Asim et al. *Journal of Inequalities and Applications* (2022) 2022:2
https://doi.org/10.1186/s13660-021-02739-z

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Weighted variable Morrey–Herz estimates for fractional Hardy operators

Muhammad Asim[1*], Amjad Hussain[1] and Naqash Sarfraz[2]

*Correspondence:
muhammad.asim2899@gmail.com
[1]Department of Mathematics, Quaid-I-Azam University 45320, Islamabad 44000, Pakistan
Full list of author information is available at the end of the article

Abstract

The present article discusses the boundedness criteria for the fractional Hardy operators on weighted variable exponent Morrey–Herz spaces $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)$.

MSC: 42B35; 26D10; 47B38; 47G10

Keywords: Hardy-type operators; Fractional integral; Variable exponent function spaces; Weights

1 Introduction

In mathematical analysis, the Hardy operator is considered a significant averaging operator and has been exercised a lot during the recent past. In [1], Hardy defined the classical Hardy operator as follows:

$$Pg(z) = \frac{1}{z}\int_0^z g(t)\,dt, \quad z > 0, \tag{1}$$

and established a sharp (q,q) inequality for it. Faris [2] introduced the n-dimensional version of (1), however, the exact value of norm of the n-dimensional Hardy operator on the Lebesgue space was obtained in [3]. Subsequently, in [4], the authors defined the fractional Hardy operator and its adjoint operator as follows:

$$Hg(z) = \frac{1}{|z|^{n-\beta}}\int_{|t|\le|z|} g(t)\,dt, \qquad H^*g(z) = \int_{|t|>|z|}\frac{g(t)}{|t|^{n-\beta}}\,dt, \quad z \in \mathbb{R}^n \setminus \{0\}, \tag{2}$$

where $|z| = \sqrt{\sum_{i=1}^n z_i^2}$. Here, we cite some important readings with regards to the study of Hardy-type operators on different function spaces which include [5–10].

The concept of generalizing function spaces started with the work presented in [11]. However, variable exponent Lebesgue spaces $L^{p(\cdot)}$ were firstly introduced by Kováčik and Rákosník in [12]. After that, the development of variable Lebesgue spaces was started along with the investigation of boundedness of several operators including the maximal operator on $L^{p(\cdot)}$ [13–16]. Recently, the theory of generalized function spaces showed deep concern in many fields of mathematical analysis like, for example, in the field of image processing [17], in the analysis of electrorheological fluids models [18], and in the theory of partial differential equations with nonstandard growth conditions [19].

Besides, Izuku introduced Herz spaces with variable exponent $\dot{K}_q^{\alpha,p(\cdot)}$ in [20, 21]. Later on, Almeida and Drihemn [22] gave a new definition of Herz spaces by taking the exponent alpha as a variable. However, the Herz space having all the exponents as variables was defined and studied in [23]. Morrey–Herz spaces with variable exponent $M\dot{K}_{q,p(\cdot)}^{\alpha,\lambda}$ first appeared in [24]. The ensuing paper [25] made some generalization in the definition of Morrey–Herz spaces given in [24] by replacing the exponent α with $\alpha(\cdot)$. A few important considerations in this regard can be found in [26–29].

Recent advancements in the field of variable exponent function spaces include the development of its weighted theory based on the Muckenhoupt weights [30]. In [31, 32], Cruz-Uribe with different co-authors gave the continuity criteria for Hardy–Littlewood maximal operator M:

$$Mg(z) = \sup_{B:\text{ball},z\in B} \frac{1}{|B|} \int_B \left|g(t)\right| dt,$$

on weighted $L^{p(\cdot)}(w)$ spaces. Equivalence between the boundedness of M on $L^{p(\cdot)}(w)$ and the Muckenhoupt condition was proved by Diening and Hästö in [33]. Izuki and Noi defined the weighted Herz spaces with variable exponents in [34]. However, weighted Morrey–Herz spaces with variable exponents were defined and studied in [35, 36].

The aim of this article is to study the continuity criteria for fractional type Hardy operators on weighted variable exponents Morrey–Herz spaces. It is worth mentioning here that our idea is based on Muckenhoupt theory and on Banach function spaces. We thus extend some results presented in [27]. Also, at an intermediate level, we use the boundedness of fractional integral to control the boundedness of the fractional Hardy operators. The fractional integral can be defined as

$$I_\beta(g)(z) = \int_{\mathbb{R}^n} \frac{g(t)}{|z-t|^{n-\beta}} \, dt.$$

The variable Lebesgue spaces boundedness property of Riesz potential was reported in [37]. On the weighted Herz spaces, the boundedness of fractional integral operator was obtained by Izuki and Noi [34].

The presentation of this paper includes four sections. The next section is full of necessary notations and definitions. In Sect. 3, we furnish key lemmas which are helpful in proving our main results in Sect. 4.

2 Notations and definitions

In the remainder of this article, the letter C will denote a constant whose value may change from line to line. A nonempty set S is considered to be a measurable set in $\mathbb{R}^n$, and χ_S represents the characteristic function of S, whereas $|S|$ represents its Lebesgue measure. Let us first define variable exponent Lebesgue spaces based on the fundamental papers and books [12, 15, 16].

Definition 2.1 Let $p(\cdot):\mathbb{R}^n\to[1,\infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of all measurable functions f such that

$$F_p(f)=\int_{\mathbb{R}^n}\left|f(x)\right|^{p(x)}\,dx<\infty.$$

The space $L^{p(\cdot)}(\mathbb{R}^n)$ turns out to be Banach function space with respect to the norm

$$\|f\|_{L^{p(\cdot)}}=\inf\left\{\sigma>0:F_p\left(\frac{f}{\sigma}\right)=\int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\sigma}\right)^{p(x)}dx\le 1\right\}.$$

Definition 2.2 We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot):\mathbb{R}^n\to(1,\infty)$ such that

$$1<p_-\le p(x)\le p_+<\infty,$$

where

$$p_-:=\operatorname*{essinf}_{x\in\mathbb{R}^n}p(x),\qquad p_+:=\operatorname*{esssup}_{x\in\mathbb{R}^n}p(x).$$

Definition 2.3 Suppose that $p(\cdot)$ is a real-valued function on $\mathbb{R}^n$. We say that

(i) $\mathcal{C}^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$ is the set of all local log-Hölder continuous functions $p(\cdot)$ satisfying

$$\left|p(x)-p(y)\right|\le\frac{-C}{\log(|x-y|)},\quad |x-y|<\frac{1}{2},x,y\in\mathbb{R}^n.$$

(ii) $\mathcal{C}^{\log}_{0}(\mathbb{R}^n)$ is the set of all local log-Hölder continuous functions $p(\cdot)$ satisfying at origin

$$\left|p(x)-p(0)\right|\lesssim\frac{C}{\log(|e+\frac{1}{|x|}|)},\quad x\in\mathbb{R}^n.$$

(iii) $\mathcal{C}^{\log}_{\infty}(\mathbb{R}^n)$ is a set of all log-Hölder continuous functions satisfying at infinity

$$\left|p(x)-p_\infty\right|\le\frac{C_\infty}{\log(e+|x|)},\quad x\in\mathbb{R}^n.$$

(iv) $\mathcal{C}^{\log}(\mathbb{R}^n)=\mathcal{C}^{\log}_{\infty}\cap\mathcal{C}^{\log}_{\mathrm{loc}}$ denotes the set of all global log-Hölder continuous functions $p(\cdot)$.

It was proved in [38] that if $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)\cap\mathcal{C}^{\log}(\mathbb{R}^n)$, then the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Suppose that $w(x)$ is a weight function on $\mathbb{R}^n$, which is a nonnegative and locally integrable function on $\mathbb{R}^n$. Let $L^{p(\cdot)}(w)$ be the space of all complex-valued functions f on $\mathbb{R}^n$ such that $fw^{\frac{1}{p(\cdot)}}\in L^{p(\cdot)}(\mathbb{R}^n)$. The space $L^{p(\cdot)}(w)$ is a Banach function space equipped with the norm

$$\|f\|_{L^{p(\cdot)}(w)}=\left\|fw^{\frac{1}{p(\cdot)}}\right\|_{L^{p(\cdot)}}.$$

Benjamin Muckenhoupt introduced the theory of A_p $(1 < p < \infty)$ weights on $\mathbb{R}^n$ in [30]. Recently, in [34, 39] Izuki and Noi generalized the Muckenhoupt A_p class by taking p as variable.

Definition 2.4 Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. A weight w is an $A_{p(\cdot)}$ weight if

$$\sup_B \frac{1}{|B|} \big\| w^{1/p(\cdot)} \chi_B \big\|_{L^{p(\cdot)}} \big\| w^{-1/p(\cdot)} \chi_B \big\|_{L^{p'(\cdot)}} < \infty.$$

In [40], authors proved that $w \in A_{p(\cdot)}$ if and only if M is bounded on the space $L^{p(\cdot)}$.

Remark 2.5 ([34]) Suppose $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap \mathcal{C}^{\log}(\mathbb{R}^n)$ and $p(\cdot) \le q(\cdot)$, then we have

$$A_1 \subset A_{p(\cdot)} \subset A_{q(\cdot)}.$$

Definition 2.6 Suppose $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\beta \in (0, n)$ such that $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\beta}{n}$. A weight w is said to be $A(p_1(\cdot), p_2(\cdot))$ weight if

$$\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \|\chi_B\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \le C|B|^{1-\frac{\beta}{n}}.$$

Definition 2.7 ([34]) Suppose $p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $\beta \in (0, n)$ such that $\frac{1}{p_2(x)} = \frac{1}{p_1(x)} - \frac{\beta}{n}$. Then $w \in A_{(p_1(\cdot), p_2(\cdot))}$ if and only if $w^{p_2(\cdot)} \in A_{1+p_2(\cdot)/p_1'(\cdot)}$.

It is well known that Herz spaces play an important role in harmonic analysis. After they have been introduced in [41], the theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the characterization of multipliers on Hardy spaces [42], in the summability of Fourier transforms [43], and in regularity theory for elliptic equations in divergence form [44]. For a detailed study of Herz-type spaces, we recommend the reader to see the book [45]. Now, we define variable exponent weighted Morrey–Herz space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)$. Let $B_k = \{x \in \mathbb{R}^n : |x| \le 2^k\}$, $A_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$.

Definition 2.8 Let w be a weight on $\mathbb{R}^n$, $\lambda \in [0, \infty)$, $q \in (0, \infty)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, and $\alpha(\cdot) : \mathbb{R}^n \to \mathbb{R}$ with $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The space $M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)$ is the set of all measurable functions f given by

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w) = \Big\{ f \in L^{p(\cdot)}_{loc}\big(\mathbb{R}^n \setminus \{0\}, w\big) : \|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)} < \infty \Big\},$$

where

$$\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q,p(\cdot)}(w)} = \sup_{k_0 \in Z} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\alpha(\cdot)q} \|f\chi_k\|^q_{L^{p(\cdot)}(w)} \right)^{1/q}.$$

Obviously, $M\dot{K}^{\alpha(\cdot),0}_{q,p(\cdot)}(w) = \dot{K}^{\alpha(\cdot)}_{q,p(\cdot)}(w)$ is the weighted Herz space with variable exponent (see [22]).

3 Key lemmas

We start this section with some useful lemmas that will be helpful in proving our main results.

Lemma 3.1 ([46]) *If X is a Banach function space, then*

(i) *The associated space X' is also a Banach function space.*

(ii) *$\|\cdot\|_{(X')'}$ and $\|\cdot\|_X$ are equivalent.*

(iii) *If $g \in X$ and $f \in X'$, then*

$$\int_{\mathbb{R}^n} \bigl|f(x)g(x)\bigr|\,dx \le \|g\|_X \|f\|_{X'}$$

is the generalized Hölder inequality.

Lemma 3.2 *Suppose that X is a Banach function space, we have that, for all balls B,*

$$1 \le \frac{1}{|B|}\|\chi_B\|_X \|\chi_B\|_{X'}.$$

Lemma 3.3 ([47]) *Consider Banach function space X. Let M be a Hardy–Littlewood maximal operator that is weakly bounded on X, that is,*

$$\|\chi_{\{Mf>\sigma\}}\|_X \lesssim \sigma^{-1}\|f\|_X$$

is true for $\sigma > 0$ and for all $f \in X$. Then we have

$$\sup_{B:\text{ball}} \frac{1}{|B|}\|\chi_B\|_X \|\chi_B\|_{X'} < \infty.$$

Lemma 3.4 ([37])

(1) *$X(\mathbb{R}^n, W)$ is a Banach function space equipped with the norm*

$$\|f\|_{X(\mathbb{R}^n,W)} = \|fW\|_X;$$

(2) *The associate space $X'(\mathbb{R}^n, W^{-1})$ of $X(\mathbb{R}^n, W)$ is also a Banach function space.*

Lemma 3.5 ([34]) *Let X be a Banach function space and M be bounded on X', then there exists a constant $\delta \in (0,1)$ for all $B \subset \mathbb{R}^n$ and $E \subset B$,*

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \le \left(\frac{|E|}{|B|}\right)^{\delta}.$$

The paper [12] shows that $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space and the associated space $L^{p'(\cdot)}(R^n)$ with equivalent norm.

Remark 3.6 Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and by comparing the Lebesgue space $L^{p(\cdot)}(w^{p(\cdot)})$ and $L^{p'(\cdot)}(w^{-p'(\cdot)})$ with the definition of $X(\mathbb{R}^n, W)$, we have:

1. If we take $W = w$ and $X = L^{p(\cdot)}(\mathbb{R}^n)$, then we get $L^{p(\cdot)}(\mathbb{R}^n, w) = L^{p(\cdot)}(w^{p(\cdot)})$.
2. If we consider $W = w^{-1}$ and $X = L^{p'(\cdot)}(\mathbb{R}^n)$, then we have $L^{p'(\cdot)}(w^{-p'(\cdot)}) = L^{p'(\cdot)}(\mathbb{R}^n, w^{-1})$.

By virtue of Lemma 3.4, we get

$$\big(L^{p(\cdot)}(\mathbb{R}^n,w)\big)' = \big(L^{p(\cdot)}(w^{p(\cdot)})\big)' = L^{p'(\cdot)}\big(w^{-p'(\cdot)}\big) = L^{p'(\cdot)}\big(\mathbb{R}^n,w^{-1}\big).$$

Lemma 3.7 ([48]) *Let $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)\cap\mathcal{C}^{\log}(\mathbb{R}^n)$ be a Log Hölder continuous function both at infinity and at origin, if $w^{p_2(\cdot)}\in A_{p_2(\cdot)}$ implies $w^{-p_2'(\cdot)}\in A_{p_2'(\cdot)}$. Thus the Hardy–Littlewood operator is bounded on $L^{p_2'(\cdot)}(w^{-p_2'(\cdot)})$ and there exist constants $\delta_1,\delta_2\in(0,1)$ such that*

$$\frac{\|\chi_E\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}{\|\chi_B\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}} = \frac{\|\chi_E\|_{(L^{p_2'(\cdot)}(w^{-p_2'(\cdot)}))'}}{\|\chi_B\|_{(L^{p_2'(\cdot)}(w^{-p_2'(\cdot)}))'}} \lesssim \left(\frac{|E|}{|B|}\right)^{\delta_1} \tag{3}$$

and

$$\frac{\|\chi_E\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_B\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}} \lesssim \left(\frac{|E|}{|B|}\right)^{\delta_2}, \tag{4}$$

for all balls B and all measurable sets $E\subset B$.

Lemma 3.8 ([34]) *Let $p_1(\cdot)\in\mathcal{P}(\mathbb{R}^n)\cap\mathcal{C}^{\log}(\mathbb{R}^n)$ and $0<\beta<\frac{n}{p_{1+}}$, and $\frac{1}{p_2(\cdot)}=\frac{1}{p_1(\cdot)}-\frac{\beta}{n}$. If $w\in A(p_1(\cdot),p_2(\cdot))$, then I^β is bounded from $L^{p_1(\cdot)}(w^{p_1(\cdot)})$ to $L^{p_2(\cdot)}(w^{p_2(\cdot)})$.*

4 Main results and proofs

The following proposition was proved in [36].

Proposition 4.1 *Let $q(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, $0<p<\infty$ and $0\le\lambda<\infty$. If $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{C}^{\log}(\mathbb{R}^n)$, then*

$$\begin{aligned}
\|f\|^p_{M\dot{K}^{\alpha(\cdot),\lambda}_{p,q(\cdot)}(w^{q(\cdot)})} &= \sup_{k_0\in Z}2^{-k_0\lambda p}\sum_{j=-\infty}^{k_0}2^{j\alpha(\cdot)p}\|f\chi_j\|^p_{L^{q(\cdot)}(w^{q(\cdot)})} \\
&\le \max\Bigg\{\sup_{\substack{k_0\in Z\\k_0<0}}2^{-k_0\lambda p}\Bigg(\sum_{j=-\infty}^{k_0}2^{j\alpha(0)p}\|f\chi_j\|^p_{L^{q(\cdot)}(w^{q(\cdot)})}\Bigg), \\
&\quad \sup_{\substack{k_0\in Z\\k_0\ge0}}\Bigg(2^{-k_0\lambda p}\Bigg(\sum_{j=-\infty}^{-1}2^{j\alpha(0)p}\|f\chi_j\|^p_{L^{q(\cdot)}(w^{q(\cdot)})}\Bigg) \\
&\quad + 2^{-k_0\lambda p}\Bigg(\sum_{j=0}^{k_0}2^{j\alpha(\infty)p}\|f\chi_j\|^p_{L^{q(\cdot)}(w^{q(\cdot)})}\Bigg)\Bigg)\Bigg\}.
\end{aligned}$$

One of the main results of this study is as follows.

Theorem 4.2 *Let $0<q_1\le q_2<\infty$, $p_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)\cap\mathcal{C}^{\log}(\mathbb{R}^n)$, and $p_1(\cdot)$ be such that $\frac{1}{p_2(\cdot)}=\frac{1}{p_1(\cdot)}-\frac{\beta}{n}$. Also, let $w^{p_2(\cdot)}\in A_1$, $\lambda>0$ and $\alpha(\cdot)\in L^\infty(\mathbb{R}^n)\cap\mathcal{C}^{\log}(\mathbb{R}^n)$ be log Hölder continuous at the origin, with $\alpha(0)\le\alpha(\infty)<\lambda+n\delta_2-\beta$, where $\delta_2\in(0,1)$ is the constant appearing in (4), then*

$$\|H_\beta f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})}\le C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.$$

Proof For any $f \in M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})$, if we represent $f_j = f \cdot \chi_j = f \cdot \chi_{A_j}$ for each $j \in \mathbb{Z}$, then we write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x) \cdot \chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

The generalized Hölder inequality yields

$$\begin{aligned} \left|H_\beta f(x) \cdot \chi_k(x)\right| &\leq \frac{1}{|x|^{n-\beta}} \int_{B_k} \left|f(t)\right| dt \cdot \chi_k(x) \\ &\leq C2^{-kn} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} 2^{k\beta} \chi_k(x). \end{aligned} \tag{5}$$

Making use of Lemmas 3.3 and 3.7, respectively, we obtain

$$\begin{aligned} &\|H_\beta f \cdot \chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ &\quad\leq C2^{k\beta} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} 2^{-kn} \|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ &\quad\leq C2^{k\beta} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_k\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'} \\ &\quad\leq C2^{k\beta} \sum_{j=-\infty}^{k} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_j\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'} \frac{\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}}{\|\chi_k\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}} \\ &\quad\leq C2^{k\beta} \sum_{j=-\infty}^{k} 2^{n\delta_2(j-k)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_j\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}. \end{aligned} \tag{6}$$

To proceed further, we take $f = \chi_{B_j}$ in the definition of I_β to get

$$I_\beta(\chi_{B_j})(x) \geq C2^{j\beta} \chi_{B_j}(x),$$

which implies that

$$\chi_{B_j}(x) \leq C2^{-j\beta} I_\beta(\chi_{B_j})(x).$$

Taking the norm on both sides and using Lemmas 3.8 and 3.3, respectively, we get

$$\begin{aligned} \|\chi_{B_j}\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\leq C2^{-j\beta} \left\|I_\beta(\chi_{B_j})\right\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ &\leq C2^{-j\beta} \left\|(\chi_{B_j})\right\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \\ &\leq 2^{j(n-\beta)} \left\|(\chi_{B_j})\right\|^{-1}_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'}. \end{aligned} \tag{7}$$

Inserting (7) into (6), we are down to

$$\begin{aligned}
&\|H_\beta f\cdot\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\quad\le C2^{k\beta}\sum_{j=-\infty}^{k}2^{n\delta_2(j-k)}2^{j(n-\beta)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\|\chi_j\|^{-1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|^{-1}_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\\
&\quad= C\sum_{j=-\infty}^{k}2^{(\beta-n\delta_2)(k-j)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\bigl(2^{-jn}\|\chi_j\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\|\chi_j\|_{(L^{p_2(\cdot)}(w^{p_2(\cdot)}))'}\bigr)^{-1}\\
&\quad\le C\sum_{j=-\infty}^{k}2^{(\beta-n\delta_2)(k-j)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}. \qquad (8)
\end{aligned}$$

In the rest of the proof, in order to estimate $\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}$, we consider two cases as below.

Case 1: We take $j < 0$ and start estimating as follows:

$$\begin{aligned}
\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} &= 2^{-j\alpha(0)}\bigl(2^{j\alpha(0)q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\bigr)^{\frac{1}{q_1}}\\
&\le 2^{-j\alpha(0)}\left(\sum_{i=-\infty}^{j}2^{i\alpha(0)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}}\\
&\le 2^{j(\lambda-\alpha(0))}2^{-j\lambda}\left(\sum_{i=-\infty}^{j}2^{i\alpha(\cdot)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}}\\
&\le C2^{j(\lambda-\alpha(0))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}. \qquad (9)
\end{aligned}$$

Case 2: For $j \ge 0$, we get

$$\begin{aligned}
\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} &= 2^{-j\alpha(\infty)}\bigl(2^{j\alpha(\infty)q_1}\|f_j\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\bigr)^{\frac{1}{q_1}}\\
&\le 2^{-j\alpha(\infty)}\left(\sum_{i=0}^{j}2^{i\alpha(\infty)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}}\\
&\le 2^{j(\lambda-\alpha(\infty))}2^{-j\lambda}\left(\sum_{i=-\infty}^{j}2^{i\alpha(\cdot)q_1}\|f_i\|^{q_1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\right)^{\frac{1}{q_1}}\\
&\le C2^{j(\lambda-\alpha(\infty))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}. \qquad (10)
\end{aligned}$$

By the definition of variable exponent Morrey–Herz space along with the use of Proposition 4.1, we arrive at the following inequality:

$$\begin{aligned}
\|H_\beta f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})} &= \sup_{k_0\in Z}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{k\alpha(\cdot)q_1}\|H_\beta f\cdot\chi_k\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\\
&\le \max\Biggl\{\sup_{\substack{k_0\in Z\\ k_0<0}}2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0}2^{k\alpha(0)q_1}\|H_\beta f\cdot\chi_k\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})},
\end{aligned}$$

$$\begin{aligned} &\sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda q_1}\Bigg(\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_1}\|H_\beta f\cdot\chi_k\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\ &\quad + \sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1}\|H_\beta f\cdot\chi_k\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}\Bigg)\Bigg\} \\ &= \max\{Y_1, Y_2+Y_3\}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} Y_1 &= \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1}\|H_\beta f\cdot\chi_k\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}, \\ Y_2 &= \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_1}\|H_\beta f\cdot\chi_k\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}, \\ Y_3 &= \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda q_1}\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1}\|H_\beta f\cdot\chi_k\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}. \end{aligned}$$

First, we approximate Y_1. Since $\alpha(0)\leq\alpha(\infty)<n\delta_2+\lambda-\beta$,

$$\begin{aligned} Y_1 &\leq C\sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1}\Bigg(\sum_{j=-\infty}^{k} 2^{(\beta-n\delta_2)(k-j)}\|f\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg)^{q_1} \\ &\leq C\sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1}\Bigg(\sum_{j=-\infty}^{k} 2^{(\beta-n\delta_2)(k-j)}2^{j(\lambda-\alpha(0))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg)^{q_1} \\ &\leq C\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1}\Bigg(\sum_{j=-\infty}^{k} 2^{(\beta-n\delta_2)(k-j)}2^{j(\lambda-\alpha(0))}\Bigg)^{q_1} \\ &\leq C\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda q_1}\sum_{k=-\infty}^{k_0} 2^{k\lambda q_1}\Bigg(\sum_{j=-\infty}^{k} 2^{(j-k)(-\beta+n\delta_2-\alpha(0)+\lambda)}\Bigg)^{q_1} \\ &\leq C\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}. \end{aligned}$$

The estimate of Y_2 is similar to that of Y_1. Lastly, we estimate Y_3

$$\begin{aligned} Y_3 &\leq C\sup_{\substack{k_0\in Z\\ K_0\geq 0}} 2^{-k_0\lambda q_1}\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1}\Bigg(\sum_{j=-\infty}^{k} 2^{(\beta-n\delta_2)(k-j)}\|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg)^{q_1} \\ &\leq C\sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda q_1}\sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1}\Bigg(\sum_{j=-\infty}^{k} 2^{(\beta-n\delta_2)(k-j)}2^{j(\lambda-\alpha(\infty))}\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}\Bigg)^{q_1} \end{aligned}$$

$$
\begin{aligned}
&\le C\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})} \sup_{\substack{k_0\in Z\\ k_0\ge 0}} 2^{-k_0\lambda q_1} \sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1} \left(\sum_{j=-\infty}^{k} 2^{(\beta-n\delta_2)(k-j)} 2^{j(\lambda-\alpha(\infty))} \right)^{q_1} \\
&\le C\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})} \sup_{\substack{k_0\in Z\\ k_0\ge 0}} 2^{-k_0\lambda q_1} \sum_{k=0}^{k_0} 2^{k\lambda q_1} \left(\sum_{j=-\infty}^{k} 2^{(j-k)(-\beta+n\delta_2-\alpha(\infty)+\lambda)} \right)^{q_1} \\
&\le C\|f\|^{q_1}_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.
\end{aligned}
$$

The desired result is obtained by inserting the approximations of Y_1, Y_2, and Y_3 into (11). □

Theorem 4.3 *Let $q_1, q_2, p_1(\cdot), p_2(\cdot), \beta, \alpha(\cdot)$ and w be as in Theorem 4.2. In addition, if $-n\delta_1 + \lambda < \alpha(0) \le \alpha(\infty)$, where $\delta_1 \in (0,1)$ is the constant appearing in (3), then*

$$
\|H^*_\beta f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_2,p_2(\cdot)}(w^{p_2(\cdot)})} \le C\|f\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q_1,p_1(\cdot)}(w^{p_1(\cdot)})}.
$$

Proof An application of the Hölder inequality gives

$$
\begin{aligned}
\left|H^*_\beta f(x)\cdot\chi_k(x)\right| &\le \int_{R^n\setminus B_k} \left|f(t)\|x\right|^{\beta-n} dt \cdot \chi_k(x) \\
&\le C\sum_{j=k+1}^{\infty} 2^{j(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \chi_k(x).
\end{aligned}
$$

Now, using Lemma 3.3, we have

$$
\begin{aligned}
\left\|H^*_\beta f\cdot\chi_k\right\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C\sum_{j=k+1}^{\infty} 2^{j(\beta-n)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|_{(L^{p_1(\cdot)}(w^{p_1(\cdot)}))'} \|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} \\
&\le C\sum_{j=k+1}^{\infty} 2^{j\beta} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_j\|^{-1}_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}. \qquad (12)
\end{aligned}
$$

In view of inequality (7), we obtain

$$
\begin{aligned}
\left\|H^*_\beta f\cdot\chi_k\right\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} &\le C\sum_{j=k+1}^{\infty} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})} \frac{\|\chi_k\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}}{\|\chi_j\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}} \\
&\le C\sum_{j=k+1}^{\infty} 2^{n\delta_1(k-j)} \|f_j\|_{L^{p_1(\cdot)}(w^{p_1(\cdot)})}, \qquad (13)
\end{aligned}
$$

where we used Lemma 3.7 in the last step.

In the remaining proof of this theorem, we follow the procedure as in Theorem 4.2 to have

$$
\left\|H^*_\beta f\cdot\chi_k\right\|_{L^{p_2(\cdot)}(w^{p_2(\cdot)})} = \max\{Z_1, Z_2 + Z_3\}, \qquad (14)
$$

where

$$Z_1 = \sup_{\substack{k_0\in Z\\ k_0<0}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{k_0} 2^{k\alpha(0)q_1} \left\| H^*_\beta f \cdot \chi_k \right\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

$$Z_2 = \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda q_1} \sum_{k=-\infty}^{-1} 2^{k\alpha(0)q_1} \left\| H^*_\beta f \cdot \chi_k \right\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

$$Z_3 = \sup_{\substack{k_0\in Z\\ k_0\geq 0}} 2^{-k_0\lambda q_1} \sum_{k=0}^{k_0} 2^{k\alpha(\infty)q_1} \left\| H^*_\beta f \cdot \chi_k \right\|^{q_1}_{L^{p_2(\cdot)}(w^{p_2(\cdot)})}.$$

The estimates of Z_i $(i = 1,2,3)$ are similar to those of Y_i $(i = 1,2,3)$ of Theorem 4.2. Here we conclude our result. □

Acknowledgements
The authors would like to thank the referees for careful reading of the paper and valuable suggestions. This research is supported by the Higher Education Commission (HEC) of Pakistan through the National Research Program for Universities (NRPU) Project No: 7098/Federal/NRPU/R&D/HEC/2017 and the Quaid-I-Azam University Research Fund (URF) Project.

Funding
No specific funding has been received for this work.

Availability of data and materials
Data sharing not applicable to this article as no data-sets were generated or analyzed during the current study.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MA, AH: Formal analysis, MA, AH: Investigation, NS: Resources, AH: Supervision All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Quaid-I-Azam University 45320, Islamabad 44000, Pakistan. [2]Department of Mathematics, University of Kotli, Azad Jammu and Kashmir, Pakistan.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 5 July 2021 Accepted: 7 December 2021 Published online: 03 January 2022

References
1. Hardy, G.H.: Note on a theorem of Hilbert. Math. Z. **6**, 314–317 (1920)
2. Faris, W.G.: Weak Lebesgue spaces and quantum mechanical binding. Duke Math. J. **43**, 365–373 (1976)
3. Chris, M., Grafakos, L.: Best constant for two nonconvolution inequalities. Math. Z. **123**, 1687–1693 (1995)
4. Fu, Z.W., Liu, Z.G., Lu, S.Z., Wong, H.: Characterization for commutators of *n*-dimensional fractional Hardy operators. Sci. China Ser. A **50**(10), 1418–1426 (2007)
5. Bliss, A.G.: An integral inequality. J. Lond. Math. Soc. **5**, 40–46 (1930)
6. Fu, Z.W., Grafakos, L., Lu, S.Z., Zhao, F.Y.: Sharp bounds for *m*-linear Hardy and Hilbert operators. Houst. J. Math. **38**(1), 225–244 (2012)
7. Persson, L.-E., Samko, S.G.: A note on the best constants in some Hardy inequalities. J. Math. Inequal. **9**(2), 437–447 (2015)
8. Hussain, A., Sarfraz, N.: Optimal weak type estimates for *p*-adic Hardy operator. p-Adic Numb. Ultrametric. Anal. Appl. **12**(1), 12–21 (2020)
9. Mizuta, Y., Nekvinda, A., Shimomura, T.: Optimal estimates for the fractional Hardy operator. Stud. Math. **227**(1), 1–19 (2015)
10. Yee, T.-L., Ho, K.-P.: Hardy's inequalities and integral operators on Herz-Morrey spaces. Open Math. **18**, 106–121 (2020)
11. Orlicz, W.: Uber konjugierete exponentenfolgen. Stud. Math. **3**, 200–212 (1931)

12. Kováčik, O., Rákosník, J.: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslov. Math. J. **41**, 592–618 (1991)
13. Cruz-Uribe, D., Fiorenza, A., Martell, J.M., Pérez, C.: The boundedness of classical operators on variable L^p spaces. Ann. Acad. Sci. Fenn., Math. **31**(1), 239–264 (2006)
14. Cruz-Uribe, D., Diening, L., Fiorenza, A.: A new proof of the boundedness of maximal operators on variable Lebesgue spaces. Boll. Unione Mat. Ital. **2**(1), 151–173 (2009)
15. Cruz-Uribe, D., Fiorenza, D.V.: Variable Lebesgue Spaces: Foundations and Harmonic Analysis. Appl. and Numerical Harmonic Anal.. Springer, Heidelberg (2013)
16. Diening, L., Harjulehto, L., Hästö, P., Rȕžicka, P.: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics, vol. 2017. Springer, Heidelberg (2011)
17. Chen, Y., Levine, S., Rao, M.: Variable exponent, linear growth functionals in image restoration. Appl. Math. **6**, 1383–1406 (2006)
18. Ruzicka, M.: Electrorheological Fluids, Modeling and Mathematical Theory. Lectures Notes in Math., vol. 1748. Springer, Berlin (2000)
19. Harjulehto, P., Hasto, P., Le, U.V., Nuortio, M.: Overview of differential equations with non-standard growth. Nonlinear Anal. **72**(12), 4551–4574 (2010)
20. Izuki, M.: Herz and amalgam spaces with variable exponent, the Haar wavelets and greediness of the wavelet system. East J. Approx. **15**, 87–109 (2009)
21. Izuki, M.: Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization. Anal. Math. **13**, 33–50 (2010)
22. Almeida, A., Drihem, D.: Maximal, potential and singular type operators on Herz spaces with variable exponents. J. Math. Anal. Appl. **394**, 781–795 (2012)
23. Samko, S.: Variable exponent Herz spaces. Mediterr. J. Math. **10**, 2007–2025 (2013)
24. Izuki, M.: Boundedness of vector-valued sublinear operators on Herz-Morrey spaces with variable exponent. Math. Sci. Res. J. **13**, 243–253 (2009)
25. Dong, B.H., Xu, J.S.: Herz-Morrey type Besov and Triebel-Lizorkin spaces with variable exponents. Banach J. Math. Anal. **9**, 75–101 (2015)
26. Hussain, A., Gao, G.: Multilinear singular integrals and commutators on Herz space with variable exponent. ISRN Math. Anal. **2014**, 1–10 (2014)
27. Wu, J.L., Zhao, W.J.: Boundedness for fractional Hardy-type operator on variable-exponent Herz-Morrey spaces. Kyoto J. Math. **56**, 831–845 (2016)
28. Meskhi, A., Rafeiro, H., Zaighum, M.A.: Central Calderón-Zygmund operators on Herz-type Hardy spaces of variable smoothness and integrability. Ann. Funct. Anal. **9**(3), 310–321 (2018)
29. Ho, K.-P.: Extrapolation to Herz spaces with variable exponents and applications. Rev. Mat. Complut. **33**, 437–463 (2020)
30. Muckenhoupt, B.: Weighted norm inequalities for the Hardy maximal function. Trans. Am. Math. Soc. **165**, 207–226 (1972)
31. Cruz-Uribe, D., Diening, L., Hästö, P.: The maximal operator on weighted variable Lebesgue spaces. Fract. Calc. Appl. Anal. **14**, 361–374 (2011)
32. Cruz-Uribe, D., Fiorenza, A., Neugebauer, C.J.: Weighted norm inequalities for the maximal operator on variable Lebesgue spaces. J. Math. Anal. Appl. **394**, 744–760 (2012)
33. Diening, L., Hästö, P.: Muckenhoupt weights in variable exponent spaces. https://www.problemsolving.fi/pp/p75_submit.pdf. Preprint, available at
34. Izuki, M., Noi, T.: Boundedness of fractional integrals on weighted Herz spaces with variable exponent. J. Inequal. Appl. **2016**, 199 (2016)
35. Wang, L., Shu, L.: Boundedness of some sublinear operators on weighted variable Herz-Morrey spaces. J. Math. Inequal. **12**, 31–42 (2018)
36. Wang, S.R., Xu, J.S.: Weighted norm inequality for bilinear Calderón-Zygmund operators on Herz-Morrey spaces with variable exponents. J. Inequal. Appl. **2019**(2019), 251 (2019)
37. Karlovich, A.Y., Spitkovsky, I.M.: The Cauchy singular integral operator on weighted variable Lebesgue spaces. In: Concrete Operators, Spectral Theory, Operators in Harmonic Analysis and Approximation. Oper. Theory Adv. Appl., vol. 236, pp. 275–291. Springer, Basel (2014)
38. Cru-Uribe, D., Fiorenza, A., Neugebauer, C.: The maximal function on variable L^p spaces. Ann. Acad. Sci. Fenn., Math. **28**, 223–238 (2003)
39. Izuki, M., Noi, T.: An intrinsic square function on weighted Herz spaces with variable exponent. J. Math. Inequal. **11**, 799–816 (2017)
40. Cru-Uribe, D., Fiorenza, A., Neugebauer, C.: Weighted norm inequalities for the maximal operator on variable Lebesgue spaces. J. Math. Anal. Appl. **394**, 744–760 (2012)
41. Herz, C.: Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms. J. Math. Mech. **18**, 283–324 (1968)
42. Baernstein II, A., Sawyer, E.T.: Embedding and multiplier theorems for $H^p(\mathbb{R}^n)$. Mem. Am. Math. Soc. **53**, 318 (1985)
43. Feichtinger, H.G., Weisz, F.: Herz spaces and summability of Fourier transforms. Math. Nachr. **281**, 309–324 (2008)
44. Ragusa, M.A.: Homogeneous Herz spaces and regularity results. Nonlinear Anal. **71**, 1909–1914 (2009)
45. Lu, S., Yang, D., Hu, G.: Herz Type Spaces and Their Applications. Science Press, Beijing (2008)
46. Bennett, C., Sharpley, R.: Interpolation of Operators. Academic Press, Boston (1988)
47. Izuki, M.: Remarks on Muckenhoupt weights with variable exponent. J. Anal. Appl. **11**, 27–41 (2013)
48. Izuki, M., Noi, T.: Two weighted Herz space variable exponent. Bull. Malays. Math. Soc. **43**, 169–200 (2020)