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A note on optimal Hermite interpolation in Sobolev spaces

Guiqiao Xu[1*] and Xiaochen Yu[1]

*Correspondence:
Xuguiqiao@aliyun.com
[1]Department of Mathematics, Tianjin Normal University, Tianjin, 300387, P.R. China

Abstract

This paper investigates the optimal Hermite interpolation of Sobolev spaces $W_\infty^n[a,b]$, $n \in \mathbb{N}$ in space $L_\infty[a,b]$ and weighted spaces $L_{p,\omega}[a,b]$, $1 \le p < \infty$ with ω a continuous-integrable weight function in (a,b) when the amount of Hermite data is n. We proved that the Lagrange interpolation algorithms based on the zeros of polynomial of degree n with the leading coefficient 1 of the least deviation from zero in L_∞ (or $L_{p,\omega}[a,b]$, $1 \le p < \infty$) are optimal for $W_\infty^n[a,b]$ in $L_\infty[a,b]$ (or $L_{p,\omega}[a,b]$, $1 \le p < \infty$). We also give the optimal Hermite interpolation algorithms when we assume the endpoints are included in the interpolation systems.

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1 Introduction and main results

Let F be a Banach space of functions defined on a compact set D that can be continuously embedded in $C^r(D)$, BF be the unit ball of F, and G ($\supseteq F$) be a normed linear space with norm $\|\cdot\|_G$. We want to approximate functions f from F by using a finite number of arbitrary Hermite data $f^{(s)}(t)$ for some $s \le r$ and $t \in D$. We use an algorithm A_n that uses exactly n Hermite data to reconstruct functions from BF. The worst-case error of the algorithm A_n for BF in G is defined by

$$e(BF, A_n, G) := \sup_{f \in BF} \left\| f - A_n(f) \right\|_G. \tag{1.1}$$

Let Λ_n be an algorithm class that uses exactly n Hermite data, $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$. If there exists an algorithm $A_n^* \in \Lambda_n$ such that

$$e\left(BF, A_n^*, G\right) = \inf_{\Lambda_n} e(BF, A_n, G), \tag{1.2}$$

then we call A_n^* the nth optimal algorithm for Λ in the norm G. The value $e(BF, A_n^*, G)$ is called the nth optimal worst-case error for BF in G and we denote it as $e(n, BF, G)$.

Let $L_\infty \equiv L_\infty[a,b]$ be the space of measurable functions defined on $[a,b]$, for which the norm

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x\in[a,b]} \bigl|f(x)\bigr|$$

is finite. Meanwhile, for $1 \le p < \infty$ and continuous-integrable $\omega(x) > 0$ on (a,b), let $L_{p,\omega} \equiv L_{p,\omega}[a,b]$ be the space of measurable functions defined on $[a,b]$, for which the norm

$$\|f\|_{p,\omega} := \left(\int_a^b \bigl|f(x)\bigr|^p \omega(x)\,dx\right)^{1/p}$$

is finite. Using $C^n \equiv C^n[a,b]$, $n = 0,1,2,\dots$ represents the spaces of functions with an nth-order continuous derivative on $[a,b]$, respectively. Denote by $W_\infty^n \equiv W_\infty^n[a,b]$, $n \in \mathbb{N}$ the class of all functions f such that $f^{(n-1)}(f^{(0)} := f)$ are absolutely continuous and $f^{(n)} \in L_\infty$.

For the construction of algorithms for approximating multivariate functions using function values, the univariate Lagrange interpolation polynomial algorithms play a key role, see [2, 5, 8, 13–15]. Recently, some papers [3, 10] have considered the algorithms for approximating multivariate functions using Hermite data. To compare the approximation errors of Lagrange interpolation and Hermite interpolation, we introduce the concept of Hermite interpolation.

Let $x_1, x_2, \dots, x_r$ be r distinct points in $[a,b]$. Let $\Delta := \{a \le x_1 < x_2 < \cdots < x_r \le b, \alpha_i \in \mathbb{N}, n = \sum_{i=1}^r \alpha_i\}$ be a Hermite interpolation system. Then, the Hermite interpolation polynomial $H_\Delta(f)$ of a function $f \in W_\infty^n$ based on Δ is defined by

$$H_\Delta(f) \in \mathcal{P}_{n-1}, \quad \text{and} \quad H_\Delta^{(j)}(f, x_i) = f^{(j)}(x_i), \quad 0 \le j \le \alpha_i - 1,\ 1 \le i \le r, \tag{1.3}$$

where, and in the following, $\mathcal{P}_n$ represents the space of all algebraic polynomials of degree at most n. The classical Hermite interpolation formula gives

$$H_\Delta(f,x) = \sum_{k=1}^r \frac{W_\Delta(x)}{(x-x_k)^{\alpha_k}} \sum_{h=0}^{\alpha_k - 1} f^{(h)}(x_k) \frac{(x-x_k)^h}{h!} \left\{\frac{(x-x_k)^{\alpha_k}}{W_\Delta(x)}\right\}_{(x_k)}^{(\alpha_k - h - 1)},$$

where, and in the following,

$$W_\Delta(x) = \prod_{k=1}^r (x - x_k)^{\alpha_k}, \tag{1.4}$$

and $\{f(x)\}_{(x_k)}^{(s)}$ is the s-degree Taylor polynomial of f at x_k. In particular, if $x_1, x_2, \dots, x_n$ are n distinct points in $[a,b]$, i.e., $\Delta := \{a \le x_1 < x_2 < \cdots < x_n \le b, \alpha_i = 1, i = 1,2,\dots,n\}$, then we obtain the well-known Lagrange interpolation

$$L_\Delta(f,x) = \sum_{k=1}^n f(x_k)\ell_k(x),$$

where

$$\ell_k(x) = \frac{W_\Delta(x)}{(x-x_k)W'_\Delta(x_k)}, \quad k = 1,2,\dots,n.$$

Choosing interpolation systems is important for interpolation algorithms. For example, given a sufficiently smooth function, if nodes are not suitably chosen, then the Lagrange interpolation polynomials do not converge to the function as the number of nodes tends to infinity. A well-known example is Runge's phenomenon. Hence, the study of optimal Lagrange interpolation nodes is a hot research topic, see [1, 6, 9, 16] and the references therein.

The most important optimal Lagrange interpolation nodes problem is for C^0 in L_∞. For $n = 3$ and $n = 4$, the results can be found in [11] and [12], respectively. For $n \geq 5$, it is still an open problem. For $r \geq 1$, it is well known that the rth optimal Lagrange interpolation nodes are the zeros of the rth Chebyshev polynomial of the first kind for $C^r[-1,1]$ in $L_\infty[-1,1]$. Recently, [16] obtained the rth optimal Lagrange interpolation nodes for $W^r_\infty[-1,1]$ in $L_{p,\omega}[-1,1]$, $1 \leq p < \infty$.

Hermite interpolation is a kind of interpolation that is wider than Lagrange interpolation. It can use not only the function value data but also the derivatives value data. Under the condition of using the same amount of data, can increasing the use of derivatives value data make the calculation result more accurate? In general the answer is no. In the following, we give the optimal Hermite interpolation systems to show this.

Let

$$E_{n,p,\omega} := \inf_{g\in\mathcal{P}_{n-1}} \left\|x^n - g(x)\right\|_{p,\omega}, \quad 1 \leq p < \infty. \tag{1.5}$$

Furthermore, let $W_{n,p,\omega} \in \mathcal{P}_n$ satisfy

$$\|W_{n,p,\omega}\|_{p,\omega} = E_{n,p,\omega} \quad \text{and} \quad W_{n,p,\omega}(x) = x^n + c_1 x^{n-1} + \cdots + c_n. \tag{1.6}$$

Then, $W_{n,p,\omega}$ is unique and has exactly n zeros (see Lemma 2.2)

$$a < \xi_{1,p,\omega} < \xi_{2,p,\omega} < \cdots < \xi_{n,p,\omega} < b. \tag{1.7}$$

Let

$$\Delta_{n,p,\omega} = \{\xi_{1,p,\omega}, \xi_{2,p,\omega}, \ldots, \xi_{n,p,\omega}\}. \tag{1.8}$$

Then, $L_{\Delta_{n,p,\omega}}(f)$ has the explicit expression

$$L_{\Delta_{n,p,\omega}}(f,x) = \sum_{k=1}^{n} f(\xi_{k,p,\omega})\ell_{k,p,\omega}(x), \tag{1.9}$$

where

$$\ell_{k,p,\omega}(x) = \frac{W_{n,p,\omega}(x)}{(x-\xi_{k,p,\omega})W'_{n,p,\omega}(\xi_{k,p,\omega})}$$

and

$$W_{n,p,\omega}(x) = \prod_{k=1}^{n}(x-\xi_{k,p,\omega}). \tag{1.10}$$

First, we obtained the following results.

Theorem 1.1

(1) *For $p = \infty$, we have*

$$e(n, BW_\infty^n, L_\infty) = e(BW_\infty^n, L_{\Delta_{n,\infty}}, L_\infty) = \frac{(b-a)^n}{n! 2^{2n-1}}, \tag{1.11}$$

where

$$\Delta_{n,\infty} = \left(\frac{a+b}{2} + \frac{b-a}{2} \cos\frac{(2n-1)\pi}{2n}, \frac{a+b}{2} + \frac{b-a}{2} \cos\frac{(2n-3)\pi}{2n}, \dots, \right. \\ \left. \frac{a+b}{2} + \frac{b-a}{2} \cos\frac{\pi}{2n} \right). \tag{1.12}$$

(2) *Let $1 \le p < \infty$ and assume that $\omega(x) > 0$ is continuous-integrable on (a, b). Then, we have*

$$e(n, BW_\infty^n, L_{p,\omega}) = e(BW_\infty^n, L_{\Delta_{n,p,\omega}}, L_{p,\omega}) = \frac{E_{n,p,\omega}}{n!}, \tag{1.13}$$

where $\Delta_{n,p,\omega}$ is given by (1.8).

From Theorem 1.1 we know that the optimal Hermite interpolation is Lagrange interpolation, i.e., increasing the use of derivatives value data can not make the calculation result more accurate.

In practice one often wants to have boundary points as interpolation systems, i.e.,

$$\Delta := \left\{ a = x_1 < x_2 < \cdots < x_r = b, \alpha_i \in \mathbb{N}, n = \sum_{i=1}^{r} \alpha_i \right\}. \tag{1.14}$$

Then, the following question arises: for which Δ^* of the form (1.14), we have

$$e(BF, H_{\Delta^*}, G) = \bar{e}(n, BF, G) := \inf_{\Delta \text{ is of the form (1.14)}} e(BF, H_\Delta, G). \tag{1.15}$$

For the Lagrange interpolation, Hoang [6] considered this problem for $C^n[-1,1]$ in $L_\infty[-1,1]$. Recently, Xu and Wang [16] considered this problem for $W_\infty^n[-1,1]$ in $L_\infty[-1,1]$ and $L_{p,\omega}[-1,1]$, $1 \le p < \infty$. In this paper, we will extend the result of [16] into Hermite interpolation and obtain the following results.

Theorem 1.2

(1) *Let $p = \infty$ and $n > 2$. Then, we have*

$$\bar{e}(n, BW_\infty^n, L_\infty) = e(BW_\infty^n, L_{\Delta^*_{n,\infty}}, L_\infty) = \frac{(b-a)^n}{(\cos\frac{\pi}{2n})^n 2^{2n-1} n!}, \tag{1.16}$$

where

$$\Delta^*_{n,\infty} = \left(a, \frac{a+b}{2} + \frac{b-a}{2} \cos\frac{(2n-3)\pi}{2n} \Big/ \cos\frac{\pi}{2n}, \dots, \right. \\ \left. \frac{a+b}{2} + \frac{b-a}{2} \cos\frac{3\pi}{2n} \Big/ \cos\frac{\pi}{2n}, b \right). \tag{1.17}$$

(2) *Let $1 \le p < \infty$, $n > 2$ and assume that $\omega(x) > 0$ is continuous-integrable on (a,b). Then, we have*

$$\bar{e}\big(n, BW_\infty^n, L_{p,\omega}\big) = e\big(BW_\infty^n, L_{\Delta_{n,p,\omega}^*}, L_{p,\omega}\big) = \frac{E_{n-2,p,\overline{\omega}}}{n!}, \tag{1.18}$$

where

$$\overline{\omega}(x) = (x-a)^p(b-x)^p\omega(x), \qquad \Delta_{n,p,\omega}^* = (a, \xi_{1,p,\overline{\omega}}, \xi_{2,p,\overline{\omega}}, \dots, \xi_{n-2,p,\overline{\omega}}, b), \tag{1.19}$$

and $\xi_{1,p,\overline{\omega}}, \xi_{2,p,\overline{\omega}}, \dots, \xi_{n-2,p,\overline{\omega}}$ are given by (1.7) with $n-2$.

The remainder of this paper is organized as follows. In Sect. 2, we give the proofs of Theorem 1.1 and Theorem 1.2. In Sect. 3, we give some examples to show the results.

2 Proofs of Theorem 1.1 and Theorem 1.2

To prove Theorem 1.1, we first give a lemma.

Lemma 2.1 *Let $f \in W_\infty^n$. Assume that $\Delta := \{a \le x_1 < x_2 < \cdots < x_r \le b, \alpha_i \in \mathbb{N}, n = \sum_{i=1}^r \alpha_i\}$ is a Hermite interpolation system. Then, the remainder $R_\Delta(f,x) := f(x) - H_\Delta(f,x)$ for the Hermite interpolation polynomial based on Δ satisfies*

$$\big|R_\Delta(f,x)\big| = \big|f(x) - H_\Delta(f,x)\big| \le \frac{\|f^{(n)}\|_\infty}{n!}\big|W_\Delta(x)\big|, \quad x \in [a,b], \tag{2.1}$$

where W_Δ is given by (1.4). In particular, if $f \in C^n$, then

$$R_\Delta(f,x) = f(x) - H_\Delta(f,x) = \frac{f^{(n)}(\xi)}{n!} W_\Delta(x), \quad x \in [a,b], \tag{2.2}$$

for some $\xi \in [-1,1]$ depending on x and Δ.

Proof Since (2.1) is trivially satisfied if x coincides with one of the interpolation points $x_1, \dots, x_r$, we need be concerned only with the case where x does not coincide with one of the interpolation nodes. Keeping x fixed, consider $g : [a,b] \to \mathbb{R}$ given by

$$g(y) := R_\Delta(f,y) - W_\Delta(y)\frac{R_\Delta(f,x)}{W_\Delta(x)}, \quad y \in [a,b]. \tag{2.3}$$

By the assumption on f we know $g \in W_\infty^n$. From (1.3) and (2.3) we conclude that g has at least $n+1$ zeros (counting multiplicity), namely single zero x and α_k fold zeros $x_k, k = 1, \dots, r$. Then, by Rolle's theorem, the derivative g' has at least n zeros. Repeating the argument, by induction we deduce that the derivative $g^{(n-1)}$ has at least two zeros in $[a,b]$, which we denote by z_1 and z_2 ($z_1 < z_2$), respectively. Since $g \in W_\infty^n$, then by the Newton–Leibniz formula we obtain

$$0 = g^{(n-1)}(z_2) - g^{(n-1)}(z_1) = \int_{z_1}^{z_2} g^{(n)}(y)\,dy. \tag{2.4}$$

It is known that $H_\Delta(f)$ is an algebraic polynomial of degree at most $n-1$. Hence, we obtain

$$\big(H_\Delta(f)\big)^{(n)}(y) = 0. \tag{2.5}$$

By a direct computation we obtain

$$(W_\triangle)^{(n)}(y) = n!. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), we obtain

$$0 = \int_{z_1}^{z_2} \left[f^{(n)}(y) - n! \frac{R_\triangle(f,x)}{W_\triangle(x)} \right] dy = \int_{z_1}^{z_2} f^{(n)}(y)\, dy - n!(z_2 - z_1) \frac{R_\triangle(f,x)}{W_\triangle(x)}. \tag{2.7}$$

From (2.7) it follows that

$$R_\triangle(f,x) = \frac{\int_{z_1}^{z_2} f^{(n)}(y)\, dy}{n!(z_2 - z_1)} W_\triangle(x). \tag{2.8}$$

Combining

$$\left| \int_{z_1}^{z_2} f^{(n)}(y)\, dy \right| \leq \int_{z_1}^{z_2} \left| f^{(n)}(y) \right| dy \leq \int_{z_1}^{z_2} \left\| f^{(n)} \right\|_\infty dy = \left\| f^{(n)} \right\|_\infty (z_2 - z_1)$$

with (2.8) we obtain (2.1). Besides, if $f \in C^n$, then $g^{(n)}$ has at least one zero ξ in $[a,b]$, i.e., $g^{(n)}(\xi) = 0$. Hence, by differentiating n times on two sides of (2.3) first, and then substituting (2.5) and (2.6) into the obtained relation, we obtain

$$0 = f^{(n)}(\xi) - n! \frac{R_\triangle(f,x)}{W_\triangle(x)}. \tag{2.9}$$

From (2.9) we obtain (2.2). This completes the proof of Lemma 2.1. □

Lemma 2.2 *Let $1 \leq p < \infty$ and assume that $\omega(x) > 0$ is continuous-integrable on $(-1,1)$. Then, there exists a unique $W_{n,p,\omega} \in \mathcal{P}_n$ for all $n \in \mathbb{N}$ such that*

$$\| W_{n,p,\omega} \|_{p,\omega} = E_{n,p,\omega} \quad \textit{and} \quad W_{n,p,\omega}(x) = x^n + c_1 x^{n-1} + \cdots + c_n,$$

where $E_{n,p,\omega}$ is given by (1.5). Furthermore, $W_{n,p,\omega}$ has exactly n zeros given by (1.7).

Proof The proof of the problem on $[-1,1]$ can be found in [16]. In general, we can use the variable substitution $x = \frac{a+b}{2} + \frac{b-a}{2} t$ to refer the problem on $[a,b]$ to this on $[-1,1]$. We omit the details. □

Proof of Theorem 1.1 We consider (1) first. Let $\Delta_{n,\infty}$ be given by (1.12). Then, for any $f \in W_\infty^n$, it follows from (2.1) that

$$\left| f(x) - L_{\Delta_{n,\infty}}(f,x) \right| \leq \frac{\| f^{(n)} \|_\infty}{n!} \left| \prod_{i=1}^{n} \left(x - \frac{a+b}{2} - \frac{b-a}{2} \cos \frac{(2i-1)\pi}{2n} \right) \right|,$$
$$x \in [a,b]. \tag{2.10}$$

Let $x = \frac{a+b}{2} + \frac{b-a}{2}t$. Then, (2.10) becomes

$$\begin{aligned} \left|f(x) - L_{\Delta_{n,\infty}}(f,x)\right| &\le \frac{\|f^{(n)}\|_\infty (b-a)^n}{n!2^n} \left|\prod_{i=1}^{n}\left(t - \cos\frac{(2i-1)\pi}{2n}\right)\right| \\ &= \frac{\|f^{(n)}\|_\infty (b-a)^n}{n!2^{2n-1}} \left|T_n(t)\right|, \quad t \in [-1,1], \end{aligned} \tag{2.11}$$

where T_n is the nth Chebyshev polynomial of the first kind, i.e., $T_n(t) = \cos(n \arccos t)$. Let $f \in BW_\infty^n$. Then, we have $\|f^{(n)}\|_\infty \le 1$. Combining this fact with $\|T_n\|_\infty = 1$ as well as (2.11), we obtain

$$e\left(BW_\infty^n, L_{\Delta_{n,\infty}}, L_\infty\right) = \sup_{f \in BW_\infty^n} \left\|f - L_{\Delta_{n,\infty}}(f)\right\|_\infty \le \frac{(b-a)^n}{n!2^{2n-1}}. \tag{2.12}$$

From (1.2) and (2.12) we obtain the upper estimate.

Now, we consider the lower estimate. Let $\Delta := \{a \le x_1 < x_2 < \cdots < x_r \le b, \alpha_i \in \mathbb{N}, n = \sum_{i=1}^{r} \alpha_i\}$ be an arbitrary Hermite interpolation system of cardinality n in $[a,b]$. Consider the function $g(x) = \frac{x^n}{n!}$. Then, from $g^{(n)}(x) = 1$ and (2.2) it follows that $g \in W_\infty^n$ and

$$g(x) - H_\Delta(g,x) = \frac{W_\Delta(x)}{n!}, \quad x \in [a,b]. \tag{2.13}$$

Let $x = \frac{a+b}{2} + \frac{b-a}{2}t$. Then, by (1.4) we obtain

$$W_\Delta(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = \frac{(b-a)^n}{2^n} h(t), \quad t \in [-1,1], \tag{2.14}$$

where

$$h(t) = t^n + b_1 t^{n-1} + b_2 t^{n-2} + \cdots + b_n. \tag{2.15}$$

Then, it follows from Theorem 6.1 in [4, Ch. 3] that

$$\|h\|_\infty \ge 2^{1-n}. \tag{2.16}$$

Combining (1.1), (2.13), (2.14) and (2.16), we obtain

$$e\left(BW_\infty^n, H_\Delta, L_\infty\right) \ge \left\|g - H_\Delta(g)\right\|_\infty = \frac{\|W_\Delta\|_\infty}{n!} = \frac{(b-a)^n}{n!2^n}\|h\|_\infty \ge \frac{(b-a)^n}{n!2^{2n-1}}. \tag{2.17}$$

From (1.2) and (2.17) we obtain the lower estimate.

Next, we consider (2). We consider the upper estimate first. Let $\Delta_{n,p,\omega}$ be given by (1.8) and $W_{n,p,\omega}$ be given by (1.6). If $f \in BW_\infty^n$, then we have $\|f^{(n)}\|_\infty \le 1$. Combining this fact with (2.1) we obtain

$$\left|f(x) - L_{\Delta_{n,p,\omega}}(f,x)\right| \le \frac{|W_{n,p,\omega}(x)|}{n!}, \quad x \in [a,b].$$

It follows that

$$\left\|f - L_{\Delta_{n,p,\omega}}(f)\right\|_{p,\omega} \le \frac{\|W_{n,p,\omega}\|_{p,\omega}}{n!} = \frac{E_{n,p,\omega}}{n!}. \tag{2.18}$$

From (1.1) and (2.18) we obtain

$$e(BW_\infty^n, L_{\Delta_{n,p,\omega}}, L_{p,\omega}) \le \frac{E_{n,p,\omega}}{n!}. \tag{2.19}$$

From (1.2) and (2.19) we obtain the upper estimate.

Now, we consider the lower estimate. Let $\Delta := \{a \le x_1 < x_2 < \cdots < x_r \le b, \alpha_i \in \mathbb{N}, n = \sum_{i=1}^r \alpha_i\}$ be an arbitrary Hermite interpolation system of cardinality n in $[a,b]$. Consider the function $g(x) = \frac{x^n}{n!}$. Then, $g \in W_\infty^n$ and (2.13) holds. From the first equality in (2.14) and (1.5) as well as (1.6) it follows that

$$\| W_\Delta \|_{p,\omega} \ge E_{n,p,\omega}. \tag{2.20}$$

From (1.1), (2.13) and (2.20) it follows that

$$e(BW_\infty^n, H_\Delta, L_{p,\omega}) \ge \| g - H_\Delta(g) \|_{p,\omega} = \frac{\| W_\Delta \|_{p,\omega}}{n!} \ge \frac{E_{n,p,\omega}}{n!}. \tag{2.21}$$

From (1.2) and (2.21) we obtain the lower estimate of (2). Theorem 1.1 is proved. □

Let $BC^n = \{f \in C^n : \|f^{(n)}\|_\infty \le 1\}$. Using the fact $BW_\infty^n \subset BC^n$ and $g(x) = \frac{x^n}{n!} \in BC^n$ for $n \in \mathbb{N}$, combining the proof of Theorem 1.1, we obtained the following results.

Corollary 2.3

(1) *For $p = \infty$, we have*

$$e(n, BC^n, L_\infty) = e(BC^n, L_{\Delta_{n,\infty}}, L_\infty) = \frac{(b-a)^n}{n! 2^{2n-1}}, \tag{2.22}$$

where $\Delta_{n,\infty}$ is given by (1.12).

(2) *Let $1 \le p < \infty$ and assume that $\omega(x) > 0$ is continuous-integrable on (a,b). Then, we have*

$$e(n, BC^n, L_{p,\omega}) = e(BC^n, L_{\Delta_{n,p,\omega}}, L_{p,\omega}) = \frac{E_{n,p,\omega}}{n!}, \tag{2.23}$$

where $\Delta_{n,p,\omega}$ is given by (1.8).

Proof of Theorem 1.2 We consider (1) first. For any $f \in BW_\infty^n$, from (2.1) it follows that

$$\left| f(x) - L_{\Delta_{n,\infty}^*}(f,x) \right| \le \frac{1}{n!} \left| \prod_{i=1}^n \left(x - \frac{a+b}{2} - \frac{b-a}{2} \cos \frac{(2i-1)\pi}{2n} \Big/ \cos \frac{\pi}{2n} \right) \right|, \quad x \in [a,b]. \tag{2.24}$$

Let $x = \frac{a+b}{2} + \frac{b-a}{2\cos\frac{\pi}{2n}} t$. Then, we have

$$\prod_{i=1}^n \left(x - \frac{a+b}{2} - \frac{b-a}{2} \cos \frac{(2i-1)\pi}{2n} \Big/ \cos \frac{\pi}{2n} \right) = \frac{(b-a)^n T_n(t)}{(\cos\frac{\pi}{2n})^n 2^{2n-1}}, \quad t \in \left[-\cos \frac{\pi}{2n}, \cos \frac{\pi}{2n} \right]. \tag{2.25}$$

From (1.1), (2.24) and (2.25) it follows that

$$\begin{aligned} e\bigl(BW_\infty^n, L_{\Delta^*_{n,\infty}}, L_\infty\bigr) &\le \frac{(b-a)^n}{(\cos\frac{\pi}{2n})^n 2^{2n-1} n!} \sup_{t\in[-\cos\frac{\pi}{2n},\cos\frac{\pi}{2n}]} \bigl|T_n(t)\bigr| \\ &= \frac{(b-a)^n}{(\cos\frac{\pi}{2n})^n 2^{2n-1} n!}. \end{aligned} \tag{2.26}$$

From (1.2) and (2.26) we obtain the upper estimate.

Now, we consider the lower estimate. Let $\Delta := \{a = x_1 < x_2 < \cdots < x_r = b, \alpha_i \in \mathbb{N}, n = \sum_{i=1}^r \alpha_i\}$ be an arbitrary Hermite interpolation system of cardinality n including the endpoints a and b. Consider the function $g(x) = \frac{x^n}{n!}$. Then, $g \in W_\infty^n$ and (2.13) holds. Let $x = \frac{a+b}{2} + \frac{b-a}{2}t$. Denote $t_i = \frac{2}{b-a}(x_i - \frac{a+b}{2})$, $i = 1,\dots,r$. Then, by (1.4) one obtains

$$W_\Delta(x) = \frac{(b-a)^n}{2^n}\prod_{i=1}^r (t-t_i)^{\alpha_i}, \quad t_1 = -1,\ t_r = 1,\ t \in [-1,1]. \tag{2.27}$$

Let

$$g(t) = \bigl(t^2-1\bigr)\prod_{i=2}^{n-1}\left(t - \cos\frac{(2i-1)\pi}{2n} \Big/ \cos\frac{\pi}{2n}\right) = \frac{T_n(t\cos\frac{\pi}{2n})}{2^{n-1}(\cos\frac{\pi}{2n})^n}.$$

Then, it is easy to verify that

$$\|g\|_\infty = \frac{1}{2^{n-1}(\cos\frac{\pi}{2n})^n} \tag{2.28}$$

and

$$g\left(\frac{\cos\frac{i\pi}{n}}{\cos\frac{\pi}{2n}}\right) = \frac{(-1)^i}{2^{n-1}(\cos\frac{\pi}{2n})^n}, \quad i = 1,\dots,n-1. \tag{2.29}$$

Assume that

$$\left\|\prod_{i=1}^r (t-t_i)^{\alpha_i}\right\|_\infty < \frac{1}{2^{n-1}(\cos\frac{\pi}{2n})^n}. \tag{2.30}$$

Let

$$R(t) = g(t) - \prod_{i=1}^r (t-t_i)^{\alpha_i}, \quad t \in [-1,1].$$

Then, it is easy to verify that $R(t)$ is a polynomial of degree at most $n-1$. Furthermore, from (2.29) and (2.30) one can check that

$$R\left(\frac{\cos\frac{i\pi}{n}}{\cos\frac{\pi}{2n}}\right)(-1)^i > 0, \quad i = 1,\dots,n-1.$$

Thus, the polynomial $R(t)$ has at least $n-2$ zeros in $(-1,1)$. As $t_1 = -1$, $t_r = 1$, it is clear that ± 1 are zeros of $R(t)$. Hence, $R(t)$ has at least n zeros in $[-1,1]$. This, and the fact that

$R(t)$ is a polynomial of degree at most $n-1$, implies that $R(t) = 0$. Therefore,

$$\left\|\prod_{i=1}^{r}(t-t_i)^{\alpha_i}\right\|_\infty = \|g\|_\infty = \frac{1}{2^{n-1}(\cos\frac{\pi}{2n})^n},$$

which contradicts (2.30). Hence, we have

$$\left\|\prod_{i=1}^{r}(t-t_i)^{\alpha_i}\right\|_\infty \geq \frac{1}{2^{n-1}(\cos\frac{\pi}{2n})^n}. \tag{2.31}$$

From (1.1), (2.13), (2.27) and (2.31) we obtain

$$e\big(BW_\infty^n, H_\Delta, L_\infty\big) \geq \big\|g-H_\Delta(g)\big\|_\infty = \frac{\|W_\Delta\|_\infty}{n!} \geq \frac{(b-a)^n}{(\cos\frac{\pi}{2n})^n 2^{2n-1} n!}. \tag{2.32}$$

From (1.2) and (2.32) we obtain the lower estimate of (1).

Next, we consider (2). Let $\overline{\omega}$ and $\Delta^*_{n,p,\omega}$ be given by (1.19). Then, for any $f \in BW_\infty^n$, from (2.1) it follows that

$$\big|f(x) - L_{\Delta^*_{n,p,\omega}}(f,x)\big| \leq \frac{(1-x^2)|W_{n-2,p,\overline{\omega}}(x)|}{n!}, \quad x \in [a,b]. \tag{2.33}$$

From (2.33) it follows that

$$\big\|f - L_{\Delta^*_{n,p,\omega}}(f)\big\|_{p,\omega} \leq \frac{\|W_{n-2,p,\overline{\omega}}\|_{p,\overline{\omega}}}{n!} = \frac{E_{n-2,p,\overline{\omega}}}{n!}. \tag{2.34}$$

From (1.1) and (2.34) we conclude that

$$e\big(BW_\infty^n, L_{\Delta^*_{n,p,\omega}}, L_{p,\omega}\big) \leq \frac{E_{n-2,p,\overline{\omega}}}{n!}. \tag{2.35}$$

On the other hand, let $\Delta := \{a = x_1 < x_2 < \cdots < x_r = b, \alpha_i \in \mathbb{N}, n = \sum_{i=1}^r \alpha_i\}$ be an arbitrary Hermite interpolation system of cardinality n including the endpoints. Consider the function $g(x) = \frac{x^n}{n!}$. Then, $g \in W_\infty^n$ and (2.13) holds. From (1.1), (2.13), (1.5) and (1.6) it follows that

$$\begin{aligned} e\big(BW_\infty^n, H_\Delta, L_{p,\omega}\big) &\geq \big\|g-H_\Delta(g)\big\|_{p,\omega} = \frac{1}{n!}\left\|\prod_{k=1}^{r}(x-x_k)^{\alpha_k}\right\|_{p,\omega} \\ &= \frac{1}{n!}\left\|(x-a)^{\alpha_1-1}(b-x)^{\alpha_r-1}\prod_{k=2}^{r-1}(x-x_k)^{\alpha_k}\right\|_{p,\overline{\omega}} \\ &\geq \frac{1}{n!}\left\|\prod_{k=1}^{n-2}(x-\xi_{k,p,\overline{\omega}})\right\|_{p,\overline{\omega}} = \frac{E_{n-2,p,\overline{\omega}}}{n!}. \end{aligned} \tag{2.36}$$

From (2.35) and (2.36) as well as (1.2) we obtain the result of (2). Theorem 1.2 is proved. □

Using the fact that $BC^n \subset BW_\infty^n$ and $g(x) = \frac{x^n}{n!} \in BC^n$ for $n \in \mathbb{N}$, combining the proof of Theorem 1.2, we obtained the following results.

Corollary 2.4

(1) *Let $p = \infty$ and $n > 2$. Then, we have*

$$\overline{e}\big(n, BC^n, L_\infty\big) = e\big(BC^n, L_{\Delta^*_{n,\infty}}, L_\infty\big) = \frac{(b-a)^n}{(\cos\frac{\pi}{2n})^n 2^{2n-1} n!},$$

where $\Delta^*_{n,\infty}$ is given by (1.17).

(2) *Let $1 \le p < \infty$, $n > 2$ and assume that $\omega(x) > 0$ is continuous-integrable on (a,b). Then, we have*

$$\overline{e}\big(n, BC^n, L_{p,\omega}\big) = e\big(BC^n, L_{\Delta^*_{n,p,\omega}}, L_{p,\omega}\big) = \frac{E_{n-2,p,\overline{\omega}}}{n!},$$

where $\overline{\omega}$ and $\Delta^*_{n,p,\omega}$ are given by (1.19).

Remark 2.5 When $n \neq r$, the nth optimal Hermite interpolation system of the problems given by (1.2) and (1.15) for BW^r_∞ in L_∞ and $L_{p,\omega}$ $(1 \le p < \infty)$ are open problems.

Remark 2.6 When $n = r$, the nth optimal Birkhoff interpolation system of the problems given by (1.2) and (1.15) for BW^n_∞ in L_∞ and $L_{p,\omega}$ $(1 \le p < \infty)$ are open problems.

3 Some examples

Example 1 Let $\omega(x) = 1$, $[a,b] = [-1,1]$. Then for $1 \le p < \infty$ we obtain the usual $L_p \equiv L_p[-1,1]$ spaces. For $p = 1$, it follows from [4, pp. 87–88] that

$$E_{n,1,1} = \frac{1}{2^{n-1}}, \qquad W_{n,1,1}(x) = \frac{U_n(x)}{2^n}, \qquad \xi_{k,1,1} = \cos\frac{k\pi}{n+1}, \quad k = 1, \dots, n,$$

where U_n is the nth Chebyshev polynomial of the second kind, i.e.,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta.$$

From Theorem 1.1 it follows that

$$e\big(n, BW^n_\infty, L_1\big) = e\big(BW^n_\infty, L_{\Delta_{n,1,1}}, L_1\big) = \frac{E_{n,1,1}}{n!} = \frac{1}{2^{n-1} n!},$$

where $\Delta_{n,1,1} = \{\cos\frac{n\pi}{n+1}, \cos\frac{(n-1)\pi}{n+1}, \dots, \cos\frac{\pi}{n+1}\}$.

Example 2 Let $p = 2$. In this case, for any continuous-integrable weight function $\omega(x) > 0$ on (a,b), there is a unique orthogonal system $\{p_{k,\omega}\}_{k\in\mathbb{Z}_+}$ in $L_{2,\omega}$ that is complete and satisfies the following conditions:

(1) $p_{k,\omega} \in \mathcal{P}_k$ for all $k \in \mathbb{Z}_+$.

(2)

$$\int_a^b p_{k,\omega}(x) p_{j,\omega}(x)\omega(x)\,dx = \begin{cases} 0, & k \neq j; \\ 1, & k = j. \end{cases} \tag{3.1}$$

(3) The coefficient $C_{k,\omega}$ of the leading term x^k of $p_{k,\omega}$ is positive.

In this case, similar to (3.18) in [16], we have $W_{k,2,\omega} = \frac{p_{k,\omega}}{C_{k,\omega}}$ and

$$E_{k,2,\omega} = \|W_{k,2,\omega}\|_{2,\omega} = 1/C_{k,\omega}. \tag{3.2}$$

Let $\omega^{(\alpha,\beta)}$ be the Jacobi weights, i.e., $\omega^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ with $\alpha,\beta > -1$ on $(-1,1)$ and we denote the corresponding orthogonal system as $\{p_k^{(\alpha,\beta)}\}_{k\in\mathbb{Z}_+}$. It is known that the coefficient of the leading term x^k of $p_k^{(\alpha,\beta)}(x)$ is (see (3.21) in [16])

$$C_{k,\omega^{(\alpha,\beta)}} = \frac{\sqrt{\alpha+\beta+2k+1}\,\Gamma(\alpha+\beta+2k+1)}{2^{k+(\alpha+\beta+1)/2}\sqrt{k!\Gamma(\alpha+\beta+k+1)\Gamma(\alpha+k+1)\Gamma(\beta+k+1)}}. \tag{3.3}$$

From Theorem 1.1, (3.2) and (3.3), it follows that

$$\begin{aligned} e\big(n, BW_\infty^n, L_{2,\omega^{(\alpha,\beta)}}\big) &= e\big(BW_\infty^n, L_{\Delta_{n,2,\omega^{(\alpha,\beta)}}}, L_{2,\omega^{(\alpha,\beta)}}\big) = \frac{1}{n!C_{n,\omega^{(\alpha,\beta)}}} \\ &= \frac{2^{n+(\alpha+\beta+1)/2}\sqrt{\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}}{\sqrt{n!(\alpha+\beta+2n+1)}\,\Gamma(\alpha+\beta+2n+1)}, \end{aligned} \tag{3.4}$$

where $\Delta_{n,2,\omega^{(\alpha,\beta)}}$ consists of the zeros of $p_n^{(\alpha,\beta)}$. From Theorem 1.2, (3.2), (3.3) and (3.4), it follows that for $n > 2$

$$\begin{aligned} \overline{e}\big(n, BW_\infty^n, L_{2,\omega^{(\alpha,\beta)}}\big) &= e\big(BW_\infty^n, L_{\Delta^*_{n,2,\omega^{(\alpha,\beta)}}}, L_{2,\omega^{(\alpha,\beta)}}\big) \\ &= \frac{E_{n-2,2,\omega^{(\alpha+2,\beta+2)}}}{n!} = \frac{1}{n!C_{n-2,\omega^{(\alpha+2,\beta+2)}}} \\ &= \sqrt{\frac{(\alpha+\beta+n+2)(\alpha+\beta+n+1)}{n(n-1)}}\, e\big(n, BW_\infty^n, L_{2,\omega^{(\alpha,\beta)}}\big), \end{aligned} \tag{3.5}$$

where $\Delta^*_{n,2,\omega^{(\alpha,\beta)}} = (-1, \xi_{1,2,\omega^{(\alpha+2,\beta+2)}}, \xi_{2,2,\omega^{(\alpha+2,\beta+2)}}, \dots, \xi_{n-2,2,\omega^{(\alpha+2,\beta+2)}}, 1)$, and $\xi_{1,2,\omega^{(\alpha+2,\beta+2)}}$, $\xi_{2,2,\omega^{(\alpha+2,\beta+2)}}, \dots, \xi_{n-2,2,\omega^{(\alpha+2,\beta+2)}}$ are the zeros of $p_{n-2}^{(\alpha+2,\beta+2)}$.

Next, we list three examples.

For $\alpha = \beta = 0$, i.e., $\omega^{(0,0)}(x) = 1$, we have (see [7, p. 205])

$$W_{n,2,1}(x) = \frac{2^n(n!)^2}{(2n)!}P_n(x) = \prod_{k=1}^n (x - \xi_{k,2,1}), \tag{3.6}$$

where P_n is the nth Legendre polynomial, i.e.,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\big(x^2-1\big)^n.$$

From (3.4) it follows that

$$e\big(n, BW_\infty^n, L_2\big) = e\big(BW_\infty^n, L_{\Delta_{n,2,1}}, L_2\big) = \frac{2^{n+1/2}n!}{(2n)!\sqrt{2n+1}}, \tag{3.7}$$

where $\Delta_{n,2,1}$ consists of the zeros of P_n. Furthermore, from (3.5) and (3.7) it follows that for $n > 2$

$$\overline{e}\big(n, BW_\infty^n, L_2\big) = \frac{2^{n+1/2}(n+2)!}{(2n)!\sqrt{(n-1)n(n+1)(n+2)(2n+1)}}. \tag{3.8}$$

For $\alpha = \beta = -1/2$, i.e., $\omega^{(-1/2,-1/2)}(x) = \frac{1}{\sqrt{1-x^2}}$, we know $W_{n,2,\omega^{(-1/2,-1/2)}}(x) = \frac{T_n(x)}{2^{n-1}}$. From (3.4) it follows that

$$e\big(n, BW_\infty^n, L_{2,\omega^{(-1/2,-1/2)}}\big) = e\big(BW_\infty^n, L_{\Delta_{n,2,\omega^{(-1/2,-1/2)}}}, L_{2,\omega^{(-1/2,-1/2)}}\big) = \frac{\sqrt{2\pi}}{2^n n!}, \tag{3.9}$$

where $\Delta_{n,2,\omega^{(-1/2,-1/2)}} = \{\cos\frac{(2n-1)\pi}{2n}, \cos\frac{(2n-3)\pi}{2n}, \ldots, \cos\frac{\pi}{2n}\}$. Furthermore, from (3.5) and (3.9) it follows that for $n > 2$

$$\overline{e}\big(n, BW_\infty^n, L_{2,\omega^{(-1/2,-1/2)}}\big) = \frac{\sqrt{2\pi(n+1)(n-1)}}{2^n(n-1)n!}.$$

For $\alpha = \beta = 1/2$, i.e., $\omega^{(1/2,1/2)}(x) = \sqrt{1-x^2}$, we know $W_{n,2,\omega^{(1/2,1/2)}}(x) = \frac{U_n(x)}{2^n}$. From (3.4) it follows that

$$e\big(n, BW_\infty^n, L_{2,\omega^{(1/2,1/2)}}\big) = e\big(BW_\infty^n, L_{\Delta_{n,2,\omega^{(1/2,1/2)}}}, L_{2,\omega^{(1/2,1/2)}}\big) = \frac{\sqrt{\pi}}{2^{n+1/2}n!}, \tag{3.10}$$

where $\Delta_{n,2,\omega^{(1/2,1/2)}} = \Delta_{n,1,1}$. From (3.5) and (3.10) it follows that for $n > 2$

$$\overline{e}\big(n, BW_\infty^n, L_{2,\omega^{(1/2,1/2)}}\big) = \frac{\sqrt{\pi(n+3)(n+2)n(n-1)}}{2^{n+1/2}n!n(n-1)}.$$

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Authors' information
Xu Guiqiao, male, professor, the winner of THE 2020 JOSEPH F. TRAUB PRIZE FOR ACHIEVEMENT IN INFORMATION-BASED COMPLEXITY. Yu Xiaochen, female, postgraduate.

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