Bodaghi *Journal of Inequalities and Applications* (2022) 2022:6
https://doi.org/10.1186/s13660-021-02740-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Equalities and inequalities for several variables mappings

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Abstract

In this paper, some special mappings of several variables such as the multicubic and the multimixed quadratic–cubic mappings are introduced. Then, the systems of equations defining a multicubic and a multimixed quadratic–cubic mapping are unified to a single equation. Under some mild conditions, it is shown that a multimixed quadratic–cubic mapping can be multiquadratic, multicubic and multiquadratic–cubic. Furthermore, by applying a known fixed-point theorem, the Hyers–Ulam stability of multimixed quadratic–cubic, multiquadratic, multicubic and multiquadratic–cubic are studied in non-Archimedean normed spaces.

MSC: 39B52; 39B72; 39B82; 46B03

Keywords: Hyers–Ulam stability; Multiquadratic mapping; Multicubic mapping; Multiquadratic–cubic mapping; Non-Archimedean normed space

1 Introduction

The stability problems of functional equations are some of the classical and practical issues in the area of mathematical analysis, physics and engineering. The story of stability for functional equations commenced with the question of Ulam [42] for group homomorphisms. Later, it was answered and developed by Hyers [28], Aoki [2], Rassias [38] and Găvruța [24] for additive and linear functional equations on Banach spaces. Indeed, a certain equation is applicable to model a physical process of a small change of the equation gives rise to a small change in the corresponding result. In this case, we say the equation is stable. In other words, a functional equation $\mathfrak{F}$ is said to be *stable* if any mapping ϕ fulfilling $\mathfrak{F}$ approximately is near to an exact solution of $\mathfrak{F}$. Moreover, $\mathfrak{F}$ is called *hyperstable* if any function ϕ satisfying $\mathfrak{F}$ approximately (under some conditions) is an exact solution of $\mathfrak{F}$. For more details and updated definitions of the stability and hyperstability of functional equations, refer to [15]. Some stability results can be available for instance in [5, 16, 26] and [30] and also references therein.

In the two last decades, the Ulam stability problem has been answered and studied for some special several variables mappings such as multiadditive, multi-Jensen, multiquadratic, multicubic and multiquartic mappings. In what follows, we state some historical notes about known functional equations used in this paper. Here, we have three famous functional equations, studied in books [1, 21, 32] and [39] and references therein.

$$A(x+y) = A(x) + A(y) \quad \text{(The Cauchy equation)}, \tag{1.1}$$

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \quad \text{(The quadratic equation)}, \tag{1.2}$$

$$C(x+2y) + C(x-2y) = 4C(x+y) + 4C(x-y) - 6C(x) \quad \text{(The cubic equation)}. \tag{1.3}$$

Indeed, Rassias was the first author who defined the cubic functional equation in [37] as follows:

$$\mathcal{C}(x+2y) = 3\mathcal{C}(x+y) + \mathcal{C}(x-y) - 3\mathcal{C}(x) + 6\mathcal{C}(y). \tag{1.4}$$

Next, Jun and Kim introduced the cubic equation

$$C(2x+y) + C(2x-y) = 2C(x+y) + 2C(x-y) + 12C(x) \tag{1.5}$$

in [29] and a different form of the cubic functional equation, namely (1.3) was introduced by them in [30]. Equations (1.3) and (1.5) were generalized by Bodaghi in [6] as follows:

$$\mathfrak{C}(rx+sy) + \mathfrak{C}(rx-sy) = rs^2\big[\mathfrak{C}(x+y) + \mathfrak{C}(x-y)\big] + 2r\big(r^2-s^2\big)\mathfrak{C}(x), \tag{1.6}$$

where r, s are fixed integers with $r \pm s \neq 0$; see also [9].

Throughout this paper, we use the notations $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ as the set of positive integers, integers and rationals, respectively, and moreover $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ := [0, \infty)$. For any $l \in \mathbb{N}_0$, $n \in \mathbb{N}$, $t = (t_1, \dots, t_n) \in A^n$ and $x = (x_1, \dots, x_n) \in V^n$ we write $lx := (lx_1, \dots, lx_n)$ and $tx := (t_1x_1, \dots, t_nx_n)$, where $A = \{-2, -1, 1, 2\}$ and lx stands, as usual, for the scaler product of l on x in the commutative group V.

Let V be a commutative group, W be a linear space over $\mathbb{Q}$, and $n \in \mathbb{Z}$ with $n \geq 2$. A mapping $f : V^n \longrightarrow W$ is called

- *multiadditive* if it satisfies (1.1) in each variable [18];
- *multiquadratic* if it satisfies (1.2) in each variable [19];
- *multicubic* if it satisfies either (1.5) or (1.6) in each variable [13, 25].

We have the following observations from [18] and [45]. Consider a mapping $f : V^n \longrightarrow W$. Then,

(i) f is multiadditive if and only if it satisfies

$$f(x_1 + x_2) = \sum_{j_1, \dots, j_n \in \{1,2\}} f(x_{1j_1}, \dots, x_{nj_n});$$

(ii) f is multiquadratic if and only if it fulfills

$$\sum_{s \in \{-1,1\}^n} f(x_1 + sx_2) = 2^n \sum_{j_1, \dots, j_n \in \{1,2\}} f(x_{1j_1}, \dots, x_{nj_n}), \tag{1.7}$$

where $x_j = (x_{1j}, x_{2j}, \dots, x_{nj}) \in V^n$ with $j \in \{1, 2\}$. More information about the structure of multiadditive mappings and their Ulam stabilities can be found in [18, 20], [34, Sects. 13.4 and 17.2] and [44]. Furthermore, some facts on multiquadratic mappings such as Jensen type and generalized forms with stabilities in various Banach spaces are available in [7, 10–12, 19, 22] and [40]).

Ghaemi et al. [25] introduced multicubic mappings for the first time. In fact, they considered a mapping $f : V^n \longrightarrow W$ that satisfies (1.6) in each variable. Next, a special case of such mappings was studied in [13]. In other words, the authors unified the system of functional equations defining a multicubic mapping to a single equation, namely, the multicubic functional equation [13]. Moreover, they showed that every multicubic functional equation is stable and such functional equations can be hyperstable (for the miscellaneous versions of multicubic mappings and their stabilities in non-Archimedean normed and modular spaces, we refer to [23] and [36], respectively). In addition, the general system of cubic functional equations has been defined in [25] and characterized as a single equation in [23]. For the definitions and the structure of multiadditive–quadratic and multiquadratic–cubic mappings, see [3] and [8].

In [17], Chang and Jung introduced the mixed-type quadratic and cubic functional equation

$$6\mathcal{C}_q(x+y) - 6\mathcal{C}_q(x-y) + 4\mathcal{C}_q(3y) = 3\mathcal{C}_q(x+2y) - 3\mathcal{C}_q(x-2y) + 9\mathcal{C}_q(2y). \tag{1.8}$$

They established the general solution of (1.8) and investigated the Hyers–Ulam–Rassias stability of this equation; for a different form of the mixed-type quadratic–cubic functional equations, see [31]. Towanlong and Nakmahachalasint [41] considered the mixed-type quadratic–cubic functional that is different (when its solution is either an even or odd mapping) from (1.8) as follows:

$$\mathfrak{C}_q(x+2y) - 3\mathfrak{C}_q(x+y) + 3\mathfrak{C}_q(x) - \mathfrak{C}_q(x-y) - 3\mathfrak{C}_q(y) + 3\mathfrak{C}_q(-y) = 0. \tag{1.9}$$

It is easily verified that the function $\mathfrak{C}_q(x) = ax^3 + bx^2 + c$ is a solution of equation (1.9); see [35] for more stability results of (1.9).

According to equation (1.9), in this paper, we define the multimixed quadratic–cubic mappings and present a characterization of such mappings. In other words, we describe the system of n equations defining a multimixed quadratic–cubic mapping as a single equation. We also show that under some mild condition, a multimixed quadratic–cubic mapping can be multiquadratic, multicubic and multiquadratic–cubic. Finally, we prove the Hyers–Ulam stability and hyperstability of the multimixed quadratic–cubic mappings in non-Archimedean normed spaces by applying a known fixed-point theorem that was introduced and studied in [14]; for more applications of this method we refer to [4, 43] and [44].

2 The structure of some several variables mappings

Recall that the mixed-type quadratic and cubic functional equation (1.9) was introduced in [41] and the authors proved the following theorem.

Theorem 2.1 *Let X and Y be vector spaces. A function $h : X \longrightarrow Y$ satisfies functional equation* (1.9) *if and only if there exist a quadratic function $A^2 : X \longrightarrow Y$, a cubic function*

$A^3 : X \longrightarrow Y$ and a constant A^0 such that

$$h(x) = A^0 + A^2(x) + A^3(x)$$

for all $x \in X$.

Next, by using an alternative method we obtain the general solution of (1.9), which is a tool to reach some of our goals in this section.

Lemma 2.2 *Let X and Y be real vector spaces. Suppose that $h : X \longrightarrow Y$ satisfies (1.9) for all $x, y \in X$.*
(i) *If h is even and $h(0) = 0$, then it is quadratic;*
(ii) *If h is odd, then it is cubic.*

Proof (i) We first note that the evenness of h converts (1.9) to

$$h(x + 2y) - 3h(x + y) + 3h(x) - h(x - y) = 0, \tag{2.1}$$

for all $x, y \in X$ (here and in the rest of the proof). Letting $x = 0$ in (2.1), we have

$$h(2y) = 4h(y). \tag{2.2}$$

Interchanging x with $x - y$ in (2.1), we find

$$h(x + y) - 3h(x) + 3h(x - y) - h(x - 2y) = 0. \tag{2.3}$$

A difference computation of (2.1) and (2.3) shows that

$$h(x + 2y) + h(x - 2y) - 4h(x + y) - 4h(x - y) + 6h(x) = 0. \tag{2.4}$$

Replacing x by $2x$ in (2.4) and using (2.2), we obtain

$$h(x + y) + h(x - y) - h(2x + y) - h(2x - y) + 6h(x) = 0. \tag{2.5}$$

Switching (x, y) by (y, x) in (2.5) and applying again the evenness of f, we obtain

$$h(x + y) + h(x - y) - h(x + 2y) - h(x - 2y) + 6h(y) = 0. \tag{2.6}$$

Inserting (2.4) into (2.6), we find

$$h(x + y) + h(x - y) = 2h(x) + 2h(y).$$

(ii) Using our assumption on (1.9), we obtain

$$h(x + 2y) - 3h(x + y) + 3h(x) - h(x - y) - 6h(y) = 0. \tag{2.7}$$

Putting $x = 0$ in (2.7) and applying from the oddness of h, we find

$$h(2y) = 8h(y). \tag{2.8}$$

On the other hand, if we replace $x - y$ instead of x in (2.6), then,

$$h(x+y) - 3h(x) + 3h(x-y) - h(x-2y) - 6h(y) = 0. \tag{2.9}$$

Now, it follows from (2.7) and (2.9) that

$$h(x+2y) + h(x-2y) = 4\big[h(x+y) + h(x-y)\big] - 6h(x).$$

This completes the proof. □

Henceforth, let V and W be vector spaces over $\mathbb{Q}$, $n \in \mathbb{N}$ and $x_i^n = (x_{i1}, x_{i2}, \ldots, x_{in}) \in V^n$, where $i \in \{1,2\}$. We denote x_i^n by x_i if there is no risk of a mistake. Let $x_1, x_2 \in V^n$ and $m \in \mathbb{N}_0$ with $0 \le m \le n$. Put $\mathcal{M}^n = \{\mathfrak{N}_n = (N_1, \ldots, N_n) \mid N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\}$, where $j \in \{1, \ldots, n\}$. Consider

$$\mathcal{M}_m^n := \big\{\mathfrak{N}_n \in \mathcal{M}^n \mid \operatorname{Card}\{N_j : N_j = x_{1j}\} = m\big\}.$$

For $r \in \mathbb{Q}$, we put $r\mathcal{M}_m^n = \{r\mathfrak{N}_n : \mathfrak{N}_n \in \mathcal{M}_m^n\}$ in which $r\mathfrak{N}_n = (rN_1, \ldots, rN_n)$.

Definition 2.3 A mapping $f : V^n \longrightarrow W$ is *n-multicubic* or *multicubic* if f satisfies (1.3) in each variable.

For a multicubic mapping f, we use the notations

$$f\big(\mathcal{M}_m^n\big) := \sum_{\mathfrak{N}_n \in \mathcal{M}_m^n} f(\mathfrak{N}_n), \tag{2.10}$$

and

$$f\big(\mathcal{M}_m^n, z\big) := \sum_{\mathfrak{N}_n \in \mathcal{M}_m^n} f(\mathfrak{N}_n, z) \quad (z \in V).$$

We recall that $\binom{n}{m} := \frac{n!}{m!(n-m)!}$ is the binomial coefficient, where $n, m \in \mathbb{N}_0$ with $n \ge m$. In the upcoming result, we find a necessary and sufficient condition for a several-variable mapping to be multicubic.

Proposition 2.4 *For a mapping $f : V^n \longrightarrow W$, the following assertions are equivalent.*

(i) *f is multicubic;*

(ii) *f satisfies*

$$\sum_{t \in \{-2,2\}^n} f(x_1 + tx_2) = \sum_{m=0}^{n} 4^{n-m}(-6)^m f\big(\mathcal{M}_m^n\big), \tag{2.11}$$

where $f(\mathcal{M}_m^n)$ is defined in (2.10).

Proof (i) ⇒ (ii) The proof of this implication is by induction on n. For $n = 1$, it is clear that f fulfills (1.3). Suppose that (2.11) holds for some positive integer $n > 1$. Then,

$$\begin{aligned}
&\sum_{t\in\{-2,2\}^{n+1}} f\big(x_1^{n+1} + tx_2^{n+1}\big) \\
&= 4\sum_{t\in\{-2,2\}^{n}}\sum_{s\in\{-1,1\}} f\big(x_1^{n} + tx_2^{n}, x_{1,n+1} + sx_{2,n+1}\big) - 6\sum_{t\in\{-2,2\}^{n}} f\big(x_1^{n} + tx_2^{n}, x_{1,n+1}\big) \\
&= 4\sum_{m=0}^{n}\sum_{s\in\{-1,1\}} 4^{n-m}(-6)^m f\big(\mathcal{M}_m^n, x_{1,n+1} + sx_{2,n+1}\big) - 6\sum_{m=0}^{n} 4^{n-m}(-6)^m f\big(\mathcal{M}_m^n, x_{1,n+1}\big) \\
&= \sum_{m=0}^{n+1} 4^{n+1-m}(-6)^m f\big(\mathcal{M}_m^{n+1}\big).
\end{aligned}$$

(ii) ⇒ (i) Let $j \in \{1,\ldots,n\}$ be arbitrary and fixed. It is enough to prove that f is cubic in the jth variable. Set

$$f_j^*(z) := f(z_1,\ldots,z_{j-1},z,z_{j+1},\ldots,z_n).$$

Assuming $x_{2m} = 0$ for all $m \in \{1,\ldots,n\}\backslash\{j\}$, $x_{2j} = w$ and $x_1 = (z_1,\ldots,z_{j-1},z,z_{j+1},\ldots,z_n)$ in (2.11), we obtain

$$\begin{aligned}
&2^{n-1}\big[f_j^*(z+2w) + f_j^*(z-2w)\big] \\
&= \sum_{m=0}^{n-1}\left[\binom{n-1}{m} 2^{n-1-m}4^{n-m}(-6)^m\right]\big[f_j^*(z+w) + f_j^*(z-w)\big] \\
&\quad + \sum_{m=1}^{n}\left[\binom{n}{m} 2^{n-m}4^{n-m}(-6)^m\right] f_j^*(z) \\
&= 4\sum_{m=0}^{n-1}\left[\binom{n-1}{m} 8^{n-1-m}(-6)^m\right]\big[f_j^*(z+w) + f_j^*(z-w)\big] \\
&\quad - 6\sum_{m=0}^{n-1}\left[\binom{n}{m} 8^{n-1-m}(-6)^m\right] f_j^*(z) \\
&= 4\times 2^{n-1}\big[f_j^*(z+w) + f_j^*(z-w)\big] - 6\times 2^{n-1} f_j^*(z).
\end{aligned} \tag{2.12}$$

It follows from (2.12) that

$$f_j^*(z+2w) + f_j^*(z-2w) = 4\big[f_j^*(z+w) + f_j^*(z-w)\big] - 6f_j^*(z).$$

Now, the proof is completed. □

Definition 2.5 Let $n \in \mathbb{N}$ and $k \in \{0,\ldots,n\}$. A mapping $f : V^n \longrightarrow W$ is called k-quadratic and $n-k$-cubic (briefly, multiquadratic–cubic) if f is quadratic (see equation (1.2)) in each of some k variables and is cubic in each of the other variables (see equation (1.3)).

In Definition 2.5, we suppose for simplicity that f is quadratic in each of the first k variables, but one can obtain analogous results without this assumption. It is obvious that for

$k = n$ (resp., $k = 0$), the above definition leads to the so-called multiquadratic (resp., multicubic) mappings; some basic facts on the mentioned mappings can be found, for instance, in [8, 13] and [45].

To reach our results in the rest of this section, we identify $x = (x_1,\dots,x_n) \in V^n$ with $(x^k,x^{n-k}) \in V^k \times V^{n-k}$, where $x^k := (x_1,\dots,x_k)$ and $x^{n-k} := (x_{k+1},\dots,x_n)$. Put $x_i^k = (x_{i1},\dots,x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1},\dots,x_{in}) \in V^{n-k}$, where $i \in \{1,2\}$. For a multiquadratic–cubic mapping f, we also recall the notation

$$f\left(x_i^k,\mathcal{M}_m^{n-k}\right) := \sum_{\mathfrak{N}_n \in \mathcal{M}_m^{n-k}} f\left(x_i^k,\mathfrak{N}_{n-k}\right),$$

where

$$\mathcal{M}_m^{n-k} := \left\{\mathfrak{N}_{n-k} \in \mathcal{M}^{n-k} \mid \operatorname{Card}\{N_j : N_j = x_{1j}\} = m\right\},$$

in which

$$\mathcal{M}^{n-k} = \left\{\mathfrak{N}_{n-k} = (N_{k+1},\dots,N_n) \mid N_j \in \{x_{1j} \pm x_{2j}, x_{1j}\}\right\}.$$

In the following result, we describe a multiquadratic–cubic mapping as an equation. The proof is similar to the proof of [8, Proposition 2.1], but we include some parts for the sake of completeness.

Proposition 2.6 *Let $n \in \mathbb{N}$ and $k \in \{0,\dots,n\}$. If a mapping $f : V^n \longrightarrow W$ is k-quadratic and $n-k$-cubic mapping, then f satisfies the equation*

$$\sum_{s\in\{-1,1\}^k}\sum_{t\in\{-2,2\}^{n-k}} f\left(x_1^k + sx_2^k, x_1^{n-k} + tx_2^{n-k}\right) = 2^k \sum_{m=0}^{n-k} 4^{n-k-m}(-6)^m \sum_{i\in\{1,2\}} f\left(x_i^k,\mathcal{M}_m^{n-k}\right) \tag{2.13}$$

for all $x_i^k = (x_{i1},\dots,x_{ik}) \in V^k$ and $x_i^{n-k} = (x_{i,k+1},\dots,x_{in}) \in V^{n-k}$, where $i \in \{1,2\}$.

Proof Since for $k \in \{0,n\}$ our assertion follows from Proposition 2.4 and [45, Theorem 3], we can assume that $k \in \{1,\dots,n-1\}$. Let $x^{n-k} \in V^{n-k}$ be arbitrary and fixed. Consider the mapping $g_{x^{n-k}} : V^k \longrightarrow W$ defined via $g_{x^{n-k}}(x^k) := f(x^k,x^{n-k})$ for $x^k \in V^k$. Similar to the proof of Proposition 2.1 from [8], one can show that

$$\sum_{s\in\{-1,1\}^k} f\left(x_1^k + sx_2^k, x^{n-k}\right) = 2^k \sum_{j_1,\dots,j_k\in\{1,2\}} f\left(x_{j_1 1},\dots,x_{j_k k},x^{n-k}\right) \tag{2.14}$$

for all $x_1^k, x_2^k \in V^k$ and $x^{n-k} \in V^{n-k}$. Similar to the above, we obtain from Proposition 2.4 that

$$\sum_{t\in\{-2,2\}^{n-k}} f\left(x^k, x_1^{n-k} + tx_2^{n-k}\right) = \sum_{m=0}^{n-k} 4^{n-k-m}(-6)^m f\left(x^k,\mathcal{M}_m^{n-k}\right) \tag{2.15}$$

for all $x_1^{n-k}, x_2^{n-k} \in V^{n-k}$ and $x^k \in V^k$. Now, equalities (2.14) and (2.15) show that (2.13) holds for f. □

It is easily seen that the mapping $f:\mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $f(r_1,\dots,r_n)=\prod_{j=1}^{k}\prod_{i=k+1}^{n} r_j^2 r_i^3$ is multiquadratic–cubic and hence (2.13) is valid for f by Proposition 2.6. Therefore, this equation is called a *multiquadratic–cubic functional equation*. Note that in the case $k=n$ and $k=0$, equation (2.13) converts to (1.7) and (2.11), respectively.

Definition 2.7 Let V and W be vector spaces over $\mathbb{Q}$, $n\in\mathbb{N}$. A mapping $f:V^n\longrightarrow W$ is said to be *n-multimixed quadratic–cubic*, or briefly *multimixed quadratic–cubic*, if f satisfies (1.9) in each variable.

Let $x_1,x_2\in V^n$ and $p_l\in\mathbb{N}_0$ with $0\le p_l\le n$, where $l\in\{1,2,3,4\}$. Set

$$\mathbb{M}^n=\bigl\{\mathfrak{M}_n=(M_1,\dots,M_n)\mid M_j\in\{x_{1j}\pm x_{2j},x_{1j},x_{2j},-x_{2j}\}\bigr\},$$

for all $j\in\{1,\dots,n\}$. Consider the subset $\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}$ of $\mathbb{M}^n$ as follows:

$$\begin{aligned}\mathbb{M}^n_{(p_1,p_2,p_3,p_4)} := \bigl\{\mathfrak{M}_n\in\mathbb{M}^n \mid{} & \operatorname{Card}\{M_j:M_j=x_{1j}\}=p_1,\\ & \operatorname{Card}\{M_j:M_j=x_{2j}\}=p_2, \operatorname{Card}\{M_j:M_j=-x_{2j}\}=p_3,\\ & \operatorname{Card}\{M_j:M_j=x_{1j}+x_{2j}\}=p_4\bigr\}.\end{aligned}$$

Hereafter, for a multimixed quadratic–cubic mapping f, we use the notations

$$f\bigl(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}\bigr):=\sum_{\mathfrak{M}_n\in\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}} f(\mathfrak{M}_n), \tag{2.16}$$

and

$$f\bigl(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)},z\bigr):=\sum_{\mathfrak{M}_n\in\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}} f(\mathfrak{M}_n,z)\quad (z\in V).$$

For each $x_1,x_2\in V^n$, we consider the equation

$$f(x_1+2x_2)=\sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}\sum_{p_4=0}^{n-p_1-p_2-p_3}(-3)^{p_1+p_3}3^{p_2+p_4} f\bigl(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}\bigr), \tag{2.17}$$

where $f(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)})$ is defined in (2.16).

Definition 2.8 Given a mapping $f:V^n\longrightarrow W$. We say f

(i) has *zero condition* if $f(x)=0$ for any $x\in V^n$ with at least one component that is equal to zero;

(ii) is even in the jth variable if

$$f(z_1,\dots,z_{j-1},-z_j,z_{j+1},\dots,z_n)=f(z_1,\dots,z_{j-1},z_j,z_{j+1},\dots,z_n);$$

(iii) is odd in the jth variable if

$$f(z_1,\dots,z_{j-1},-z_j,z_{j+1},\dots,z_n)=-f(z_1,\dots,z_{j-1},z_j,z_{j+1},\dots,z_n).$$

In what follows, it is assumed that every mapping $f : V^n \longrightarrow W$ satisfying (2.17) has zero condition. With this assumption, we unify the general system of mixed-type quadratic and cubic functional equations defining a multimixed quadratic–cubic mapping to an equation and indeed this functional equation describe a multimixed quadratic–cubic mapping as well.

Proposition 2.9 *A mapping $f : V^n \longrightarrow W$ is multimixed quadratic–cubic if and only if it satisfies equation* (2.17).

Proof Suppose that f is a multimixed quadratic–cubic mapping. We proceed with the proof by induction on n. For $n = 1$, it is obvious that f satisfies equation (1.9). Let (2.17) be true for some fixed and positive integer $n > 1$. Then,

$$\begin{aligned} &f\left(x_1^{n+1} + 2x_2^{n+1}, z\right) \\ &\quad= \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \sum_{p_3=0}^{n-p_1-p_2} \sum_{p_4=0}^{n-p_1-p_2-p_3} (-3)^{p_1+p_3} 3^{p_2+p_4} f\left(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}, z\right), \end{aligned} \tag{2.18}$$

for all $x_1, x_2 \in V^n$ and $z \in V$. Using (2.18) and the fact that (2.17) holds for the case $n = 1$, we obtain

$$\begin{aligned} &f\left(x_1^{n+1} + 2x_2^{n+1}\right) \\ &\quad= f\left(x_1^n + 2x_2^n, x_{1,n+1} + 2x_{2,n+1}\right) \\ &\quad= 3f\left(x_1^n + 2x_2^n, x_{1,n+1} + x_{2,n+1}\right) + f\left(x_1^n + 2x_2^n, x_{1,n+1} - x_{2,n+1}\right) \\ &\qquad - 3f\left(x_1^n + 2x_2^n, x_{1,n+1}\right) + 3f\left(x_1^n + 2x_2^n, x_{2,n+1}\right) - 3f\left(x_1^n + 2x_2^n, -x_{2,n+1}\right) \\ &\quad= 3\sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \sum_{p_3=0}^{n-p_1-p_2} \sum_{p_4=0}^{n-p_1-p_2-p_3} (-3)^{p_1+p_3} 3^{p_2+p_4} f\left(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}, x_{1,n+1} + x_{2,n+1}\right) \\ &\qquad + \sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \sum_{p_3=0}^{n-p_1-p_2} \sum_{p_4=0}^{n-p_1-p_2-p_3} (-3)^{p_1+p_3} 3^{p_2+p_4} f\left(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}, x_{1,n+1} - x_{2,n+1}\right) \\ &\qquad - 3\sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \sum_{p_3=0}^{n-p_1-p_2} \sum_{p_4=0}^{n-p_1-p_2-p_3} (-3)^{p_1+p_3} 3^{p_2+p_4} f\left(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}, x_{1,n+1}\right) \\ &\qquad + 3\sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \sum_{p_3=0}^{n-p_1-p_2} \sum_{p_4=0}^{n-p_1-p_2-p_3} (-3)^{p_1+p_3} 3^{p_2+p_4} f\left(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}, x_{2,n+1}\right) \\ &\qquad - 3\sum_{p_1=0}^{n} \sum_{p_2=0}^{n-p_1} \sum_{p_3=0}^{n-p_1-p_2} \sum_{p_4=0}^{n-p_1-p_2-p_3} (-3)^{p_1+p_3} 3^{p_2+p_4} f\left(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}, -x_{2,n+1}\right) \\ &\quad= \sum_{p_1=0}^{n+1} \sum_{p_2=0}^{n+1-p_1} \sum_{p_3=0}^{n+1-p_1-p_2} \sum_{p_4=0}^{n+1-p_1-p_2-p_3} (-3)^{p_1+p_3} 3^{p_2+p_4} f\left(\mathbb{M}^{n+1}_{(p_1,p_2,p_3,p_4)}\right). \end{aligned}$$

This means that (2.17) holds for $n + 1$.

Conversely, let $j \in \{1, \ldots, n\}$ be arbitrary and fixed. Set

$$f_j^*(z) := f(z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_n).$$

Putting $x_{2k} = 0$ for all $k \in \{1, \dots, n\} \backslash \{j\}$, $x_{2j} = w$ and $x_1 = (z_1, \dots, z_{j-1}, z, z_{j+1}, \dots, z_n)$ in (2.17), we obtain

$$\begin{aligned}
f_j^*(z+2w) &= \left[\sum_{p_1=0}^{n-1}\sum_{p_4=1}^{n-p_1}\binom{n-1}{p_1}\binom{n-1-p_1}{p_4-1}(-3)^{p_1}3^{p_4}\right]f_j^*(z+w)\\
&\quad+\left[\sum_{p_1=0}^{n-1}\sum_{p_4=0}^{n-1-p_1}\binom{n-1}{p_1}\binom{n-1-p_1}{p_4}(-3)^{p_1}3^{p_4}\right]f_j^*(z-w)\\
&\quad+\left[\sum_{p_1=1}^{n}\sum_{p_4=0}^{n-p_1}\binom{n-1}{p_1-1}\binom{n-p_1}{p_4}(-3)^{p_1}3^{p_4}\right]f_j^*(z)\\
&\quad+3\left[\sum_{p_1=0}^{n-1}\sum_{p_4=0}^{n-1-p_1}\binom{n-1}{p_1}\binom{n-1-p_1}{p_4}(-3)^{p_1}3^{p_4}\right]f_j^*(w)\\
&\quad-3\left[\sum_{p_1=0}^{n-1}\sum_{p_4=0}^{n-1-p_1}\binom{n-1}{p_1}\binom{n-1-p_1}{p_4}(-3)^{p_1}3^{p_4}\right]f_j^*(-w). \qquad (2.19)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sum_{p_1=0}^{n-1}\binom{n-1}{p_1}(-3)^{p_1}\sum_{p_4=0}^{n-1-p_1}\binom{n-1-p_1}{p_4}3^{p_4+1}\times 1^{n-1-p_4}\\
&\quad= 3\sum_{p_1=0}^{n-1}\binom{n-1}{p_1}(-3)^{p_1}4^{n-1-p_1} = 3(4-3)^{n-1} = 3. \qquad (2.20)
\end{aligned}$$

Similarly,

$$\sum_{p_1=0}^{n-1}\binom{n-1}{p_1}(-3)^{p_1}\sum_{p_4=0}^{n-1-p_1}\binom{n-1-p_1}{p_4}3^{p_4}\times 1^{n-1-p_4} = 1. \qquad (2.21)$$

Moreover,

$$\sum_{p_1=0}^{n-1}\binom{n-1}{p_1}(-3)^{p_1+1}\sum_{p_4=0}^{n-1-p_1}\binom{n-1-p_1}{p_4}3^{p_4}\times 1^{n-1-p_4} = -3. \qquad (2.22)$$

It follows from (2.19), (2.20), (2.21) and (2.22) that

$$f_j^*(z+2w) = 3f_j^*(z+w) + f_j^*(z-w) - 3f_j^*(z) + 3f_j^*(w) - 3f_j^*(-w).$$

This completes the proof. □

Corollary 2.10 *Suppose that a mapping $f : V^n \longrightarrow W$ satisfies equation* (2.17).

(i) *If f is even in each variable, then it is multiquadratic. Moreover, f satisfies equation* (1.7);

(ii) *If f is odd in each variable, then it is multicubic. In addition, equation* (2.11) *is true for f*;

(iii) *If f is even in each of some k variables and is odd in each of the other variables, then it is multiquadratic–cubic. In particular, f fulfilling equation* (2.13).

Proof (i) It is shown in Proposition 2.9 that for each j, f_j^* satisfies (1.9). Since it is assumed that $f_j^*(0) = 0$, the result follows from part (i) of Lemma 2.2.

(ii) This is a direct consequence of part (ii) of Lemma 2.2.

(iii) The result follows from the previous parts. □

3 Stability results

In this section, we prove the Hyers–Ulam stability of equation (2.17) in non-Archimedean normed spaces. The method of proof is taken from a fixed point result that was proved in [14, Theorem 1]. Before that, we state some basic facts concerning non-Archimedean spaces and some preliminary results. A metric d on a nonempty set X is called *non-Archimedean* if $d(x,z) \leq \max\{d(x,y), d(y,z)\}$ for $x, y, z \in X$. By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0,\infty)$ such that

(i) $|a| = 0$ if and only if $a = 0$;

(ii) $|ab| = |a||b|$ for all $a, b \in \mathbb{K}$;

(iii) $|a + b| \leq \max\{|a|, |b|\}$ for all $a, b \in \mathbb{K}$.

It is obvious that $|1| = |-1| = 1$ and $|n| \leq 1$ for all integers n. A trivial valuation on any field $\mathbb{K}$ is defined by the following for $a \in \mathbb{K}$

$$|a| := \begin{cases} 0, & a = 0, \\ 1, & a \neq 0. \end{cases}$$

For a nontrivial non-Archimedean valuation on $\mathbb{Q}$, assume that p is a prime number. It is known that any non-zero rational number r can be uniquely written as $r = \frac{m}{n}p^s$, where $m, n, s \in \mathbb{Z}$ in which m and n are not divisible by p. It is easily verified that the function $|\cdot|_p : \mathbb{Q} \longrightarrow [0,\infty)$ given through

$$|r|_p := \begin{cases} 0, & a = 0, \\ p^{-s}, & a \neq 0, \end{cases}$$

is a nontrivial non-Archimedean valuation on $\mathbb{Q}$.

Let V be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : V \longrightarrow \mathbb{R}$ is a non-Archimedean norm if it satisfies the next conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|ax\| = |a|\|x\|$, $(x \in V, a \in \mathbb{K})$;

(iii) the strong triangle inequality; namely,

$$\|x + y\| \leq \max\big\{\|x\|, \|y\|\big\} \quad (x, y \in V).$$

Then, $(V, \|\cdot\|)$ is said to be a *non-Archimedean normed space.*

A sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean normed space $\mathcal{X}$. Indeed, the above definition is taken from the fact that

$$\|x_n - x_m\| \leq \max\big\{\|x_{j+1} - x_j\|; m \leq j \leq n-1\big\} \quad (n \geq m).$$

A non-Archimedean normed space is complete if every Cauchy sequence is convergent. If $(V, \|\cdot\|)$ is a non-Archimedean normed space, then it is easy to check that the function $d_V : V \times V \longrightarrow \mathbb{R}_+$, defined via $d_V(x,y) := \|x-y\|$, is a non-Archimedean metric on V that is invariant, that is $d_V(x+z, y+z) = d_V(x,y)$ for $x,y,z \in X$. In other words, every non-Archimedean normed space is a special case of a metric space with invariant metrics; see [27] and [33] for more information and details of p-adic numbers as an example of non-Archimedean normed spaces.

We recall that for a field $\mathbb{K}$ with multiplicative identity 1, the characteristic of $\mathbb{K}$ is the smallest positive number n such that $\overbrace{1+\cdots+1}^{n\text{-times}} = 0$. For two sets A and B, the set of all mappings from A to B is denoted by B^A. Here, we state a theorem that is an important result in fixed-point theory [14, Theorem 1] and use this result in obtaining our purpose in this section.

Theorem 3.1 *Suppose that the following hypotheses hold.*

(H1) *E is a nonempty set, Y is a complete non-Archimedean normed space over a non-Archimedean field of the characteristic different from* 2, $j \in \mathbb{N}$, $g_1, \dots, g_j : E \longrightarrow E$ *and* $L_1, \dots, L_j : E \longrightarrow \mathbb{R}_+$,

(H2) $\mathcal{T} : Y^E \longrightarrow Y^E$ *is an operator satisfying the inequality*

$$\big\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\big\| \leq \max_{i\in\{1,\dots,j\}} L_i(x)\big\|\lambda\big(g_i(x)\big) - \mu\big(g_i(x)\big)\big\|,$$

for all $\lambda, \mu \in Y^E$, $x \in E$,

(H3) $\Lambda : \mathbb{R}_+^E \longrightarrow \mathbb{R}_+^E$ *is an operator defined through*

$$\Lambda\delta(x) := \max_{i\in\{1,\dots,j\}} L_i(x)\delta\big(g_i(x)\big)\delta \in \mathbb{R}_+^E, \quad x \in E.$$

If a function $\theta : E \longrightarrow \mathbb{R}_+$ *and a mapping* $\varphi : E \longrightarrow Y$ *fulfills the following two conditions*:

$$\big\|\mathcal{T}\varphi(x) - \varphi(x)\big\| \leq \theta(x), \qquad \lim_{l\to\infty} \Lambda^l\theta(x) = 0 \quad (x \in E),$$

then for every $x \in E$, *the limit* $\lim_{l\to\infty} \mathcal{T}^l\varphi(x) =: \psi(x)$ *exists and the mapping* $\psi \in Y^E$, *defined in this way, is a fixed point of* $\mathcal{T}$ *with*

$$\big\|\varphi(x) - \psi(x)\big\| \leq \sup_{l\in\mathbb{N}_0} \Lambda^l\theta(x) \quad (x \in E).$$

Here and subsequently, for the mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathbf{D_{qc}}f : V^n \times V^n \longrightarrow W$ by

$$\mathbf{D_{qc}}f(x_1,x_2) := f(x_1+2x_2) - \sum_{p_1=0}^{n}\sum_{p_2=0}^{n-p_1}\sum_{p_3=0}^{n-p_1-p_2}\sum_{p_4=0}^{n-p_1-p_2-p_3} (-3)^{p_1+p_3}3^{p_2+p_4} f\big(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)}\big),$$

where $f(\mathbb{M}^n_{(p_1,p_2,p_3,p_4)})$ is defined in (2.16).

We remember henceforth, all mappings $f : V^n \longrightarrow W$ are assumed that satisfy the zero condition. In addition, all non-Archimedean fields have the characteristic different from 2. With these assumptions, we have the next stability result for equation (2.17).

Theorem 3.2 *Let $\beta \in \{-1,1\}$ be fixed, V be a linear space and W be a complete non-Archimedean normed space. Suppose that $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying the equality*

$$\lim_{l\to\infty}\left(\frac{1}{|2|^{(3n-k)\beta}}\right)^l \varphi\left(2^{l\beta}x_1, 2^{l\beta}x_2\right) = 0, \tag{3.1}$$

for all $x_1, x_2 \in V^n$. Assume also $f : V^n \longrightarrow W$ is a mapping even in each of some k variables and is odd in each of the other variables, and moreover satisfies the inequality

$$\left\|\mathbf{D}_{\mathbf{qc}}f(x_1,x_2)\right\| \leq \varphi(x_1,x_2), \tag{3.2}$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique multiquadratic–cubic mapping $\mathcal{F}_{qc} : V^n \longrightarrow W$ such that

$$\left\|f(x) - \mathcal{F}_{qc}(x)\right\| \leq \sup_{l\in\mathbb{N}_0} \frac{1}{|2|^{(3n-k)\frac{\beta+1}{2}}}\left(\frac{1}{|2|^{(3n-k)\beta}}\right)^l \varphi\left(0, 2^{l\beta+\frac{\beta-1}{2}}x\right), \tag{3.3}$$

for all $x \in V^n$.

Proof Without loss of generality, we assume for simplicity that f is even in each of the first k variables. Putting $x_1 = 0$ in (3.2), we have

$$\left\|f(2x) - \mathbb{S}\mathbb{T}f(x)\right\| \leq \varphi(0,x) \tag{3.4}$$

for all $x_2 = x \in V^n$ (here and in the rest of the proof), where

$$\mathbb{S} = \sum_{p_2=0}^{k}\sum_{p_3=0}^{k-p_2}\sum_{p_4=0}^{k-p_2-p_3}\binom{k}{p_2}\binom{k-p_2}{p_3}\binom{k-p_2-p_3}{p_4}(-3)^{p_3}3^{p_2+p_4},$$

and

$$\begin{aligned}\mathbb{T} = &\sum_{p_2=0}^{n-k}\sum_{p_3=0}^{n-k-p_2}\sum_{p_4=0}^{n-k-p_2-p_3}\binom{n-k}{p_2}\binom{n-k-p_2}{p_3}\binom{n-k-p_2-p_3}{p_4}\\ &\times 3^{p_2+p_3+p_4}(-1)^{n-k-p_2-p_3-p_4}.\end{aligned}$$

We compute $\mathbb{S}$ as follows:

$$\begin{aligned}\mathbb{S} &= \sum_{p_2=0}^{k}\binom{k}{p_2}3^{p_2}\sum_{p_3=0}^{k-p_2}\binom{k-p_2}{p_3}(-3)^{p_3}\sum_{p_4=0}^{k-p_2-p_3}\binom{k-p_2-p_3}{p_4}1^{k-p_2-p_3-p_4}3^{p_4}\\ &= \sum_{p_2=0}^{k}\binom{k}{p_2}3^{p_2}\sum_{p_3=0}^{k-p_2}\binom{k-p_2}{p_3}(-3)^{p_3}4^{k-p_2-p_3}\\ &= \sum_{p_2=0}^{k}\binom{k}{p_2}3^{p_2}1^{k-p_2} = (3+1)^k = 2^{2k}.\end{aligned} \tag{3.5}$$

Similarly, one can show that

$$\mathbb{T} = 2^{3(n-k)}. \tag{3.6}$$

Relations (3.4), (3.5) and (3.6) imply that

$$\left\|f(2x) - 2^{3n-k}f(x)\right\| \le \varphi(0,x). \tag{3.7}$$

Set

$$\theta(x) := \frac{1}{|2|^{(3n-k)\frac{\beta+1}{2}}}\varphi\left(0,2^{\frac{\beta-1}{2}}x\right), \qquad \mathcal{T}\xi(x) := \frac{1}{2^{(3n-k)\beta}}\xi\left(2^{\beta}x\right),$$

for all $\xi \in W^{V^n}$. Here, we rewrite (3.7) as follows:

$$\left\|f(x) - \mathcal{T}f(x)\right\| \le \theta(x). \tag{3.8}$$

For each $\eta \in \mathbb{R}_+^{V^n}$, we define $\Lambda\eta(x) := \frac{1}{|2|^{(3n-k)\beta}}\eta(r^{\beta}x)$. Considering $E = V^n$, $g_1(x) := 2^{\beta}x$ and $L_1(x) = \frac{1}{|2|^{(3n-k)\beta}}$, we see that Λ fulfils (H3). Furthermore, for any $\lambda, \mu \in W^{V^n}$, we obtain

$$\left\|\mathcal{T}\lambda(x) - \mathcal{T}\mu(x)\right\| = \left\|\frac{1}{2^{(3n-k)\beta}}\lambda\left(2^{\beta}x\right) - \frac{1}{2^{(3n-k)\beta}}\mu\left(2^{\beta}x\right)\right\| \le L_1(x)\left\|\lambda\left(g_1(x)\right) - \mu\left(g_1(x)\right)\right\|.$$

It follows from the above relation that hypothesis (H2) holds. One can argue by induction on $l \in \mathbb{N}$ that

$$\Lambda^l\theta(x) := \left(\frac{1}{|2|^{(3n-k)\beta}}\right)^l\theta\left(2^{l\beta}x\right) = \frac{1}{|2|^{(3n-k)\frac{\beta+1}{2}}}\left(\frac{1}{|2|^{(3n-k)\beta}}\right)^l\varphi\left(0,2^{l\beta+\frac{\beta-1}{2}}\right). \tag{3.9}$$

Now, all the assumptions of Theorem 3.1 hold by (3.8) and (3.9) and therefore there exists a unique mapping $\mathcal{F}_{qc} : V^n \longrightarrow W$ such that $\mathcal{F}_{qc}(x) = \lim_{l\to\infty}(\mathcal{T}^lf)(x)$ and (3.3) is valid as well. In addition, an induction argument on l leads us to

$$\left\|\mathbf{D_{qc}}\left(\mathcal{T}^lf\right)(x_1,x_2)\right\| \le \left(\frac{1}{|2|^{(3n-k)\beta}}\right)^l\varphi\left(2^{l\beta}x_1,2^{l\beta}x_2\right), \tag{3.10}$$

for all $x_1,x_2 \in V^n$. Letting $l \to \infty$ in (3.10) and applying (3.1), we obtain $\mathbf{D_{qc}}\mathcal{F}_{qc}(x_1,x_2) = 0$ for all $x_1,x_2 \in V^n$. This means that the mapping $\mathcal{F}_{qc}$ satisfies equation (2.17) and the proof is now completed by part (iii) of Corollary 2.10. □

In what follows, it is assumed that the non-Archimedean field has the characteristic different from 2 and $|2| < 1$. The following corollaries are some direct applications of Theorem 3.2 concerning the stability of (2.17).

Corollary 3.3 *Given $\delta > 0$. Let V be a normed space and W be a complete non-Archimedean normed space. Suppose that $f : V^n \longrightarrow W$ is a mapping even in each of some k variables and is odd in each of the other variables and moreover satisfies the inequality*

$$\left\|\mathbf{D_{qc}}f(x_1,x_2)\right\| \le \delta,$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique multiquadratic–cubic mapping $\mathcal{F}_{qc} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{F}_{qc}(x)\| \leq \delta,$$

for all $x \in V^n$.

Proof Note that $|2| < 1$. Given $\varphi(x_1, x_2) = \delta$ in the case $\beta = -1$ of Theorem 3.2, we obtain $\lim_{l\to\infty} |2|^{(3n-k)l}\delta = 0$. Therefore, one can obtain the desired result. □

Corollary 3.4 *Let $\alpha \in \mathbb{R}$ fulfill $\alpha \neq 3n - k$. Let V be a non-Archimedean normed space and W be a complete non-Archimedean normed space. Suppose that $f : V^n \longrightarrow W$ is a mapping even in each of some k variables and is odd in each of the other variables and also satisfies the inequality*

$$\|\mathbf{D_{qc}} f(x_1, x_2)\| \leq \sum_{i=1}^{2}\sum_{j=1}^{n} \|x_{ij}\|^{\alpha},$$

for all $x_1, x_2 \in V^n$. Then, there exists a unique multiquadratic–cubic mapping $\mathcal{F}_{qc} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{F}_{qc}(x)\| \leq \begin{cases} \frac{1}{|2|^{3n-k}} \sum_{j=1}^{n} \|x_{1j}\|^{\alpha}, & \alpha > 3n - k, \\ \frac{1}{|2|^{\alpha}} \sum_{j=1}^{n} \|x_{1j}\|^{\alpha}, & \alpha < 3n - k \end{cases}$$

for all $x = x_1 \in V^n$.

Proof Set $\varphi(x_1, x_2) := \sum_{i=1}^{2}\sum_{j=1}^{n} \|x_{ij}\|^{\alpha}$. It now follows from Theorem 3.2 the first and second inequalities in the cases $\beta = 1$ and $\beta = -1$, respectively. □

Here, note that in Corollary 3.4 if we change non-Archimedean normed space V with a normed space, then in the case $\alpha < 3n - k$ there exists a unique multiquadratic–cubic mapping $\mathcal{F}_{qc} : V^n \longrightarrow W$ such that

$$\|f(x) - \mathcal{F}_{qc}(x)\| \leq \frac{1}{|2|^{\alpha}} \sum_{j=1}^{n} \|x_{1j}\|^{\alpha}.$$

Under some conditions, equation (2.17) can be hyperstable as follows.

Corollary 3.5 *Suppose that $\alpha_{ij} > 0$ for $i \in \{1, 2\}$ and $j \in \{1, \dots, n\}$ fulfill $\sum_{i=1}^{2}\sum_{j=1}^{n} \alpha_{ij} \neq 3n - k$. Let V be a normed space and W be a complete non-Archimedean normed space. If $f : V^n \longrightarrow W$ is a mapping even in each of some k variables and is odd in each of the other variables and moreover satisfies the inequality*

$$\|\mathbf{D_{qc}} f(x_1, x_2)\| \leq \prod_{i=1}^{2}\prod_{j=1}^{n} \|x_{ij}\|^{\alpha_{ij}},$$

for all $x_1, x_2 \in V^n$, then f is multiquadratic–cubic.

Next, we have the stability result for functional equation (2.17) when f is either even or odd in each variable.

Theorem 3.6 *Let $\beta \in \{-1,1\}$ be fixed. Let V be a linear space and W be a complete non-Archimedean normed space. Suppose that $\varphi : V^n \times V^n \longrightarrow \mathbb{R}_+$ is a mapping satisfying*

$$\left\| \mathbf{D_{qc}} f(x_1,x_2) \right\| \leq \varphi(x_1,x_2),$$

for all $x_1,x_2 \in V^n$.

(i) *If $f : V^n \longrightarrow W$ is a mapping even in each variable and fulfilling*

$$\lim_{l\to\infty} \left(\frac{1}{|2|^{2n\beta}}\right)^l \varphi\left(2^{l\beta}x_1, 2^{l\beta}x_2\right) = 0,$$

then there exists a unique multiquadratic mapping $\mathcal{Q} : V^n \longrightarrow W$ such that

$$\left\| f(x) - \mathcal{Q}(x) \right\| \leq \sup_{l\in\mathbb{N}_0} \frac{1}{|2|^{2n\frac{\beta+1}{2}}} \left(\frac{1}{|2|^{2n\beta}}\right)^l \varphi\left(0, 2^{l\beta+\frac{\beta-1}{2}}x\right),$$

for all $x \in V^n$.

(ii) *If $f : V^n \longrightarrow W$ is a mapping odd in each variable and*

$$\lim_{l\to\infty} \left(\frac{1}{|2|^{3n\beta}}\right)^l \varphi\left(2^{l\beta}x_1, 2^{l\beta}x_2\right) = 0,$$

then there exists a unique multicubic mapping $\mathcal{C} : V^n \longrightarrow W$ such that

$$\left\| f(x) - \mathcal{C}(x) \right\| \leq \sup_{l\in\mathbb{N}_0} \frac{1}{|2|^{3n\frac{\beta+1}{2}}} \left(\frac{1}{|2|^{3n\beta}}\right)^l \varphi\left(0, 2^{l\beta+\frac{\beta-1}{2}}x\right),$$

for all $x \in V^n$.

Proof The result follows from Theorem 3.2 by putting $k = 0, n$. □

The upcoming corollaries are some direct consequences of Theorem 3.6 concerning the stability of multiquadratic and multicubic mappings. We include them without the proofs.

Corollary 3.7 *Given $\delta > 0$. Let V be a normed space and W be a complete non-Archimedean normed space. Suppose that $f : V^n \longrightarrow W$ is a mapping even in each variable satisfying the inequality*

$$\left\| \mathbf{D_{qc}} f(x_1,x_2) \right\| \leq \delta,$$

for all $x_1,x_2 \in V^n$.

(i) *If f is even in each variable, then there exists a unique multiquadratic mapping $\mathcal{Q} : V^n \longrightarrow W$ such that*

$$\left\| f(x) - \mathcal{Q}(x) \right\| \leq \delta,$$

for all $x \in V^n$.

(ii) *If f is odd in each variable, then there exists a unique multicubic mapping $\mathcal{C}: V^n \longrightarrow W$ such that*

$$\|f(x) - \mathcal{C}(x)\| \leq \delta,$$

for all $x \in V^n$.

Corollary 3.8 *Given $\alpha \in \mathbb{R}$ fulfills $\alpha \neq 2n, 3n$. Let V be a non-Archimedean normed space and W be a complete non-Archimedean normed space. Suppose that $f: V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathbf{D_{qc}}f(x_1, x_2)\| \leq \sum_{i=1}^{2}\sum_{j=1}^{n}\|x_{ij}\|^{\alpha},$$

for all $x_1, x_2 \in V^n$.

(i) *If f is even in each variable, then there exists a unique multiquadratic mapping $\mathcal{Q}: V^n \longrightarrow W$ such that*

$$\|f(x) - \mathcal{Q}(x)\| \leq \begin{cases} \frac{1}{|2|^{2n}}\sum_{j=1}^{n}\|x_{1j}\|^{\alpha}, & \alpha > 2n, \\ \frac{1}{|2|^{\alpha}}\sum_{j=1}^{n}\|x_{1j}\|^{\alpha}, & \alpha < 2n, \end{cases}$$

for all $x = x_1 \in V^n$.

(ii) *If f is odd in each variable, then there exists a unique multicubic mapping $\mathcal{C}: V^n \longrightarrow W$ such that*

$$\|f(x) - \mathcal{C}(x)\| \leq \begin{cases} \frac{1}{|2|^{3n}}\sum_{j=1}^{n}\|x_{1j}\|^{\alpha}, & \alpha > 3n, \\ \frac{1}{|2|^{\alpha}}\sum_{j=1}^{n}\|x_{1j}\|^{\alpha}, & \alpha < 3n, \end{cases}$$

for all $x = x_1 \in V^n$.

Corollary 3.9 *Given $\alpha_{ij} > 0$ for $i \in \{1, 2\}$ and $j \in \{1, \dots, n\}$ fulfill $\sum_{i=1}^{2}\sum_{j=1}^{n}\alpha_{ij} \neq 2n, 3n$. Let V be a normed space and W be a complete non-Archimedean normed space. Suppose that $f: V^n \longrightarrow W$ is a mapping satisfying the inequality*

$$\|\mathbf{D_{qc}}f(x_1, x_2)\| \leq \prod_{i=1}^{2}\prod_{j=1}^{n}\|x_{ij}\|^{\alpha_{ij}},$$

for all $x_1, x_2 \in V^n$.

(i) *If f is even in each variable, then it is multiquadratic*;

(ii) *If f is odd in each variable, then it is multicubic.*

Acknowledgements
The author sincerely thanks the anonymous reviewers for their careful reading, constructive comments and suggesting some related references to improve the quality of the first draft of the paper.

Funding
Not applicable.

Availability of data and materials
Not applicable. In fact, all results are obtained without any software and found by manual computations. In other words, the manuscript is in the pure mathematics (Mathematical Analysis) field.

Declarations

Competing interests
The author declares that they have no competing interests.

Authors' contributions
The author conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Received: 13 September 2021 Accepted: 22 December 2021 Published online: 04 January 2022

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