilon, \\ \|f(e_1) - f(e_2)\| - \sqrt{2} &\leq \varepsilon, & \|f(e_2) - f(e_3)\| - \sqrt{2} &\leq \varepsilon, \\ \|f(e_3) - f(e_1)\| - \sqrt{2} &\leq \varepsilon \end{aligned}$$

for any ε with $0 < \varepsilon \leq \frac{1}{13}$. Therefore, from the inequalities above, we obtain the following inequalities along with (3.2), (3.3) and (3.4):

$$(1 - \varepsilon)^2 \leq e_{31}^2 + e_{32}^2 + e_{33}^2 \leq (1 + \varepsilon)^2, \quad (3.19)$$

$$(\sqrt{2} - \varepsilon)^2 \leq (e'_{21} - e'_{31})^2 + (e'_{22} - e'_{32})^2 + e_{33}^2 \leq (\sqrt{2} + \varepsilon)^2, \quad (3.20)$$

$$(\sqrt{2} - \varepsilon)^2 \leq (e'_{31} - e'_{11})^2 + e_{32}^2 + e_{33}^2 \leq (\sqrt{2} + \varepsilon)^2. \quad (3.21)$$

Moreover, using (3.21), we have

$$(\sqrt{2} - \varepsilon)^2 - e_{11}^2 - (e_{31}^2 + e_{32}^2 + e_{33}^2) \leq -2e'_{11}e'_{31} \leq (\sqrt{2} + \varepsilon)^2 - e_{11}^2 - (e_{31}^2 + e_{32}^2 + e_{33}^2).$$

In view of (3.2) and (3.19), we obtain

$$\frac{-(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - \varepsilon)} \leq e'_{31} \leq \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - \varepsilon)}$$

and we solve the following inequalities

$$-c_{31}\varepsilon \leq \frac{-(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - \varepsilon)} \quad \text{and} \quad \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2}{2(1 - \varepsilon)} \leq c_{31}\varepsilon$$

and we find

$$c_{31} = \frac{53 + 26\sqrt{2}}{24} \approx 3.7403981 \quad (3.22)$$

as the smallest possible positive real number that satisfies the preceding inequalities.

By (3.20), we obtain

$$\begin{aligned} 2 - 2\sqrt{2}\varepsilon + \varepsilon^2 - (e_{21}'^2 + e_{22}'^2) - (e_{31}'^2 + e_{32}'^2 + e_{33}'^2) \\ \leq -2e_{21}'e_{31}' - 2e_{22}'e_{32}' \leq 2 + 2\sqrt{2}\varepsilon + \varepsilon^2 - (e_{21}'^2 + e_{22}'^2) - (e_{31}'^2 + e_{32}'^2 + e_{33}'^2). \end{aligned}$$

Moreover, we use (3.3) and (3.19) to obtain

$$-(4 + 2\sqrt{2})\varepsilon + \varepsilon^2 - 2e_{21}'e_{31}' \leq 2e_{22}'e_{32}' \leq (4 + 2\sqrt{2})\varepsilon + \varepsilon^2 - 2e_{21}'e_{31}',$$

or one time using (3.1) with $i = 2$ and $j = 1$ and the next time using (3.1) with $i = 3$ and $j = 1$,

$$-(4 + 2\sqrt{2})\varepsilon + \varepsilon^2 - 2c_{21}c_{31}\varepsilon^2 \leq 2e_{22}'e_{32}' \leq (4 + 2\sqrt{2})\varepsilon + \varepsilon^2 + 2c_{21}c_{31}\varepsilon^2.$$

Due to (3.1) with $i = j = 2$, we have

$$\frac{-(4 + 2\sqrt{2})\varepsilon + \varepsilon^2 - 2c_{21}c_{31}\varepsilon^2}{2(1 - c_{22}\varepsilon)} \leq e_{32}' \leq \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2 + 2c_{21}c_{31}\varepsilon^2}{2(1 - c_{22}\varepsilon)}$$

and we solve the following inequalities

$$-c_{32}\varepsilon \leq \frac{-(4 + 2\sqrt{2})\varepsilon + \varepsilon^2 - 2c_{21}c_{31}\varepsilon^2}{2(1 - c_{22}\varepsilon)} \quad \text{and} \quad \frac{(4 + 2\sqrt{2})\varepsilon + \varepsilon^2 + 2c_{21}c_{31}\varepsilon^2}{2(1 - c_{22}\varepsilon)} \leq c_{32}\varepsilon.$$

We use (3.5), (3.7), and (3.22) and put $\varepsilon = \frac{1}{13}$ in the second inequality to obtain

$$c_{32} = \frac{4 + 2\sqrt{2} + \frac{1}{13}(1 + 2 \times 3.7403981^2)}{2(1 - \frac{1}{13} \times 1.5978326)} \approx 5.1635231 \quad (3.23)$$

as the smallest possible positive number that satisfies the last inequalities.

Finally, by (3.1) with $i = 3$ and $j = 1$, (3.1) with $i = 3$ and $j = 2$, and by (3.19), we have

$$1 - 2\varepsilon - (c_{31}^2 + c_{32}^2 - 1)\varepsilon^2 \leq e_{33}'^2 \leq (1 + \varepsilon)^2.$$

Moreover, we solve the following inequality

$$(1 - c_{33}\varepsilon)^2 \leq 1 - 2\varepsilon - (c_{31}^2 + c_{32}^2 - 1)\varepsilon^2,$$

whose solution is given as

$$\frac{1}{\varepsilon}(1 - \sqrt{1 - 2\varepsilon - (c_{31}^2 + c_{32}^2 - 1)\varepsilon^2}) \leq c_{33} \leq \frac{1}{\varepsilon}(1 + \sqrt{1 - 2\varepsilon - (c_{31}^2 + c_{32}^2 - 1)\varepsilon^2}).$$

In addition, we use (3.22) and (3.23) and put $\varepsilon = \frac{1}{13}$ in the lower bound for c_{33} to obtain

$$c_{33} = 13 - \sqrt{144 - c_{31}^2 - c_{32}^2} \approx 2.8340052 \quad (3.24)$$

as small a positive real number as possible that satisfies the last inequalities. \square

In the following theorem, we set $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$ such that $\{e_1, e_2, e_3\}$ is the standard basis for the 3-dimensional Euclidean space \mathbb{R}^3 . We denote by $B_d(0)$ the closed ball of radius d and centered at the origin of \mathbb{R}^3 , i.e., $B_d(0) = \{x \in \mathbb{R}^3 : \|x\| \leq d\}$.

Theorem 3.4 *Let D be a subset of the 3-dimensional Euclidean space \mathbb{R}^3 such that $\{0, e_1, e_2, e_3\} \subset D \subset B_d(0)$ for some $d \geq 1$ and let $f : D \rightarrow \mathbb{R}^3$ be a function that satisfies $f(0) = 0$ and the inequality (1.3) for all $x, y \in D$ and for some constant ε with $0 < \varepsilon \leq \frac{1}{13}$. Then, there exists an isometry $U : D \rightarrow \mathbb{R}^3$ such that*

$$\|f(x) - U(x)\| \leq (16d + 5)\varepsilon \quad (3.25)$$

for all $x \in D$.

Proof Considering Lemma 2.2, we can assume that $f(e_1) = (e'_{11}, 0, 0)$, $f(e_2) = (e'_{21}, e'_{22}, 0)$, and $f(e_3) = (e'_{31}, e'_{32}, e'_{33})$, where $e'_{11} \geq 0$, $e'_{22} \geq 0$, and $e'_{33} \geq 0$.

For any point $x = (x_1, x_2, x_3)$ of D , let $f(x) = x' = (x'_1, x'_2, x'_3)$. It then follows from (1.3) that

$$|\sqrt{x_1'^2 + x_2'^2 + x_3'^2} - \sqrt{x_1^2 + x_2^2 + x_3^2}| \leq \varepsilon, \quad (3.26)$$

$$|\sqrt{(x'_1 - e'_{11})^2 + x_2'^2 + x_3'^2} - \sqrt{(x_1 - 1)^2 + x_2^2 + x_3^2}| \leq \varepsilon, \quad (3.27)$$

$$|\sqrt{(x'_1 - e'_{21})^2 + (x'_2 - e'_{22})^2 + x_3'^2} - \sqrt{x_1^2 + (x_2 - 1)^2 + x_3^2}| \leq \varepsilon, \quad (3.28)$$

$$|\sqrt{(x'_1 - e'_{31})^2 + (x'_2 - e'_{32})^2 + (x'_3 - e'_{33})^2} - \sqrt{x_1^2 + x_2^2 + (x_3 - 1)^2}| \leq \varepsilon. \quad (3.29)$$

It follows from (3.26) that

$$\begin{aligned} & |(x_1'^2 + x_2'^2 + x_3'^2) - (x_1^2 + x_2^2 + x_3^2)| \\ &= |\sqrt{x_1'^2 + x_2'^2 + x_3'^2} - \sqrt{x_1^2 + x_2^2 + x_3^2}| |\sqrt{x_1'^2 + x_2'^2 + x_3'^2} + \sqrt{x_1^2 + x_2^2 + x_3^2}| \\ &\leq \varepsilon(d + \varepsilon + d) \leq \left(2d + \frac{1}{13}\right)\varepsilon, \end{aligned} \quad (3.30)$$

since $\sqrt{x_1'^2 + x_2'^2 + x_3'^2} = \|f(x)\| \leq \|x\| + \varepsilon \leq d + \varepsilon$, $\sqrt{x_1^2 + x_2^2 + x_3^2} \leq d$ and $0 < \varepsilon \leq \frac{1}{13}$. Similarly, it follows from (3.27) that

$$|((x'_1 - e'_{11})^2 + x_2'^2 + x_3'^2) - ((x_1 - 1)^2 + x_2^2 + x_3^2)| \leq \left(2d + \frac{27}{13}\right)\varepsilon, \quad (3.31)$$

since $\sqrt{(x'_1 - e'_{11})^2 + x_2'^2 + x_3'^2} = \|f(x) - f(e_1)\| \leq \|x - e_1\| + \varepsilon \leq d + 1 + \varepsilon$, $\sqrt{(x_1 - 1)^2 + x_2^2 + x_3^2} = \|x - e_1\| \leq d + 1$ and $0 < \varepsilon \leq \frac{1}{13}$. Analogously, by (3.28) and (3.29), we obtain

$$|((x'_1 - e'_{21})^2 + (x'_2 - e'_{22})^2 + x_3'^2) - (x_1^2 + (x_2 - 1)^2 + x_3^2)| \leq \left(2d + \frac{27}{13}\right)\varepsilon \quad (3.32)$$

and

$$|((x'_1 - e'_{31})^2 + (x'_2 - e'_{32})^2 + (x'_3 - e'_{33})^2) - (x_1^2 + x_2^2 + (x_3 - 1)^2)| \leq \left(2d + \frac{27}{13}\right)\varepsilon, \quad (3.33)$$

since $\sqrt{x_1^2 + (x_2 - 1)^2 + x_3^2} \leq d + 1$, $\sqrt{x_1^2 + x_2^2 + (x_3 - 1)^2} \leq d + 1$, and $0 < \varepsilon \leq \frac{1}{13}$.

It follows from (3.31) that

$$|(x_1'^2 + x_2'^2 + x_3'^2) - (x_1^2 + x_2^2 + x_3^2) - 2e'_{11}x'_1 + 2x_1 + e_{11}'^2 - 1| \leq \left(2d + \frac{27}{13}\right)\varepsilon.$$

By using (3.30), we obtain

$$-\left(4d + \frac{28}{13}\right)\varepsilon \leq -2e'_{11}x'_1 + 2x_1 + e_{11}'^2 - 1 \leq \left(4d + \frac{28}{13}\right)\varepsilon. \quad (3.34)$$

Similarly, it follows from (3.32) that

$$|(x_1'^2 + x_2'^2 + x_3'^2) - (x_1^2 + x_2^2 + x_3^2) - 2e'_{21}x'_1 - 2e'_{22}x'_2 + 2x_2 + e_{21}'^2 + e_{22}'^2 - 1| \leq \left(2d + \frac{27}{13}\right)\varepsilon.$$

Using (3.30), we obtain

$$-\left(4d + \frac{28}{13}\right)\varepsilon \leq -2e'_{21}x'_1 - 2e'_{22}x'_2 + 2x_2 + e_{21}'^2 + e_{22}'^2 - 1 \leq \left(4d + \frac{28}{13}\right)\varepsilon. \quad (3.35)$$

Analogously, we use (3.30) and (3.33) to obtain

$$\begin{aligned} -\left(4d + \frac{28}{13}\right)\varepsilon &\leq -2e'_{31}x'_1 - 2e'_{32}x'_2 - 2e'_{33}x'_3 + 2x_3 + e_{31}'^2 + e_{32}'^2 + e_{33}'^2 - 1 \\ &\leq \left(4d + \frac{28}{13}\right)\varepsilon. \end{aligned} \quad (3.36)$$

Moreover, putting $x = e_1$ and $y = 0$ in (1.3) and using (3.1), we have

$$|e'_{11} - 1| \leq \varepsilon \quad \text{and} \quad |e_{11}'^2 - 1| = |e'_{11} - 1||e'_{11} + 1| \leq \frac{27}{13}\varepsilon.$$

Therefore, it follows from (3.34) that

$$|x_1 - x'_1| \leq \left(3d + \frac{57}{26}\right)\varepsilon, \quad (3.37)$$

since $\max\{|x'_1|, |x'_2|, |x'_3|\} \leq \|f(x)\| \leq \|x\| + \varepsilon \leq d + \frac{1}{13}$ and

$$-2e'_{11}x'_1 + 2x_1 + e_{11}'^2 - 1 = 2(1 - e'_{11})x'_1 + 2(x_1 - x'_1) + e_{11}'^2 - 1.$$

On the other hand, it follows from (3.1) that

$$|2e'_{21}x'_1| \leq 2c_{21}|x'_1|\varepsilon \leq 2c_{21}\left(d + \frac{1}{13}\right)\varepsilon.$$

By (3.1), we obtain

$$-\varepsilon \leq 1 - e'_{22} \leq c_{22}\varepsilon.$$

Furthermore, using (3.35) together with (3.3), we have

$$-\left((2 + c_{21})d + \frac{55 + 2c_{21}}{26}\right)\varepsilon \leq x_2 - e'_{22}x'_2 \leq \left((2 + c_{21})d + \frac{27 + c_{21}}{13}\right)\varepsilon$$

or

$$\begin{aligned} & -\left((2 + c_{21} + c_{22})d + \frac{55 + 2c_{21} + 2c_{22}}{26}\right)\varepsilon \\ & \leq x_2 - x'_2 \leq \left((2 + c_{21} + c_{22})d + \frac{27 + c_{21} + c_{22}}{13}\right)\varepsilon, \end{aligned}$$

since

$$x_2 - e'_{22}x'_2 = (x_2 - x'_2) + (1 - e'_{22})x'_2.$$

Thus, we see that

$$|x_2 - x'_2| \leq \left((2 + c_{21} + c_{22})d + \frac{55 + 2c_{21} + 2c_{22}}{26}\right)\varepsilon. \quad (3.38)$$

Analogously, by using (3.1), we obtain

$$|2e'_{31}x'_1| \leq 2c_{31}\left(d + \frac{1}{13}\right)\varepsilon \quad \text{and} \quad |2e'_{32}x'_2| \leq 2c_{32}\left(d + \frac{1}{13}\right)\varepsilon.$$

Moreover, due to (3.36), we obtain

$$\begin{aligned} & -\left(4d + \frac{28}{13}\right)\varepsilon + 2e'_{31}x'_1 + 2e'_{32}x'_2 - (e'^2_{31} + e'^2_{32} + e'^2_{33}) + 1 \\ & \leq 2(x_3 - e'_{33}x'_3) \leq \left(4d + \frac{28}{13}\right)\varepsilon + 2e'_{31}x'_1 + 2e'_{32}x'_2 - (e'^2_{31} + e'^2_{32} + e'^2_{33}) + 1. \end{aligned}$$

In view of (3.19), we have

$$\begin{aligned} & -\left((4 + 2c_{31} + 2c_{32})d + \frac{55 + 2c_{31} + 2c_{32}}{13}\right)\varepsilon \\ & \leq 2(x_3 - e'_{33}x'_3) \leq \left((4 + 2c_{31} + 2c_{32})d + \frac{54 + 2c_{31} + 2c_{32}}{13}\right)\varepsilon. \end{aligned}$$

By (3.1), we see that $1 - c_{33}\varepsilon \leq e'_{33} \leq 1 + \varepsilon$ and hence,

$$|x_3 - x'_3| \leq \left((2 + c_{31} + c_{32} + c_{33})d + \frac{55 + 2c_{31} + 2c_{32} + 2c_{33}}{26} \right) \varepsilon. \quad (3.39)$$

Finally, we may define an isometry $U : D \rightarrow \mathbb{R}^3$ by $U(x) = x = (x_1, x_2, x_3)$. Then, we use (3.37), (3.38), (3.39), and Lemma 3.3 to obtain

$$\begin{aligned} \|f(x) - U(x)\| &= \|f(x) - x\| = \|(x'_1 - x_1, x'_2 - x_2, x'_3 - x_3)\| \\ &= \sqrt{(x'_1 - x_1)^2 + (x'_2 - x_2)^2 + (x'_3 - x_3)^2} \\ &\leq \sqrt{252d^2 + 134d + 21}\varepsilon \\ &\leq (16d + 5)\varepsilon, \end{aligned}$$

which completes our proof. \square

By c_{ij} in Lemma 3.3, (3.37), (3.38), and (3.39), we can express the upper bound of inequality (3.25) more precisely using rational numbers as:

$$\begin{aligned} \|f(x) - U(x)\| &= \sqrt{251.5802517d^2 + 133.1572732d + 20.2971267}\varepsilon \\ &\leq \sqrt{251.5802517d^2 + 134.9643999d + 18.4900000}\varepsilon \\ &\leq (15.8612816d + 4.3000000)\varepsilon. \end{aligned}$$

On the other hand, it follows from [8] that

$$\begin{aligned} \|f(x) - U(x)\| &= \sqrt{451d^2 + 1026d + 587}\varepsilon \\ &\leq \sqrt{451d^2 + 1029.0519930d + 587}\varepsilon \\ &\leq (21.2367606d + 24.2280829)\varepsilon. \end{aligned}$$

Therefore, one can see that the result of this paper is better than that of the previous paper [8].

4 Discussion

This paper deals with Fickett's theorem, which can also be called the stability problem of isometries, with a focus on the bounded subsets of \mathbb{R}^2 or \mathbb{R}^3 .

Fickett provided the authors with a decisive motivation to study this subject. The upper bound introduced in Fickett's theorem for the difference between ε -isometry and exact isometry is unfortunately very large compared to ε for any sufficiently small ε . This is a big drawback of Fickett's theorem. Thus, the work of further improving Fickett's theorem has to be attractive.

Moreover, the present paper not only proved the local stability of isometries for the case $n = 2$, which was not dealt with in the recent paper [8], but also improved the result of [8] when $n = 3$.

Finally, it is unclear whether the upper bound for the difference between ε -isometry and exact isometry is independent of the 'radius' d of the domain D . It seems unlikely, but it

would be very surprising if it could be proved that the upper bound is independent of the radius of the domain.

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Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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